On strong continuity of weak solutions to the compressible Euler equations

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Prologue

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0,T]; L^p(\Omega; R^d)),\ t \mapsto \int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \ {\rm d}x \in C[0,T]$$

$$\boldsymbol{\varphi} \in L^{p'}(\Omega; R^d)$$

Strong continuity

$$\tau \in [0,T],\ \| \mathbf{U}(t,\cdot) - \mathbf{U}(\tau, \cdot) \|_{L^p(\Omega;R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong} \ \Rightarrow \text{weak},\ \text{weak} \not\Rightarrow \text{strong}$$

Euler system for a barotropic inviscid fluid

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

Momentum equation

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = 0,\ p(\varrho) = a\varrho^{\gamma},\ a > 0,\ \gamma > 1$$

Impermeability boundary conditions

$$\mathbf{m} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0,\ \mathbf{m}(0, \cdot) = \mathbf{m}_0$$

First and Second law – energy

Energy

$$\mathcal{E} = \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho),\ \ P'(\varrho)\varrho - P(\varrho) = p(\varrho)$$

$$p' \geq 0 \Rightarrow [\varrho, \mathbf{m}] \mapsto \left\{ \begin{array}{l} \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \text{ if } \varrho > 0 \\ P(\varrho) \text{ if } |\mathbf{m}| = 0 \\ \infty \text{ if } \varrho = 0, |\mathbf{m}| \neq 0 \end{array} \right. \quad \text{is convex l.s.c}$$

Energy balance (conservation)

$$\partial_t \mathcal{E} + \mathrm{div}_x \left(\mathcal{E} \frac{\mathbf{m}}{\varrho} \right) + \mathrm{div}_x \left(p \frac{\mathbf{m}}{\varrho} \right) = 0$$

Energy dissipation

$$\partial_t \mathcal{E} + \mathrm{div}_x(\mathcal{E}\mathbf{u}) + \mathrm{div}_x(p\mathbf{u}) \boxed{\leq} 0$$

$$E = \int_\Omega \mathcal{E}\ \mathrm{d}x,\ \ \partial_t E \leq 0,\ \ E(0+) = \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right]\ \mathrm{d}x$$

Weak solutions

Field equations

$$\int_0^\infty \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \ \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x, \ \varphi \in C^1_c([0,\infty) \times \overline{\Omega})$$

$$\int_0^\infty \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} : \nabla_x \boldsymbol{\varphi} + p(\varrho) \mathrm{div}_x \boldsymbol{\varphi} \right] \ \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \mathbf{m}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x, \ \boldsymbol{\varphi} \in C^1_c([0,T) \times \overline{\Omega}; R^N), \ \boldsymbol{\varphi} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Admissible weak solutions

$$\int_0^\infty \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x \ \partial_t \psi \ \mathrm{d}t \geq 0$$

$$\psi \in C^1_c(0,\infty), \ \psi \geq 0$$

“Typical” convex integration results(ignoring Riemann problem)

Result A: (De Lellis-Székelyhidy, Chiodaroli)
For any smooth initial data there exist infinitely many solutions satisfying the energy inequality on the open interval $(0, T)$ but experiencing initial energy “jump”

Result B: (De Lellis-Székelyhidy, Chiodaroli, Xin et al., EF)
For any smooth initial density ϱ_0 there exists $\mathbf{m}_0$ (not enecessarily regular) such that there are infinitely many weak solutions satisfying the energy inequality on the open interval $(0, T)$ and with the energy continous at $t = 0$

Result C (Giri and Kwon) :
There is a set of smooth initial densities ϱ_0 and Hölder $\mathbf{m}_0$ such that there are infinitely many solutions satisfying the energy equation on the open interval $(0, T)$ (with the energy continous at $t = 0$)

Class of Riemann integrable functions

Class $\mathcal{R}$

The complement of the points of continuity of **U** is of zero Lebesgue measure in a domain Q

Riemann integrability

A function **U** is Riemann integrable in Q only if **U** belongs to the class $\mathcal{R}$

Oscillations

$$\mathrm{osc}[v](y) = \lim_{s \searrow 0} \left[\sup_{B((y),s)\cap\overline{Q}} v - \inf_{B((y),s)\cap\overline{Q}} v \right],$$

$$A_\eta = \left\{ (y) \in \overline{Q} \;\middle|\; \mathrm{osc}[v](y) \geq \eta \right\} \text{ is closed and of zero content}$$

$$A_\eta \subset \cup_{i\in\mathrm{fin}} Q_i,\ \sum_i |Q_i| < \delta \text{ for any } \delta > 0,\ Q_i \text{ - a box}$$

Main result

Theorem

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$, and E be given such that

$$\varrho_0 \in \mathcal{R}(\Omega),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$
$$0 \leq E \leq \overline{E},\ E \in \mathcal{R}(0, T).$$

Then there exists a positive constant E_∞ (large) such that the Euler problem admits infinitely many weak solutions with the energy profile

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] (t, \cdot)\ \mathrm{d}x = E_\infty + E(t) \text{ for a.a. } t \in (0, T)$$

Strongly discontinuous solutions, I

Let $d = 2, 3$. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R}(\Omega),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.

Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)]\ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i,\ i = 1, 2, \dots.$$

Strongly discontinuous solutions, II

Let $d = 2, 3$. Let ϱ_0,

$$\varrho_0 \in C^\infty(\overline{\Omega}),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

be given, together with an F_σ subset G of Ω, $|G| = 0$, and an arbitrary (countable dense) set of times $\{\tau_i\}_{i=1}^\infty \subset (0, T)$

Then there exists

$$\mathbf{m}_0 \in \mathcal{R}(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

such that the Euler problem admits infinitely many weak solution ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that ϱ is not continuous at any point

$$t > 0,\ x \in G,$$

and

$$\varrho \in C_{\mathrm{weak}}([0, T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\mathrm{weak}}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

with

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)] \text{ not strongly continuous at any } \tau_i,\ i = 1, 2, \ldots.$$

Strongly discontinuous solutions, III

Let $d = 2, 3$. Let ϱ_0,

$$\varrho_0 \in C^\infty(\overline{\Omega}),\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

be given, together with an F_σ subset G of Ω, $|G| = 0$, an arbitrary (countable dense) set of times $\{\tau_i\}_{i=1}^\infty \subset (0, T)$, and a number $\delta > 0$.

Then there exists

$$\mathbf{m}_0 \in L^\infty(\Omega; R^d),\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R}(\Omega),\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0$$

such that the Euler problem admits infinitely many weak solution ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile *continuous* at $t = 0$ such that ϱ is not continuous at any point

$$t > \delta,\ x \in G,$$

$$\varrho \in C_{\rm weak}([0, T]; L^\gamma(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

with

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)] \text{ not strongly continuous at any } \tau_i,\ i = 1, 2, \dots,\ \tau_i > \delta$$

Convex integration ansatz

Helmholtz decomposition of the initial data

$$\mathbf{m}_0 = \mathbf{v}_0 + \nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v}_0 = 0,\ \Delta_x \Phi_0 = \mathrm{div}_x \mathbf{m}_0,\ (\nabla_x \Phi_0 - \mathbf{m}_0) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Convex integration ansatz

$$\varrho(t,x) = \varrho_0 + h(t)\Delta_x \Phi_0,\ h(0) = 0,\ h'(0) = -1$$

$$\mathbf{m}(t,x) = \mathbf{v} - h'(t)\nabla_x \Phi_0,\ \mathrm{div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{v}(0,\cdot) = \mathbf{v}_0$$

“Overdetermined” Euler system

Given quantities

$$h,\ \Phi_0 \varrho$$

Balance of momentum

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \frac{1}{d} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \mathbb{I} \right)$$

$$= 0$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0,\ \mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Energy

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} = \Lambda(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t)\Phi_0$$

Subsolutions

Energy profile

$$e = e(t,x) = \frac{E(t)}{|\Omega|} + \Lambda_0(t) - \frac{d}{2} p(\varrho) + \frac{d}{2} h''(t)\Phi_0, \ e \in \mathcal{R}([0,T] \times \overline{\Omega}).$$

Field equations

$$\mathrm{div}_x \mathbf{v} = 0, \ \partial_t \mathbf{v} + \mathrm{div}_x \mathbb{U} = 0, \ \mathbf{v}(0,\cdot) = \mathbf{v}_0, \ \mathbb{U}(t,x) \in R^{d\times d}_{\mathrm{sym},0}$$

Convex constraint

$$\frac{d}{2} \sup_{[0,T]\times\overline{\Omega}} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right] < \inf_{[0,T]\times\overline{\Omega}} e$$

Algebraic inequality

$$\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} \le \frac{d}{2} \lambda_{\max} \left[\frac{(\mathbf{v} - h'(t)\nabla_x \Phi_0) \otimes (\mathbf{v} - h'(t)\nabla_x \Phi_0)}{\varrho} - \mathbb{U} \right]$$

Critical points (De Lellis- Székelyhidi)

Convex functional

$$I[\mathbf{v}] = \int_0^T \int_\Omega \left(\frac{1}{2} \frac{|\mathbf{v} - h'(t)\nabla_x \Phi_0|^2}{\varrho} - e \right) \mathrm{d}x\mathrm{d}t \text{ for } \mathbf{v} \in X.$$

Zero points

$$I[\mathbf{v}] = 0 \ \Rightarrow \ \mathbf{v} \text{ is a weak solution of the problem}$$

Points of continuity

$$\mathbf{v} \text{ – a point of continuity of } I \text{ on } X \ \Rightarrow I[\mathbf{v}] = 0$$

Oscillatory Lemma (De Lellis, Székelyhidi)

