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Abstract

We propose a stochastic collocation method based on the piecewise constant interpolation on the probability space combined with a finite volume method to solve the compressible Navier–Stokes system at the nodal points. We show convergence of numerical solutions to a statistical solution of the Navier–Stokes system on condition that the numerical solutions are bounded in probability. The analysis uses the stochastic compactness method based on the Skorokhod/Jakubowski representation theorem and the criterion of convergence in probability due to Gyöngy and Krylov.

Keywords: stochastic collocation, multi–element probabilistic collocation method, random compressible Navier–Stokes system, finite volume method, random solution

1 Introduction

Mathematical models arising in science and engineering inherit several sources of uncertainties, such as model parameters, initial and/or boundary conditions. Prominent examples are the fluid

flow models in meteorology, where the initial distribution of the pressure (density), the velocity and the temperature, and even some rheological parameters, for instance the transport coefficients, may be viewed as random data, see e.g. [8]. Consequently, in order to predict reliable results, deterministic models are insufficient and more sophisticated methods are needed to analyse the influence of uncertainties on numerical solutions. In the recent years a wide variety of uncertainty quantification methods has been proposed and investigated. Although the standard Monte Carlo method is often used in practical applications, it may become prohibitively expensive due to its slow convergence and large number of required samples. Alternatively, stochastic spectral methods, such as the stochastic Galerkin and stochastic collocation methods, can be applied in order to efficiently compute numerical solutions to systems with parametric uncertainty. Stochastic Galerkin method is based on a spectral element approximation in the probability space. It belongs to the class of intrusive methods, where the corresponding deterministic numerical scheme needs to be adjusted to calculate moment statistics of a solution. The stochastic collocation methods are non-intrusive and only require solving the underlying deterministic system at the certain collocation nodes together with application of a suitable interpolation method in the probability space. Further details can be found in the monographs by Le Maître and Knio [25], Pettersson et al. [33], Xiu [39], Zhang and Karniadakis [41].

Rigorous convergence analysis of these uncertainty quantification methods leans on uniqueness and continuous dependence of solutions on random parameters (stochastic regularity of the solution). Relevant convergence results for the stochastic collocation methods can be found, for example, in Babuška et al. [1], Nobile et al. [32], Tang and Zhou [36] and the references therein. Convergence analysis of the stochastic Galerkin method was presented, e.g., by Babuška et al. [2], Bespalov et al. [4], Cohen et al. [9], Ernst et al. [11]. For the convergence analysis of the Monte Carlo-type methods we refer to Charrier et al. [7], Herrmann and Schwab [21], Koley at al. [23], Kuo et al. [24], Leonardi et al. [28], Mishra and Schwab [31].

Apparently, much less seems to be known for the nonlinear evolutionary equations arising in fluid flow modelling, where well posedness even in the deterministic setting represents a largely open problem. Our aim is to show convergence of a stochastic collocation method for the random *barotropic (isentropic) Navier–Stokes system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \boldsymbol{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \boldsymbol{u}) + \mathrm{div}_x(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}), \tag{1.2}$$

$$\mathbb{S}(\nabla_x \boldsymbol{u}) = \mu \left(\nabla_x \boldsymbol{u} + \nabla_x^t \boldsymbol{u} - \frac{2}{d} \mathrm{div}_x \boldsymbol{u} \mathbb{I} \right) + \eta \, \mathrm{div}_x \boldsymbol{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0, \tag{1.3}$$

$$p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1, \tag{1.4}$$

describing the time evolution of the mass density $\varrho = \varrho(t, x)$ and the velocity field $\boldsymbol{u} = \boldsymbol{u}(t, x)$ of

a viscous, compressible barotropic fluid. For the sake of simplicity, we suppose the space periodic boundary conditions, meaning the fluid domain may be identified with the flat torus,

$$t \in [0, T], \ x \in \mathbb{T}^d, \ d = 2, 3. \tag{1.5}$$

Randomness is enforced through the initial data

$$\varrho(0, \cdot) = \varrho_0, \ \boldsymbol{m}(0, \cdot) \equiv \varrho \boldsymbol{u}(0, \cdot) = \boldsymbol{m}_0, \tag{1.6}$$

and the viscosity coefficients μ and η that are random variables defined on a probability basis

$$\left(\Omega, \mathfrak{B}[\Omega], \mathbb{P} \right),$$

where Ω is a compact metric space of events, $\mathfrak{B}[\Omega]$ the $\sigma-$field of Borel subsets of Ω, and $\mathbb{P}$ a complete Borel probability measure on Ω. To the best of our knowledge, this paper presents the first result available on convergence analysis of a numerical scheme approximating the random system (1.1)–(1.6).

Opposite to the above mentioned results, the principal difficulty is due to the fact that the Navier–Stokes system (1.1)–(1.6) is not known to be solvable in the class of smooth solutions on a possibly large time interval $(0, T)$ even for smooth initial data. The weak solutions exist globally in time for $\gamma > \frac{d}{2}$ (see [12] and the pioneering work of Lions [29]), however, their uniqueness in terms of the initial data remains an open problem. To approximate differential equations with low regularity of exact solutions, several methods were proposed in the literature: The multi-element probabilistic collocation methods, see Foo and Karniadakis [17], Foo et al. [18], the multi-resolution analysis methods using stochastic finite elements Le Maître et al. [27] and multi-wavelet expansions Le Maître et al. [26].

Our stochastic collocation method for the random Navier-Stokes system is based on a piece-wise constant interpolation of numerical solutions evaluated at specific collocation nodes in the probability space and can be seen as a low order *multi-element probabilistic collocation method*. Specifically, we consider a decomposition of the probabilistic basis Ω,

$$\Omega = \bigcup_{m=1}^{\nu(M)} \Omega_m^M, \ \Omega_m^M \in \mathfrak{B}[\Omega], \ \Omega_i^M \cap \Omega_j^M = \emptyset \text{ for } i \neq j,$$

and choose the nodal points

$$\omega_m^M \in \Omega_m^M, \quad m = 1, \dots, \nu(M).$$

Given *deterministic* initial data $(\varrho_{0,m}^M, \boldsymbol{m}_{0,m}^M)$ evaluated at each nodal point,

$$\varrho_{0,m}^M(x) = \varrho_0(x, \omega_m^M), \ \boldsymbol{m}_{0,m}^M(x) = \boldsymbol{m}_0(x, \omega_m^M), \ x \in \mathbb{T}^d,$$

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along with the associated viscosity coefficients

$$\mu_m^M = \mu(\omega_m^M), \ \eta_m^M = \eta(\omega_m^M),$$

the exact solution $(\varrho_m^M, \boldsymbol{u}_m^M)_{m=1}^{\nu(M)}$ of the Navier–Stokes system will be approximated through a finite volume numerical scheme specified in Section 2.2 yielding a family of approximate solutions,

$$(\varrho_{h,m}^M, \boldsymbol{u}_{h,m}^M), \ \text{where } h > 0 \text{ denotes the mesh size.}$$

For the sake of simplicity, we consider h independent of m, however, $h = h(M)$,

$$h(M) \to 0 \text{ as } M \to \infty.$$

This process gives rise to a sequence of discrete random variables $(\varrho_h^M(t, x, \omega), \boldsymbol{u}_h^M(t, x, \omega))_{M=1}^{\infty}$ obtained using a suitable interpolation of the approximate solutions $(\varrho_{h,m}^M, \boldsymbol{u}_{h,m}^M)$, $m = 1, \ldots, \nu(M)$. Our goal is to show that for

$$\max_{m \leq \nu(M)} \text{diam}[\Omega_m^M] \to 0, \ h(M) \to 0 \text{ as } M \to \infty,$$

the sequence of approximate solutions converges, specifically

$$\varrho_{h(M)}^M \to \varrho, \ \boldsymbol{u}_{h(M)}^M \to \boldsymbol{u} \ \text{ as } M \to \infty \text{ in probability,} \tag{1.7}$$

where $(\varrho, \boldsymbol{u})$ is a statistical (random) solution to the Navier–Stokes system (1.1)–(1.6), with the random initial data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients μ and η. For the time being, we leave unspecified the topology in which the convergence (1.7) takes place.

To avoid the well–posedness problem in the class of weak solutions, we focus on regular initial data, specifically,

$$\varrho_0 \in W^{3,2}(\mathbb{T}^d), \ \inf_{x \in \mathbb{T}^d} \varrho_0(x) > 0, \tag{1.8}$$

$$\boldsymbol{m}_0 \in W^{3,2}(\mathbb{T}^d; R^d), \ \mathbb{P}\text{-a.s.} \tag{1.9}$$

Under these circumstances, the Navier–Stokes system admits a regular local in time solution, with a random life–span, see Matsumura and Nishida [30], Tani [37], among others. As shown by Sun, Wang, and Zhang [35], the eventual blow up of a smooth solution is conditioned by some concentration of the density. The leading idea of the present paper is therefore to exclude blow up of smooth solutions at least at a statistical level. Accordingly, our main working hypothesis is that the approximate solutions $(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)$ are *bounded in probability*:

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$$\text{For any } \varepsilon > 0, \text{ there exists } N = N(\varepsilon) \text{ such that}$$
$$\limsup_{M \to \infty} \mathbb{P}\left[\left(\|\varrho^M_{h(M)}\|_{L^\infty((0,T)\times\mathbb{T}^d)} + \|\boldsymbol{u}^M_{h(M)}\|_{L^\infty((0,T)\times\mathbb{T}^d;R^d)}\right) > N\right] \leq \varepsilon. \qquad (1.10)$$

Hypothesis (1.10) is quite mild as boundedness is required only for a statistical significant number of approximate solutions. Indeed (1.10) follows from a stronger hypothesis

$$\mathbb{E}\left[\|\varrho^M_{h(M)}\|_{L^\infty((0,T)\times\mathbb{T}^d)} + \|\boldsymbol{u}^M_{h(M)}\|_{L^\infty((0,T)\times\mathbb{T}^d;R^d)}\right] \overset{<}{\sim} 1, \qquad (1.11)$$

where $\mathbb{E}$ denotes the expected value with the respect to the measure $\mathbb{P}$. Note carefully that we do not impose any uniform bounds on the pointwise values (for fixed ω) of the random variables $(\varrho^M_{h(M)}, \boldsymbol{u}^M_{h(M)})$. We also remark that boundedness of a sequence of approximate solutions is anticipated frequently in analysis of many numerical methods. Finally, boundedness in probability (1.10) represents an *a posteriori* condition verifiable directly in the course of a numerical simulation. In addition, we show that the satisfaction of (1.10) for a specific choice of collocation nodal points associated to a specific partition of the probability space implies convergence of the method for *any* choice of collocation nodal points and *any* partition of Ω.

Our approach is based on the stochastic compactness method, notably Skorokhod/Jakubowski representation theorem developed in the context of weak (distributional) topologies in the monograph [6]. We proceed as follows:

- Using a particular sample of random fields, the choice of which is motivated by [6], we use Skorokhod theorem to pass to a family of problems defined on a new probability space, where all quantities in question share the same law with the original ones. As an added benefit, the boundedness in probability stated in (1.10) is transformed to a.s. boundedness on the new probability space.

- We perform the limit $M \to \infty$ on the new probability space. First we observe that the sequence of approximate numerical solutions converges, up to a subsequence, to a dissipative measure–valued solution in the sense of [14, Chapter 11] a.s. in the new probability space. Using the fact that the limit is bounded a.s. and the regularity criterion of Sun, Wang, and Zhang [35] we conclude that the limit is a smooth solution of the Navier–Stokes system and the convergence is unconditional.

- We use Gyöngy–Krylov criterion to show convergence in probability of the approximate solutions on the original probability space.

