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On Nash's conjecture for models of viscous, compressible, and heat conducting fluids

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ON NASH'S CONJECTURE FOR MODELS OF VISCOUS, COMPRESSIBLE, AND HEAT CONDUCTING FLUIDS

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Abstract. We show a new blow up criterion for regular solutions of the Navier–Stokes–Fourier system in terms of uniform bounds on the density and integral bounds on the absolute temperature. In comparison with the existing results, we remove the technical conditions relating the values of the shear and bulk viscosity coefficients. The result can be seen as a rigorous justification of Nash's conjecture concerning the character of possibly singularities in the equations of fluid dynamics.

Key Words: Compressible Navier-Stokes equations; Nash's conjecture; global classical solutions

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1. Introduction

As pointed out by Nash in his seminal paper [21], mathematical problems arising in continuum fluid dynamics consist in vast majority of systems

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of parabolic/hyperbolic *nonlinear* equations. Nash also realized that solvability of these problem is intimately related to available *a priori* bounds. Standard examples among these models are the Euler and Navier–Stokes equations describing the motion of an inviscid and viscous fluid, respectively. In this paper, we focus on the Navier–Stokes–Fourier system governing the time evolution of a general compressible, heat conducting, and linearly viscous fluid. Here again, it is Nash's truly pioneering contribution [21], [22] that represents the very first step in understanding the well posedness of this problem, see also [8]. Nash also makes a remarkable statement that might be interpreted as *Nash's conjecture*, see [21]:

> *Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density.*

The results of the present paper can be seen as the ultimate step in the proof of Nash's conjecture in the context of compressible, viscous Newtonian flows. It is interesting to note that possible singularities must first appear at the level of thermodynamic variables - the density and the temperature - and not for the fluid velocity as often conjectured in the context of incompressible fluids, see e.g. Prodi [23], Serrin [24]. Moreover, in view of the recent results by Merle et al. [20] and Buckmaster et al. [3] on blow up for the *isentropic* Navier–Stokes system, the regularity criterion proved below seems sharp.

The time evolution of the density $\rho = \rho(x,t)$, the (bulk) velocity $u = (u_1, u_2, u_3)(x,t)$ and the total energy $E = E(x,t)$ of a viscous, compressible, and heat conducting fluid is governed by the following system of field equations:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) = \operatorname{div}(\mathcal{T}), \\ (\rho E)_t + \operatorname{div}(\rho E u) = \operatorname{div}(\mathcal{T} u) - \operatorname{div}(q). \end{cases} \tag{1.1}$$

For the sake of simplicity, we have deliberately ignored the effect of external mechanical and heat sources.

For linearly viscous fluids, the Cauchy stress $\mathcal{T}$ is given by Newton's rheological law

$$\mathcal{T} = \mu \left(\nabla u + (\nabla u)' \right) + \lambda \operatorname{div} u I_3 - P I_3,$$

where I_3 is a 3×3 unit matrix, and $P = P(\rho, \theta)$ is the pressure determined in terms of the density ρ and the (absolute) temperature $\theta = \theta(x,t)$. Accordingly, the heat flux q is given by Fourier's law

$$q = -\kappa \nabla \theta.$$

The shear viscosity coefficient μ, the bulk viscosity coefficient $\lambda + \frac{2\mu}{3}$, and the heat conductivity coefficient κ are supposed to be constant satisfying

$$\mu > 0, \quad \lambda + \frac{2\mu}{3} \geq 0, \ \kappa > 0. \tag{1.2}$$

Finally, we write the total energy E as the sum of the kinetic and internal energy,

$$E = e + \frac{|u|^2}{2}.$$

For definiteness, we consider Boyle's law of a perfect gas,

$$P = \rho\theta.$$

Similarly, the internal energy is a linear function of the temperature,

$$e = C_\nu \theta,$$

where C_ν is a positive constant representing the specific heat at constant volume.

There has been a long way in understanding the precise meaning of "certain gross types of singularity" suggested in Nash's seminal work. It turns out that the analysis depends considerably on the type of physical domain $\Omega \subset R^3$ occupied by the fluid. There are essentially two types considered in the literature: (i) $\Omega = R^3$ representing a mathematical idealization of a fluid not influenced by the effects of the kinematic boundary and complying with suitable far field conditions, (ii) a more realistic situation Ω a bounded/exterior domain supplement with suitable boundary conditions.

- One of the first results due to Cho, Choe, and Kim ([4]) states a blow up criterion:

  $$\limsup_{t \nearrow T^\star} \left(\|\rho\|_{W^{1,2} \cap W^{1,q}} + \|u\|_{D_0^1} \right) = 0.$$

  This and several other blow up criteria (see [14, 27] for instance), however, refer to possible gradient singularity and therefore remain far from the original Nash statement.
- Fan, Jiang, Ou ([9]) obtained the following blowup criterion for the strong solution to (1.1) in three dimensions:

  $$\limsup_{t \nearrow T^\star} \left(\|\theta\|_{L^\infty(0,t;L^\infty)} + \|\nabla u\|_{L^1(0,t;L^\infty)} \right) = \infty. \tag{1.3}$$

  Obviously, a bound on the amplitude of the velocity gradient implies boundness of the fluid density as well. The result is conditioned by a technical but physically irrelevant restriction

  $$7\mu > \lambda. \tag{1.4}$$

  It is worth–noting, however, that (1.4) is still compatible with Newton's hypothesis of vanishing bulk viscosity relevant to the monoatomic gas.

- Sun, Wang, and Zhang ([26]) obtained a blow up criterion of strong solutions in terms of the density and the temperature for the initial-boundary value problem in three dimensions, where $u|_{\partial\Omega} = 0$ and $\frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0$:

$$\limsup_{t\nearrow T^\star} \left(\|\theta\|_{L^\infty(0,t;L^\infty)} + \|\rho\|_{L^\infty(0,t;L^\infty)} + \left\|\frac{1}{\rho}\right\|_{L^\infty(0,t;L^\infty)}\right) = \infty, \tag{1.5}$$

still under the technical condition (1.4).
- The term $\|1/\rho\|_{L^\infty(0,t;L^\infty)}$ has been removed from (1.5) by Wen, Zhu ([29]) for the Cauchy problem with vanishing far field conditions $\tilde{\rho} = \tilde{\theta} = 0$ under even more restrictive condition

$$3\mu > \lambda.$$

The condition "$7\mu > \lambda$" in the criterion (1.5) for the initial-boundary value problem and "$3\mu > \lambda$" for the Cauchy problem used in [26] and [29], respectively, are crucial for the bound on $\int_\Omega \rho|u|^{3+\sigma}\,\mathrm{d}x$ with "$\sigma > 0$" necessary for controlling some super-critical nonlinear terms.

Our main goal in this work is to remove completely any technical assumption relating the two viscosity coefficients and relax slightly the blow up conditions in terms of the temperature. From this perspective, the result gives an ultimate affirmative answer to Nash's conjecture. Besides, it is interesting to note that the blow up results obtained recently by Merle et al. [20] and Buckmaster et al. [3] in the context of *isentropic flows* assert a simultaneous blow up of the density and the velocity in the L^∞-norm for the Cauchy problem with zero/positive far field density. As the isentropic flow in the context of *viscous* fluids seems physically less realistic but still a widely used approximation, the effect of temperature changes in possibly blow up mechanism represents a challenging open problem.

Last but not least, removing the hypothesis on smallness of the bulk viscosity coefficient is not only academic. As observed by Graves and Argrow [7](cf. also Cramer [6]): "*Several fluids, including common diatomic gases, are seen to have bulk viscosities which are hundreds or thousands of times larger than their shear viscosities.*"

In the context of smooth solutions considered in the present paper, system (1.1) can be written in the form:

$$\begin{cases} \rho_t + \nabla\cdot(\rho u) = 0, \\ \rho u_t + \rho u\cdot\nabla u + \nabla P(\rho,\theta) = \mu\Delta u + (\mu+\lambda)\nabla \mathrm{div} u, \\ C_v\left(\rho\theta_t + \rho u\cdot\nabla\theta\right) + \rho\theta \mathrm{div} u = \frac{\mu}{2}\left|\nabla u + (\nabla u)'\right|^2 + \lambda(\mathrm{div} u)^2 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad +\kappa\Delta\theta, \end{cases} \tag{1.6}$$

in $\Omega \times (0, \infty)$, where μ, λ, and κ are constants satisfying (1.2). System (1.6) is supplemented with the initial conditions:

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0), \quad x \in \Omega, \tag{1.7}$$

and one of the following boundary/far field conditions:

- $\Omega \subset \mathbb{R}^3$ is a bounded and smooth domain:

$$u|_{\partial\Omega} = 0, \quad \left.\frac{\partial \theta}{\partial n}\right|_{\partial\Omega} = 0 \text{ for } t \geq 0, \tag{1.8}$$

  where n denotes the outer normal vector.
- $\Omega = \mathbb{R}^3$:

$$(\rho, u, \theta) \to (\tilde{\rho}, 0, \tilde{\theta}), \quad \text{as } |x| \to \infty, \tag{1.9}$$

  with constants $\tilde{\rho}$, $\tilde{\theta} \geq 0$;

Remark 1.1. *Note that the above boundary conditions correspond to an energetically closed fluid system, where the boundary of the physical space is both mechanically and thermally insulated. This fact facilitates considerably the analysis, in particular obtaining the uniform bounds, performed below. Extension to more complicated boundary conditions would definitely require a more elaborate treatment notably of the estimates presented in Section 3 below.*

Notation:

- $\displaystyle\int f = \int_\Omega f \, \mathrm{d}x.$
- For $1 \leq l \leq \infty$, we use the following notation for the standard Lebesgue and Sobolev spaces:

$$L^l = L^l(\Omega), \quad D^{k,l} = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) : \|\nabla^k u\|_{L^l} < \infty \right\},$$

$$W^{k,l} = L^l \cap D^{k,l}, \quad H^k = W^{k,2}, \quad D^k = D^{k,2},$$

$$D^1_0 = \Big\{ u \in L^6 : \|\nabla u\|_{L^2} < \infty, \ u|_{\partial\Omega} = 0 \},$$

$$\|u\|_{D^{k,l}} = \|\nabla^k u\|_{L^l}.$$

- For 3×3 matrices $E = (E_{ij})$, $F = (F_{ij})$, we denote the scalar product of E with F by

$$E : F = \sum_{i,j=1}^{3} E_{ij} F_{ij}.$$

1.1. **Main result.** Before presenting our main result, we introduce the concept of strong solution to (1.6) used throughout the paper.

Definition 1.2. *(Strong solution) Given a time $T > 0$, a trio (ρ, u, θ) is called* strong solution *to the Navier-Stokes-Fourier equations (1.6), (1.7) , (1.9), or (1.6), (1.7) (1.8) in $\Omega \times [0,T]$ if:*

$$\rho \geq 0,\ \ \rho - \tilde{\rho} \in C\big([0,T]; W^{1,q}(\Omega) \cap H^1(\Omega)\big),\ \ \rho_t \in C\big([0,T]; L^2(\Omega) \cap L^q(\Omega)\big),$$

$$(u, \theta - \tilde{\theta}) \in C\big([0,T]; D^2(\Omega) \cap D^1_0(\Omega)\big) \cap L^2\big(0,T; D^{2,q}(\Omega)\big),$$

$$(u_t, \theta_t) \in L^2\big(0,T; D^1_0(\Omega)\big), \quad (\sqrt{\rho} u_t, \sqrt{\rho}\theta_t) \in L^\infty\big(0,T; L^2(\Omega)\big),$$

for some $q \in (3,6]$, and (ρ, u, θ) satisfies (1.6) a.a. in $\Omega \times (0,T]$, together with the associated initial and boundary conditions.

Initial data. In agreement with the regularity class specified in Definition 1.2, the initial data satisfy $\rho_0 \geq 0$, $\rho_0 - \tilde{\rho} \in W^{1,q}(\Omega) \cap H^1(\Omega)$ for some $q \in (3,6]$, $(u_0, \theta_0 - \tilde{\theta}) \in D^2(\Omega) \cap D^1_0(\Omega)$. In addition, we suppose $\rho_0|u_0|^2 + \rho_0|\theta_0 - \tilde{\theta}|^2 \in L^1(\Omega)$, and that the following compatibility conditions:

$$\begin{cases} \mu\Delta u_0 + (\mu+\lambda)\nabla \mathrm{div} u_0 - \nabla P(\rho_0, \theta_0) = \sqrt{\rho}_0 g_1, \\ \kappa\Delta\theta_0 + \frac{\mu}{2}\left|\nabla u_0 + (\nabla u_0)'\right|^2 + \lambda(\mathrm{div} u_0)^2 = \sqrt{\rho_0} g_2, \quad x \in \Omega \end{cases} \tag{1.10}$$

for some $g_i \in L^2(\Omega)$, $i = 1, 2$. Finally, we require (u_0, θ_0) to satisfy the relevant boundary condition specified in (1.8) if Ω is bounded.

Remark 1.3. *Under the above stated assumption on the initial data, the local existence of strong solutions was obtained in* [5, 31] *except for the boundary condition $\frac{\partial\theta}{\partial n}|_{\partial\Omega} = 0$. However, it turns out that the local existence in this case can be established in a way similar to* [5, 31]*. In particular, the strong solution always exists on a non–empty time interval for the initial data belonging to the class specified above. Moreover, the life span can be always extended beyond the existing one as long as uniform bounds are available. Thus any strong solution is defined up a maximal existence time $T^* > 0$.*

Now we are in a position to state our main result:

Theorem 1.4. *Let (ρ, u, θ) be a strong solution to the initial-boundary value problem (1.6), (1.7), (1.8), or to the Cauchy problem (1.6), (1.7), (1.9) defined on a maximal existence time interval $[0, T^*)$.*

If $T^* < +\infty$, then

$$\limsup_{t \nearrow T^*} \left(\|\rho\|_{L^\infty\left(0,t;L^\infty(\Omega)\right)} + \|\theta - \tilde{\theta}\|_{L^s\left(0,t;L^r(\Omega)\right)} \right) = \infty \tag{1.11}$$

for any $r \in (\frac{3}{2}, \infty]$ and $s \in [1, \infty]$ satisfying $\frac{2}{s} + \frac{3}{r} \leq 2$.