Oscillatory Lemma, basic constant coefficients form

Let $Q = (0,1) \times (0,1)^d$, $d = 2,3$. Suppose that $\mathbf{v} \in R^d$, $\mathbb{U} \in R^{d \times d}_{0,\mathrm{sym}}$, $e \leq \overline{e}$ are given constant quantities such that

$$\frac{d}{2} \lambda_{\max} \left[\mathbf{v} \otimes \mathbf{v} - \mathbb{U} \right] < e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences of vector functions $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C^\infty_c(Q; R^d), \ \mathbb{V}_n \in C^\infty_c(Q; R^{d \times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0, \ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \lambda_{\max} \left[(\mathbf{v} + \mathbf{w}_n) \otimes (\mathbf{v} + \mathbf{w}_n) - (\mathbb{U} + \mathbb{V}_n) \right] < e \text{ in } Q \text{ for all } n = 1, 2, \dots,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,1]; L^2((0,1)^d; R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n \to \infty} \int_Q |\mathbf{w}_n|^2 \mathrm{d}x \mathrm{d}t \geq c(d, \overline{e}) \int_Q \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 \mathrm{d}x \mathrm{d}t$$

Oscillatory Lemma, continuous form

$$\mathbf{v} \in C(\overline{Q}; R^d),\ \mathbb{U} \in C(\overline{Q}; R^{d\times d}_{0,\mathrm{sym}}),\ e \in C(\overline{Q}),\ r \in \mathcal{C}(\overline{Q}),\ Q = (0,T)\times\Omega$$

$$0 < \underline{r} \le r(t,x) \le \overline{r},\ e(t,x) \le \overline{e} \text{ for all } (t,x) \in \overline{Q},$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{\mathbf{v}\otimes\mathbf{v}}{r} - \mathbb{U}\right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C_c^{\infty}(Q; R^d),\ \mathbb{V}_n \in C_c^{\infty}(Q; R^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{w}_n)\otimes(\mathbf{v}+\mathbf{w}_n)}{r} - (\mathbb{U}+\mathbb{V}_n)\right] < \inf_{\overline{Q}} e,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,T]; \Omega; R^d)) \text{ as } n\to\infty,$$

$$\liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{r} \mathrm{d}x\mathrm{d}t \ge c(d,\overline{e}) \int_Q \left(e - \frac{1}{2}\frac{|\mathbf{v}|^2}{r}\right)^2 \mathrm{d}x\mathrm{d}t$$

Oscillatory Lemma, proof via decomposition

- Domain decomposition

$$Q = \cup_{i \in \mathrm{fin}} Q_i, \;\; Q_i \text{ boxes}$$

- Replace the functions by constants (integral means) on each Q_i. The difference is small if the functions are continuous and $\mathrm{diam}[Q_i]$ is small so that all relevant inequalities remain valid
- Use the fact that the constant version of oscillatory lemma is invariant under scaling and apply it on each Q_i
- Sum up the results

Oscillatory Lemma, "Riemann" form

$$\mathbf{v} \in \mathcal{R}(\overline{Q}; R^d),\ \mathbb{U} \in \mathcal{R}(\overline{Q}; R^{d\times d}_{0,\mathrm{sym}}),\ e \in \mathcal{R}(\overline{Q}),\ r \in \mathcal{R}(\overline{Q}),\ Q = (0,T)\times\Omega$$

$$0 < \underline{r} \leq r(t,x) \leq \overline{r},\ \ e(t,x) \leq \overline{e} \text{ for all } (t,x) \in \overline{Q},$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{\mathbf{v}\otimes\mathbf{v}}{r} - \mathbb{U}\right] < \inf_{\overline{Q}} e.$$

Then there is a constant $c = c(d, \overline{e})$ and sequences $\{\mathbf{w}_n\}_{n=1}^{\infty}$, $\{\mathbb{V}_n\}_{n=1}^{\infty}$,

$$\mathbf{w}_n \in C_c^{\infty}(Q; R^d),\ \mathbb{V}_n \in C_c^{\infty}(Q; R^{d\times d}_{0,\mathrm{sym}})$$

satisfying

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{V}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } Q,$$

$$\frac{d}{2} \sup_{\overline{Q}} \lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{w}_n)\otimes(\mathbf{v}+\mathbf{w}_n)}{r} - (\mathbb{U}+\mathbb{V}_n)\right] < \inf_{\overline{Q}} e,$$

$$\mathbf{w}_n \to 0 \text{ in } C_{\mathrm{weak}}([0,T];\Omega;R^d)) \text{ as } n \to \infty,$$

$$\liminf_{n\to\infty} \int_Q \frac{|\mathbf{w}_n|^2}{r} \mathrm{d}x\mathrm{d}t \geq c(d,\overline{e}) \int_Q \left(e - \frac{1}{2}\frac{|\mathbf{v}|^2}{r}\right)^2 \mathrm{d}x\mathrm{d}t$$