Finally, let us comment briefly on our choice of the stochastic collocation method. Here, we adopt the point of view that the real distribution of the data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients

μ, η is *a priori* not known and the only piece of information available are their values at the nodes ω_m^M. In contrast with the collocation methods based on a global polynomial approximation, we do not anticipate continuity of the random fields with respect to the parameter ω, which may be particularly relevant for the rheological parameters μ and η that may attain only a specific *finite* number of values. Instead we only assume that the piecewise constant interpolations

$$(\varrho_0^M, \boldsymbol{m}_0^M)(x, \omega) = \sum_{m=1}^{\nu(M)} (\varrho_0(x, \omega_m^M), \boldsymbol{m}_0(x, \omega_m^M)) \mathbb{1}_{\Omega_m^M}(\omega),$$

$$(\mu^M, \eta^M)(\omega) = \sum_{m=1}^{\nu(M)} (\mu(\omega_m^M), \eta(\omega_m^M)) \mathbb{1}_{\Omega_m^M}(\omega), \ \nu(M) \to \infty \text{ as } M \to \infty, \qquad (1.12)$$

approach, as $M \to \infty$, the initial data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients (μ, η) in probability for any choice of the nodal points ω_m^M and the volumes Ω_m^M. As we shall see below, this is equivalent to the assumption that the random variables in question are bounded and continuous (in ω) with the exception of a set of zero probability. Such a property is in fact equivalent to the Riemann integrability of the random data in Ω. In Section 3, we give a detailed proof of this statement that may be of independent interest. Let us point out that most of the real world application are based on *finite* dimensional probability space isomorphic to a cube. There is, however, a new rather elegant theory of Riemann integration on general measured compact metric spaces developed by Taylor [38]. We shall systematically refer to Taylor's theory in the present paper. Note that such a choice of the probability space Ω goes beyond the standard Riemann/Darboux theory developed mostly on finite–dimensional spaces.

The ansatz (1.12) corresponds to the piecewise constant interpolation obtained using (deterministic) numerical solutions,

$$\varrho_h^M(t, x, \omega) = \sum_{m=1}^{\nu(M)} \varrho_{h,m}^M(t, x) \mathbb{1}_{\Omega_m^M}(\omega), \ \boldsymbol{u}_h^M(t, x, \omega) = \sum_{m=1}^{\nu(M)} \boldsymbol{u}_{h,m}^M(t, x) \mathbb{1}_{\Omega_m^M}(\omega).$$

Our goal is to show convergence of the resulting random numerical solutions independently of the choice of the collocation nodes under the sole assumption

$$\max_{m=1,\dots,\nu(M)} \text{diam}[\Omega_m^M] \to 0 \text{ as } M \to \infty.$$

Note that our approximation method coincides with the *nearest neighbour interpolation* in the case the partition of Ω is formed by a Voronoi tessellation. We point out that our choice of this relatively simple interpolation method is motivated by the anticipated low regularity of the data with respect to the random parameter. The convergence of the sparse grids approximation in the spirit of Smolyak [34], Xiu [40], Nobile et al. [32] can be handled for the Navier–Stokes system in a similar manner.

The paper is organized as follows. In Section 2, we present the preliminary material including the exact formulation of the numerical scheme and state our main results. In Section 3, we show

that unconditional convergence of the piecewise constant data approximation is in fact equivalent to the Riemann integrability of the data. Section 4 summarizes the properties of numerical solutions obtained in the monograph [14]. In Section 5, we apply the stochastic compactness method to transform the problem to a new probability space. The convergence proof is completed in Section 6 by means of the Gyöngy–Krylov criterion. The paper is concluded by a short discussion in Section 7.

2 Preliminaries, main results

To state our main result in a rigorous way, we have to specify the probability basis as well as the finite volume (FV) numerical method used to construct the approximate solutions.

2.1 Probability basis

Our choice of probability space Ω –a compact metric space– is quite general and includes, in particular, the finite–dimensional case $\Omega \approx [0,1]^N$ with $\mathbb{P} = \rho dy$, considered frequently in the literature, see e.g. Babuška et al. [1].

Definition 2.1 (Partition). A *partition* of Ω is a (finite) family of Borel sets $(\Omega_m^M)_{m=1}^{\nu(M)}$,

$$\Omega_i^M \cap \Omega_j^M = \emptyset \text{ for } i \neq j, \ \Omega = \bigcup_{m=1}^{\nu(M)} \Omega_m^M.$$

The *diameter* of the partition $(\Omega_m^M)_{m=1}^{\nu(M)}$ is

$$\text{diam}[(\Omega_m^M)_{m=1}^{\nu(M)}] = \max_{m=1,\dots,\nu(M)} \text{diam}[\Omega_m^M].$$

The *collocation nodes* associated to a partition $(\Omega_m^M)_{m=1}^{\nu(M)}$ are the points ω_m^M,

$$\omega_m^M \in \Omega_m^M, \ m = 1,\dots,\nu(M).$$

2.2 Finite volume numerical scheme

We introduce a finite volume (FV) method to approximate the (deterministic) Navier-Stokes system (1.1)-(1.4). The physical domain $\mathbb{T}^d$ is decomposed into finite volumes (cuboids for simplicity)

$$\mathbb{T}^d = \bigcup_{K \in \mathcal{T}_h} K.$$

The set of all faces $\sigma \in \partial K$, $K \in \mathcal{T}_h$ is denoted by Σ. We suppose $|K| \approx h^d$, $|\sigma| \approx h^{d-1}$ for any $K \in \mathcal{T}_h$, and $\sigma \in \Sigma$, where the parameter $h \in (0,1)$ denotes the size of the mesh $\mathcal{T}_h$.

The space of functions constant on each element $K \in \mathcal{T}_h$ is denoted Q_h. The associated projection reads

$$\Pi_h : L^1(\mathbb{T}^d) \to Q_h, \ \Pi_h v = \sum_{K \in \mathcal{T}_h} \mathbb{1}_K \frac{1}{|K|} \int_K v \, \mathrm{d}x.$$

The average and jump of $v \in Q_h$ on a face $\sigma \in \Sigma$ are denoted

$$\{\!\{v\}\!\} = \frac{v^{\mathrm{in}} + v^{\mathrm{out}}}{2}, \quad [\![v]\!] = v^{\mathrm{out}} - v^{\mathrm{in}},$$

where $v^{\mathrm{out}}, v^{\mathrm{in}}$ are respectively the outward, inward limits with respect to a given normal $\boldsymbol{n}$ to $\sigma \in \Sigma$.

Moreover, the following discrete differential operators for piecewise constant functions $r_h \in Q_h$, $\boldsymbol{v}_h \in \boldsymbol{Q}_h \equiv (Q_h)^d$ will be used:

$$\nabla_h r_h = \sum_{K \in \mathcal{T}_h} (\nabla_h r_h)_K \mathbb{1}_K, \quad (\nabla_h r_h)_K = \sum_{\sigma \in \partial K} \frac{|\sigma|}{|K|} \{\!\{r_h\}\!\} \, \boldsymbol{n},$$

$$\Delta_h r_h = \sum_{K \in \mathcal{T}_h} (\Delta_h r_h)_K \mathbb{1}_K, \quad (\Delta_h r_h)_K = \sum_{\sigma \in \partial K} \frac{|\sigma|}{|K|} \frac{[\![r_h]\!]}{h},$$

$$\nabla_{\mathcal{D}} r_h = \sum_{\sigma \in \partial K} (\nabla_{\mathcal{D}} r_h)_\sigma \mathbb{1}_\sigma, \quad (\nabla_{\mathcal{D}} r_h)_\sigma = \frac{[\![r_h]\!]}{h} \boldsymbol{n}$$

$$\mathrm{div}_h \boldsymbol{v}_h = \sum_{K \in \mathcal{T}_h} (\mathrm{div}_h \boldsymbol{v}_h)_K \mathbb{1}_K, \quad (\mathrm{div}_h \boldsymbol{v}_h)_K = \sum_{\sigma \in \partial K} \frac{|\sigma|}{|K|} \{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n}.$$

Convective terms will be approximated by a dissipative upwind numerical flux denoted by F_h; specifically

$$Up[r_h, \boldsymbol{v}_h] = \{\!\{r_h\}\!\} \ \{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n} - \frac{1}{2} |\{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n}| [\![r_h]\!]$$

$$F_h(r_h, \boldsymbol{v}_h) = Up[r_h, \boldsymbol{v}_h] - h^\varepsilon [\![r_h]\!] = \{\!\{r_h\}\!\} \ \{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n} - \left(h^\varepsilon + \frac{1}{2} |\{\!\{\boldsymbol{v}_h\}\!\} \cdot \boldsymbol{n}| \right) [\![r_h]\!], \ -1 < \varepsilon.$$

Analogously, we define the vector-valued numerical flux $\mathbf{F}_h(\mathbf{r}_h, \boldsymbol{v}_h)$ componentwisely.

In order to discretize the time evolution in $[0, T]$ we introduce a time step $\Delta t > 0$, $\Delta t \approx h$, and denote

$$t_k = k\Delta t, \ k = 1, 2, \ldots, N_T.$$

Furthermore, we set

$$v^k(x) = v(t^k, x) \ \text{for all} \ x \in \mathbb{T}^d, \ t^k = k \, \Delta t \ \text{for} \ k = 0, 1, \ldots, N_T.$$

The time derivative $\frac{\partial v}{\partial t}$ is approximated by the backward Euler finite difference

$$\frac{\partial v}{\partial t} \approx D_t v^k \equiv \frac{v^k - v^{k-1}}{\Delta t}.$$

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Finally, we introduce a piecewise constant interpolation in time of the discrete values v^k,

$$v_h(t, \cdot) = v_0 \text{ for } t < \Delta t, \ v_h(t, \cdot) = v^k \text{ for } t \in [k\Delta t, (k+1)\Delta t), \ k = 1, 2, \ldots, N_T. \tag{2.1}$$

Definition 2.2 (FV numerical scheme).

- Given the initial data $(\varrho_0, \boldsymbol{m}_0) \in L^1(\mathbb{T}^d; R^{d+1})$, we set

$$\varrho_{0,h} = \Pi_h \varrho_0, \ \boldsymbol{m}_{0,h} = \Pi_h \boldsymbol{m}_0, \ \varrho_{0,h} \boldsymbol{u}_{0,h} = \boldsymbol{m}_{0,h}.$$

- A pair $(\varrho_h, \boldsymbol{u}_h)$ of piecewise constant functions (in space and time) is a numerical approximation of the Navier-Stokes system (1.1)-(1.4) if the following system of discrete equations holds:

$$\int_{\mathbb{T}^d} D_t \varrho_h \varphi_h \ \mathrm{d}x - \sum_{\sigma \in \Sigma} \int_\sigma F_h(\varrho_h, \boldsymbol{u}_h) \, [\![\varphi_h]\!] \ \mathrm{d}\sigma = 0 \quad \text{for all } \varphi_h \in Q_h, \tag{2.2a}$$

$$\int_{\mathbb{T}^d} D_t(\varrho_h \boldsymbol{u}_h) \cdot \boldsymbol{\varphi}_h \ \mathrm{d}x - \sum_{\sigma \in \Sigma} \int_\sigma \mathbf{F}_h(\varrho_h \boldsymbol{u}_h, \boldsymbol{u}_h) \cdot [\![\boldsymbol{\varphi}_h]\!] \ \mathrm{d}\sigma - \sum_{\sigma \in \Sigma} \int_\sigma \{\!\!\{p(\varrho_h)\}\!\!\} \, \boldsymbol{n} \cdot [\![\boldsymbol{\varphi}_h]\!] \ \mathrm{d}\sigma$$

$$= -\mu \frac{1}{h} \sum_{\sigma \in \Sigma} \int_\sigma [\![\boldsymbol{u}_h]\!] \cdot [\![\boldsymbol{\varphi}_h]\!] \ \mathrm{d}\sigma - \lambda \int_{\mathbb{T}^d} \mathrm{div}_h \boldsymbol{u}_h \mathrm{div}_h \boldsymbol{\varphi}_h \ \mathrm{d}x \quad \text{for all } \boldsymbol{\varphi}_h \in \boldsymbol{Q}_h. \tag{2.2b}$$

where $\lambda = \frac{1}{d}\mu + \eta$.