Remark 1.5. *Apparently at odds with the basic physical principles, we do not require the (absolute) temperature θ to be strictly positive. Similarly, although the Navier–Stokes–Fourier system is derived as a model of non–dilute fluids, we allow the density to vanish at some parts of the physical space. From the pure analysis point of view, however, omitting these physically grounded hypotheses we obtain a mathematically stronger result. In addition, it is easy to see that positivity of both the density ρ and the temperature θ is inherited by any strong solution from the initial/boundary data.*

Remark 1.6. *In combination with a suitable weak–strong uniqueness result in the spirit of* [12], *condition* (1.11) *can be interpreted as a regularity criterion for a weak or even more general dissipative measure valued solution introduced in* [2]. *Note, however that the* existence *of a weak solution for the present constitutive relations is a largely open problem that persists even in the larger class of dissipative weak solutions due to the lack of suitable a priori bounds notably on the entropy flux.*

1.2. **Main result and Nash's conjecture.**

- In the particular case $s = r = \infty$, Theorem 1.4 yields the no blow up criterion
$$\limsup_{t \nearrow T^*} \left(\|\rho(t,\cdot)\|_{L^\infty(\Omega)} + \|\theta(t,\cdot)\|_{L^\infty(\Omega)} \right) < \infty$$
that may be interpreted as an affirmative solution of Nash's conjecture. In contrast with all previously known results, the conclusion holds without any non–physical restriction imposed on the viscosity coefficients.

- Theorem 1.4 provides a general criterion on the life span of strong solutions. Specifically, if there exist $r \in (\frac{3}{2}, \infty]$ and $s \in [1, \infty]$ satisfying $\frac{2}{s} + \frac{3}{r} \leq 2$ such that
$$\|\rho\|_{L^\infty\left(0,t;L^\infty(\Omega)\right)} + \|\theta - \tilde{\theta}\|_{L^s\left(0,t;L^r(\Omega)\right)} \tag{1.12}$$
remains bounded for $t \nearrow T$, then the life span of the strong solution can be extended beyond T. In fact, condition (1.12) has been verified for any positive T in some special cases such as the Cauchy problem for vacuum solutions with small initial energy or small mass, giving rise to the global existence results obtained in [16, 30]. However, validity of (1.12) for the initial-boundary value problem (i.e. (1.8)) with the same smallness assumptions is not known.

- Hypothesis $\rho_0|u_0|^2 \in L^1(\Omega)$ on boundedness of the initial kinetic energy is relevant only for the Cauchy problem with strictly positive far field temperature $\tilde{\theta} > 0$ (see Lemma 4.3). If $\tilde{\theta} = 0$, this condition may be replaced by $\rho_0|u_0|^4 \in L^1(\Omega)$.

- Theorem 1.4 also holds for classical solution in the sense introduced in [16, 30]. As we shall see, given the estimates in Sections 3 and 4, the higher-order estimates for the Cauchy problem can be obtained following step by step the arguments of [16, 30]. For the initial-boundary value problem, one may use the decomposition of velocity introduced in Section 3.

1.3. **Main challenges and principal ideas.**

Main challenge. The main challenge here is to deduce sufficiently strong *a priori* bounds for a (hypothetical) regular solution under the mere assumption that both the temperature and and the density are bounded. This may be seen as a counterpart of Nash's celebrated conditional regularity statement $L^\infty \to C^\alpha$ in the context of parabolic equations.

The method originally used in [21] is nowadays known as Nash's iteration. Nash naturally conjectured that his new method (see [8] for Klainerman's comments on Nash's work [21]) or some suitable extension, would apply to more complex systems such as the Navier-Stokes equations in fluid dynamics. The problem turned out to be more delicate, however, due to the limited applicability of De Giorgi-Nash-Moser techniques to general *systems* of equations. In particular, the compressible Navier–Stokes system is of mixed type of a transport and parabolic equations. In addition, strict parabolicity of the momentum and internal energy equations may become degenerate in the nearly vacuum state of very low density.

Main ideas. Let us explain the principal ideas of the proof of Theorem 1.4 that allow us to remove the technical restrictions imposed on the viscosity coefficients omnipresent in the existing literature.

On condition that the density and the temperature remain bounded, the higher order *a priori* bounds depend in a crucial way on boundedness of the quantity $\int_\Omega \rho|u|^{3+\sigma}$ for $\sigma > 0$. In particular, this estimate is necessary to control certain super-critical quantities arising in the convective terms. The problem is definitely more delicate than for a simple parabolic equation. To understand the principal stumbling blocks suppose, for a while, that the velocity solves a linear parabolic "system" of equations:

$$\frac{\partial u_i}{\partial t} = \mu \Delta u_i$$

for $i = 1, 2, 3$. Multiplying on $(3+\sigma)|u|^{1+\sigma} u_i$, $u = (u_1, u_2, u_3)$, we get

$$\begin{aligned}\frac{\partial (|u|^{3+\sigma})}{\partial t} =& \mu\Delta(|u|^{3+\sigma}) - (3+\sigma)\mu|u|^{1+\sigma}|\nabla u|^2 \\ & - (3+\sigma)(1+\sigma)\mu|u|^{1+\sigma}\big|\nabla|u|\big|^2.\end{aligned} \tag{1.13}$$

The desired $L^{3+\sigma}$ estimate can be derived by integrating (1.13) over $\Omega\times(0,t)$ using the boundary conditions, specifically,

$$\int_\Omega |u|^{3+\sigma}\,dx + D_0 = \int_\Omega |u_0|^{3+\sigma}\,dx$$

where $u_0(x)=u(x,0)$ for $x\in\Omega$ and

$$D_0 = (3+\sigma)\mu\int_0^t\int_\Omega |u|^{1+\sigma}\left[|\nabla u|^2 + (1+\sigma)\big|\nabla|u|\big|^2\right]dx\,ds \geq 0.$$

The problem becomes more difficult for the linear parabolic system

$$\frac{\partial u_i}{\partial t} = \mu\Delta u_i + (\mu+\lambda)\partial_i \mathrm{div} u, \tag{1.14}$$

where $u=(u_1,u_2,u_3)$ and $\partial_i=\frac{\partial}{\partial x_i}$ and the viscosity coefficients satisfy (1.2). Multiplying (1.14) by $(3+\sigma)|u|^{1+\sigma}u_i$, and integrating the result over $\Omega\times(0,t)$, we obtain

$$\int_\Omega |u|^{3+\sigma}\,dx + D_1 = \int_\Omega |u_0|^{3+\sigma}\,dx \tag{1.15}$$

where

$$\begin{aligned} D_1 =&(3+\sigma)\int_0^t\int_\Omega |u|^{1+\sigma}\left[\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div} u|^2 + (1+\sigma)\mu\big|\nabla|u|\big|^2\right]dx\,ds \\ &+ (3+\sigma)(1+\sigma)(\mu+\lambda)\int_\Omega |u|^\sigma u\cdot\nabla|u|\,\mathrm{div}u\,\mathrm{d}x. \end{aligned}$$

Unlike D_0 the integral D_1 may not be positive depending on the specific values of the viscosity coefficients. The simplest solution is imposing the technical condition $7\mu>\lambda$. Accordingly, D_1 becomes non-negative and the desired $L^{3+\sigma}$ estimate of u can be obtained.

The counterpart of (1.15) in the momentum equation reads

$$\int_\Omega \rho|u|^4\,dx + D_1 = \int_\Omega \rho_0|u_0|^4\,dx + 4\int_0^t\int_\Omega \mathrm{div}(|u|^2u)P\,\mathrm{d}x\,\mathrm{d}s \tag{1.16}$$

where we choose $\sigma=1$ for simplicity. To deduce from (1.16) the desired estimate without imposing any extra restriction on the viscosity coefficients, it is crucial to control the div$u$-related terms in D_1, see Lemma 3.1. To achieve this, we introduce a new quantity

$$\rho|u|^4 - \frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde\theta),$$

for $\lambda>0$. It turns out that the integral-in-space of the quantity $\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde\theta)$ satisfies a new inequality containing div$u$ with an enhanced weight via a nonlinear term containing velocity in the temperature equation, see Lemmas 3.2 and 3.3. Such a combination produces the desired cancellation in the div$u$-related terms in D_1 yielding the estimate of $\int_\Omega\rho|u|^4\,dx$ without any technical restriction on μ and λ. To see this, a series of new associated *a priori* estimates need to be derived, see Lemmas 3.4 and 3.5, and Corollary 3.6.

In the case $\lambda \leq 0$, D_1 is non-negative, which can absorb the corresponding terms on the right-hand side of (1.16) by virtue of Cauchy inequality.

2. Preliminaries

In this section, we recall some useful results used throughout the rest of the paper.

Lemma 2.1. *([18, 28]) Let $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ be a bounded domain with piecewise smooth boundary. Then the following inequality is valid for every function $u \in W^{1,p}(\Omega)$:*

$$\|u\|_{L^{p'}(\Omega)} \leq C_2(\|u\|_{L^1(\Omega)} + \|\nabla u\|^{\alpha}_{L^p(\Omega)}\|u\|^{1-\alpha}_{L^{r'}(\Omega)}), \tag{2.1}$$

where $\alpha = (1/r' - 1/p')(1/r' - 1/p + 1/N)^{-1}$. If, moreover, $p < N$, then $p' \in [r', pN/(N-p)]$ for $r' \leq pN/(N-p)$, and $p' \in [pN/(N-p), r']$ for $r' > pN/(N-p)$. If $p \geq N$, then $p' \in [r', \infty)$ is arbitrary. The positive constant C_2 in inequality (2.1) depends on N, p, r', α and the geometry of the domain Ω.

Remark 2.2. *The first term on the right-hand side of (2.1), specifically, $\|u\|_{L^1(\Omega)}$, can be omitted if $u \in W^{1,p}_0(\Omega)$. In this case, (2.1) is the well-known Gagliardo-Nirenberg inequality.*

Lemma 2.3. *([11]) Let $v \in W^{1,2}(\Omega)$, and let ρ be a non-negative function such that*

$$0 < M \leq \int_\Omega \rho \,\mathrm{d}x, \qquad \int_\Omega \rho^\gamma \,\mathrm{d}x \leq E_0,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain for $N \geq 1$ and $\gamma > 1$.

Then there exists a constant c depending solely on M, E_0 such that

$$\|v\|^2_{L^2(\Omega)} \leq c(E_0, M)\left\{\|\nabla_x v\|^2_{L^2(\Omega)} + \Big(\int_\Omega \rho|v| \,\mathrm{d}x\Big)^2\right\}.$$

Remark 2.4. *For the boundary condition (1.8), the solution in Theorem 1.4 satisfies the mass conservation,*

$$\int \rho = \int \rho_0 := M_0 > 0. \tag{2.2}$$

Thus under the conditions of Lemma 2.3, the following estimate can be obtained by using the Hölder inequality and (2.2):

$$\|v\|^2_{L^2(\Omega)} \leq c(E_0, M_0)\Big(\|\nabla_x v\|^2_{L^2(\Omega)} + \int_\Omega \rho|v|^2 \,\mathrm{d}x\Big). \tag{2.3}$$

3. Initial-boundary value problem

Assume that $T^* < \infty$ and that there exist constants $r \in (\frac{3}{2}, \infty]$ and $s \in [1, \infty]$ satisfying

$$\frac{2}{s} + \frac{3}{r} \leq 2.$$

such that

$$\|\rho\|_{L^\infty\left(0,T;L^\infty(\Omega)\right)} + \|\theta - \tilde{\theta}\|_{L^s\left(0,T;L^r(\Omega)\right)} \leq M^* < \infty \tag{3.1}$$

for any $T \in (0, T^*)$. Our aim is to show that under the assumption (3.1) and the hypotheses of Theorem 1.4, there is a bound $C > 0$ depending only on $M^*, \rho_0, u_0, \theta_0, \mu, \lambda, \kappa$, and T^* such that

$$\begin{aligned} &\sup_{0\leq t<T^*} \Big(\|\rho\|_{W^{1,q}} + \|(u,\theta)\|_{H^2} + \|\rho_t\|_{L^q} + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t)\|_{L^2} \Big) \\ &+ \int_0^{T^*} \left(\|(u_t,\theta_t)\|_{H^1}^2 + \|(u,\theta)\|_{W^{2,q}}^2 \right) \mathrm{d}t \leq C. \end{aligned} \tag{3.2}$$

In view of the available local existence results specified in Remark 1.3, it is easy to check (see for instance [27]) that (3.2) implies the strong solution can be extended beyond T^*, meaning T^* is not the maximal existence time, which yields the desired contradiction.

Throughout the rest of the paper, we denote by C a generic constant that may depend on $M^*, \rho_0, u_0, \theta_0, \tilde{\rho}, \tilde{\theta}, \mu, \lambda, \kappa$, and T^* but independent of the other parameters ϵ, ϵ_1 and δ specified below. The symbols C_ϵ and C_δ denote constants that may depend on ϵ and δ, respectively.

As in [25], we denote $w = u - h$, where h is the unique solution to

$$\begin{cases} Lh = \nabla P, \quad \text{in } \Omega \times (0,T], \\ h|_{\partial\Omega} = 0, \quad \text{if } \Omega \text{ is bounded}, \\ h \to 0 \quad \text{as } |x| \to \infty, \quad \text{if } \Omega = \mathbb{R}^3, \end{cases} \tag{3.3}$$

where $Lh = \mu\Delta h + (\mu + \lambda)\nabla \mathrm{div} h$. Then we have

$$\begin{cases} Lw = \rho\dot{u}, \quad \text{in } \ \Omega \times (0,T], \\ w|_{\partial\Omega} = 0, \quad \text{if } \Omega \text{ is bounded}, \\ w \to 0 \quad \text{as } |x| \to \infty, \quad \text{if } \Omega = \mathbb{R}^3, \end{cases} \tag{3.4}$$

where $\dot{u} = u_t + u \cdot \nabla u$. Relations (3.3) and (3.4) yield

$$\begin{cases} \|\nabla h\|_{L^p} \leq C\|P(\rho,\theta) - P(\tilde{\rho},\tilde{\theta})\|_{L^p}, \\ \|\nabla^2 h\|_{L^p} \leq C\|\nabla P\|_{L^p}, \end{cases} \tag{3.5}$$

and

$$\|\nabla^2 w\|_{L^p} \le C\|\rho\dot{u}\|_{L^p}, \tag{3.6}$$

for any $p \in (1,\infty)$, see for instance [1, 25].

