

# On the relevance of stochastic models in turbulence

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# Obstacle problem

## Fluid domain and obstacle

$$Q = R^d \setminus B, \quad d = 2, 3$$

$B$  compact, convex

## Navier–Stokes system

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$p(\varrho) \approx a\varrho^\gamma, \quad \gamma > 1, \quad \mathbb{S} = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \operatorname{div}_x \mathbf{u} \mathbb{I},$$

## Boundary and far field conditions

$$\mathbf{u}|_{\partial Q} = 0, \quad \varrho \rightarrow \varrho_\infty, \quad \mathbf{u} \rightarrow \mathbf{u}_\infty \quad \text{as } |x| \rightarrow \infty$$

# High Reynolds number (vanishing viscosity) limit

## Vanishing viscosity

$$\varepsilon_n \searrow 0, \mu_n = \varepsilon_n \mu, \mu > 0, \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

## Questions

- Identify the limit of the corresponding solutions  $(\varrho_n, \mathbf{u}_n)$  as  $n \rightarrow \infty$  in the fluid domain  $Q$
- **Yakhot and Orszak [1986]:** *“The effect of the boundary in the turbulence regime can be modeled in a **statistically equivalent way** by fluid equations driven by stochastic forcing”*

Clarify the meaning of “statistically equivalent way”

Is the (compressible) Euler system driven by a general cylindrical white noise force adequate to describe the limit of  $(\varrho_n, \mathbf{u}_n)$ ?

## Bounded energy solutions

(Relative) energy

$$E(\varrho, \mathbf{u} \mid \varrho_\infty, \mathbf{u}_\infty) = \frac{1}{2} \varrho |\mathbf{u} - \mathbf{u}_\infty|^2 + P(\varrho) - P'(\varrho_\infty)(\varrho - \varrho_\infty) - P(\varrho_\infty)$$

$$P(\varrho) = \frac{a}{\gamma - 1} \varrho^\gamma, \quad \mathbf{u}_\infty = 0 \text{ for } |x| < R_1, \quad \mathbf{u}_\infty = \mathbf{u}_\infty \text{ for } |x| > R_2$$

Energy inequality

$$\begin{aligned} & \frac{d}{dt} \int_Q E(\varrho, \mathbf{u} \mid \varrho_\infty, \mathbf{u}_\infty) \, dx + \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, dx \\ & \leq - \int_Q (\varrho \mathbf{u} \otimes \mathbf{u} + p(\varrho) \mathbb{I}) : \nabla_x \mathbf{u}_\infty \, dx + \frac{1}{2} \int_Q \varrho \mathbf{u} \cdot \nabla_x |\mathbf{u}_\infty|^2 \, dx \\ & \quad + \int_Q \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}_\infty \, dx. \end{aligned}$$

# Statistical limit

## Energy bounds

$$\mathbf{m} \equiv \varrho \mathbf{u}$$

$$\frac{1}{N} \sum_{n=1}^N \left[ \sup_{0 \leq \tau \leq T} \int_Q E(\varrho_n, \mathbf{m}_n | \varrho_\infty, \mathbf{u}_\infty)(\tau, \cdot) dx + \varepsilon_n \int_0^T \int_Q \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n dx dt \right] \leq \bar{\mathcal{E}}$$

uniformly for  $N \rightarrow \infty$

## Trajectory space

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} \equiv C_{\text{weak}}([0, T]; L_{\text{loc}}^\gamma(Q) \times L_{\text{loc}}^{\frac{2\gamma}{\gamma+1}}(Q; R^d))$$

## Statistical limit

$$\mathcal{V}_N = \frac{1}{N} \sum_{n=1}^N \delta_{(\varrho_n, \mathbf{m}_n)}, \quad \mathbf{m}_n = \varrho_n \mathbf{u}_n$$

**Prokhorov theorem**  $\Rightarrow \mathcal{V}_N \rightarrow \mathcal{V}$  narrowly in  $\mathfrak{P}[\mathcal{T}]$

$(\varrho, \mathbf{m}) \approx \mathcal{V}$  a random process with paths in  $\mathcal{T}$

# Limit problem

## Statistical dissipative solutions to the Euler system

$$\begin{aligned}\partial_t \varrho + \operatorname{div}_x \mathbf{m} &= 0 \\ \partial_t \mathbf{m} + \operatorname{div}_x \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) &= -\operatorname{div}_x \mathfrak{R}\end{aligned}$$

$\mathcal{V}$  a.s.