Definition 2.3 (Approximate statistical solution).
Given a partition $\left(\Omega_m^M\right)_{m=1}^{\nu(M)}$ of Ω and a set of the collocation nodes $\omega_m^M \in \Omega_m^M$, *approximate statistical solution* of the (random) Navier–Stokes system is a pair of random variables,

$$\varrho_h^M(t, x, \omega) = \sum_{m=1}^{\nu(M)} \varrho_{h,m}^M(t, x) \mathbb{1}_{\Omega_m^M}(\omega), \ \boldsymbol{u}_h^M(t, x, \omega) = \sum_{m=1}^{\nu(M)} \boldsymbol{u}_{h,m}^M(t, x) \mathbb{1}_{\Omega_m^M}(\omega),$$

where $(\varrho_{h,m}^M, \boldsymbol{u}_{h,m}^M)$ is a solution of the FV method (2.2), with the (deterministic) initial data $(\varrho_0(\omega_m^M), \boldsymbol{m}_0(\omega_m^M))$, and the viscosity coefficients $\mu(\omega_m^M), \lambda(\Omega_m^M) = \frac{1}{d}\mu(\omega_m^M) + \eta(\omega_m^M)$.

Remark 2.4. Statistical solutions given in Definition 2.3 are sometimes called *random solutions* in literature.

2.3 Main result

We start by introducing the class of admissible data. Let

$$\mathcal{R}(\Omega, \mathbb{P}) = \left\{ f : \Omega \to R \ \middle| \ f \text{ bounded, } \mathbb{P}\{\omega \in \Omega \ \middle| \ f \text{ is not continuous at } \omega\} = 0 \right\}$$

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denote the class of Riemann integrable functions on Ω, cf. Taylor [38]. Moreover, we introduce the total energy as

$$E(\varrho, \boldsymbol{m}) \equiv \begin{cases} \frac{1}{2}\frac{|\boldsymbol{m}|^2}{\varrho} + P(\varrho) & \text{if } \varrho > 0, \\ 0 & \text{if } \varrho = 0, \boldsymbol{m} = 0, \\ \infty & \text{if } \varrho = 0, \boldsymbol{m} \neq 0 \text{ or } \varrho < 0, \end{cases} \tag{2.3}$$

where $P(\varrho)$ is the pressure potential, $P'(\varrho)\varrho - P(\varrho) = p(\varrho)$. Specifically, we may consider $P(\varrho) = \frac{a}{\gamma-1}\varrho^\gamma$ if $p(\varrho) = a\varrho^\gamma$. Note that E is convex l.s.c. for $(\varrho, \boldsymbol{m}) \in R^{d+1}$. Alternatively, we also denote

$$E(\varrho, \boldsymbol{u}) = \frac{1}{2}\varrho|\boldsymbol{u}|^2 + P(\varrho).$$

Definition 2.5 (Admissible data). We say that the data ϱ_0, $\boldsymbol{m}_0$, μ, η are *admissible* if:

-
$$\varrho_0(\cdot, \omega) \in W^{3,2}(\mathbb{T}^d), \ \inf_{\mathbb{T}^d} \varrho_0(\cdot, \omega) > 0, \ \boldsymbol{m}_0(\cdot, \omega) \in W^{3,2}(\mathbb{T}^d; R^3) \text{ for } \omega \in \Omega;$$

-
$$\mu(\omega) \geq \underline{\mu} > 0, \ \eta(\omega) \geq 0 \text{ for } \omega \in \Omega, \text{ with deterministic constant } \underline{\mu} > 0;$$

- the functions

$$\omega \in \Omega \to \int_{\mathbb{T}^d} \varrho_0(x, \omega)\varphi(x) \ \mathrm{d}x, \ \varphi \in C^\infty(\mathbb{T}^d),$$

$$\omega \in \Omega \to \int_{\mathbb{T}^d} \boldsymbol{m}_0(x, \omega) \cdot \boldsymbol{\varphi}(x) \ \mathrm{d}x, \ \boldsymbol{\varphi} \in C^\infty(\mathbb{T}^d),$$

$$\omega \in \Omega \to \int_{\mathbb{T}^d} E(\varrho_0(x, \omega), \boldsymbol{m}_0(x, \omega)) \ \mathrm{d}x,$$

$$\omega \in \Omega \to \mu(\omega), \ \omega \in \Omega \to \eta(\omega),$$

belong to $\mathcal{R}(\Omega, \mathbb{P})$.

As we shall see in Section 3, admissibility of the data implies convergence in probability of their piecewise constant interpolation independent of the choice partition of the probability space and the position of the collocation nodes. Moreover, it can be shown that these two properties are in fact *equivalent* provided the measure $\mathbb{P}$ of any open ball in Ω and of its closure coincides, see Beer [3]. In view of this argument, admissibility in the sense of Definition 2.5 is the weakest condition for a collocation method to be correctly defined.

Our main result reads as follows.

Theorem 2.6 (Convergence of stochastic collocation FV method).
Let the initial data $(\varrho_0, \boldsymbol{m}_0)$ as well as the viscosity coefficients μ, η be admissible in the sense of Definition 2.5.

Let $\left(\Omega_m^M\right)_{m=1}^{\nu(M)}$, $M = 1, 2, \ldots$ be a sequence of partitions of Ω with a family of nodal points $\omega_m^M \in \Omega_m^M$ such that

$$\nu(M) \to \infty, \ \operatorname{diam}[(\Omega_m^M)_{m=1}^{\nu(M)}] \to 0 \ as \ M \to \infty.$$

Suppose that for $h = h(M) \to 0$ as $M \to \infty$, the associated sequence of approximate statistical solutions introduced in Definition 2.3 is bounded in probability:

For any $\varepsilon > 0$, there exists $N = N(\varepsilon)$ such that
$$\limsup_{M \to \infty} \mathbb{P}\left[\left(\|\varrho_{h(M)}^M\|_{L^\infty((0,T) \times \mathbb{T}^d)} + \|\boldsymbol{u}_{h(M)}^M\|_{L^\infty((0,T) \times \mathbb{T}^d; R^d)}\right) > N\right] \leq \varepsilon. \tag{2.4}$$

Then

$$\varrho_{h(M)}^M \to \varrho \ \ as \ M \to \infty \ in \ L^q((0,T) \times \mathbb{T}^d), \ q < \infty, \ in \ probability,$$
$$\boldsymbol{u}_{h(M)}^M \to \boldsymbol{u} \ \ as \ M \to \infty \ in \ L^q((0,T) \times \mathbb{T}^d; R^d), \ q < \infty, \ in \ probability, \tag{2.5}$$

where $(\varrho, \boldsymbol{u})$ is the classical statistical solution of the Navier–Stokes system with the initial data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients μ, η.

Remark 2.7 (Convergence in expectation). As the initial energy $\int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \ \mathrm{d}x$ is uniformly bounded in $\omega \in \Omega$, and the numerical solutions satisfy the energy inequality (4.9) below, we have

$$\sup_{t \in [0,T]} \|\varrho_{h(M)}^M\|_{L^\gamma(\mathbb{T}^d)}^\gamma + \int_0^T \int_{\mathbb{T}^d} |\boldsymbol{u}_{h(M)}^M|^2 \ \mathrm{d}x \lesssim 1 \text{ uniformly for } \omega \in \Omega.$$

Consequently, the convergence (2.5) yields

$$\mathbb{E}\left[\|\varrho_{h(M)}^M - \varrho\|_{L^\gamma((0,T) \times \mathbb{T}^d)}^q\right] \to 0, \ \mathbb{E}\left[\|\boldsymbol{u}_{h(M)}^M - \boldsymbol{u}\|_{L^2((0,T) \times \mathbb{T}^d; R^d)}^q\right] \to 0$$

for any $1 \leq q < \infty$.

We point out that the above result is conditioned only by the satisfaction of (2.4). In particular, it is independent of the position of the collocation nodes and the choice of the "elements" Ω_m^M. We also do not anticipate continuity of the data with respect to the random variable. As we have seen, hypothesis (2.4) follows from a weaker stipulation (1.11) that can be reformulated as

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$$\sum_{m=1}^{\nu(M)} \mathbb{P}[\Omega_m^M] \left(\|\varrho_{h(M),m}^M\|_{L^\infty((0,T)\times\mathbb{T}^d)} + \|\boldsymbol{u}_{h(M),m}^M\|_{L^\infty((0,T)\times\mathbb{T}^d;R^d)} \right) \stackrel{<}{\sim} 1 \text{ uniformly for } M \to \infty, \quad (2.6)$$

which can be explicitly controlled during numerical simulations.

Finally, we realize that the conclusion of Theorem 2.6 implies that the Navier–Stokes system admits a unique classical solution for the given set of random data $\mathbb{P}-$a.s. This yields the following corollary.

Corollary 2.8 (Unconditional convergence). *Let the initial data $(\varrho_0, \boldsymbol{m}_0)$ as well as the viscosity coefficients μ, η be admissible in the sense of Definition 2.5. Suppose there there exists a sequence of partitions $(\Omega_m^M)_{m=1}^{\nu(M)}$ with a family of nodal points $\omega_m^M \in \Omega_m^M$ such that*

$$\nu(M) \to \infty, \ \text{diam}[(\Omega_m^M)_{m=1}^{\nu(M)}] \to 0 \ \text{as } M \to \infty.$$

In addition, suppose there is $h(M) \to \infty$ such that the associated sequence of approximate statistical solutions is bounded in probability in the sense of (2.4).

Then for any sequence of partitions $(\Omega_n^N)_{n=1}^{\nu(N)}$, $N = 1, 2, \dots$ satisfying

$$\nu(N) \to \infty, \ \text{diam}[(\Omega_n^N)_{n=1}^{\nu(N)}] \to 0 \ \text{as } N \to \infty,$$

any family of nodal points $\omega_m^N \in \Omega_n^N$, and any $h(N) \to 0$ as $N \to \infty$, the associated family of approximate statistical solutions $(\varrho_{h(N)}^N, \boldsymbol{u}_{h(N)}^N)_{n=1}^{\nu(N)}$ converges, specifically,

$$\varrho_{h(N)}^N \to \varrho \ \text{as } N \to \infty \ \text{in } L^\gamma((0,T) \times \mathbb{T}^d) \ \text{in probability,}$$
$$\boldsymbol{u}_{h(N)}^N \to \boldsymbol{u} \ \text{as } N \to \infty \ \text{in } L^2((0,T) \times \mathbb{T}^d; R^d) \ \text{in probability,}$$

where $(\varrho, \boldsymbol{u})$ is the classical statistical solution of the Navier–Stokes system with the initial data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients μ, η.

The rest of the paper is devoted to the proof of Theorem 2.6 and Corollary 2.8.

3 Data convergence

We start with the following result on convergence of admissible data that can be of independent interest.

Proposition 3.1 (**Convergence of Riemann approximations**). *Let $f \in \mathcal{R}(\Omega; \mathbb{P})$. Let $(\Omega_m^M)_{m=1}^{\nu(M)}$, $M = 1, 2, \dots$ be a sequence of partitions of Ω such that*

$$\nu(M) \to \infty, \;\; \mathrm{diam}[(\Omega_m^M)_{m=1}^{\nu(M)}] \to 0 \;\; as \; M \to \infty.$$

Let

$$f^M(\omega) = \sum_{m=1}^{\nu(M)} f(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega), \;\; \omega_m^M \in \Omega_m^M.$$

Then

$$\mathbb{E}\left[\left|f^M - f\right|^q\right] \to 0 \;\; as \; M \to \infty \tag{3.1}$$

for any $1 \leq q < \infty$.