The following results (Lemmas 3.1-3.4) hold for both the initial-boundary value problem and the Cauchy problem. For the sake of simplicity, we include the constants $\tilde\rho$ and $\tilde\theta$ even in the context of the initial–boundary value problem. These results will be used in the next section.

Lemma 3.1. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^4 + \mu\int |u|^2|\nabla u|^2\\
\le & C\epsilon\int|\nabla\theta|^2 + C_\epsilon\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(\rho|\theta-\tilde\theta|^2+\rho|u|^4\right)\\
&+C\int|\nabla u|^2 + C\int|\mathrm{div}u|^2|u|^2,
\end{aligned}\tag{3.7}$$

for any $\tilde\rho,\tilde\theta\ge 0$ and any sufficiently small $\epsilon>0$ specified in (3.12) below.

Proof. Multiplying $(1.6)_2$ by $4|u|^2u$, and integrating by parts over Ω, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^4 + \int 4|u|^2\left(\mu|\nabla u|^2+(\lambda+\mu)|\mathrm{div}u|^2+2\mu|\nabla|u||^2\right)\\
=&4\int \mathrm{div}(|u|^2u)P - 8(\mu+\lambda)\int \mathrm{div}u|u|u\cdot\nabla|u|\\
\le& C\int\rho|\theta-\tilde\theta||u|^2|\nabla u| + C\tilde\theta\int\rho|u|^2|\nabla u| + 2\mu\int|u|^2\big|\nabla|u|\big|^2\\
&+C\int|\mathrm{div}u|^2|u|^2\\
\le& C\int\rho^2|\theta-\tilde\theta|^2|u|^2 + C\int\rho^2|u|^4 + C\int|\nabla u|^2 + 2\mu\int|u|^2\big|\nabla u\big|^2\\
&+2\mu\int|u|^2\big|\nabla|u|\big|^2 + C\int|\mathrm{div}u|^2|u|^2.
\end{aligned}\tag{3.8}$$

The fourth and the fifth term on the right-hand side of (3.8) can be absorbed by the integrals on the left–hand side; whence we have

$$\begin{aligned}
&\frac{d}{dt}\int \rho|u|^4 + 2\mu\int |u|^2\left(|\nabla u|^2 + \left|\nabla|u|\right|^2\right)\\
\le& C\int \rho^2|\theta-\tilde\theta|^2|u|^2 + C\int \rho^2|u|^4 + C\int|\nabla u|^2 + C\int |\mathrm{div}u|^2|u|^2\\
\le& C\|\rho(\theta-\tilde\theta)\|_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|_{L^{\frac{2r}{r-1}}}\left\|\sqrt{\rho}|u|^2\right\|_{L^{\frac{2r}{r-1}}} + C\int\rho|u|^4\\
&+C\int|\nabla u|^2 + C\int|\mathrm{div}u|^2|u|^2\\
\le& C\|\theta-\tilde\theta\|_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}} + C\|\theta-\tilde\theta\|_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^2_{L^{\frac{2r}{r-1}}}\\
&+C\int\rho|u|^4 + C\int|\nabla u|^2 + C\int|\mathrm{div}u|^2|u|^2, \qquad (3.9)
\end{aligned}$$

for $r \in (\frac{3}{2},\infty]$[4], where we have used Hölder inequality, Cauchy inequality and (3.1).

Using the standard interpolation inequality and (3.1), we have

$$\begin{aligned}
\|\sqrt{\rho}(\theta-\tilde\theta)\|_{L^{\frac{2r}{r-1}}} &\le \|\sqrt{\rho}(\theta-\tilde\theta)\|^{\alpha}_{L^2}\|\sqrt{\rho}(\theta-\tilde\theta)\|^{1-\alpha}_{L^6}\\
&\le C\|\sqrt{\rho}(\theta-\tilde\theta)\|^{\alpha}_{L^2}\|(\theta-\tilde\theta)\|^{1-\alpha}_{L^6},
\end{aligned}$$

where $\alpha = 1-\frac{3}{2r}$. This yields

$$\begin{aligned}
&\|\theta-\tilde\theta\|_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}}\\
\le& C\|\theta-\tilde\theta\|_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^{2\alpha}_{L^2}\|\theta-\tilde\theta\|^{2(1-\alpha)}_{L^6}\\
\le& \epsilon\|\theta-\tilde\theta\|^2_{L^6} + C_\epsilon\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^2}\\
\le& C\epsilon\left(\int\rho|\theta-\tilde\theta|^2 + \int|\nabla\theta|^2\right) + C_\epsilon\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^2}, \qquad (3.10)
\end{aligned}$$

for any $\epsilon>0$, where we have used Young inequality, the Sobolev inequality, and (2.3) if Ω is bounded.

Similarly to (3.10), for the second term on the right-hand side of (3.9), we have

$$\begin{aligned}
&\|\theta-\tilde\theta\|_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^2_{L^{\frac{2r}{r-1}}}\\
\le& \|\theta-\tilde\theta\|_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^{2-\frac{3}{r}}_{L^2}\left\|\sqrt{\rho}|u|^2\right\|^{\frac{3}{r}}_{L^6}\\
\le& C_\epsilon\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^2_{L^2} + \epsilon\left\|\sqrt{\rho}|u|^2\right\|^2_{L^6}\\
\le& C_\epsilon\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^2_{L^2} + \epsilon C\left\|u\nabla|u|\right\|^2_{L^2}. \qquad (3.11)
\end{aligned}$$

[4]Here $\frac{2r}{r-1} = 2$ if $r=\infty$.

Substituting (3.10) and (3.11) into (3.9), and choosing ϵ small enough so that

$$\epsilon \leq \frac{\mu}{C}, \tag{3.12}$$

we get (3.7). The proof of Lemma 3.1 is complete.

$\square$

Lemma 3.2. *Under the hypotheses of Theorem 1.4 and (3.1), we have, for any given $\lambda > 0$,*

$$\begin{aligned}
&\frac{\mu}{2}\int \left|\nabla u + (\nabla u)'\right|^2 |u|^2 + \frac{\lambda}{2}\int (\mathrm{div} u)^2 |u|^2 \\
\leq & \frac{\mathrm{d}}{\mathrm{d}t}\int C_v \rho |u|^2(\theta - \tilde{\theta}) + \epsilon \int \rho |u_t|^2 + 2(\epsilon + \delta C_\epsilon)\int |u|^2 |\nabla u|^2 \\
&+ C_\epsilon C_\delta \left(\|\theta - \tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1 \right) \int \left[\rho(\theta - \tilde{\theta})^2 + \rho |u|^4\right] \\
&+ C_\epsilon \int |\nabla \theta|^2 + C \int |\nabla u|^2
\end{aligned} \tag{3.13}$$

for any $\tilde{\rho}, \tilde{\theta} \geq 0$, any small $\delta > 0$, and $\epsilon > 0$ specified in (3.26) below.

Proof. Multiplying $(1.6)_3$ by $|u|^2$, and integrating the resulting equation over Ω, we have

$$\begin{aligned}
&\frac{\mu}{2}\int \left|\nabla u + (\nabla u)'\right|^2 |u|^2 + \lambda \int (\mathrm{div} u)^2 |u|^2 \\
=& C_v \int \rho |u|^2 \theta_t + C_v \int \rho |u|^2 u \cdot \nabla \theta + \int \rho \theta \mathrm{div} u |u|^2 - \kappa \int \Delta \theta |u|^2 \\
=& I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{3.14}$$

For I_1, we have

$$I_1 = C_v \int \rho |u|^2 (\theta - \tilde{\theta})_t = \frac{\mathrm{d}}{\mathrm{d}t}\int C_v \rho |u|^2 (\theta - \tilde{\theta}) - \int C_v [\rho |u|^2]_t (\theta - \tilde{\theta}). \tag{3.15}$$

For I_2, using integration by parts we have

$$I_2 = \int C_v \rho |u|^2 u \cdot \nabla(\theta - \tilde{\theta}) = -\int C_v (\theta - \tilde{\theta}) \nabla \cdot [\rho |u|^2 u]. \tag{3.16}$$

For I_3, using Cauchy inequality and (3.1) we have

$$\begin{aligned}
I_3 =& \int \rho(\theta - \tilde{\theta}) \mathrm{div} u |u|^2 + \tilde{\theta}\int \rho \mathrm{div} u |u|^2 \\
\leq & \epsilon \int (\mathrm{div} u)^2 |u|^2 + C_\epsilon \int \rho^2 (\theta - \tilde{\theta})^2 |u|^2 \\
&+ C\int \rho |u|^4 + C \int |\nabla u|^2,
\end{aligned} \tag{3.17}$$

where the second term on the right-hand side of (3.17) is estimated by virtue of (3.9), (3.10) and (3.11) as follows:

$$\begin{aligned}\int \rho^2(\theta-\tilde\theta)^2|u|^2 \le & C\|\theta-\tilde\theta\|_{L^r}\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}} \\ & + C\|\theta-\tilde\theta\|_{L^r}\left\|\sqrt{\rho}|u|^2\right\|^2_{L^{\frac{2r}{r-1}}} \\ \le & \delta\left(\int |\nabla\theta|^2 + \int |u|^2\big|\nabla|u|\big|^2\right) \\ & + C_\delta\left(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\right)\int\left[\rho(\theta-\tilde\theta)^2+\rho|u|^4\right] \end{aligned} \tag{3.18}$$

for any $\delta>0$. Hence (3.17) and (3.18) yield

$$\begin{aligned} I_3 \le & \epsilon\int(\mathrm{div}u)^2|u|^2 + \delta C_\epsilon\Big(\int|\nabla\theta|^2+\int|u|^2\big|\nabla u\big|^2\Big) \\ & + C_\delta C_\epsilon\Big(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\Big)\int\left[\rho(\theta-\tilde\theta)^2+\rho|u|^4\right] + C\int|\nabla u|^2. \end{aligned} \tag{3.19}$$

For I_4, using integration by parts and Cauchy inequality we have

$$I_4 = 2\kappa\int\nabla\theta|u|\nabla|u| \le \epsilon\int|u|^2|\nabla u|^2 + C_\epsilon\int|\nabla\theta|^2. \tag{3.20}$$

Inserting (3.15), (3.16), (3.19) and (3.20) in (3.14), we have

$$\begin{aligned} & \frac{\mu}{2}\int\left|\nabla u+(\nabla u)'\right|^2|u|^2 + \lambda\int(\mathrm{div}u)^2|u|^2 \\ \le & \frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|u|^2(\theta-\tilde\theta) - \int C_v(\theta-\tilde\theta)\left[(\rho|u|^2)_t+\nabla\cdot(\rho|u|^2u)\right] \\ & + \epsilon\int(\mathrm{div}u)^2|u|^2 + (\epsilon+\delta C_\epsilon)\int|u|^2\big|\nabla u\big|^2 + C_\epsilon\int|\nabla\theta|^2 \\ & + C\int|\nabla u|^2 + C_\delta C_\epsilon\left(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\right)\int\left[\rho(\theta-\tilde\theta)^2+\rho|u|^4\right]. \end{aligned} \tag{3.21}$$

Recalling that $\rho_t+\nabla\cdot(\rho u)=0$ we have

$$\begin{aligned} (\rho|u|^2)_t+\nabla\cdot(\rho u|u|^2) &= \rho(|u|^2)_t+\rho u\cdot\nabla(|u|^2) \\ &= 2\rho u\cdot u_t + 2\rho\nabla u : u\otimes u. \end{aligned} \tag{3.22}$$

Substituting (3.22) for the second term on the right-hand side of (3.21), and using Cauchy inequality and (3.1), we have

$$\begin{aligned} & -\int C_v(\theta-\tilde\theta)\left[(\rho|u|^2)_t+\nabla\cdot(\rho|u|^2u)\right] \\ & = -2C_v\int(\theta-\tilde\theta)\rho u\cdot u_t - 2C_v\int(\theta-\tilde\theta)\rho\nabla u : u\otimes u \\ & \le \epsilon\int\rho|u_t|^2 + \epsilon\int|u|^2|\nabla u|^2 + C_\epsilon\int\rho|u|^2(\theta-\tilde\theta)^2 \end{aligned} \tag{3.23}$$

for any $\epsilon>0$.

As the density is bounded, relation (3.18) remains valid for ρ^2 on the left-hand side replaced by ρ, and (3.23) yields

$$
\begin{aligned}
&-\int C_v(\theta-\tilde{\theta})\big[(\rho|u|^2)_t+\nabla\cdot(\rho|u|^2u)\big]\\
\le&\epsilon\int\rho|u_t|^2+(\epsilon+\delta C_\epsilon)\int|u|^2|\nabla u|^2+\delta C_\epsilon\int|\nabla\theta|^2\\
&+C_\epsilon C_\delta\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\big[\rho(\theta-\tilde{\theta})^2+\rho|u|^4\big],
\end{aligned}\tag{3.24}
$$

for any $\epsilon,\delta>0$.

Plugging (3.24) in (3.21) we have

$$
\begin{aligned}
&\frac{\mu}{2}\int\left|\nabla u+(\nabla u)'\right|^2|u|^2+\lambda\int(\mathrm{div}u)^2|u|^2\\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|u|^2(\theta-\tilde{\theta})+\epsilon\int\rho|u_t|^2+2(\epsilon+\delta C_\epsilon)\int|u|^2|\nabla u|^2\\
&+C_\epsilon C_\delta\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\big[\rho(\theta-\tilde{\theta})^2+\rho|u|^4\big]\\
&+\epsilon\int|\mathrm{div}u|^2|u|^2+C_\epsilon\int|\nabla\theta|^2+C\int|\nabla u|^2.
\end{aligned}\tag{3.25}
$$

Choosing $\epsilon>0$ in (3.25) so that

$$\epsilon\le\frac{\lambda}{2},\tag{3.26}$$

we get (3.13). The proof of Lemma 3.2 is complete.