## Reynolds stress

$$\mathfrak{R} \in L_{\text{weak}-(*)}^{\infty}(0, T; \mathcal{M}^+(Q; R_{\text{sym}}^{d \times d}))$$

$$\mathfrak{R} : (\xi \otimes \xi) \geq 0, \quad \xi \in R^d$$

$$\mathbb{E} \left[ \int_0^T \psi \int_Q \varphi \, d \operatorname{trace}[\mathfrak{R}] dt \right] \leq c \bar{\mathcal{E}} \|\psi\|_{L^1(0, T)} \|\varphi\|_{BC(Q)}$$

# Reynolds stress

## Skorokhod–Jakubowski representation theorem

$$\varrho_N \approx \tilde{\varrho}_N, \mathbf{m}_N \approx \tilde{\mathbf{m}}_N \text{ (equivalence in law)}$$

a.s. weak convergence

$$(\tilde{\varrho}_N, \tilde{\mathbf{m}}_N) \rightarrow (\varrho, \mathbf{m}) \text{ in } C_{\text{weak}}([0, T]; L_{\text{loc}}^\gamma(Q) \times L_{\text{loc}}^{\frac{2\gamma}{\gamma+1}}(Q; R^d))$$

$$\frac{\tilde{\mathbf{m}}_N \otimes \tilde{\mathbf{m}}_N}{\tilde{\varrho}_N} + p(\varrho_N)\mathbb{I} \rightarrow \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} + p(\varrho)\mathbb{I}$$

$$\text{weakly-} (*) \text{ in } L_{\text{weak-} (*)}^\infty(0, T; \mathcal{M}(Q; R_{\text{sym}}^{d \times d}))$$

## Reynolds stress

$$\mathfrak{R} \equiv \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} + p(\varrho) - \left( \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p(\varrho)\mathbb{I} \right)$$

$$\text{convexity of } (\varrho, \mathbf{m}) \mapsto \left( \frac{|\mathbf{m} \cdot \xi|^2}{\varrho} + p(\varrho)|\xi|^2 \right) \Rightarrow \mathfrak{R} : (\xi \otimes \xi) \geq 0$$

# Stochastic Euler system

Euler system with stochastic forcing

$$\begin{aligned}d\tilde{\varrho} + \operatorname{div}_x \tilde{\mathbf{m}} dt &= 0 \\d\tilde{\mathbf{m}} + \operatorname{div}_x \left( \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right) dt + \nabla_x p(\tilde{\varrho}) dt &= \mathbf{F} dW\end{aligned}$$

$W = (W_k)_{k \geq 1}$  cylindrical Wiener process

$\mathbf{F} = (\mathbf{F}_k)_{k \geq 1}$  – diffusion coefficient

$$\mathbb{E} \left[ \int_0^T \sum_{k \geq 1} \|\mathbf{F}_k\|_{W^{-\ell, 2}(Q; \mathbb{R}^d)}^2 dt \right] < \infty$$

we allow  $\mathbf{F} = \mathbf{F}(\varrho, \mathbf{m})$



# Statistical equivalence

statistical equivalence  $\Leftrightarrow$  identity in expectation of some quantities

$(\varrho, \mathbf{m})$  statistically equivalent to  $(\tilde{\varrho}, \tilde{\mathbf{m}})$

$\Leftrightarrow$

## ■ density and momentum

$$\mathbb{E} \left[ \int_D \varrho \right] = \mathbb{E} \left[ \int_D \tilde{\varrho} \right], \quad \mathbb{E} \left[ \int_D \mathbf{m} \right] = \mathbb{E} \left[ \int_D \tilde{\mathbf{m}} \right]$$

## ■ kinetic and internal energy

$$\mathbb{E} \left[ \int_D \frac{|\mathbf{m}|^2}{\varrho} \right] = \mathbb{E} \left[ \int_D \frac{|\tilde{\mathbf{m}}|^2}{\tilde{\varrho}} \right], \quad \mathbb{E} \left[ \int_D \rho(\varrho) \right] = \mathbb{E} \left[ \int_D \rho(\tilde{\varrho}) \right]$$

## ■ angular energy

$$\mathbb{E} \left[ \int_D \frac{1}{\varrho} (\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m} \right] = \mathbb{E} \left[ \int_D \frac{1}{\tilde{\varrho}} (\mathbb{J}_{x_0} \cdot \tilde{\mathbf{m}}) \cdot \tilde{\mathbf{m}} \right]$$

$$D \subset (0, T) \times Q, \quad x_0 \in R^d, \quad \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

## Results

### Hypothesis:

$(\varrho, \mathbf{m})$  statistically equivalent to a solution of the stochastic Euler system  $(\tilde{\varrho}, \tilde{\mathbf{m}})$