Proof. We start by a result of Taylor [38, Section 1, Section 3, Proposition 3.2]:

$$f \in \mathcal{R}(\Omega; \mathbb{P}) \;\; \Rightarrow \;\; f \text{ is } \mathbb{P} - \text{measurable, and}$$

$$\int_\Omega f \; \mathrm{d}\mathbb{P} = \lim_{M \to \infty} \sum_{m=1}^{\nu(M)} f(\omega_m^M) \mathbb{P}[\Omega_m^M] = \lim_{M \to \infty} \int_\Omega f^M \; \mathrm{d}\mathbb{P}. \tag{3.2}$$

Next, observe that

$$(f^M)^2 = \left(\sum_{m=1}^{\nu(M)} f(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega)\right)^2 = \sum_{m=1}^{\nu(M)} f^2(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega).$$

Seeing that

$$f \in \mathcal{R}(\Omega; \mathbb{P}) \;\; \Rightarrow f^2 \in \mathcal{R}(\Omega; \mathbb{P}),$$

we may apply (3.2) to f^2 obtaining

$$\int_\Omega (f^M)^2 \; \mathrm{d}\mathbb{P} = \int_\Omega \sum_{m=1}^{\nu(M)} f^2(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega) \; \mathrm{d}\mathbb{P} = \sum_{m=1}^{\nu(M)} f^2(\omega_m^M) \mathbb{P}[\Omega_m^M] \to \int_\Omega f^2 \; \mathrm{d}\mathbb{P} \;\; \text{as } M \to \infty.$$

Now, we examine the limit

$$\lim_{M \to \infty} \int_\Omega f^M \phi \; \mathrm{d}\mathbb{P}, \;\; \phi \in \mathrm{Lip}(\Omega).$$

Set

$$\phi^M(\omega) = \sum_{m=1}^{\nu(M)} \phi(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega).$$

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Similarly to the above

$$f^M \phi^M = \left(\sum_{m=1}^{\nu(M)} f(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega) \right) \left(\sum_{m=1}^{\nu(M)} \phi(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega) \right) = \sum_{m=1}^{\nu(M)} f(\omega_m^M) \phi(\omega_m^M) \mathbb{1}_{\Omega_m^M}(\omega).$$

As $f\phi \in \mathcal{R}(\Omega; \mathbb{P})$, we may infer, using (3.2) once more,

$$\int_\Omega f^M \phi^M \, \mathrm{d}\mathbb{P} \to \int_\Omega f\phi \, \mathrm{d}\mathbb{P} \text{ as } M \to \infty \tag{3.3}$$

for any $\phi \in \mathrm{Lip}(\Omega)$. In addition, as ϕ is Lipschitz,

$$\sup_\Omega |\phi^M - \phi| = \max_{m=1,\dots,\nu(M)} \sup_{\omega \in \Omega_m^M} |\phi(\omega_m^M) - \phi(\omega)| \leq L \max_{m=1,\dots,\nu(M)} \mathrm{diam}[\Omega_m^M] \to 0 \text{ as } M \to \infty,$$

where L denotes the Lipschitz constant of ϕ. As the sequence f^M is uniformly bounded, the convergence (3.3) yields

$$\int_\Omega f^M \phi \, \mathrm{d}\mathbb{P} \to \int_\Omega f\phi \, \mathrm{d}\mathbb{P} \text{ as } M \to \infty \text{ for any } \phi \in \mathrm{Lip}(\Omega). \tag{3.4}$$

Our ultimate goal is to extend (3.4) to any $\phi \in L^1(\Omega; \mathbb{P})$. Since $(f^M)_{M=1}^\infty$ is uniformly bounded we need the set of Lipschitz functions to be dense in $L^1(\Omega; \mathbb{P})$. For measurable compact metric spaces this was proved by Hanneke et al. [20, Appendix A, Lemma A1].

Consequently, we have

$$f^M \to f \text{ weakly in } L^2(\Omega; \mathbb{P}), \ \|f^M\|_{L^2(\Omega; \mathbb{P})}^2 \to \|f\|_{L^2(\Omega; \mathbb{P})}^2$$

yielding

$$\mathbb{E}\left[|f^M - f|^2\right] \to 0 \text{ as } M \to \infty.$$

As f^M are uniformly bounded, the desired conclusion (3.1) follows.

$\square$

Since the data are admissible, Proposition 3.1 yields

$$\mathbb{E}\left[\left|\int_{\mathbb{T}^d} \left(\sum_{m=1}^{\nu(M)} \varrho_0(\cdot, \omega_M^m) \mathbb{1}_{\Omega_M^m} - \varrho_0 \right) \varphi \, \mathrm{d}x \right|^q\right] \to 0 \text{ as } M \to \infty \text{ for any } \varphi \in C^\infty(\mathbb{T}^d),$$

$$\mathbb{E}\left[\left|\int_{\mathbb{T}^d} \left(\sum_{m=1}^{\nu(M)} \boldsymbol{m}_0(\cdot, \omega_M^m) \mathbb{1}_{\Omega_M^m} - \boldsymbol{m}_0 \right) \boldsymbol{\varphi} \, \mathrm{d}x \right|^q\right] \to 0 \text{ as } M \to \infty \text{ for any } \boldsymbol{\varphi} \in C^\infty(\mathbb{T}^d; R^3),$$

$$\mathbb{E}\left[\left|\sum_{m=1}^{\nu(M)} \int_{\mathbb{T}^d} E\Big(\varrho_0(\cdot, \omega_M^m); \boldsymbol{m}_0(\cdot, \omega_M^m)\Big) \, \mathrm{d}x \mathbb{1}_{\Omega_M^m} - \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \, \mathrm{d}x \right|^q\right] \to 0 \text{ as } M \to \infty, \tag{3.5}$$

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and, similarly,

$$\mathbb{E}\left[\left|\sum_{m=1}^{\nu(M)}\mu(\omega_M^m)\mathbb{1}_{\Omega_M^m}-\mu\right|^q\right]\to 0, \quad \mathbb{E}\left[\left|\sum_{m=1}^{\nu(M)}\eta(\omega_M^m)\mathbb{1}_{\Omega_M^m}-\eta\right|^q\right]\to 0 \text{ as } M\to\infty, \quad (3.6)$$

for any $1\le q<\infty$.

Now we use the following result proved in [13, Section 4.3].

Lemma 3.2. *Let*

$$\varrho_M\to\varrho \text{ weakly in } L^\gamma(\mathbb{T}^d)$$
$$\boldsymbol{m}_M\to\boldsymbol{m} \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d)$$

and

$$\int_{\mathbb{T}^d}E(\varrho_M,\boldsymbol{m}_M) \text{ d}x\to\int_{\mathbb{T}^d}E(\varrho,\boldsymbol{m}) \text{ d}x \text{ as } M\to\infty.$$

Then

$$\varrho_M\to\varrho \text{(strongly) in } L^\gamma(\mathbb{T}^d)$$
$$\boldsymbol{m}_M\to\boldsymbol{m} \text{ (strongly) in } L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d) \text{ as } M\to\infty.$$

Obviously,

$$\sum_{m=1}^{\nu(M)}\int_{\mathbb{T}^d}E\Big(\varrho_0(x,\omega_m^M);\boldsymbol{m}_0(x,\omega_M^m)\Big) \text{ d}x\mathbb{1}_{\Omega_m^M}(\omega)$$

$$=\int_{\mathbb{T}^d}E\left(\sum_{m=1}^{\nu(M)}\varrho_0(x,\omega_m^M)\mathbb{1}_{\Omega_{\nu(M)}^m}(\omega);\sum_{m=1}^{\nu(M)}\boldsymbol{m}_0(x,\omega_m^M)\mathbb{1}_{\Omega_m^M}(\omega)\right) \text{ d}x,$$

therefore it follows from (3.5) and Lemma 3.2 that

$$\sum_{m=1}^{\nu(M)}\varrho_0(\cdot,\omega_m^M)\mathbb{1}_{\Omega_m^M}\to\varrho_0 \text{ in } L^\gamma(\mathbb{T}^d) \text{ } \mathbb{P}-\text{a.s.};$$

$$\sum_{m=1}^{\nu(M)}\boldsymbol{m}_0(\cdot,\omega_m^M)\mathbb{1}_{\Omega_m^M}\to\boldsymbol{m}_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^3) \text{ } \mathbb{P}-\text{a.s.}$$

passing to a subsequence as the case may be. As the energy

$$\int_{\mathbb{T}^d}E\left(\sum_{m=1}^{\nu(M)}\varrho_0(x,\omega_m^M)\mathbb{1}_{\Omega_m^M}(\omega);\sum_{m=1}^{\nu(M)}\boldsymbol{m}_0(x,\omega_m^M)\mathbb{1}_{\Omega_m^M}(\omega)\right) \text{ d}x$$

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is bounded, we conclude

$$\mathbb{E}\left[\left\|\sum_{m=1}^{\nu(M)}\varrho_0(\cdot,\omega_m^M)\mathbb{1}_{\Omega_m^M}-\varrho_0\right\|_{L^\gamma(\mathbb{T}^d)}+\left\|\sum_{m=1}^{\nu(M)}\boldsymbol{m}_0(\cdot,\omega_m^M)\mathbb{1}_{\Omega_m^M}-\boldsymbol{m}_0\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d)}\right]\to 0 \qquad (3.7)$$

as $M\to\infty$. Finally, as the limit is unique, the result is unconditional – no need of subsequence.

4 Properties of FV method

The crucial feature of FV method (2.2) is the fact that the approximate solutions $(\varrho_h,\boldsymbol{u}_h)$ give rise to a stable and consistent approximation of the Navier–Stokes system, see [14, 15, 16]. Specifically, the following structure preserving properties hold:

- **Conservation of discrete mass**

$$\int_{\mathbb{T}^d}\varrho_h(\tau,\cdot)\ \mathrm{d}x=\int_{\mathbb{T}^d}\varrho_{0,h}\ \mathrm{d}x>0,\ \ \tau\geq 0.$$

- **Positivity of the discrete density**

$$\varrho_h(\tau)>0\ \ \text{for any}\ \tau>0\ \ \text{provided}\ \ \varrho_{0,h}>0.$$

- **Discrete total energy dissipation**

$$\int_{\mathbb{T}^d}E(\varrho_h,\boldsymbol{u}_h)(\tau,\cdot)\ \mathrm{d}x+\int_0^T\mu\left\|\nabla_{\mathcal{D}}\boldsymbol{u}_h\right\|_{L^2(\mathbb{T}^d)}^2+\lambda\left\|\mathrm{div}_h\boldsymbol{u}_h\right\|_{L^2(\mathbb{T}^d)}^2\ \mathrm{d}t+\int_0^\tau\mathfrak{E}(\varrho_h,\boldsymbol{u}_h)\,\mathrm{d}t$$
$$\leq\int_{\mathbb{T}^d}E(\varrho_{0,h},\boldsymbol{m}_{0,h})\ \mathrm{d}x,$$

where the term $\mathfrak{E}(\varrho_h,\boldsymbol{u}_h)\geq 0$ represents numerical dissipation, see [14] for its specific form.