□

Lemma 3.3. *Under the hypotheses of Theorem 1.4 and (3.1), the following estimates hold depending on the sign of the bulk viscosity coefficient λ.*

1. $\lambda>0$:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int[\rho|u|^4-\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta})]+\frac{\mu}{2}\int|u|^2|\nabla u|^2\\
\le&C_\epsilon\int|\nabla\theta|^2+C\int|\nabla u|^2+\epsilon C\int\rho|u_t|^2\\
&+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int(\rho|\theta-\tilde{\theta}|^2+\rho|u|^4),
\end{aligned}\tag{3.27}
$$

for any $\tilde{\rho},\tilde{\theta}\ge0$, and any small $\epsilon>0$ satisfying (3.12), (3.26) and (3.30) below;

2. $\lambda\le0$:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^4+2\mu\int|u|^2|\nabla u|^2\le&C\int|\nabla u|^2+C\epsilon\int|\nabla\theta|^2\\
&+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\big[\rho(\theta-\tilde{\theta})^2+\rho|u|^4\big],
\end{aligned}\tag{3.28}
$$

for any $\tilde\rho, \tilde\theta \ge 0$, and any small $\epsilon > 0$ satisfying (3.34) below.

Proof. For any given $\lambda > 0$, multiplying (3.13) by $\frac{4C}{\lambda}$, adding the resulting equation to (3.7), and noticing that the last term on the right-hand side of (3.7) can be absorbed by the second term on the left-hand side of the updated (3.13), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left[\rho|u|^4 - \frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde\theta)\right] + \mu\int |u|^2|\nabla u|^2 \\
&+ \frac{2C\mu}{\lambda}\int \left|\nabla u + (\nabla u)'\right|^2|u|^2 + C\int (\mathrm{div}u)^2|u|^2 \\
\le& \left(C\epsilon + C_\epsilon\frac{4C}{\lambda}\right)\int|\nabla\theta|^2 + \left(C + \frac{4C^2}{\lambda}\right)\int|\nabla u|^2 \\
&+ \epsilon\frac{4C}{\lambda}\int\rho|u_t|^2 + \frac{8C}{\lambda}(\epsilon+\delta C_\epsilon)\int|u|^2|\nabla u|^2 \\
&+ \left(C_\epsilon + \frac{4C}{\lambda}C_\epsilon C_\delta\right)\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}} + 1\right)\int\left(\rho|\theta-\tilde\theta|^2 + \rho|u|^4\right).
\end{aligned}\tag{3.29}$$

In addition to the smallness assumptions (3.12) and (3.26), let ϵ and δ be chosen small enough so that

$$\frac{8C}{\lambda}(\epsilon+\delta C_\epsilon) \le \frac{\mu}{2}.\tag{3.30}$$

Then the fourth term on the right-hand side of (3.29) can be absorbed by the second term on the left-hand side. As δ in (3.30) depends, in fact, on ϵ, the constant C_δ can be replaced by C_ϵ. This completes the proof of (3.27).

For any given $\lambda \le 0$, noticing that $\mu+\lambda = \frac{\mu}{3} + \frac{2\mu}{3} + \lambda > 0$ and using (3.8) and Cauchy inequality, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^4 + \int 4|u|^2\left(\mu|\nabla u|^2 + (\lambda+\mu)|\mathrm{div}u|^2 + 2\mu|\nabla|u||^2\right) \\
=&4\int \mathrm{div}(|u|^2u)P - 8(\mu+\lambda)\int \mathrm{div}u|u|u\cdot\nabla|u| \\
\le& C\int\rho|\theta|\,|u|^2|\nabla u| + 4(\mu+\lambda)\int|\mathrm{div}u|^2|u|^2 \\
&+ 4(\mu+\lambda)\int|u|^2\big|\nabla|u|\big|^2.
\end{aligned}\tag{3.31}$$

The second term on the right-hand side of (3.31) can be absorbed by the left. Hence we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^4 + \int|u|^2\left[4\mu|\nabla u|^2 + 4(\mu-\lambda)|\nabla|u||^2\right] \le C\int\rho|\theta|\,|u|^2|\nabla u|.$$

Since $\lambda \le 0$, we have $\mu - \lambda \ge \mu > 0$ and thus

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^4 + 4\mu\int|u|^2|\nabla u|^2 \le C\int\rho|\theta|\,|u|^2|\nabla u|.\tag{3.32}$$

Applying Cauchy inequality to the term on the right-hand side of (3.32), and using (3.1) and (3.18), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^4 + 4\mu\int |u|^2|\nabla u|^2\\
\leq & C\int \rho|\theta-\tilde{\theta}||u|^2|\nabla u| + C\tilde{\theta}\int \rho|u|^2|\nabla u|\\
\leq & \mu\int |u|^2|\nabla u|^2 + C\int \rho^2|\theta-\tilde{\theta}|^2|u|^2 + C\int \rho|u|^4 + C\int |\nabla u|^2\\
\leq & \mu\int |u|^2|\nabla u|^2 + C\int |\nabla u|^2 + C\epsilon\left(\int |\nabla\theta|^2 + \int |u|^2\big|\nabla|u|\big|^2\right)\\
& + C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\right)\int \left[\rho(\theta-\tilde{\theta})^2 + \rho|u|^4\right],
\end{aligned}
\tag{3.33}
$$

for any small $\epsilon > 0$. The second term on the left-hand side of (3.33) can absorb the corresponding terms on the right-hand side provided ϵ satisfies

$$
C\epsilon \leq \mu. \tag{3.34}
$$

Thus we have shown (3.28). □

Lemma 3.4. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \left(C_v\rho|\theta-\tilde{\theta}|^2 + |\rho-\tilde{\rho}|^2\right) + \frac{\kappa}{2}\int |\nabla\theta|^2\\
\leq & C_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\right)\left(\|\nabla u\|_{L^2}^2 + \|\sqrt{\rho}(\theta-\tilde{\theta})\|_{L^2}^2 + \|\rho-\tilde{\rho}\|_{L^2}^2\right)\\
& + \epsilon_1\int \rho|u_t|^2 + \epsilon_1\int |u|^2|\nabla u|^2 + C\|\theta-\tilde{\theta}\|_{L^r},
\end{aligned}
\tag{3.35}
$$

for any $\tilde{\rho},\tilde{\theta} \geq 0$, and any small $\epsilon_1 > 0$ satisfying (3.40) below.

Proof. Multiplying $(1.6)_3$ by $\theta-\tilde\theta$, and integrating by parts over Ω, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|\theta-\tilde\theta|^2+\kappa\int|\nabla\theta|^2\\
=&-\int\rho(\theta-\tilde\theta)^2\mathrm{div}u-\tilde\theta\int\rho(\theta-\tilde\theta)\mathrm{div}u\\
&+\int\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2(\theta-\tilde\theta)+\int\lambda(\mathrm{div}u)^2(\theta-\tilde\theta)\\
\le& C\|\theta-\tilde\theta\|_{L^r}\|\rho(\theta-\tilde\theta)\|_{L^{\frac{2r}{r-1}}}\|\nabla u\|_{L^{\frac{2r}{r-1}}}+C\int\rho|\theta-\tilde\theta|^2\\
&+C\int|\nabla u|^2+C\|\theta-\tilde\theta\|_{L^r}\|\nabla u\|^2_{L^{\frac{2r}{r-1}}}\\
\le& C\|\theta-\tilde\theta\|_{L^r}\|\rho(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}}+C\|\theta-\tilde\theta\|_{L^r}\|\nabla u\|^2_{L^{\frac{2r}{r-1}}}\\
&+C\int\rho|\theta-\tilde\theta|^2+C\int|\nabla u|^2
\end{aligned}
\tag{3.36}
$$

for $r\in(\frac{3}{2},\infty]$, where we have used (3.1), Hölder inequality, and Cauchy inequality.

Recalling $u=w+h$, where h, w satisfy (3.3), (3.4), respectively, and using (3.5) and (3.6), we have

$$
\begin{aligned}
\|\nabla u\|_{L^{\frac{2r}{r-1}}}&\le\|\nabla h\|_{L^{\frac{2r}{r-1}}}+\|\nabla w\|_{L^{\frac{2r}{r-1}}}\\
&\le C\|\rho\theta-\tilde\rho\tilde\theta\|_{L^{\frac{2r}{r-1}}}+C\|\nabla w\|_{L^2}+C\|\nabla w\|^{\alpha}_{L^2}\|\nabla^2 w\|^{1-\alpha}_{L^2}\\
&\le C\|\rho(\theta-\tilde\theta)\|_{L^{\frac{2r}{r-1}}}+C\|\tilde\theta(\rho-\tilde\rho)\|_{L^{\frac{2r}{r-1}}}\\
&\quad+C\|\nabla w\|_{L^2}+C\|\nabla w\|^{\alpha}_{L^2}\|\rho\dot u\|^{1-\alpha}_{L^2},
\end{aligned}
\tag{3.37}
$$

where $\alpha=1-\frac{3}{2r}$. In addition, we have the interpolation inequality in terms of ∇w,

$$
\|\nabla w\|_{L^{\frac{2r}{r-1}}}\le C\|\nabla w\|_{L^2}+C\|\nabla w\|^{\alpha}_{L^2}\|\nabla^2 w\|^{1-\alpha}_{L^2},
$$

see Lemma 2.1 if Ω is bounded, or Gagliardo-Nirenberg inequality if $\Omega=\mathbb{R}^3$.

Implementing (3.37) in (3.36), and using Young inequality and (3.1), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|\theta-\tilde{\theta}|^2+\kappa\int|\nabla\theta|^2\\
\leq& C\|\theta-\tilde{\theta}\|_{L^r}\|\rho(\theta-\tilde{\theta})\|^2_{L^{\frac{2r}{r-1}}}+C\|\theta-\tilde{\theta}\|_{L^r}\|\tilde{\theta}(\rho-\tilde{\rho})\|^2_{L^{\frac{2r}{r-1}}}\\
&+C\|\theta-\tilde{\theta}\|_{L^r}\|\nabla w\|^2_{L^2}+C\|\theta-\tilde{\theta}\|_{L^r}\|\nabla w\|^{2-\frac{3}{r}}_{L^2}\|\rho\dot{u}\|^{\frac{3}{r}}_{L^2}\\
&+C\int\rho|\theta-\tilde{\theta}|^2+C\int|\nabla u|^2\\
\leq& C\|\theta-\tilde{\theta}\|_{L^r}\|\rho(\theta-\tilde{\theta})\|^2_{L^{\frac{2r}{r-1}}}+C\|\theta-\tilde{\theta}\|_{L^r}\|\tilde{\theta}(\rho-\tilde{\rho})\|^2_{L^{\frac{2r}{r-1}}}\\
&+\epsilon_1\int\rho|u_t|^2+\epsilon_1\int|u|^2|\nabla u|^2+C_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|^{\frac{2r}{2r-3}}_{L^r}+1\right)\|\nabla w\|^2_{L^2}\\
&+C\int\rho|\theta-\tilde{\theta}|^2+C\int|\nabla u|^2, \qquad (3.38)
\end{aligned}
$$

for any $\epsilon_1>0$. In view of the fact that $r\in(\frac{3}{2},\infty]$, we get $\frac{2r}{r-1}\geq 2$. Then using (3.1) and Young inequality, we obtain

$$
\|\tilde{\theta}(\rho-\tilde{\rho})\|^2_{L^{\frac{2r}{r-1}}}\leq C(\|\rho-\tilde{\rho}\|^2_{L^2}+1). \tag{3.39}
$$

Inserting (3.10) and (3.39) in (3.38), and choosing ϵ_1 small enough such that

$$
C\epsilon_1\leq\frac{\kappa}{2}, \tag{3.40}
$$

we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|\theta-\tilde{\theta}|^2+\frac{\kappa}{2}\int|\nabla\theta|^2\\
\leq& C_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|^{\frac{2r}{2r-3}}_{L^r}+1\right)\|\sqrt{\rho}(\theta-\tilde{\theta})\|^2_{L^2}+C\|\theta-\tilde{\theta}\|_{L^r}\left(\|\rho-\tilde{\rho}\|^2_{L^2}+1\right)\\
&+\epsilon_1\int\rho|u_t|^2+\epsilon_1\int|u|^2|\nabla u|^2+C_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|^{\frac{2r}{2r-3}}_{L^r}+1\right)\|\nabla w\|^2_{L^2}\\
&+C\int|\nabla u|^2. \qquad (3.41)
\end{aligned}
$$

Recalling that $u=w+h$ again, and using (3.1) and (3.5), we have

$$
\begin{aligned}
\|\nabla w\|_{L^2}\leq&\|\nabla u\|_{L^2}+\|\nabla h\|_{L^2}\\
\leq&\|\nabla u\|_{L^2}+C\|\rho\theta-\tilde{\rho}\tilde{\theta}\|_{L^2}\\
\leq&\|\nabla u\|_{L^2}+C\|\sqrt{\rho}(\theta-\tilde{\theta})\|_{L^2}+C\|\tilde{\theta}(\rho-\tilde{\rho})\|_{L^2}. \qquad (3.42)
\end{aligned}
$$

Inserting (3.42) in (3.41), and using Young inequality, we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int C_v\rho|\theta-\tilde\theta|^2+\frac{\kappa}{2}\int|\nabla\theta|^2\\
\le& C_{\epsilon_1}\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\left(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}(\theta-\tilde\theta)\|_{L^2}^2+\|\rho-\tilde\rho\|_{L^2}^2\right)\\
&+\epsilon_1\int\rho|u_t|^2+\epsilon_1\int|u|^2|\nabla u|^2+C\|\theta-\tilde\theta\|_{L^r},
\end{aligned}\tag{3.43}
$$

for any small $\epsilon_1>0$ satisfying (3.40).