### Conclusion:

- **Noise inactive**

$\mathfrak{R} = 0$ ,  $(\varrho, \mathbf{m})$  is a statistical solution to a **deterministic** Euler system

- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N} \sum_{n=1}^N b(\varrho_n, \mathbf{m}_n) \rightarrow \mathbb{E} [b(\varrho, \mathbf{m})] \text{ strongly in } L^1_{\text{loc}}((0, T) \times Q)$$

for any  $b \in C_c(R^{d+1})$ ,  $\varphi \in C_c^\infty((0, T) \times Q)$

- **Conditional statistical convergence**

barycenter  $(\bar{\varrho}, \bar{\mathbf{m}}) \equiv \mathbb{E}[(\varrho, \mathbf{m})]$  solves the Euler system

$\Rightarrow$

$$\frac{1}{N} \# \left\{ n \leq N \mid \|\varrho_n - \bar{\varrho}\|_{L^\gamma(K)} + \|\mathbf{m}_n - \bar{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K; R^d)} > \varepsilon \right\} \rightarrow 0$$

as  $N \rightarrow \infty$  for any  $\varepsilon > 0$ , and any compact  $K \subset [0, T] \times Q$

## Main ideas

- Use statistical equivalence of  $(\varrho, \mathbf{m})$  to  $(\tilde{\varrho}, \tilde{\mathbf{m}})$  and the fact that the Itô integral is a martingale to obtain the identity

$$\mathbb{E} [\operatorname{div}_x \mathfrak{R}] = \mathbb{E} \left[ \operatorname{div}_x \left( \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) \right] \quad (1)$$

in  $\mathcal{D}'((0, T) \times Q)$

- Show that if  $Q$  is exterior to a ball and  $(\varrho, \mathbf{m})$  statistically equivalent to  $(\tilde{\varrho}, \tilde{\mathbf{m}})$ , then

$$\mathfrak{R} = 0 \text{ a.s.}$$

Hint: Use test functions of the form

$$\phi_L(x) = \chi \left( \frac{|x|}{L} \right) \nabla_x F(|x|^2), \quad \phi \in C_c^1(Q), \quad L \geq 1$$

$$\chi \in C_c^\infty[0, \infty), \quad \chi(Z) = 1 \text{ for } Z \leq 1, \quad \chi(Z) = 0 \text{ for } Z \geq 2$$

$$F \text{ convex, } F(Z) = 0 \text{ for } 0 \leq Z \leq R^2, \quad 0 < F'(Z) \leq \bar{F} \text{ for } R^2 < Z < R^2 + 1$$

$$F'(Z) = \bar{F} \text{ if } Z \geq R^2 + 1,$$

and let  $L \rightarrow \infty$  to conclude  $\mathbb{E} \left[ \int_0^T \int_Q \operatorname{tr}[\mathfrak{R}] \right] = 0$

- Extend the result to  $Q = R^d \setminus B$ ,  $B$  compact, convex.

# Stratonovich drift

## Stochastic Euler system

$$\begin{aligned}d\tilde{\varrho} + \operatorname{div}_x \tilde{\mathbf{m}} dt &= 0 \\d\tilde{\mathbf{m}} + \operatorname{div}_x \left( \frac{\tilde{\mathbf{m}} \otimes \tilde{\mathbf{m}}}{\tilde{\varrho}} \right) dt + \nabla_x \rho(\tilde{\varrho}) dt &= \boxed{(\sigma \cdot \nabla_x) \tilde{\mathbf{m}} \circ dW_1} + \mathbf{F} dW_2\end{aligned}$$

## Additional hypotheses

- $Q = R^d$
- If  $d = 2$ , we need  $\varrho_\infty = 0$ ; if  $d = 3$ , we need  $\varrho_\infty = 0$ ,  $\mathbf{u}_\infty = 0$ , and  $1 < \gamma \leq 3$

Similar type of noise used recently by Flandoli et al to produce a regularizing effect in the incompressible Navier–Stokes system

# Conclusion

- Stochastically driven Euler system **irrelevant** in the description of compressible turbulence (slightly extrapolated statement)

## Possible scenarios:

- **Oscillatory limit.** The sequence  $(\varrho_n, \mathbf{m}_n)$  generates a Young measure. Its barycenter (weak limit of  $(\varrho_n, \mathbf{m}_n)$ ) **is not** a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is **compatible** with the hypothesis that the limit is independent of the choice of  $\varepsilon_n \searrow 0 \Rightarrow$  computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario **is not compatible** with the hypothesis that the limit is independent of  $\varepsilon_n \searrow 0$  ( $\Rightarrow$  numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.