As the energy is a convex function of $(\varrho,\boldsymbol{m})$, Jensen's inequality yields

$$\int_{\mathbb{T}^d}E\left(\Pi_h[\varrho_0],\Pi_h[\boldsymbol{m}_0]\right)\ \mathrm{d}x=\sum_{K\in\mathcal{T}_h}\int_K E\left[\frac{1}{|K|}\int_K\varrho_0(s)\mathrm{d}s,\frac{1}{|K|}\int_K\boldsymbol{m}_0(s)\mathrm{d}s\right]\mathbb{1}_K\,\mathrm{d}x$$
$$\leq\sum_{K\in\mathcal{T}_h}\int_K E\left(\varrho_0,\boldsymbol{m}_0\right)\ \mathrm{d}s=\int_{\mathbb{T}^d}E\left(\varrho_0,\boldsymbol{m}_0\right)\ \mathrm{d}x.$$

Thus the discrete energy inequality finally gives rise to

$$\int_{\mathbb{T}^d}E(\varrho_h,\boldsymbol{u}_h)(\tau,\cdot)\ \mathrm{d}x+\int_0^\tau\int_{\mathbb{T}^d}\left[\mu|\nabla_{\mathcal{D}}\boldsymbol{u}_h|^2+\lambda|\mathrm{div}_h\boldsymbol{u}_h|^2\right]\ \mathrm{d}x\,\mathrm{d}t+\int_0^\tau\mathfrak{E}(\varrho_h,\boldsymbol{u}_h)\,\mathrm{d}t$$

$$\le \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \ \mathrm{d}x, \qquad (4.1)$$

where the right–hand side is independent of h.

- **Consistent approximation**
  As shown in [14, 15] the numerical dissipation basically controls all consistency errors. Accordingly, the FV method (2.2) can be rewritten as follows

$$\int_0^T \int_{\mathbb{T}^d} [\varrho_h \partial_t \varphi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \varphi] \ \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_{0,h} \varphi(0, \cdot) \ \mathrm{d}x + e_1[h, \varphi],$$

$$\int_0^T \int_{\mathbb{T}^d} [\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + p(\varrho_h) \mathrm{div}_x \boldsymbol{\varphi}] \ \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^d} \mu \nabla_{\mathcal{D}} \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \lambda \, \mathrm{div}_h \boldsymbol{u}_h \mathrm{div} \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t$$

$$- \int_{\mathbb{T}^d} \boldsymbol{m}_{0,h} \cdot \boldsymbol{\varphi} \ \mathrm{d}x + e_2[h, \boldsymbol{\varphi}] \qquad (4.2)$$

with the test functions $\varphi \in C^2([0,T] \times \mathbb{T}^d)$, $\boldsymbol{\varphi} \in C^2([0,T] \times \mathbb{T}^d; R^d)$, respectively. The consistency errors are controlled in terms of the norm of the test functions and the initial energy. Specifically, there is a function B such that

$$B : R^2 \to R \text{ locally bounded,} \ o = o(h), \ o(h) \to 0 \text{ as } h \to 0,$$

$$|e_1[h, \phi]| + |e_2[h, \boldsymbol{\varphi}]| \le B \left(\|\varphi\|_{C^2([0,T] \times \mathbb{T}^d)} + \|\boldsymbol{\varphi}\|_{C^2([0,T] \times \mathbb{T}^d; R^d)}, \ \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \ \mathrm{d}x \right) o(h).$$

$$(4.3)$$

Moreover, we write the initial data integral as

$$\int_{\mathbb{T}^d} \varrho_{0,h} \phi \ \mathrm{d}x = \int_{\mathbb{T}^d} \varrho_0 \phi \ \mathrm{d}x + \int_{\mathbb{T}^d} (\Pi_h[\varrho_0] - \varrho_0) \phi \ \mathrm{d}x,$$

where

$$\left| \int_{\mathbb{T}^d} (\Pi_h[\varrho_0] - \varrho_0) \phi \ \mathrm{d}x \right| = \left| \int_{\mathbb{T}^d} (\Pi_h[\varrho_0] - \varrho_0)(\phi - \Pi_h[\phi]) \ \mathrm{d}x \right| \lesssim h \|\varrho_0\|_{L^\gamma(\mathbb{T}^d)} \|\phi\|_{C^1(\mathbb{T}^d)}.$$

Similarly, we can control the initial data in the momentum equation. Accordingly, consistency equations (4.2) can be rewritten in the form

$$\int_0^T \int_{\mathbb{T}^d} [\varrho_h \partial_t \varphi + \varrho_h \boldsymbol{u}_h \cdot \nabla_x \varphi] \ \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^d} \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x + e_1[h, \varphi],$$

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$$\int_0^T \int_{\mathbb{T}^d} [\varrho_h \boldsymbol{u}_h \cdot \partial_t \boldsymbol{\varphi} + \varrho_h \boldsymbol{u}_h \otimes \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} + p(\varrho_h) \mathrm{div}_x \boldsymbol{\varphi}] \ \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^d} \mu \nabla_{\mathcal{D}} \boldsymbol{u}_h : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \lambda \, \mathrm{div}_h \boldsymbol{u}_h \mathrm{div} \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t$$

$$- \int_{\mathbb{T}^d} \boldsymbol{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x + e_2[h, \boldsymbol{\varphi}] \tag{4.4}$$

with the consistency error estimate (4.3) still valid.

4.1 Dissipative solutions

Any consistent approximation (4.1), (4.4) generates a dissipative weak solution introduced in [15]. We say that $(\varrho, \boldsymbol{u})$ is a dissipative weak solutions of the Navier-Stokes system with the initial data ϱ_0, $\boldsymbol{m}_0$ if

$$\varrho \in C_{\mathrm{weak}}([0, T]; L^\gamma(\mathbb{T}^d)), \quad \varrho(0, \cdot) = \varrho_0$$
$$\boldsymbol{u} \in L^2(0, T; W^{1,2}(\mathbb{T}^d; R^d)),$$
$$\varrho\boldsymbol{u} \in C_{\mathrm{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d)), \quad \varrho\boldsymbol{u}(0, \cdot) = \boldsymbol{m}_0 \tag{4.5}$$

and the following relations are satisfied in the sense of distributions:

$$\partial_t \varrho + \mathrm{div}_x(\varrho\boldsymbol{u}) = 0,$$
$$\partial_t(\varrho\boldsymbol{u}) + \mathrm{div}_x(\varrho\boldsymbol{u} \otimes \boldsymbol{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \boldsymbol{u}) - \mathrm{div}_x \mathfrak{R},$$
$$\int_{\mathbb{T}^d} \left[\frac{1}{2}\varrho|\boldsymbol{u}|^2 + P(\varrho) \right](\tau, \cdot) \ \mathrm{d}x + \overline{d} \int_{\mathbb{T}^d} \mathrm{tr}[\mathfrak{R}](\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} [\mathbb{S}(\nabla_x \boldsymbol{u}) : \nabla_x \boldsymbol{u}] \ \mathrm{d}x$$
$$\leq \int_{\mathbb{T}^d} \left[\frac{1}{2} \frac{|\boldsymbol{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x, \tag{4.6}$$

where $\overline{d} > 0$ is a positive constant. The extra term $\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\mathbb{T}^d; R_{\mathrm{sym}}^{d \times d}))$ called the Reynolds stress is a product of possible concentrations and/or oscillations in a generating sequence. In the context of FV approximation, the relation $\mathfrak{R} = 0$ is essentially equivalent to strong convergence of $(\varrho_h, \varrho_h \boldsymbol{u}_h)$ as $h \to 0$.

The following conditional regularity shown in [15, Corollary 6.4] is essential:

Suppose that the initial data belong to the class

$$\varrho_0 \in W^{3,2}(\mathbb{T}^d), \ \inf_{\mathbb{T}^d} \varrho_0 > 0, \quad \boldsymbol{m}_0 \in W^{3,2}(\mathbb{T}^d; R^d), \tag{4.7}$$

and the dissipative weak solution satisfies

$$\varrho \in L^\infty((0, T) \times \mathbb{T}^d), \quad \boldsymbol{u} \in L^\infty((0, T) \times \mathbb{T}^d; R^d).$$

Then $(\varrho, \boldsymbol{u})$ is a classical solution, and

$$\mathfrak{R} \equiv 0. \tag{4.8}$$

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4.2 Convergence of FV method

Summarizing the above discussion we report the following result proved in [15, Theorem 11.3].

Proposition 4.1 (Convergence of FV method). *Let $(\varrho_h, \boldsymbol{u}_h)_{h>0}$ be a FV solution (2.2) with the initial data $(\varrho_0, \boldsymbol{m}_0)$,*

$$\operatorname*{ess\,inf}_{\mathbb{T}^d} \varrho_0 > 0, \quad \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \ \mathrm{d}x < \infty.$$

Then the following holds:

1. *There exists $h_n \to 0$ such that*

$$\varrho_{h_n} \to \varrho \ \text{weakly-}(^*) \ \text{in} \ L^\infty(0, T; L^\gamma(\mathbb{T}^d)),$$

$$\boldsymbol{u}_{h_n} \to \boldsymbol{u} \ \text{weakly in} \ L^2(0, T; L^r(\mathbb{T}^d, R^d)), \ r = 6 \ \text{if} \ d = 3, \ 1 \le r < \infty \ \text{if} \ d = 2,$$

$$\boldsymbol{m}_{h_n} = \varrho_{h_n} \boldsymbol{u}_{h_n} \to \varrho \boldsymbol{u} \ \text{weakly-}(^*) \ \text{in} \ L^\infty(0, T; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d)),$$

   where $(\varrho, \boldsymbol{u})$ is a dissipative weak solution of the Navier–Stokes system specified in Section 4.1.

2. *If, in addition,*

   - *the initial data $(\varrho_0, \boldsymbol{u}_0)$ belong to the regularity class (4.7),*
   - *$\sup_{h>0} \|\varrho_h\|_{L^\infty((0,T)\times\mathbb{T}^d)} < \infty, \quad \sup_{h>0} \|\boldsymbol{u}_h\|_{L^\infty((0,T)\times\mathbb{T}^d; R^d)} < \infty,$*

   then the limit is the unique classical solution of the Navier–Stokes system and the convergence is unconditional, no need to subtract a subsequence. Moreover,

$$\varrho_h \to \varrho \ \text{in} \ L^q((0, T) \times \mathbb{T}^d),$$

$$\boldsymbol{u}_h \to \boldsymbol{u} \ \text{in} \ L^q((0, T) \times \mathbb{T}^d; R^d) \ \ \text{for any} \ 1 \le q < \infty.$$

3. *If the Navier–Stokes system admits a strong solution in the class*

$$\varrho > 0, \ \varrho \in C([0, T] \times \mathbb{T}^d), \ \nabla_x \varrho, \ \boldsymbol{u} \in C([0, T] \times \mathbb{T}^d; R^d),$$

$$\nabla_x \boldsymbol{u} \in C([0, T] \times \mathbb{T}^d; R^{d\times d}), \ \partial_t \boldsymbol{u} \in C([0, T] \times \mathbb{T}^d; R^d),$$

   then

$$\varrho_h \to \varrho \ \text{in} \ L^\gamma((0, T) \times \mathbb{T}^d),$$

$$\boldsymbol{u}_h \to \boldsymbol{u} \ \text{in} \ L^2((0, T) \times \mathbb{T}^d; R^d).$$

Remark 4.2. As a matter of fact, the conclusion of Proposition 4.1 remains valid for any consistent approximation by piecewise constant functions satisfying (4.4) together with the energy inequality (4.1) perturbed by a small consistency error:

$$\int_{\mathbb{T}^d} E(\varrho_h, \boldsymbol{u}_h)(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \left[\mu |\nabla_\mathcal{D} \boldsymbol{u}_h|^2 + \lambda |\mathrm{div}_h \boldsymbol{u}_h|^2 \right] \ \mathrm{d}x \, \mathrm{d}t$$

$$\le \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \ \mathrm{d}x + e_3[h], \ e_3[h] \to 0 \ \text{as} \ h \to 0. \tag{4.9}$$