The term $\|\rho-\tilde\rho\|_{L^2}$[5] on the right-hand side of (3.43) does not appear if $\tilde\theta=0$, see (3.39) and (3.42). To handle this term, we rewrite $(1.6)_1$ as an equation for $\rho-\tilde\rho$, which, multiplied by $2(\rho-\tilde\rho)$, yields

$$
\left[(\rho-\tilde\rho)^2\right]_t+\nabla\cdot\left[(\rho-\tilde\rho)^2u\right]+(\rho-\tilde\rho)^2\mathrm{div}u+2\tilde\rho(\rho-\tilde\rho)\mathrm{div}u=0.\tag{3.44}
$$

Integrating (3.44) over Ω, and using (3.1) and Cauchy inequality, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho-\tilde\rho|^2\le C\int|\rho-\tilde\rho|^2+C\int|\nabla u|^2.\tag{3.45}
$$

Adding (3.45) to (3.43), we get (3.35).

□

The next lemma is not valid if $\Omega=\mathbb{R}^3$ and $\tilde\theta>0$. Here, we prove the result for a bounded domain, while its counterpart for $\Omega=\mathbb{R}^3$ will be shown in the next section. The generic constant C in Lemma 3.5 may depend on the size of the domain.

Lemma 3.5. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with smooth boundary. Under the hypotheses of Theorem 1.4 and (3.1), we have*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\Big[\mu\big(|\nabla u|^2+|\nabla h|^2\big)+(\mu+\lambda)\big(|\mathrm{div}u|^2+|\mathrm{div}h|^2\big)\Big]+\int\rho|u_t|^2\\
\le&\frac{\mathrm{d}}{\mathrm{d}t}\int 2(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u+C\int|\nabla\theta|^2+C\int|u|^2|\nabla u|^2+C\int|\nabla u|^2\\
&+C\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\Big(\int\rho(\theta-\tilde\theta)^2+\int\rho|u|^4\Big)\\
&+C\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big).
\end{aligned}\tag{3.46}
$$

[5]This term is obviously bounded if the density and Ω are bounded, however, this may not be true on unbounded domains.

Proof. Multiplying $(1.6)_2$ by u_t, and integrating by parts over Ω, we have

$$\begin{aligned}
&\int \rho|u_t|^2 + \frac{1}{2}\frac{d}{dt}\int \big(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2\big) \\
&= -\int \nabla P\cdot u_t - \int \rho u\cdot\nabla u\cdot u_t \\
&\le -\int \nabla P\cdot u_t + C\int |u|^2|\nabla u|^2 + \frac{1}{4}\int \rho|u_t|^2, \qquad (3.47)
\end{aligned}$$

where we have used Cauchy inequality and (3.1).

For the first term on the right-hand side of (3.47), replacing u by $w+h$ and applying (3.3) and (3.4), we have

$$\begin{aligned}
-\int \nabla P\cdot u_t =& \frac{\mathrm{d}}{\mathrm{d}t}\int (\rho\theta-\tilde\rho\tilde\theta)\mathrm{div} u - \int P_t \mathrm{div} u \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int (\rho\theta-\tilde\rho\tilde\theta)\mathrm{div} u - \int P_t \mathrm{div} w - \int P_t \mathrm{div} h \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int (\rho\theta-\tilde\rho\tilde\theta)\mathrm{div} u - \int P_t \mathrm{div} w + \int L h_t\cdot h \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int (\rho\theta-\tilde\rho\tilde\theta)\mathrm{div} u - \int P_t \mathrm{div} w \\
& - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int [\mu|\nabla h|^2 + (\mu+\lambda)|\mathrm{div} h|^2]. \qquad (3.48)
\end{aligned}$$

Recalling from $(1.6)_3$ that

$$\begin{aligned}
P_t =& -\nabla\cdot\Big[(\rho\theta-\tilde\rho\tilde\theta)u\Big] - \tilde\rho\tilde\theta(1+\frac{1}{C_v})\mathrm{div} u - \frac{1}{C_v}\Big(\rho\theta-\tilde\rho\tilde\theta\Big)\,\mathrm{div} u \\
& + \frac{\mu}{C_v}\nabla u\cdot\big[\nabla u + (\nabla u)'\big] + \frac{\lambda}{C_v}\mathrm{div} u\,\mathrm{div} u + \frac{\kappa}{C_v}\Delta\theta, \qquad (3.49)
\end{aligned}$$

we get

$$\begin{aligned}
-\int P_t \mathrm{div} w =& -\int \Big[(\rho\theta-\tilde\rho\tilde\theta)u\Big]\cdot\nabla\mathrm{div} w + \tilde\rho\tilde\theta(1+\frac{1}{C_v})\int \mathrm{div} u\,\mathrm{div} w \\
& + \frac{1}{C_v}\int \Big(\rho\theta-\tilde\rho\tilde\theta\Big)\,\mathrm{div} u\,\mathrm{div} w - \frac{\mu}{C_v}\int \nabla u\cdot[\nabla u + (\nabla u)']\mathrm{div} w \\
& - \frac{\lambda}{C_v}\int \mathrm{div} u\,\mathrm{div} u\,\mathrm{div} w + \frac{\kappa}{C_v}\int \nabla\theta\cdot\nabla\mathrm{div} w \\
\le& C\big(\|\rho(\theta-\tilde\theta)u\|_{L^2} + \|(\rho-\tilde\rho)u\|_{L^2} + \|\nabla\theta\|_{L^2}\big)\|\nabla\mathrm{div} w\|_{L^2} \\
& + C\|\nabla u\|_{L^2}\|\nabla w\|_{L^2} + \frac{1}{C_v}\int (\rho\theta-\tilde\rho\tilde\theta)\mathrm{div} u\,\mathrm{div} w \\
& - \frac{\mu}{C_v}\int \nabla u\cdot[\nabla u + (\nabla u)']\mathrm{div} w - \frac{\lambda}{C_v}\int \mathrm{div} u\,\mathrm{div} u\,\mathrm{div} w, \qquad (3.50)
\end{aligned}$$

where we have used integration by parts and Hölder inequality. Note that we have used the hypothesis that θ satisfies the homogeneous Neumann boundary conditions.

For the last three terms on the right-hand side of (3.50), using integration by parts and the momentum equation, we have

$$\begin{aligned}
&\frac{1}{C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,\mathrm{div}w-\frac{\mu}{C_v}\int\nabla u\cdot\left[\nabla u+(\nabla u)'\right]\mathrm{div}w-\frac{\lambda}{C_v}\int\mathrm{div}u\,\mathrm{div}u\,\mathrm{div}w\\
=&\frac{1}{C_v}\int u\cdot\left[\mu\Delta u+(\mu+\lambda)\nabla\mathrm{div}u-\nabla P\right]\mathrm{div}w+\frac{\mu}{C_v}\int\left[\nabla u+(\nabla u)'\right]:\nabla\mathrm{div}w\otimes u\\
&+\frac{\lambda}{C_v}\int u\cdot\nabla\mathrm{div}w\,\mathrm{div}u-\frac{1}{C_v}\int(\rho\theta-\tilde\rho\tilde\theta)u\cdot\nabla\mathrm{div}w\\
=&\frac{1}{C_v}\int\rho u\cdot\dot u\,\mathrm{div}w+\frac{\mu}{C_v}\int\left[\nabla u+(\nabla u)'\right]:\nabla\mathrm{div}w\otimes u\\
&+\frac{\lambda}{C_v}\int u\cdot\nabla\mathrm{div}w\,\mathrm{div}u-\frac{1}{C_v}\int(\rho\theta-\tilde\rho\tilde\theta)u\cdot\nabla\mathrm{div}w,
\end{aligned}$$

which combined with Cauchy inequality and (3.1), yields

$$\begin{aligned}
&\frac{1}{C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,\mathrm{div}w-\frac{\mu}{C_v}\int\nabla u\cdot\left[\nabla u+(\nabla u)'\right]\mathrm{div}w\\
&-\frac{\lambda}{C_v}\int\mathrm{div}u\,\mathrm{div}u\,\mathrm{div}w\\
\le&\frac{1}{8}\int\rho|u_t|^2+C\int\rho|u|^2|\mathrm{div}w|^2+C_\epsilon\int|u|^2|\nabla u|^2+\epsilon\int|\nabla\mathrm{div}w|^2\\
&+C_\epsilon\int|\rho\theta-\tilde\rho\tilde\theta|^2|u|^2,
\end{aligned}\tag{3.51}$$

for any $\epsilon>0$.

By using (3.1), (3.6), (3.42) and Cauchy inequality, (3.50) and (3.51) yield

$$\begin{aligned}
-\int P_t\mathrm{div}w\le&C(\|\rho(\theta-\tilde\theta)u\|_{L^2}+\|(\rho-\tilde\rho)u\|_{L^2}+\|\nabla\theta\|_{L^2})\|\rho\dot u\|_{L^2}\\
&+C\int|\nabla u|^2+C\int\left(\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2\right)\\
&+\frac{1}{8}\int\rho|u_t|^2+C_\epsilon\int|u|^2|\nabla u|^2+\epsilon C\int\rho|u_t|^2\\
&+C\int\rho|u|^2|\mathrm{div}w|^2+C_\epsilon\int|\rho\theta-\tilde\rho\tilde\theta|^2|u|^2\\
\le&\frac{1}{4}\int\rho|u_t|^2+C\int|\nabla\theta|^2+C\int|u|^2|\nabla u|^2+C\int\rho|u|^2|\mathrm{div}w|^2\\
&+C\int\rho^2|\theta-\tilde\theta|^2|u|^2+C\|(\rho-\tilde\rho)u\|_{L^2}^2\\
&+C\int|\nabla u|^2+C\int\left(\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2\right),
\end{aligned}\tag{3.52}$$

for some small $\epsilon>0$.

Noticing that $w = u - h$, and using Hölder inequality, (3.1) and (3.5), we have

$$
\begin{aligned}
&\int \rho|u|^2|\mathrm{div} w|^2 \\
\le& C\int \rho|u|^2|\mathrm{div} u|^2 + C\int \rho|u|^2|\mathrm{div} h|^2 \\
\le& C\int |u|^2|\nabla u|^2 + C\|\mathrm{div} h\|_{L^r}\|\mathrm{div} h\|_{L^{\frac{2r}{r-1}}}\big\|\sqrt{\rho}|u|^2\big\|_{L^{\frac{2r}{r-1}}} \\
\le& C\int |u|^2|\nabla u|^2 + C\|\rho\theta-\tilde\rho\tilde\theta\|_{L^r}\|\rho\theta-\tilde\rho\tilde\theta\|_{L^{\frac{2r}{r-1}}}\big\|\sqrt{\rho}|u|^2\big\|_{L^{\frac{2r}{r-1}}} \\
\le& C\int |u|^2|\nabla u|^2 + C\|\rho\theta-\tilde\rho\tilde\theta\|_{L^r}\|\rho\theta-\tilde\rho\tilde\theta\|^2_{L^{\frac{2r}{r-1}}} \\
&+ C\|\rho\theta-\tilde\rho\tilde\theta\|_{L^r}\big\|\sqrt{\rho}|u|^2\big\|^2_{L^{\frac{2r}{r-1}}} := II_1 + II_2 + II_3.
\end{aligned}
\tag{3.53}
$$

For II_2, we have

$$
\begin{aligned}
II_2 \le& C\|\rho(\theta-\tilde\theta)\|_{L^r}\|\rho\theta-\tilde\rho\tilde\theta\|^2_{L^{\frac{2r}{r-1}}} + C\|\tilde\theta(\rho-\tilde\rho)\|_{L^r}\|\rho\theta-\tilde\rho\tilde\theta\|^2_{L^{\frac{2r}{r-1}}} \\
\le& C\|\rho(\theta-\tilde\theta)\|_{L^r}\|\rho(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}} + C\|\rho(\theta-\tilde\theta)\|_{L^r}\|\tilde\theta(\rho-\tilde\rho)\|^2_{L^{\frac{2r}{r-1}}} \\
&+ C\|\tilde\theta(\rho-\tilde\rho)\|_{L^r}\|\rho(\theta-\tilde\theta)\|^2_{L^{\frac{2r}{r-1}}} + C\|\tilde\theta(\rho-\tilde\rho)\|_{L^r}\|\tilde\theta(\rho-\tilde\rho)\|^2_{L^{\frac{2r}{r-1}}}.
\end{aligned}
\tag{3.54}
$$

As Ω is a bounded domain and the density is supposed to be bounded, we apply the standard interpolation inequality and Young inequality to (3.54) to get

$$
\begin{aligned}
II_2 \le& C\Big(\|\theta-\tilde\theta\|_{L^r}+1\Big)\|\rho(\theta-\tilde\theta)\|^{2-\frac{3}{r}}_{L^2}\|\rho(\theta-\tilde\theta)\|^{\frac{3}{r}}_{L^6} + C\|\theta-\tilde\theta\|_{L^r} + C \\
\le& C\Big(\|\theta-\tilde\theta\|_{L^r}+1\Big)\|\rho(\theta-\tilde\theta)\|^{2-\frac{3}{r}}_{L^2}\|\theta-\tilde\theta\|^{\frac{3}{r}}_{L^6} + C\|\theta-\tilde\theta\|_{L^r} + C \\
\le& C\int|\nabla\theta|^2 + C\Big(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\Big)\|\sqrt{\rho}(\theta-\tilde\theta)\|^2_{L^2} \\
&+ C\Big(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\Big),
\end{aligned}
\tag{3.55}
$$

where we have used (3.1), Sobolev inequality, and (2.3).

