Long time behavior of open fluid systems

Eduard Feireisl
based on joint work with N. Chaudhuri (London), F. Fanelli (Lyon), M. Hofmanová (TU Bielefeld), A. Novotný (Toulon), Y.-S. Kwon (Busan)

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

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Field equations

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Constitutive relations

Gibbs' equation, thermodynamics stability

$$\vartheta Ds = De + pD \left(\frac{1}{\varrho} \right), \ \frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Newton's rheological law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x \mathbf{u}^t - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa(\vartheta) \nabla_x \vartheta$$

Open fluid systems, boundary conditions

Dissipative systems

Physical space occupied by the fluid – $\Omega \subset R^d$, $d = 1, 2, 3$ bounded domain
Mechanically open - mass interchange with the outer world allowed
Energetically open - energy (heat) interchange with the outer world allowed

Dirichlet boundary condition for the velocity

$$\mathbf{u} = \mathbf{u}_B \text{ on } \partial\Omega,\ \Gamma_{\rm in} = \{x \in \partial\Omega \mid \mathbf{u}_B \cdot \mathbf{n} < 0\}$$

Mass inflow boundary condition

$$\varrho = \varrho_B \text{ on } \Gamma_{\rm in}$$

Dirichlet boundary conditions for the temperature

$$\vartheta = \vartheta_B \text{ on } \partial\Omega$$

Alternatively: Heat flow through the boundary

$$(\varrho_B e(\varrho_B, \vartheta)\mathbf{u}_B + \mathbf{q}) \cdot \mathbf{n} = F_B \text{ on } \Gamma_{\rm in},\ \mathbf{q} \cdot \mathbf{n} = 0 \text{ otherwise}$$

Long time behavior of open systems

Total energy

$$E\left(\varrho,\mathbf{u},\vartheta\right)=\frac{1}{2}\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)$$

Velocity relative energy

$$E_V\left(\varrho,\mathbf{u},\vartheta\Big|\mathbf{u}_B\right)=\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2+\varrho e(\varrho,\vartheta)$$

Ballistic energy

$$E_B\left(\varrho,\mathbf{u},\vartheta\Big|\mathbf{u}_B,\vartheta_B\right)=\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2+\varrho e(\varrho,\vartheta)-\vartheta_B\varrho s(\varrho,\vartheta)$$

Main goals:

Dissipativity - bounded absorbing sets

$$\limsup_{t\to\infty}\int_\Omega E(\varrho,\mathbf{u},\vartheta)(t,\cdot)\ \mathrm{d}x\leq\mathcal{E}_\infty$$

Convergence of ergodic averages

$$\frac{1}{T}\int_0^T\mathcal{F}(\varrho,\mathbf{u},\vartheta)\mathrm{d}t\to\int_{\mathcal{P}}\mathcal{F}(z)\ \mathrm{d}\mu(z)\ \text{as}\ T\to\infty,\ \mathcal{P}-\text{phase space}$$

Why weak solutions?

far from equilibrium (not "small")
global in time solutions ≈ weak solutions

Possible formulation of the energy balance:

Internal energy balance ≈ "heat equation"

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{p \mathrm{div}_x \mathbf{u}}$$

Energy balance ≈ First law

$$\partial_t E + \boxed{\mathrm{div}_x(E\mathbf{u})} + \mathrm{div}_x(p\mathbf{u}) + \mathrm{div}_x \mathbf{q} - \boxed{\mathrm{div}_x(\mathbb{S} \cdot \mathbf{u})} = \varrho \mathbf{g} \cdot \mathbf{u}$$

Entropy balance ≈ Second law

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \boxed{=} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Weak solutions – basic idea

Entropy inequality ≈ Second law

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\left(\frac{\mathbf{q}}{\vartheta}\right) \boxed{\geq} \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)$$

Total energy balance ≈ First law

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E \ \mathrm{d}x \leq \int_\Omega \varrho \mathbf{g} \ \mathrm{d}x + \boxed{\text{boundary energy flux}}$$

Weak solutions - basic definition

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{g}$$

Entropy **inequality**

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Some form of total energy balance must be added
for the system to be (formally) well posed

Energy balance – flux b.c. for temperature

Relative (velocity) energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2+\varrho e\right]\ \mathrm{d}x \\
&\quad + \int_{\partial\Omega} F_B \ \mathrm{sgn}\ [\mathbf{u}_B\cdot\mathbf{n}]^- \mathrm{d}\sigma_x + \int_{\partial\Omega}\Big[\varrho e(\varrho,\vartheta_B)\Big]\ [\mathbf{u}_B\cdot\mathbf{n}]^+ \mathrm{d}\sigma_x \\
&\quad \le -\int_\Omega \Big[\varrho\mathbf{u}\otimes\mathbf{u}+p\mathbb{I}-\mathbb{S}\Big] : \nabla_x\mathbf{u}_B\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_B|^2\ \mathrm{d}x \\
&\qquad + \int_\Omega \varrho(\mathbf{u}-\mathbf{u}_B)\cdot(\mathbf{g}-\partial_t\mathbf{u}_B)\ \mathrm{d}x
\end{aligned}
$$

Main problem with the Dirichlet b.c. for the temperature

Boundary heat flux in the energy balance

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}\sigma_x$$

Solution – compensation with the entropy flux

$$\int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}\sigma_x = \int_{\partial\Omega} \frac{\mathbf{q} \cdot \mathbf{n}}{\vartheta} \vartheta_B \mathrm{d}\sigma_x, \; \vartheta|_{\partial\Omega} = \vartheta_B$$

$$\Leftrightarrow$$

Replace energy by ballistic energy!