5 Stochastic compactness method

Let $(\Omega_m^M)_{m=1}^{\nu(M)}$ be the partition considered in Theorem 2.6, with the corresponding sequence of approximate statistical solutions $(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)_{M=1}^\infty$, with the viscosity coefficients

$$\mu^M = \sum_{m=1}^{\nu(M)} \mu(\omega_m^M) \mathbb{1}_{\Omega_m^M}, \quad \eta^M = \sum_{m=1}^{\nu(M)} \eta(\omega_m^M) \mathbb{1}_{\Omega_m^M},$$

and the initial data

$$\varrho_0^M = \sum_{m=1}^{\nu(M)} \varrho_0(\omega_m^M) \mathbb{1}_{\Omega_m^M}, \quad \boldsymbol{m}_0^M = \sum_{m=1}^{\nu(M)} \boldsymbol{m}_0(\omega_m^M) \mathbb{1}_{\Omega_m^M}.$$

As we have shown in Section 3, formula (3.7),

$$\mathbb{E}\left[\|\varrho_0^M - \varrho_0\|_{L^\gamma(\mathbb{T}^d)}\right] \to 0,$$

$$\mathbb{E}\left[\|\boldsymbol{m}_0^M - \boldsymbol{m}_0\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d)}\right] \to 0,$$

$$\mathbb{E}\left[\left|\int_{\mathbb{T}^d} E(\varrho_0^M, \boldsymbol{m}_0^M) \; \mathrm{d}x - \int_{\mathbb{T}^d} E(\varrho_0, \boldsymbol{m}_0) \; \mathrm{d}x\right|\right] \to 0,$$

$$\mathbb{E}\left[|\mu^M - \mu|\right] \to 0, \quad \mathbb{E}\left[|\eta^M - \eta|\right] \to 0 \quad \text{as } M \to \infty. \tag{5.1}$$

In addition, as the initial data are admissible, cf. (2.5), in particular the initial energy is bounded, the discrete energy dissipation (4.1) yields

$$\|\varrho_{h(M)}^M\|_{L^\infty(0,T;L^\gamma(\mathbb{T}^d))} + \|\varrho_{h(M)}^M \boldsymbol{u}_{h(M)}^M\|_{L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d))} \lesssim 1$$

$$\|\boldsymbol{u}_{h(M)}^M\|_{L^2(0,T;L^r(\mathbb{T}^d;R^d))} \lesssim 1, \; r = 6 \text{ if } d = 3, \; 1 \le r < \infty \text{ if } d = 2, \tag{5.2}$$

for all $M = 1, \ldots$, and uniformly in $\omega \in \Omega$.

5.1 Application of Skorokhod's theorem

Our goal is to convert hypothesis (2.4) stated in probability on Ω into a pointwise bound. To this end, we recall the celebrated Skorokhod theorem [5, Theorem 5.1, 5.2], [10, Theorem 11.7.2].

Theorem 5.1 (Skorokhod theorem). *Let $(\boldsymbol{U}^M)_{M=1}^\infty$ be a sequence of random variables ranging in a Polish space X. Suppose that their laws are tight in X, meaning for any $\varepsilon > 0$, there exists a compact set $K(\varepsilon) \subset X$ such that*

$$\mathbb{P}[\boldsymbol{U}^M \in X \setminus K(\varepsilon)] \le \varepsilon \text{ for all } M = 1, 2, \ldots.$$

Then there is a subsequence $M_n \to \infty$ and a sequence of random variables $(\widetilde{\boldsymbol{U}}^{M_n})_{n=1}^{\infty}$ defined on the standard probability space

$$\left(\widetilde{\Omega} = [0,1], \mathfrak{B}[0,1], \mathrm{d}y\right)$$

satisfying:

- $$\widetilde{\boldsymbol{U}}^{M_n} \approx_X \boldsymbol{U}^{M_n} \text{ (they are equally distributed random variables)},$$

- $$\widetilde{\boldsymbol{U}}^{M_n} \to \widetilde{\boldsymbol{U}} \text{ in } X \text{ for every } y \in [0,1].$$

Remark 5.2. The conclusion of Theorem 5.1 are slightly stronger than the original version of Skorokhod and my be found in Jakubowski [22].

Our goal is to apply the Skorokhod theorem to the sequence of random variables

$$\boldsymbol{U}^M = \left(\varrho_0, \ \varrho_0^M, \ \boldsymbol{m}_0, \ \boldsymbol{m}_0^M, \mu, \ \eta, \ \mu^M, \ \eta^M, \ \varrho_{h(M)}^M, \ \boldsymbol{u}_{h(M)}^M, \Lambda^M\right), \ M = 1, 2, \dots$$

where

$$\Lambda^M = \|(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)\|_{L^{\infty}((0,T)\times\mathbb{T}^d;R^4)}$$

ranging in the Polish space

$$X = (L^{\gamma}(\mathbb{T}^d))^2 \times (L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d))^2 \times R^4 \times W^{-k,2}((0,T)\times\mathbb{T}^d) \times W^{-k,2}((0,T)\times\mathbb{T}^d;R^d) \times R,$$

where $k \geq 4$ is an integer. In particular $W^{k,2}((0,T)\times\mathbb{T}^d) \hookrightarrow\hookrightarrow C([0,T]\times\mathbb{T}^d)$, $d = 2, 3$. The negative Sobolev spaces could be replaced by more natural Lebesgue L^p spaces endowed with the weak topology compatible with Jakubowski's extension of Skorokhod's theorem, see [22].

Now, tightness of the laws of $(\varrho_0^M, \boldsymbol{m}_0^M, \mu^M, \eta^M)_{M=1}^{\infty}$ in $L^{\gamma}(\mathbb{T}^d) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d;R^d) \times R^2$ follows from the convergence stated in (5.1). Tightness of the laws of $(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)_{m=1}^{M}$ in $W^{-k,2}((0,T)\times \mathbb{T}^d) \times W^{-k,2}((0,T)\times\mathbb{T}^d;R^d)$ is a consequence of the bounds (5.2). Finally, tightness of the laws of $(\Lambda^M)_{M=1}^{\infty}$ in R follows from the hypothesis of boundedness in probability (2.4). As a product of a final number of compacts is compact, we conclude that $(\boldsymbol{U}^M)_{M=1}^{\infty}$, X complies with the hypotheses of the Skorokhod theorem (Theorem 5.1).

The conclusion of Theorem 5.1 can be stated in a simple form as the random variables in question are finitely distributed. Keeping the original labelling for a possible subsequence, we may infer that for each M, there exists a partition of $[0,1]$ in $\nu(M)$ Lebesgue measurable sets $\widetilde{\Omega}_m^M \subset [0,1]$ such that the following holds:

- $$\mathbb{P}[\Omega_m^M] = |\widetilde{\Omega}_m^M|, \ m = 1, \dots, \nu(M);$$

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$$\tilde{\varrho}_0^M(x,y) = \sum_{m=1}^{\nu(M)} \varrho_0(x, \omega_m^M) \mathbb{1}_{\widetilde{\Omega}_m^M}(y), \ \widetilde{\boldsymbol{m}}_0^M = \sum_{m=1}^{\nu(M)} \boldsymbol{m}_0(x, \omega_m^M) \mathbb{1}_{\widetilde{\Omega}_m^M}(y), \ x \in \mathbb{T}^d, \ y \in [0,1];$$

-

$$\widetilde{\varrho}_{h(M)}^{\nu(M)}(t,x,y) = \sum_{m=1}^{M} \varrho_{h(m),m}^M(t,x) \mathbb{1}_{\widetilde{\Omega}_m^M}(y), \ \widetilde{\boldsymbol{u}}_{h(M)}^{\nu(M)}(t,x,y) = \sum_{m=1}^{M} \boldsymbol{u}_{h(m),m}^M(t,x) \mathbb{1}_{\widetilde{\Omega}_m^M}(y),$$
$$t \in [0,T], \ x \in \mathbb{T}^d, \ y \in [0,1]. \tag{5.3}$$

In addition, there are random variables

$$\tilde{\varrho}_0 \in L^\infty(0,1; L^\gamma(\mathbb{T}^d)), \ \widetilde{\boldsymbol{m}}_0 \in L^\infty(0,1; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d)), \ \varrho_0 \approx \tilde{\varrho}_0, \ \boldsymbol{m}_0 \approx \widetilde{\boldsymbol{m}}_0,$$
$$\tilde{\varrho}_0 \in W^{3,2}(\mathbb{T}^d), \ \inf_{\mathbb{T}^d} \tilde{\varrho}_0 > 0, \ \widetilde{\boldsymbol{m}}_0 \in W^{3,2}(\mathbb{T}^d; R^d) \text{ for any } y \in [0,1]$$

such that

$$\mathbb{E}\left[\left\|\tilde{\varrho}_0^M - \tilde{\varrho}_0\right\|_{L^\gamma(\mathbb{T}^d)}\right] \to 0,$$
$$\mathbb{E}\left[\left\|\widetilde{\boldsymbol{m}}_0^M - \widetilde{\boldsymbol{m}}_0\right\|_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d)}\right] \to 0,$$
$$\mathbb{E}\left[\left|\int_{\mathbb{T}^d} E(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M) \ \mathrm{d}x - \int_{\mathbb{T}^d} E(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0) \ \mathrm{d}x\right|\right] \to 0 \text{ as } M \to \infty. \tag{5.4}$$

Next,

$$\widetilde{\mu}^{\nu(M)} = \sum_{m=1}^{M} \mu(\omega_m^M) \mathbb{1}_{\widetilde{\Omega}_m^M}(y), \ \widetilde{\eta}^{\nu(M)} = \sum_{m=1}^{M} \eta(\omega_m^M) \mathbb{1}_{\widetilde{\Omega}_m^M}(y),$$
$$\mathbb{E}\left[|\widetilde{\mu}^M - \widetilde{\mu}|\right] \to 0, \ \mathbb{E}\left[|\widetilde{\eta}^M - \widetilde{\eta}|\right] \to 0, \ \widetilde{\mu} \approx \mu, \ \widetilde{\eta} \approx \eta. \tag{5.5}$$

Finally,

$$\widetilde{\Lambda}^M = \|(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)\|_{L^\infty((0,T)\times\mathbb{T}^d; R^4)} \text{ converges as } M \to \infty \text{ for any } y,$$

which yields the desired bound

$$\|(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)\|_{L^\infty((0,T)\times\mathbb{T}^d; R^4)} \text{ bounded as } M \to \infty \text{ for any } y \in [0,1]. \tag{5.6}$$

Thus we may infer, combining (5.6) with the convergence claimed by Theorem 5.1,

$$\tilde{\varrho}_0^M \to \tilde{\varrho}_0 \text{ in } L^\gamma(\mathbb{T}^d), \ \widetilde{\boldsymbol{m}}_0^M \to \widetilde{\boldsymbol{m}}_0 \text{ in } L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^d; R^d) \text{ as } M \to \infty \text{ for any } y \in [0,1],$$
$$\widetilde{\mu}^M \to \widetilde{\mu}, \ \widetilde{\mu}^M \geq \underline{\mu} > 0, \ \widetilde{\eta}^M \to \widetilde{\eta} \text{ as } M \to \infty \text{ for any } y \in [0,1],$$
$$\tilde{\varrho}_{h(M)}^M \to \tilde{\varrho} \text{ weakly-}(*) \text{ in } L^\infty((0,T)\times\mathbb{T}^d),$$
$$\tilde{\boldsymbol{u}}_{h(M)}^M \to \tilde{\boldsymbol{u}} \text{ weakly-}(*) \text{ in } L^\infty((0,T)\times\mathbb{T}^d; R^d) \text{ as } M \to \infty \text{ for any } y \in [0,1]. \tag{5.7}$$

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5.2   Limit system

Our ultimate goal in this section is to identify the system of equations satisfied by the limit $(\tilde{\varrho}, \tilde{\boldsymbol{u}})$ obtained in (5.7). To this end, we fix $y \in [0, 1]$, keeping in mind that each time we consider a subsequence, where the latter may depend on y. Fortunately, as we shall see below, the limit will be independent of the choice of the subsequence therefore unconditional.