For II_3, we have

$$
\begin{aligned}
II_3 \le& C\Big(\|\rho(\theta-\tilde\theta)\|_{L^r}+\|\rho-\tilde\rho\|_{L^r}\Big)\big\|\sqrt{\rho}|u|^2\big\|^2_{L^{\frac{2r}{r-1}}} \\
\le& C\Big(\|\rho(\theta-\tilde\theta)\|_{L^r}+1\Big)\big\|\sqrt{\rho}|u|^2\big\|^{2-\frac{3}{r}}_{L^2}\big\|\sqrt{\rho}|u|^2\big\|^{\frac{3}{r}}_{L^6} \\
\le& C\Big(\|\theta-\tilde\theta\|^{\frac{2r}{2r-3}}_{L^r}+1\Big)\big\|\sqrt{\rho}|u|^2\big\|^2_{L^2} + C\big\||u|\nabla u\big\|^2_{L^2}.
\end{aligned}
\tag{3.56}
$$

Inserting (3.55) and (3.56) in (3.53), we have

$$\begin{aligned}&\int \rho|u|^2|\mathrm{div}w|^2\\ \le& C\int |u|^2|\nabla u|^2 + C\int |\nabla\theta|^2 + C\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big)\|\sqrt{\rho}(\theta-\tilde\theta)\|_{L^2}^2\\ &+C\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big)+C\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big)\big\|\sqrt{\rho}|u|^2\big\|_{L^2}^2.\end{aligned}\tag{3.57}$$

In addition, by virtue of Hölder inequality and (3.1), it is easy to get

$$\|(\rho-\tilde\rho)u\|_{L^2}^2\le C\|\nabla u\|_{L^2}^2.\tag{3.58}$$

The estimates of the fourth, the fifth, and the sixth term on the right-hand side of (3.52) are similar to (3.57), (3.18), and (3.58), respectively. Hence (3.52) yields

$$\begin{aligned}-\int P_t\mathrm{div}w \le& \frac14\int\rho|u_t|^2 + C\int|\nabla\theta|^2 + C\int |u|^2|\nabla u|^2 + C\int|\nabla u|^2\\ &+C\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\left(\int\rho(\theta-\tilde\theta)^2+\int\rho|u|^4\right)\\ &+C\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right).\end{aligned}\tag{3.59}$$

Inserting (3.59) and (3.48) in (3.47), we have

$$\begin{aligned}&\frac12\frac{\mathrm d}{\mathrm dt}\int\Big[\mu\big(|\nabla u|^2+|\nabla h|^2\big)+(\mu+\lambda)\big(|\mathrm{div}u|^2+|\mathrm{div}h|^2\big)\Big]+\int\rho|u_t|^2\\ \le&\frac{\mathrm d}{\mathrm dt}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u + C\int|\nabla\theta|^2 + C\int|u|^2|\nabla u|^2 + C\int|\nabla u|^2\\ &+C\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\left(\int\rho(\theta-\tilde\theta)^2+\int\rho|u|^4\right)\\ &+C\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)+\frac12\int\rho|u_t|^2.\end{aligned}$$

The last term on the right-hand side can be absorbed by the integrals on the left–hand side. We have finished the proof.

□

Corollary 3.6. *Let Ω be a bounded and smooth domain in $\mathbb{R}^3$. Under the hypotheses of Theorem 1.4 and (3.1), we have*

$$\int(\rho|u|^4+\rho|\theta-\tilde\theta|^2+|\nabla u|^2)+\int_0^T\int[\rho|u_t|^2+|u|^2|\nabla u|^2+|\nabla\theta|^2]\le C,\tag{3.60}$$

for any $T\in(0,T^*)$.

Proof. Let $\lambda>0$ be given. Multiplying (3.35) by a sufficiently large positive constant M, and adding the resulting equation to (3.27), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int[\rho|u|^4-\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta})+MC_v\rho|\theta-\tilde{\theta}|^2+M|\rho-\tilde{\rho}|^2]\\
&+\frac{\mu}{2}\int|u|^2|\nabla u|^2+\frac{M\kappa}{2}\int|\nabla\theta|^2\\
\leq& MC_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\left(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}(\theta-\tilde{\theta})\|_{L^2}^2+\|\rho-\tilde{\rho}\|_{L^2}^2\right)\\
&+(M\epsilon_1+\epsilon C)\int\rho|u_t|^2+M\epsilon_1\int|u|^2|\nabla u|^2\\
&+MC\|\theta-\tilde{\theta}\|_{L^r}+C_\epsilon\int|\nabla\theta|^2+C\int|\nabla u|^2\\
&+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(\rho|\theta-\tilde{\theta}|^2+\rho|u|^4\right).
\end{aligned}\tag{3.61}
$$

Given $\epsilon>0$, we may choose $M=M(\epsilon)>0$ big enough and $\epsilon_1=\epsilon_1(M)>0$ small enough so that

$$
C_\epsilon\leq\frac{M\kappa}{4},\quad\text{and } M\epsilon_1\leq\frac{\mu}{4}.\tag{3.62}
$$

Consequently, the third term and the fifth term on the right-hand side of (3.61) can be absorbed by the left–hand side. Hence we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int[\rho|u|^4-\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta})+MC_v\rho|\theta-\tilde{\theta}|^2+M|\rho-\tilde{\rho}|^2]\\
&+\frac{\mu}{4}\int|u|^2|\nabla u|^2+\frac{M\kappa}{4}\int|\nabla\theta|^2\\
\leq& MC_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(|\nabla u|^2+\rho|\theta-\tilde{\theta}|^2+|\rho-\tilde{\rho}|^2\right)\\
&+(M\epsilon_1+\epsilon C)\int\rho|u_t|^2+MC\|\theta-\tilde{\theta}\|_{L^r}+C\int|\nabla u|^2\\
&+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(\rho|\theta-\tilde{\theta}|^2+\rho|u|^4\right).
\end{aligned}\tag{3.63}
$$

Multiplying (3.63) by a positive constant M_1, and then adding the resulting equation to (3.46), we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int G(\rho,u,\theta,h)+\frac{\mu M_1}{4}\int |u|^2|\nabla u|^2+\frac{M_1M\kappa}{4}\int|\nabla\theta|^2+\int\rho|u_t|^2\\
\le& M_1MC_{\epsilon_1}\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(|\nabla u|^2+\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2\right)\\
&+M_1(M\epsilon_1+\epsilon C)\int\rho|u_t|^2+C\int|\nabla\theta|^2+C\int|u|^2|\nabla u|^2\\
&+(M_1C_\epsilon+C)\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(\rho|\theta-\tilde\theta|^2+\rho|u|^4\right)\\
&+C(M_1+1)\int|\nabla u|^2+C(M_1M+1)\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big),
\end{aligned}
\tag{3.64}
$$

where

$$
\begin{aligned}
G(\rho,u,\theta,h)=&M_1\Big[\rho|u|^4-\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde\theta)+MC_v\rho|\theta-\tilde\theta|^2+M|\rho-\tilde\rho|^2\Big]\\
&+\mu\big(|\nabla u|^2+|\nabla h|^2\big)+(\mu+\lambda)\big(|\mathrm{div}u|^2+|\mathrm{div}h|^2\big)\\
&-2(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u
\end{aligned}
$$

and

$$
G(\rho,u,\theta,h)\sim\rho|u|^4+\rho|\theta-\tilde\theta|^2+|\nabla u|^2+|\nabla h|^2+|\rho-\tilde\rho|^2
$$

for M big enough.

For $M_1>1$ big enough so that

$$
\frac{M_1\mu}{8}\ge C,
\tag{3.65}
$$

the fourth term on the right-hand side of (3.64) can be absorbed by the left-hand side. Noticing that M_1 and C are independent of ϵ and ϵ_1, we choose $\epsilon>0$ small enough such that

$$
M_1\epsilon C\le\frac14.
\tag{3.66}
$$

Moreover, in view of (3.62), we may choose $\epsilon_1>0$ so small that

$$
\max\{2C,C_\epsilon\}\le\frac{M\kappa}{4},\quad\text{and } M\epsilon_1\le\min\left\{\frac{1}{4M_1},\frac{\mu}{4}\right\}.
\tag{3.67}
$$

Note that the order for fixing the corresponding parameters is

$$
M_1\to\epsilon\to M\to\epsilon_1.
$$

By virtue of (3.65), (3.66) and (3.67), the second term, the third term and the fourth term on the right-hand side of (3.64) can be absorbed by the left-hand side. Hence we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int G(\rho,u,\theta,h)+\frac{\mu M_1}{8}\int |u|^2|\nabla u|^2\\
&+\frac{M_1M\kappa}{8}\int|\nabla\theta|^2+\frac{1}{2}\int\rho|u_t|^2\\
\le&(M_1MC_{\epsilon_1}+M_1C_\epsilon+CM_1+C)\left(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int G(\rho,u,\theta,h)\\
&+C(M_1M+1)\Big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\Big).
\end{aligned}\tag{3.68}$$

Applying Gronwall inequality to (3.68) yields (3.60).

Suppose now $\lambda\le 0$. Multiplying (3.28) and (3.35) by $\frac{N_1}{\mu}$ and $\frac{4N_2}{\kappa}$, respectively, and adding the results to (3.46), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int G_1(\rho,u,\theta,h)+2N_1\int|u|^2|\nabla u|^2+2N_2\int|\nabla\theta|^2+\int\rho|u_t|^2\\
\le&\big(C+\frac{N_1}{\mu}C\epsilon\big)\int|\nabla\theta|^2+\big(C+\frac{4N_2\epsilon_1}{\kappa}\big)\int|u|^2|\nabla u|^2+\\
&\big(C+\frac{CN_1}{\mu}\big)\int|\nabla u|^2+\\
&\big(\frac{N_1}{\mu}C_\epsilon+C\big)\big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\big)\int\big[\rho(\theta-\tilde\theta)^2+\rho|u|^4\big]+\\
&\frac{4N_2C_{\epsilon_1}}{\kappa}\big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\big)\big(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}(\theta-\tilde\theta)\|_{L^2}^2+\|\rho-\tilde\rho\|_{L^2}^2\big)+\\
&\big(\frac{4N_2C}{\kappa}+C\big)\big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\big)+\frac{4N_2\epsilon_1}{\kappa}\int\rho|u_t|^2,
\end{aligned}\tag{3.69}$$

where

$$\begin{aligned}
G_1(\rho,u,\theta,h)=&\mu\big(|\nabla u|^2+|\nabla h|^2\big)+(\mu+\lambda)\big(|\mathrm{div}u|^2+|\mathrm{div}h|^2\big)+\frac{N_1}{\mu}\rho|u|^4\\
&+\frac{4N_2}{\kappa}\Big(C_v\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2\Big)-2(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u
\end{aligned}$$

and

$$G_1(\rho,u,\theta,h)\sim\rho|u|^4+\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2+|\nabla u|^2+|\nabla h|^2,$$

for N_2 big enough.

By choosing N_1, N_2 big enough and ϵ, ϵ_1 small enough such that

$$N_1\ge 2C,\quad N_2\ge 2C,$$

and

$$\frac{N_1}{\mu}C\epsilon\le\frac{N_2}{2},\qquad\frac{4N_2}{\kappa}\epsilon_1\le\min\left\{\frac12,\ \frac{N_1}{2}\right\},$$

the first two terms and the last term on the right-hand side of (3.69) can be absorbed by the left-hand side. Finally, applying Gronwall inequality in (3.69), we get (3.60).

□

Corollary 3.6 combined with the Sobolev inequality yields

$$\|u\|^2_{L^4\left(0,T;L^{12}(\Omega)\right)} = \big\||u|^2\big\|_{L^2\left(0,T;L^6(\Omega)\right)} \le C\big\||u|\nabla u\big\|_{L^2\left(0,T;L^2(\Omega)\right)} \le C,$$

which together with (3.1) gives

$$\|\rho\|_{L^\infty\left(0,T;L^\infty(\Omega)\right)} + \|u\|_{L^4\left(0,T;L^{12}(\Omega)\right)} \le C. \tag{3.70}$$

Since

$$\frac{2}{4} + \frac{3}{12} = \frac{3}{4} < 1$$

is compatible with Serrin's condition, the rest of the estimates in (3.2) for the initial-boundary value problem (1.6), (1.7), and (1.8) can be performed exactly as in [15]. The proof of Theorem 1.4 in the case of a bounded domain is complete.

4. Cauchy problem

Assume that $\Omega = \mathbb{R}^3$ and $T^* < \infty$ and that there exist constants $r \in (\frac{3}{2}, \infty]$ and $s \in [1, \infty]$ satisfying

$$\frac{2}{s} + \frac{3}{r} \le 2,$$

such that (3.1) holds. Our aim is to show that under the assumption (3.1) and the hypotheses of Theorem 1.4, there is a constant $C > 0$ depending only on $M^*, \rho_0, u_0, \theta_0, \tilde\rho, \tilde\theta, \mu, \lambda, \kappa$, and T^* such that

$$\begin{aligned}&\max_{l=2,q}(\|\rho-\tilde\rho\|_{W^{1,l}} + \|\rho_t\|_{L^l}) + \|(\sqrt{\rho}u_t, \sqrt{\rho}\theta_t)\|_{L^2}\\ &+ \|(u,\theta-\tilde\theta)\|_{D_0^1\cap D^2} + \int_0^{T^*}\left(\|(u_t,\theta_t)\|^2_{D^1} + \|(u,\theta)\|^2_{D^{2,q}}\right)\mathrm{d}t \le C.\end{aligned} \tag{4.1}$$

Relation (4.1), together with the available local existence results, implies the desired contradiction.

The proofs of the next two lemmas are the same as their counterparts stated in Lemmas 3.3 and 3.4.