Energy balance – Dirichlet b.c. for temperature

Ballistic energy inequality

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}-\mathbf{u}_B|^2+\varrho e-\vartheta_B\varrho s\right]\ \mathrm{d}x\\
&\quad+\int_{\partial\Omega}\Big[\varrho_B e(\varrho_B,\vartheta_B)-\vartheta_B\varrho_B s(\varrho_B,\vartheta_B)\Big]\ [\mathbf{u}_B\cdot\mathbf{n}]^-\mathrm{d}\sigma_x\\
&\quad+\int_{\partial\Omega}\Big[\varrho e(\varrho,\vartheta_B)-\vartheta_B\varrho s(\varrho,\vartheta_B)\Big]\ [\mathbf{u}_B\cdot\mathbf{n}]^+\mathrm{d}\sigma_x\\
&\quad+\int_\Omega \frac{\vartheta_B}{\vartheta}\left(\mathbb{S}:\nabla_x\mathbf{u}-\frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right)\ \mathrm{d}x\\
&\quad\leq-\int_\Omega\Big[\varrho\mathbf{u}\otimes\mathbf{u}+p\mathbb{I}-\mathbb{S}\Big]:\nabla_x\mathbf{u}_B\ \mathrm{d}x+\frac{1}{2}\int_\Omega\varrho\mathbf{u}\cdot\nabla_x|\mathbf{u}_B|^2\ \mathrm{d}x\\
&\qquad+\int_\Omega\varrho(\mathbf{u}-\mathbf{u}_B)\cdot(\mathbf{g}-\partial_t\mathbf{u}_B)\ \mathrm{d}x\\
&\qquad-\int_\Omega\left[\varrho s\left(\partial_t\vartheta_B+\mathbf{u}\cdot\nabla_x\vartheta_B\right)+\frac{\mathbf{q}}{\vartheta}\cdot\nabla_x\vartheta_B\right]\ \mathrm{d}x.
\end{aligned}
$$

Results I, existence of weak solutions

Sufficient conditions for global existence of weak solutions

$$p \approx \varrho e,\ p \approx \underbrace{q(\varrho)}_{\text{elastic component}} + p_m(\varrho,\vartheta) + \underbrace{a\vartheta^4}_{\text{radiation component}},\ q(\varrho) \gtrsim \varrho^\gamma,\ \gamma > \frac{d}{2}$$

$$\mu(\vartheta) \approx 1 + \vartheta^\Lambda,\ \eta(\vartheta) \lesssim 1 + \vartheta^\Lambda,\ \frac{1}{2} \leq \Lambda \leq 1$$

$$\kappa(\vartheta) \approx 1 + \vartheta^\beta,\ \beta \gtrsim 3 \text{ for temperature flux b.c.}$$

$$\beta \gtrsim 6 \text{ Dirichlet b.c. for temperature}$$

Results:

- **Existence.** Weak solutions exist globally in time for any finite energy initial data
- **Compatibility.** Any sufficiently smooth weak solution is a strong (classical) solution
- **Weak–strong uniqueness ($\beta \approx 3$).** A weak solution coincides with the strong solutions corresponding to the same initial/boundary data as long as the latter exists

Results II, bounded absorbing set

Hard sphere pressure EOS

$$q(\varrho) \approx (\overline{\varrho} - \varrho)^{-\alpha}, \ \overline{\varrho} > 0$$

Results:

- **Bounded absorbing set.** There is a bounded absorbing set
- **Asymptotic compactness.** Positive time shifts of any global in time solution

$$S_T(\varrho, \mathbf{u}, \vartheta)(t, \cdot) = (\varrho, \mathbf{u}, \vartheta)(T + t, \cdot)$$

  are precompact in the strong L^p topology. In particular, their asymptotic limit is another solution of the same problem (for autonomous boundary data)

Results III, ergodic averages, statistical solutions

Trajectory space [idea of Sell, Nečas]

$$\mathcal{P} = \left\{ t \in R \middle| \ (\varrho, \mathbf{m} = \varrho, \mathbf{u}, S = \varrho s(\varrho, \vartheta) \right\}$$

Ergodic averages

$$\frac{1}{T} \int_0^T \mathcal{F} \left(S_\tau [\varrho, \mathbf{m}, S] \right) \mathrm{d}\tau$$

- **Krylov – Bogolyubov method** $\Rightarrow$ any bounded global trajectory generates a stationary statistical solution $\approx$ a shift invariant measure $\mathcal{V}$ on the trajectory space $\approx$ a stationary stochastic process solving the problem $\mathcal{V}$ ma.s.
- **Birkhoff – Khinchin theorem** $\Rightarrow$ the ergodic averages converge for μ- a.a. trajectory

References

- N. Chaudhuri and E. Feireisl.
  Navier–Stokes–Fourier system with Dirichlet boundary conditions.
  arxiv preprint No. 2106.05315, 2021.

- F. Fanelli, E. Feireisl and M. Hofmanová
  Ergodic theory for energetically open compressible fluid flows.
  Phys. D, **423**, paper 132914, 2021.

- E. Feireisl and A. Novotný.
  Mathematics of open fluid systems.
  Monograph to appear by Birkhäuser/Springer

- E. Feireisl and A. Novotný.
  Navier-Stokes-Fourier System with General Boundary Conditions.
  Comm. Math. Phys., **386**(2):975–1010, 2021.

- E. Feireisl and Y.-S. Kwon
  Asymptotic stability of solutions to the Navier–Stokes–Fourier system driven by inhomogeneous Dirichlet boundary conditions
  Work in progress