First, observe that (5.4) yields

$$\int_{\mathbb{T}^d} E(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M) \ \mathrm{d}x \to \int_{\mathbb{T}^d} E(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0) \ \mathrm{d}x \text{ as } M \to \infty \text{ a.s. in } [0, 1] \tag{5.8}$$

at least for a suitable subsequence.

In accordance with (5.3), the pair $(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)$ is a solution of FV scheme (2.2) with the initial data $(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M)$ for any $y \in [0, 1]$. Specifically, the consistency formulation (4.1), (4.4),

$$\int_{\mathbb{T}^d} E(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \left[\tilde{\mu}^M |\nabla_{\mathcal{D}} \tilde{\boldsymbol{u}}_{h(M)}^M|^2 + \tilde{\lambda}^M |\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M|^2 \right] \ \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_{\mathbb{T}^d} E(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M) \ \mathrm{d}x, \tag{5.9}$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\tilde{\varrho}_{h(M)}^M \partial_t \varphi + \tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \cdot \nabla_x \varphi \right] \ \mathrm{d}x \, \mathrm{d}t = - \int_{\mathbb{T}^d} \tilde{\varrho}_0^M \varphi(0, \cdot) \ \mathrm{d}x + e_1[h(M), \varphi],$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \cdot \partial_t \boldsymbol{\varphi} + \tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \otimes \tilde{\boldsymbol{u}}_{h(M)}^M : \nabla_x \boldsymbol{\varphi} + p(\tilde{\varrho}_{h(M)}^M) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^d} \tilde{\mu}^M \nabla_{\mathcal{D}} \tilde{\boldsymbol{u}}_{h(M)}^M : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \tilde{\lambda}^M \ \mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M \mathrm{div} \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t$$

$$- \int_{\mathbb{T}^d} \widetilde{\boldsymbol{m}}_0^M \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x + e_2[h(M), \boldsymbol{\varphi}], \tag{5.10}$$

holds for any $y \in [0, 1]$.

Now, we use the fact that the viscosity coefficients $\tilde{\mu}^M$ are bounded below by a positive deterministic constant $\underline{\mu}$ to deduce from the energy inequality (5.9) a uniform bound

$$\int_0^T \int_{\mathbb{T}^d} \left[|\nabla_{\mathcal{D}} \tilde{\boldsymbol{u}}_{h(M)}^M|^2 + |\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M|^2 \right] \ \mathrm{d}x \, \mathrm{d}t \overset{<}{\sim} 1 \text{ uniformly for } y \in [0, 1]. \tag{5.11}$$

This allows us to rewrite the consistency formulation (5.9), (5.10) in the form:

$$\int_{\mathbb{T}^d} E(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)(\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \left[\tilde{\mu} |\nabla_{\mathcal{D}} \tilde{\boldsymbol{u}}_{h(M)}^M|^2 + \tilde{\lambda} |\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M|^2 \right] \ \mathrm{d}x \, \mathrm{d}t$$

$$\leq \int_{\mathbb{T}^d} E(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0) \ \mathrm{d}x$$

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$$+ \int_0^\tau \int_{\mathbb{T}^d} \left[(\widetilde{\mu} - \widetilde{\mu}^M) |\nabla_\mathcal{D} \tilde{\boldsymbol{u}}_{h(M)}^M|^2 + (\widetilde{\lambda} - \widetilde{\lambda}^M) |\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M|^2 \right] \mathrm{d}x \,\mathrm{d}t$$

$$+ \int_{\mathbb{T}^d} E(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M) - E(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0) \,\mathrm{d}x,$$

$$(5.12)$$

where, by virtue of the bound (5.11) and the convergence stated in (5.7), (5.8),

$$e_3[h(M)] \equiv \int_0^T \int_{\mathbb{T}^d} \left[(\widetilde{\mu} - \widetilde{\mu}^M) |\nabla_\mathcal{D} \tilde{\boldsymbol{u}}_{h(M)}^M|^2 + (\widetilde{\lambda} - \widetilde{\lambda}^M) |\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M|^2 \right] \mathrm{d}x \,\mathrm{d}t$$

$$+ \int_{\mathbb{T}^d} E(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M) - E(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0) \,\mathrm{d}x \to 0 \text{ as } M \to \infty \text{ a.s. in } [0,1].$$

Using similar arguments, we can deduce from (5.10):

$$\int_0^T \int_{\mathbb{T}^d} \left[\tilde{\varrho}_{h(M)}^M \partial_t \varphi + \tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \cdot \nabla_x \varphi \right] \mathrm{d}x \,\mathrm{d}t = - \int_{\mathbb{T}^d} \tilde{\varrho}_0 \varphi(0, \cdot) \,\mathrm{d}x + e_1[h(M), \varphi],$$

$$\int_0^T \int_{\mathbb{T}^d} \left[\tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \cdot \partial_t \boldsymbol{\varphi} + \tilde{\varrho}_{h(M)}^M \tilde{\boldsymbol{u}}_{h(M)}^M \otimes \tilde{\boldsymbol{u}}_{h(M)}^M : \nabla_x \boldsymbol{\varphi} + p(\tilde{\varrho}_{h(M)}^M) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \,\mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^d} \widetilde{\mu} \nabla_\mathcal{D} \tilde{\boldsymbol{u}}_{h(M)}^M : \nabla_x \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \widetilde{\lambda} \,\mathrm{div}_h \tilde{\boldsymbol{u}}_{h(M)}^M \mathrm{div} \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}t$$

$$- \int_{\mathbb{T}^d} \widetilde{\boldsymbol{m}}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \,\mathrm{d}x + e_2[h(M), \boldsymbol{\varphi}] \quad \text{a.s. in } [0,1], \tag{5.13}$$

where the errors resulting from replacing $(\tilde{\varrho}_0^M, \widetilde{\boldsymbol{m}}_0^M)$ by $(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0)$, and $(\widetilde{\mu}^M, \widetilde{\eta}^M)$ by $(\widetilde{\mu}, \widetilde{\eta})$ can be incorporated in e_1, e_2. Thus, we may infer that $(\tilde{\varrho}_{h(M)}^M, \tilde{\boldsymbol{u}}_{h(M)}^M)_{M=1}^\infty$ is a consistent approximation of the Navier–Stokes system with the viscosity coefficients $\widetilde{\mu}$, $\widetilde{\lambda}$ and the initial data $(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0)$ in the sense specified in (4.4), (4.9) for a.a. $y \in [0,1]$.

In view of (5.6), the approximate solutions are uniformly bounded; whence we may use the convergence result stated in Proposition 4.1 to conclude that

$$\tilde{\varrho}_{h(M)}^M \to \tilde{\varrho} \quad \text{in } L^q((0,T) \times \mathbb{T}^d)$$

$$\tilde{\boldsymbol{u}}_{h(M)}^M \to \tilde{\boldsymbol{u}} \quad \text{in } L^q((0,T) \times \mathbb{T}^d; R^d) \quad \text{a.s. in } [0,1], \ 1 \le q < \infty,$$

where $(\tilde{\varrho}, \tilde{\boldsymbol{u}})$ is the unique classical solution of the Navier-Stokes system with the initial data $(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0)$ and viscosity coefficients $\widetilde{\mu}, \widetilde{\lambda}$. As the limit is unique, the convergence is unconditional.

6 Application of Gyöngy – Krylov theorem

The ultimate goal is to show the convergence of the numerical method in terms of the original probability space. To this end we apply the following result, see Gyöngy, Krylov [19, Lemma 1.1].

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Theorem 6.1 (Gyöngy–Krylov theorem). *Let X be a Polish space and $(\boldsymbol{U}^M)_{M \geq 1}$ a sequence of $X-$valued random variables.*

Then $(\boldsymbol{U}^M)_{M=1}^\infty$ converges in probability if and only if for any sequence of joint laws of

$$(\boldsymbol{U}^{M_k}, \boldsymbol{U}^{N_k})_{k=1}^\infty$$

there exists further subsequence that converge weakly to a probability measure μ on $X \times X$ such that

$$\mu\left[(x, y) \in X \times X, \ x = y\right] = 1.$$

Under the hypotheses of the previous section, we consider the sequence

$$\left(\varrho_0, \ \varrho_0^{M_k}, \ \boldsymbol{m}_0, \ \boldsymbol{m}_0^{M_k}, \mu, \ \eta, \ \mu^{M_k}, \ \eta^{M_k}, \ \varrho_{h(M_k)}^{M_k}, \ \boldsymbol{u}_{h(M_k)}^{M_k}, \ \Lambda^{M_k}, \right.$$

$$\left. \varrho_0^{N_k}, \ \boldsymbol{m}_0^{N_k}, \ \mu^{N_k}, \ \eta^{N_k}, \ \varrho_{h(N_k)}^{N_k}, \ \boldsymbol{u}_{h(N_k)}^{N_k}, \ \Lambda^{N_k} \right)_{k=1}^\infty.$$

Similarly to the preceding section, we obtain the Skorokhod representation

$$\tilde{\varrho}_{h(M_k)}^{M_k}, \ \tilde{\varrho}_{h(N_k)}^{N_k}, \ \widetilde{\boldsymbol{m}}_{h(M_k)}^{M_k}, \ \widetilde{\boldsymbol{m}}_{h(N_k)}^{N_k}$$

satisfying

$$(\tilde{\varrho}_{h(M_k)}^{M_k}, \tilde{\varrho}_{h(N_k)}^{N_k}) \to (\tilde{\varrho}, \tilde{\varrho}) \text{ in } L^q((0,T) \times \mathbb{T}^d) \text{ a.s. in } [0,1],$$

$$(\tilde{\boldsymbol{u}}_{h(M_k)}^{M_k}, \tilde{\boldsymbol{u}}_{h(N_k)}^{N_k}) \to (\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}}) \text{ in } L^q((0,T) \times \mathbb{T}^d; R^3) \text{ a.s. in } [0,1], \ 1 \leq q < \infty, \quad (6.1)$$

where $(\tilde{\varrho}, \tilde{\boldsymbol{u}})$ is the unique classical solution of the Navier-Stokes system with the initial data $(\tilde{\varrho}_0, \widetilde{\boldsymbol{m}}_0)$ and viscosity coefficients $\widetilde{\mu}, \widetilde{\lambda}$.

Thus applying the Gyöngy–Krylov theorem to the sequence

$$(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)_{M=1}^\infty \text{ ranging in the Polish space } L^q(\mathbb{T}^d; R^4),$$

we may infer that

$$\varrho_{h(M)}^M \to \varrho, \ \boldsymbol{u}_{h(M)}^M \to \boldsymbol{u} \text{ in } L^q(\mathbb{T}^d; R^4), \ 1 \leq q < \infty, \text{ in probability}, \quad (6.2)$$

where $(\varrho, \boldsymbol{u})$ is the unique solution of the Navier–Stokes system with the initial data $(\varrho_0, \boldsymbol{m}_0)$ and the viscosity coefficients μ, η.

We have shown Theorem 2.6.

6.1 Unconditional convergence

It remains to show Corollary 2.8. As a byproduct of the conclusion of Theorem 2.6, we already know that the Navier–Stokes system admits a classical solution in $([0,T] \times \mathbb{T}^d)$ for $\mathbb{P}-$ a.a. data. Consequently, Corollary 2.8 follows directly from Proposition 4.1, part 3. Indeed the convergence is now unconditional and there is no need of the stochastic compactness method.

7 Concluding remarks

In this paper we have studied convergence of a stochastic collocation FV method for the random compressible Navier–Stokes system. We have clarified the necessary conditions for the stochastic collocation to be meaningful, meaning arbitrary choice of the collocation points and the "elements" in the probability space gives rise to the same asymptotic limit. We also eliminated the ambiguity in the limit of approximate numerical solutions by requiring that the approximate solutions are bounded in probability. The result can be seen as the first rigorous proof of convergence of a statistical method for the compressible Navier–Stokes system.