Lemma 4.1. *Under the hypotheses of Theorem 1.4 and (3.1), the following estimates depending on the sign of the viscosity coefficient λ hold:*

1. $\lambda > 0$*:*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left[\rho|u|^4-\frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta})\right]+\frac{\mu}{2}\int|u|^2|\nabla u|^2\\ \le& C_\epsilon\int|\nabla\theta|^2+C\int|\nabla u|^2+\epsilon C\int\rho|u_t|^2\\ &+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left(\rho|\theta-\tilde{\theta}|^2+\rho|u|^4\right),\end{aligned}\tag{4.2}$$

for any small $\epsilon > 0$ *satisfying (3.12), (3.26) and (3.30);*

2. $\lambda \le 0$*:*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|u|^4+2\mu\int|u|^2|\nabla u|^2\\ \le& C\int|\nabla u|^2+C\epsilon\int|\nabla\theta|^2\\ &+C_\epsilon\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\left[\rho(\theta-\tilde{\theta})^2+\rho|u|^4\right],\end{aligned}\tag{4.3}$$

for any small $\epsilon > 0$ *satisfying (3.34).*

Lemma 4.2. *Under the conditions of Theorem 1.4 and (3.1), it holds that*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(C_v\rho|\theta-\tilde{\theta}|^2+|\rho-\tilde{\rho}|^2\right)+\frac{\kappa}{2}\int|\nabla\theta|^2\\ \le& C_{\epsilon_1}\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\left(\|\nabla u\|_{L^2}^2+\|\sqrt{\rho}(\theta-\tilde{\theta})\|_{L^2}^2+\|\rho-\tilde{\rho}\|_{L^2}^2\right)\\ &+\epsilon_1\int\rho|u_t|^2+\epsilon_1\int|u|^2|\nabla u|^2+C\|\theta-\tilde{\theta}\|_{L^r},\end{aligned}\tag{4.4}$$

for any small $\epsilon_1 > 0$ *satisfying (3.40).*

Lemma 4.3. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\mu|\nabla u|^2+(\mu+\lambda)|\mathrm{div}u|^2-2(\rho\theta-\tilde{\rho}\tilde{\theta})\mathrm{div}u+\frac{|\rho\theta-\tilde{\rho}\tilde{\theta}|^2}{2\mu+\lambda}\right)\\ &+\int\rho|u_t|^2\\ \le& C\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}}+1\right)\int\bar{G}_1+C\int|\nabla\theta|^2+C\int|u|^2|\nabla u|^2,\end{aligned}\tag{4.5}$$

where

$$\bar{G}_1=|\nabla u|^2+\rho|\theta-\tilde{\theta}|^2+|\rho-\tilde{\rho}|^2+\rho|u|^4+\rho|u|^2.\tag{4.6}$$

Proof. Similarly to (3.47), we have

$$
\begin{aligned}
&\int \rho|u_t|^2 + \frac{1}{2}\frac{d}{dt}\int \left(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div} u|^2\right) \\
&\le -\int \nabla P\cdot u_t + C\int |u|^2|\nabla u|^2 + \frac{1}{4}\int \rho|u_t|^2, \qquad (4.7)
\end{aligned}
$$

where the first term on the right-hand side of (4.7) reads

$$
\begin{aligned}
-\int \nabla P\cdot u_t =& \frac{\mathrm{d}}{\mathrm{d}t}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u - \int P_t \mathrm{div}u \\
=& \frac{\mathrm{d}}{\mathrm{d}t}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u - \frac{1}{2\mu+\lambda}\int(\rho\theta)_t F \\
&- \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int|\rho\theta-\tilde\rho\tilde\theta|^2, \qquad (4.8)
\end{aligned}
$$

with $F=(2\mu+\lambda)\mathrm{div}u-\rho\theta+\tilde\rho\tilde\theta$.

Exactly as in (3.49),

$$
\begin{aligned}
P_t =& -\nabla\cdot\Big[(\rho\theta-\tilde\rho\tilde\theta)u\Big] - \tilde\rho\tilde\theta(1+\frac{1}{C_v})\mathrm{div}u - \frac{1}{C_v}\Big(\rho\theta-\tilde\rho\tilde\theta\Big)\,\mathrm{div}u \\
&+ \frac{\mu}{C_v}\nabla u\cdot\big[\nabla u+(\nabla u)'\big] + \frac{\lambda}{C_v}\mathrm{div}u\,\mathrm{div}u + \frac{\kappa}{C_v}\Delta\theta;
\end{aligned}
$$

whence we get

$$
\begin{aligned}
&-\frac{1}{2\mu+\lambda}\int(\rho\theta)_t F \\
=& -\frac{1}{2\mu+\lambda}\int\Big[(\rho\theta-\tilde\rho\tilde\theta)u\Big]\cdot\nabla F + \frac{\tilde\rho\tilde\theta(1+\frac{1}{C_v})}{2\mu+\lambda}\int\mathrm{div}u\,F + \\
&\frac{1}{(2\mu+\lambda)C_v}\int\Big(\rho\theta-\tilde\rho\tilde\theta\Big)\,\mathrm{div}u\,F - \frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F \\
&-\frac{\lambda}{(2\mu+\lambda)C_v}\int\mathrm{div}u\,\mathrm{div}u\,F + \frac{\kappa}{(2\mu+\lambda)C_v}\int\nabla\theta\cdot\nabla F,
\end{aligned}
$$

where we have used integration by parts. This combined with Hölder inequality yields

$$
\begin{aligned}
&-\frac{1}{2\mu+\lambda}\int(\rho\theta)_t F \\
\le& C\Big(\|\rho(\theta-\tilde\theta)u\|_{L^2}+\|\nabla\theta\|_{L^2}\Big)\|\nabla F\|_{L^2} + C\|\nabla u\|_{L^2}\|F\|_{L^2} \\
&-\frac{1}{2\mu+\lambda}\int\tilde\theta(\rho u-\tilde\rho u)\cdot\nabla F + \frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,F \\
&-\frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F - \frac{\lambda}{(2\mu+\lambda)C_v}\int\mathrm{div}u\,\mathrm{div}u\,F. \qquad (4.9)
\end{aligned}
$$

The first three terms on the right-hand side of (4.9) can be handled as

$$
\begin{aligned}
& C\Big(\|\rho(\theta-\tilde\theta)u\|_{L^2}+\|\nabla\theta\|_{L^2}\Big)\|\nabla F\|_{L^2}+C\|\nabla u\|_{L^2}\|F\|_{L^2} \\
& \quad -\frac{1}{2\mu+\lambda}\int\tilde\theta(\rho u-\tilde\rho u)\cdot\nabla F \\
\le & C\int|\rho(\theta-\tilde\theta)u|^2+C\int|\nabla\theta|^2+C\int\rho|u|^2+\frac18\int\rho|u_t|^2 \\
& \quad +C\int|u|^2|\nabla u|^2+C\int\left(|\nabla u|^2+\rho|\theta-\tilde\theta|^2+|\rho-\tilde\rho|^2\right) \\
\le & C\big(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1\big)\int\bar G_1+C\int|\nabla\theta|^2 \\
& \quad +C\int|u|^2|\nabla u|^2+\frac18\int\rho|u_t|^2,
\end{aligned}
\tag{4.10}
$$

where $\bar G_1$ is given by (4.6), and we have used integration by parts, Cauchy inequality, Hölder inequality, (3.1), (3.18), and the standard elliptic estimate:

$$
\|\nabla F\|_{L^2}\le C\|\rho\dot u\|_{L^2}\le C\|\sqrt{\rho}\dot u\|_{L^2},
\tag{4.11}
$$

as

$$
\Delta F=\operatorname{div}(\rho\dot u).
$$

For the last three terms on the right-hand side of (4.9), we have

$$
\begin{aligned}
& \frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\operatorname{div}u\,F-\frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F \\
& \quad -\frac{\lambda}{(2\mu+\lambda)C_v}\int\operatorname{div}u\operatorname{div}u\,F \\
= & \frac{1}{(2\mu+\lambda)C_v}\int u\cdot[\mu\Delta u+(\mu+\lambda)\nabla\operatorname{div}u-\nabla P]\,F \\
& \quad +\frac{\mu}{(2\mu+\lambda)C_v}\int\left[\nabla u+(\nabla u)'\right]:\nabla F\otimes u \\
& \quad +\frac{\lambda}{(2\mu+\lambda)C_v}\int u\cdot\nabla F\operatorname{div}u-\frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)u\cdot\nabla F,
\end{aligned}
\tag{4.12}
$$

where we have used integration by parts. Plugging the momentum equation in the first term on the right-hand side of (4.12), we have

$$\begin{aligned}
&\frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,F-\frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F\\
&-\frac{\lambda}{(2\mu+\lambda)C_v}\int\mathrm{div}u\,\mathrm{div}u\,F\\
=&\frac{1}{(2\mu+\lambda)C_v}\int\rho\dot u\cdot uF+\frac{\mu}{(2\mu+\lambda)C_v}\int\big[\nabla u+(\nabla u)'\big]:\nabla F\otimes u\\
&+\frac{\lambda}{(2\mu+\lambda)C_v}\int u\cdot\nabla F\mathrm{div}u-\frac{1}{(2\mu+\lambda)C_v}\int\rho(\theta-\tilde\theta)u\cdot\nabla F\\
&-\frac{\tilde\theta}{(2\mu+\lambda)C_v}\int(\rho-\tilde\rho)u\cdot\nabla F.
\end{aligned}$$

This, combined with Cauchy inequality and (3.1), yields

$$\begin{aligned}
&\frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,F-\frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F\\
&-\frac{\lambda}{(2\mu+\lambda)C_v}\int\mathrm{div}u\,\mathrm{div}u\,F\\
\le&\frac{1}{8}\int\rho|u_t|^2+C\int|u|^2|\nabla u|^2+C\int\rho|u|^2|F|^2\\
&+C\int\rho^2(\theta-\tilde\theta)^2|u|^2+C\int\rho|u|^2-\frac{\tilde\rho\tilde\theta}{(2\mu+\lambda)C_v}\int F\mathrm{div}u\\
\le&\frac{1}{8}\int\rho|u_t|^2+C\int|u|^2|\nabla u|^2+C\int\rho|u|^2|\rho-\tilde\rho|^2\\
&+C\int\rho^2(\theta-\tilde\theta)^2|u|^2+C\int\Big[\rho|u|^2+\rho(\theta-\tilde\theta)^2+|\nabla u|^2+|\rho-\tilde\rho|^2\Big].
\end{aligned}\tag{4.13}$$

Inserting (3.18) in (4.13), we have

$$\begin{aligned}
&\frac{1}{(2\mu+\lambda)C_v}\int(\rho\theta-\tilde\rho\tilde\theta)\mathrm{div}u\,F-\frac{\mu}{(2\mu+\lambda)C_v}\int\nabla u\cdot[\nabla u+(\nabla u)']F\\
&-\frac{\lambda}{(2\mu+\lambda)C_v}\int\mathrm{div}u\,\mathrm{div}u\,F\\
\le&\frac{1}{8}\int\rho|u_t|^2+C\int|u|^2|\nabla u|^2+C\int|\nabla\theta|^2\\
&+C(\|\theta-\tilde\theta\|_{L^r}^{\frac{2r}{2r-3}}+1)\int\bar G_1.
\end{aligned}\tag{4.14}$$

Relations (4.8), (4.9), (4.10), together with (4.14), (4.7), give rise to

$$\begin{aligned}&\int \rho|u_t|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int \left(\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2\right)\\ \le&\frac{\mathrm{d}}{\mathrm{d}t}\int (\rho\theta - \tilde{\rho}\tilde{\theta})\mathrm{div}u - \frac{1}{2(2\mu+\lambda)}\frac{\mathrm{d}}{\mathrm{d}t}\int |\rho\theta - \tilde{\rho}\tilde{\theta}|^2\\ &+ C\int |\nabla\theta|^2 + C\int |u|^2|\nabla u|^2 + \frac{1}{2}\int \rho|u_t|^2 + C\left(\|\theta - \tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\right)\int \bar{G}_1.\end{aligned}$$

Seeing that the fifth term on the right-hand side can be absorbed by the left-hand side, we have finished the proof of Lemma 4.3.

□

Lemma 4.4. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2 + \mu\int |\nabla u|^2 \le C\int \left(\rho|\theta - \tilde{\theta}|^2 + |\rho - \tilde{\rho}|^2\right). \tag{4.15}$$

Proof. Multiplying $(1.6)_2$ by $2u$, integrating by parts over $\mathbb{R}^3$, and using Cauchy inequality and (3.1), we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int \rho|u|^2 + 2\int \left[\mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2\right]\\ =&2\int (\rho\theta - \tilde{\rho}\tilde{\theta})\mathrm{div}u\\ \le&\mu\int |\nabla u|^2 + C\int \rho|\theta - \tilde{\theta}|^2 + C\int |\rho - \tilde{\rho}|^2.\end{aligned}$$

As the first term on the right-hand side can be absorbed by the integral on the left-hand side, the proof of Lemma 4.4 is complete.

□

Corollary 4.5. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\begin{aligned}&\sup_{0\le t\le T}\int \left(|\nabla u|^2 + \rho|\theta - \tilde{\theta}|^2 + |\rho - \tilde{\rho}|^2 + \rho|u|^4 + \rho|u|^2\right)\\ &+ \int_0^T\int \left(\rho|u_t|^2 + |u|^2|\nabla u|^2 + |\nabla\theta|^2\right) \le C,\end{aligned} \tag{4.16}$$

for any $T \in (0, T^*)$.

Proof. Let $\lambda > 0$ be given. Multiplying (4.4) by a large positive constant $\bar{M}$, and adding the resulting inequality to (4.2), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[\rho|u|^4 - \frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta}) + \bar{M}\Big(C_v\rho|\theta-\tilde{\theta}|^2 + |\rho-\tilde{\rho}|^2\Big)\Big] \\
&\quad + \frac{\mu}{2}\int |u|^2|\nabla u|^2 + \frac{\bar{M}\kappa}{2}\int |\nabla\theta|^2 \\
&\leq C_\epsilon \int |\nabla\theta|^2 + C\int |\nabla u|^2 + (\epsilon C + \bar{M}\epsilon_1)\int \rho|u_t|^2 \\
&\quad + (C_\epsilon + \bar{M}C_{\epsilon_1})\Big(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\Big)\int \bar{G}_1 \\
&\quad + \bar{M}\epsilon_1 \int |u|^2|\nabla u|^2 + \bar{M}C\|\theta-\tilde{\theta}\|_{L^r}, \qquad (4.17)
\end{aligned}$$

where $\bar{G}_1$ is given by (4.6).