We have made several constitutive restrictions that can be easily removed. In particular, the specific isentropic form of the pressure is not really necessary, any EOS preserving monotonicity of p as a functions of ϱ and the asymptotic behaviour $p(\varrho) \approx \varrho^\gamma$ for $\varrho \to \infty$ can be handled by the same method.

The Navier–Stokes system can be augmented by a driving force that can be random similarly to the viscosity coefficients μ and η.

The results actually holds for any consistent approximation of the Navier–Stokes system not necessarily generated by a numerical method.

Finally, let us comment shortly on *boundedness in probability* postulated in hypothesis (2.4). The situation is particularly simple if we assume some pointwise bounds though not on a very large set. As we have observed in Corollary 2.8, it is enough to find *one* particular sequence of partitions $(\Omega_m^M)_{m=1}^{\nu(M)}$, $M = 1, 2, \ldots$, with the associated collocation nodes ω_m^M such that (2.4) holds to guarantee convergence of the method for *any* sequence of partitions. The following result (Taylor [38, Proposition 5.1]) comes handy.

Lemma 7.1. *For any $\varepsilon > 0$, there exists a partition $(\Omega_m^M)_{m=1}^{\nu(M)}$ such that*

$$\mathrm{diam}[\Omega_m^M] < \varepsilon, \ \mathbb{P}[\partial \Omega_m^M] = 0 \ \textit{for all } m = 1, \ldots, \nu(M).$$

Now, given $K > 0$ consider the set $R(K) \subset \Omega$ of "regular events",

$$R(K) = \Big\{ \omega \in \Omega \ \Big| \ \text{there exists } \overline{h} > 0 \text{ and a family of FV solutions } (\varrho_h, \boldsymbol{u}_h)_{0 < h < \overline{h}}$$
$$\text{with the initial data } (\varrho_0, \boldsymbol{m}_0)(\omega) \text{ and the viscosity coefficients} (\mu, \eta)(\omega)$$
$$\text{such that} \| (\varrho_h, \boldsymbol{u}_h) \|_{L^\infty(\mathbb{T}^d; R^{d+1})} \le K \text{ for all } 0 < h < \overline{h} \Big\}.$$

Consider the following hypothesis:

$$\text{There exists } K > 0 \text{ such that } R(K) \text{ is dense in } \Omega. \tag{7.1}$$

Note that a dense set may not be "big" in the measure sense, in particular, we may have

$$\mathbb{P}[R(K)] = 0.$$

We claim that (7.1) yields boundedness in probability for a suitable choice of partitions. Indeed, for any $\varepsilon = \varepsilon(M) \to 0$, Lemma 7.1 yields a partition $(\Omega_m^M)_{m=1}^{\nu(M)}$ such that

$$\sup_{m=1,\dots,\nu(M)} \text{diam}[\Omega_m^M] < \varepsilon(M), \ \mathbb{P}[\partial\Omega_m^M] = 0 \text{ for all } m = 1,\dots,\nu(M).$$

In particular,

$$\mathbb{P}[\Omega_m^M] = \mathbb{P}[\text{int}[\Omega_m^M]].$$

Now, we choose the collocation nodes:

$$\omega_m^M \in R(K) \text{ if } \text{int}[\Omega_m^M] \neq \emptyset, \ \omega_m^M \text{ arbitrary otherwise.}$$

Moreover, as the partition is finite, we may fix $h = h(M)$ so that

$$\|(\varrho_{h(M)}^M, \boldsymbol{u}_{h(M)}^M)\|_{L^\infty(\mathbb{T}^d; R^{d+1})} \leq K \ \ \mathbb{P} - \text{a.s.}$$

As this construction can be repeated for any $M = \varepsilon(M) \to 0$, the desired conclusion follows.

References

[1] I. Babuška, F. Nobile, and R. Tempone. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM Rev.*, **52**(2):317–355, 2010.

[2] I. Babuška, R. Tempone, and G.E. Zouraris. Galerkin finite element approximations of stochastic elliptic partial differential equations. *SIAM J. Numer. Anal.*, **42**(2):800–825, 2004.

[3] G. A. Beer. A note on Riemann integrability. *Internat. J. Math. Math. Sci.*, **1**(1):69–73, 1978.

[4] A. Bespalov, C.E. Powell, and D. Silvester. A priori error analysis of stochastic Galerkin mixed approximations of elliptic PDEs with random data. *SIAM J. Numer. Anal.*, **50**(4):2039–2063, 2012.

[5] P. Billingsley. *Convergence of Probability Measures*. Wiley, New York, 1999.

[6] D. Breit, E. Feireisl, and M. Hofmanová. *Stochastically forced compressible fluid flows*. De Gruyter Series in Applied and Numerical Mathematics 3. De Gruyter, Berlin, 2018.

[7] J. Charrier, R. Scheichl, and A. L. Teckentrup. Finite element error analysis of elliptic PDEs with random coefficients and its application to multilevel Monte Carlo methods. *SIAM J. Numer. Anal.*, **51**(1):322–352, 2013.

[8] A. Chertock, A. Kurganov, M. Lukáčová-Medviďová, P. Spichtinger, and B. Wiebe. Stochastic Galerkin method for cloud simulation. *Math. Clim. Weather Forecast.*, **5**(1):65–106, 2019.

[9] A. Cohen, R. Devore, and C. Schwab. Analytic regularity and polynomial approximation of parametric and stochastic elliptic PDE's. *Anal. Appl. (Singap.)*, **9**(1):11–47, 2011.

[10] R. M. Dudley. *Real analysis and probability*, volume 74 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Revised reprint of the 1989 original.

[11] O. G. Ernst, A. Mugler, H.-J. Starkloff, and E. Ullmann. On the convergence of generalized polynomial chaos expansions. *ESAIM Math. Model. Numer. Anal.*, **46**(2):317–339, 2012.

[12] E. Feireisl. *Dynamics of viscous compressible fluids*. Oxford University Press, Oxford, 2004.

[13] E. Feireisl and M. Hofmanová. On convergence of approximate solutions to the compressible Euler system. *Ann. PDE*, **6**(2):11, 2020.

[14] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová, and B. She. *Numerical analysis of compressible fluid flows*. Springer-Verlag. To appear.

[15] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová, and B. She. Convergence of a finite volume scheme for the compressible Navier–Stokes system. *ESAIM Math. Model. Numer. Anal.*, **53**(6):1957–1979, 2019.

[16] E. Feireisl, M. Lukáčová-Medviďová, and H. Mizerová. A finite volume scheme for the Euler system inspired by the two velocities approach. *Numer. Math.*, **144**(1):89–132, 2020.

[17] J. Foo and G.E. Karniadakis. Multi-element probabilistic collocation method in high dimensions. *J. Comput. Phys.*, **229**(5):1536–1557, 2010.

[18] J. Foo, X. Wan, and G.E. Karniadakis. The multi-element probabilistic collocation method (ME-PCM): error analysis and applications. *J. Comput. Phys.*, **227**(22):9572–9595, 2008.

[19] I. Gyöngy and N. Krylov. Existence of strong solutions for Itô's stochastic equations via approximations. *Probab. Theory Related Fields*, **105**(2):143–158, 1996.

[20] S. Hanneke, A. Kontorovich, S. Sabato, and R. Weiss. Universal Bayes consistency in metric spaces. *Ann. Statist.*, **49**(4):2129–2150, 2021.

[21] L. Herrmann and C. Schwab. Multilevel quasi-Monte Carlo integration with product weights for elliptic PDEs with lognormal coefficients. *ESAIM Math. Model. Numer. Anal.*, **53**(5):1507–1552, 2019.

[22] A. Jakubowski. The almost sure Skorokhod representation for subsequences in nonmetric spaces. *Teor. Veroyatnost. i Primenen.*, **42**(1):209–216, 1997.

[23] U. Koley, N.H. Risebro, C. Schwab, and F. Weber. A multilevel Monte Carlo finite difference method for random scalar degenerate convection-diffusion equations. *J. Hyperbolic Differ. Equ.*, **14**(3):415–454, 2017.

[24] F. Y. Kuo, R. Scheichl, C. Schwab, I. H. Sloan, and E. Ullmann. Multilevel quasi-Monte Carlo methods for lognormal diffusion problems. *Math. Comp.*, **86**(308):2827–2860, 2017.

[25] O. P. Le Maître and O. M. Knio. *Spectral methods for uncertainty quantification.* Scientific Computation. Springer, New York, 2010. With applications to computational fluid dynamics.

[26] O.P. Le Maître, O.M. Knio, H.N. Najm, and R.G. Ghanem. Uncertainty propagation using Wiener-Haar expansions. *J. Comput. Phys.*, **197**(1):28–57, 2004.

[27] O.P. Le Maître, H.N. Najm, P.P. Pébay, R.G. Ghanem, and O.M. Knio. Multi-resolution-analysis scheme for uncertainty quantification in chemical systems. *SIAM J. Sci. Comput.*, **29**(2):864–889, 2007.

[28] F. Leonardi, S. Mishra, and C. Schwab. Numerical approximation of statistical solutions of planar, incompressible flows. *Math. Models Methods Appl. Sci.*, **26**(13):2471–2523, 2016.

[29] P.-L. Lions. *Mathematical topics in fluid dynamics, Vol.2, Compressible models.* Oxford Science Publication, Oxford, 1998.

[30] A. Matsumura and T. Nishida. The initial value problem for the equations of motion of compressible and heat conductive fluids. *Comm. Math. Phys.*, **89**:445–464, 1983.

[31] S. Mishra and Ch. Schwab. Sparse tensor multi-level Monte Carlo finite volume methods for hyperbolic conservation laws with random initial data. *Math. Comp.*, **81**(280):1979–2018, 2012.

[32] F. Nobile, R. Tempone, and C.G. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM J. Numer. Anal.*, **46**(5):2309–2345, 2008.

[33] M.P. Pettersson, G. Iaccarino, and J. Nordström. *Polynomial chaos methods for hyperbolic partial differential equations.* Mathematical Engineering. Springer, Cham, 2015. Numerical techniques for fluid dynamics problems in the presence of uncertainties.

[34] S.A. Smolyak. Quadrature and interpolation formulas for tensor products of certain classes of functions. *Soviet Math. Dokl.*, **4**:240–243, 1963.

[35] Y. Sun, C. Wang, and Z. Zhang. A Beale-Kato-Majda criterion for the 3-D compressible Navier-Stokes equations. *J. Math. Pures Appl.*, **95**(1):36–47, 2011.

[36] T. Tang and T. Zhou. Convergence analysis for stochastic collocation methods to scalar hyperbolic equations with a random wave speed. *Commun. Comput. Phys.*, **8**(1):226–248, 2010.

[37] A. Tani. On the first initial-boundary value problem of compressible viscous fluid motion. *Publ. RIMS Kyoto Univ.*, **13**:193–253, 1977.

[38] M. Taylor. Riemann integrable functions on a compact measured metric space. *https://mtaylor.web.unc.edu/notes/functional-analysis-course*, 2021. Functional analysis course, Univ. of North Carolina.

[39] D. Xiu. *Numerical methods for stochastic computations*. Princeton University Press, Princeton, NJ, 2010. A spectral method approach.

[40] D. Xiu. Stochastic collocation methods: a survey. In *Handbook of uncertainty quantification. Vol. 1, 2, 3*, pages 699–716. Springer, Cham, 2017.

[41] Z. Zhang and G.E. Karniadakis. *Numerical methods for stochastic partial differential equations with white noise*, volume 196 of *Applied Mathematical Sciences*. Springer, Cham, 2017.