Given $\epsilon > 0$, we may choose $\bar{M} = \bar{M}(\epsilon) > 0$ big enough and $\epsilon_1 = \epsilon_1(\bar{M}) > 0$ small enough so that

$$C_\epsilon \leq \frac{\bar{M}\kappa}{4}, \quad \text{and } \bar{M}\epsilon_1 \leq \frac{\mu}{4}. \qquad (4.18)$$

Consequently, the first term and the fifth term on the right-hand side of (4.17) can be absorbed by the left-hand side. Hence we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \Big[\rho|u|^4 - \frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta}) + \bar{M}\Big(C_v\rho|\theta-\tilde{\theta}|^2 + |\rho-\tilde{\rho}|^2\Big)\Big] \\
&\quad + \frac{\mu}{4}\int |u|^2|\nabla u|^2 + \frac{\bar{M}\kappa}{4}\int |\nabla\theta|^2 \\
&\leq C\int |\nabla u|^2 + (\epsilon C + \bar{M}\epsilon_1)\int \rho|u_t|^2 + \bar{M}C\|\theta-\tilde{\theta}\|_{L^r} \\
&\quad + (C_\epsilon + \bar{M}C_{\epsilon_1})\Big(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\Big)\int \bar{G}_1. \qquad (4.19)
\end{aligned}$$

Multiplying (4.19) by a positive constant $\bar{M}_1$, and then adding the resulting equation to (4.5) and (4.15), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int \bar{G} + \int \rho|u_t|^2 + \frac{\mu\bar{M}_1}{4}\int |u|^2|\nabla u|^2 + \frac{\bar{M}\bar{M}_1\kappa}{4}\int |\nabla\theta|^2 \\
&\leq C\bar{M}_1\int |\nabla u|^2 + \bar{M}_1(\epsilon C + \bar{M}\epsilon_1)\int \rho|u_t|^2 + C\int |\nabla\theta|^2 \\
&\quad + C\int |u|^2|\nabla u|^2 + \bar{M}\bar{M}_1 C\|\theta-\tilde{\theta}\|_{L^r} \\
&\quad + (\bar{M}\bar{M}_1C_{\epsilon_1} + C_\epsilon\bar{M}_1 + C)\Big(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\Big)\int \bar{G}_1, \qquad (4.20)
\end{aligned}$$

where

$$\bar{G} = \bar{M}_1\Big[\rho|u|^4 - \frac{4CC_v}{\lambda}\rho|u|^2(\theta-\tilde{\theta}) + \bar{M}\Big(C_v\rho|\theta-\tilde{\theta}|^2 + |\rho-\tilde{\rho}|^2\Big)\Big]$$
$$+ \mu|\nabla u|^2 + (\mu+\lambda)|\mathrm{div}u|^2 - 2(\rho\theta-\tilde{\rho}\tilde{\theta})\mathrm{div}u + \frac{|\rho\theta-\tilde{\rho}\tilde{\theta}|^2}{2\mu+\lambda} + \rho|u|^2.$$

For $\bar{M}_1 > 1$ large enough so that

$$\frac{\bar{M}_1\mu}{8} \geq C, \tag{4.21}$$

the fourth term on the right-hand side of (4.20) can be absorbed by the left-hand side. Noticing that $\bar{M}_1$ and C are independent of ϵ and ϵ_1, we choose $\epsilon > 0$ small enough so that

$$\bar{M}_1\epsilon C \leq \frac{1}{4}. \tag{4.22}$$

Moreover, in view of (4.18), $\epsilon_1 > 0$ can be chosen so small that

$$\max\{2C, C_\epsilon\} \leq \frac{\bar{M}\kappa}{4}, \quad \text{and } \bar{M}\epsilon_1 \leq \min\{\frac{1}{4\bar{M}_1}, \frac{\mu}{4}\}. \tag{4.23}$$

By virtue of (4.22) and (4.23), the second and the third term on the right-hand side of (4.20) can be absorbed by the left-hand side. Hence we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int\bar{G} + \frac{1}{2}\int\rho|u_t|^2 + \frac{\mu\bar{M}_1}{8}\int|u|^2|\nabla u|^2 + \frac{\bar{M}\bar{M}_1\kappa}{8}\int|\nabla\theta|^2 \\
\leq & C\bar{M}_1\int|\nabla u|^2 + \bar{M}\bar{M}_1C\|\theta-\tilde{\theta}\|_{L^r} \\
&+ (\bar{M}\bar{M}_1C_{\epsilon_1} + C_\epsilon\bar{M}_1 + C)\left(\|\theta-\tilde{\theta}\|_{L^r}^{\frac{2r}{2r-3}} + 1\right)\int\bar{G}_1,
\end{aligned} \tag{4.24}$$

where

$$\bar{G}(\rho,u,\theta,h) \sim |\nabla u|^2 + \rho|\theta-\tilde{\theta}|^2 + |\rho-\tilde{\rho}|^2 + \rho|u|^4 + \rho|u|^2 = \bar{G}_1,$$

for $\bar{M}$ big enough. Applying Gronwall inequality to (4.24) yields (4.16).

Similarly to the case $\lambda > 0$, relation (4.3) combined with (4.4), (4.5) and (4.15) yields (4.16) for $\lambda \leq 0$.

□

If $\tilde{\rho} = \tilde{\theta} = 0$, Corollary 4.5, together with (3.1), and the standard interpolation inequality, gives rise to

$$\begin{aligned}
\int_0^T \|\rho\theta\|^4_{L^{\frac{12}{5}}(\mathbb{R}^3)}\,dt &\leq C\int_0^T \|\rho\theta\|^3_{L^2(\mathbb{R}^3)}\|\rho\theta\|_{L^6(\mathbb{R}^3)}\,dt \\
&\leq C\int_0^T \|\nabla\theta\|_{L^2(\mathbb{R}^3)}\,dt \\
&\leq C.
\end{aligned}$$

This together with (3.1) and (4.16) yields

$$\begin{aligned}&\|\rho\|_{L^\infty(0,T;L^\infty)}+\|\rho\theta\|_{L^4(0,T;L^{\frac{12}{5}})}+\|\rho^{\frac14}u\|_{L^\infty(0,T;L^4)}\\ &+\big\||u||\nabla u|\big\|_{L^2(0,T;L^2)}\le C,\end{aligned}\tag{4.25}$$

for any $T\in(0,T^*)$. By virtue of Remark 2.4 in [30], we obtain (4.1).

In the case $\tilde\rho>0$ and $\tilde\theta=0$, the remaining estimates in (4.1) may be obtained following step by step [15]. The proof for the last two cases $\tilde\rho=0$, $\tilde\theta>0$, and $\tilde\rho>0$, $\tilde\theta>0$ is sketched in Appendix modifying the relevant estimates in [29].

5. Appendix

The estimates presented below lean on the results obtained in Section 4, assumption (3.1), and the hypotheses of Theorem 1.4.

Lemma 5.1. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\sup_{0\le t\le T}\int(|\nabla\theta|^2+\rho|\dot u|^2)+\int_0^T\int(\rho|\dot\theta|^2+|\nabla\dot u|^2)\le C.\tag{5.1}$$

Proof. From (4.35) in [29], we have

$$\begin{aligned}&\frac12\frac{d}{dt}\int\rho|\dot u|^2+\int\big(\mu|\nabla\dot u|^2+(\mu+\lambda)|\mathrm{div}\,\dot u|^2\big)\\ =&\int(P_t\mathrm{div}\,\dot u+u\otimes\nabla P:\nabla\dot u)+\mu\int\Big(\mathrm{div}\,(\Delta u\otimes u)-\Delta(u\cdot\nabla u)\Big)\cdot\dot u\\ &+(\mu+\lambda)\int\Big(\mathrm{div}\,(\nabla\mathrm{div}\,u\otimes u)-\nabla\mathrm{div}\,(u\cdot\nabla u)\Big)\cdot\dot u=\sum_{i=1}^3 III_i.\end{aligned}\tag{5.2}$$

For III_1, using $(1.6)_3$ and integration by parts (see also (4.36) in [29]), we have

$$III_1=\int\Big(\rho\dot\theta\mathrm{div}\,\dot u-\rho\theta(\nabla u)':\nabla\dot u\Big),$$

where $\dot\theta=\theta_t+u\cdot\nabla\theta$. Then by virtue of Hölder inequality, Sobolev inequality, (3.1), and Corollary 4.5, we have

$$\begin{aligned}III_1\le&C\|\sqrt\rho\dot\theta\|_{L^2}\|\mathrm{div}\,\dot u\|_{L^2}-\int\rho(\theta-\tilde\theta)(\nabla u)':\nabla\dot u-\int\rho\tilde\theta(\nabla u)':\nabla\dot u\\ \le&C\|\sqrt\rho\dot\theta\|_{L^2}\|\mathrm{div}\,\dot u\|_{L^2}+C\|\theta-\tilde\theta\|_{L^6}\|\nabla u\|_{L^3}\|\nabla\dot u\|_{L^2}\\ &+C\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}\\ \le&C\|\sqrt\rho\dot\theta\|_{L^2}\|\mathrm{div}\,\dot u\|_{L^2}+C\|\nabla\theta\|_{L^2}\|\nabla u\|_{L^3}\|\nabla\dot u\|_{L^2}+C\|\nabla\dot u\|_{L^2}.\end{aligned}\tag{5.3}$$

Taking curl on both side of $(1.6)_2$, we get

$$\mu\Delta(\mathrm{curl}u)=\mathrm{curl}(\rho\dot u).\tag{5.4}$$

In addition, one has

$$-\Delta u = \nabla \times (\mathrm{curl} u) - \nabla \mathrm{div} u. \tag{5.5}$$

Then using the standard elliptic estimates, the interpolation inequality, Sobolev inequality, (3.1), (4.11), and Corollary 4.5, we get

$$\begin{aligned}
\|\nabla u\|_{L^3} \leq & C\|\mathrm{curl} u\|_{L^3} + C\|\mathrm{div} u\|_{L^3} \\
\leq & C\|\mathrm{curl} u\|_{L^2}^{\frac{1}{2}}\|\mathrm{curl} u\|_{L^6}^{\frac{1}{2}} + C\|\mathrm{div} u\|_{L^2}^{\frac{1}{2}}\|\mathrm{div} u\|_{L^6}^{\frac{1}{2}} \\
\leq & C\|\nabla \mathrm{curl} u\|_{L^2}^{\frac{1}{2}} + C\|\nabla F\|_{L^2}^{\frac{1}{2}} + C\|\rho\theta - \tilde{\rho}\tilde{\theta}\|_{L^6}^{\frac{1}{2}} \\
\leq & C\|\sqrt{\rho}\dot{u}\|_{L^2}^{\frac{1}{2}} + C\|\nabla \theta\|_{L^2}^{\frac{1}{2}} + C.
\end{aligned} \tag{5.6}$$

Inserting (5.6) in (5.3), and using Cauchy inequality, we have

$$\begin{aligned}
III_1 \leq & \frac{\mu}{4}\|\nabla \dot{u}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 \\
& + C(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)\|\nabla\theta\|_{L^2}^2 + C.
\end{aligned} \tag{5.7}$$

For III_2 and III_3, we obtain (see for instance [25, 26])

$$III_2 + III_3 \leq C\|\nabla \dot{u}\|_{L^2}\|\nabla u\|_{L^4}^2 \leq \frac{\mu}{4}\|\nabla \dot{u}\|_{L^2}^2 + C\|\nabla u\|_{L^4}^4. \tag{5.8}$$

Similarly to (5.6), we have

$$\begin{aligned}
\|\nabla u\|_{L^4}^4 \leq & C\|\mathrm{curl} u\|_{L^4}^4 + C\|\mathrm{div} u\|_{L^4}^4 \\
\leq & C\|\mathrm{curl} u\|_{L^2}\|\mathrm{curl} u\|_{L^6}^3 + C\|\mathrm{div} u\|_{L^2}\|\mathrm{div} u\|_{L^6}^3 \\
\leq & C\|\sqrt{\rho}\dot{u}\|_{L^2}^3 + C\|\nabla\theta\|_{L^2}^3 + C.
\end{aligned} \tag{5.9}$$

Relation (5.8), combined with (5.9) and Young inequality, yields

$$III_2 + III_3 \leq \frac{\mu}{4}\|\nabla \dot{u}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{u}\|_{L^2}^4 + C\|\nabla\theta\|_{L^2}^4 + C. \tag{5.10}$$

Substituting (5.7) and (5.10) into (5.2), and using Cauchy inequality and (3.1), we have

$$\begin{aligned}
& \frac{d}{dt}\int \rho|\dot{u}|^2 + \int \left(\mu|\nabla \dot{u}|^2 + (\mu+\lambda)|\mathrm{div}\, \dot{u}|^2\right) \\
\leq & C\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)^2 + C.
\end{aligned} \tag{5.11}$$

Multiplying $(1.6)_3$ by $\dot\theta$, and integrating by parts over $\mathbb{R}^3$ (see also (4.41) in [29]), we have

$$\begin{aligned}
&\int C_v\rho|\dot\theta|^2+\frac{\kappa}{2}\frac{d}{dt}\int|\nabla\theta|^2\\
=&-\int\rho\,\theta\,\mathrm{div}u\,\dot\theta+\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]\theta_t\\
&+\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]u\cdot\nabla\theta+\kappa\int\Delta\theta u\cdot\nabla\theta\\
=&\sum_{i=1}^4 IV_i. \qquad (5.12)
\end{aligned}$$

For IV_1, using Cauchy inequality, Hölder inequality, Sobolev inequality, (3.1), Corollary 4.5, and (5.6), we have

$$\begin{aligned}
IV_1\le&\frac{C_v}{4}\int\rho|\dot\theta|^2+C\|\theta-\tilde\theta\|_{L^6}^2\|\mathrm{div}u\|_{L^3}^2+C\|\nabla u\|_{L^2}^2\\
\le&\frac{C_v}{4}\int\rho|\dot\theta|^2+C\|\nabla\theta\|_{L^2}^2(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)+C. \qquad (5.13)
\end{aligned}$$

As in [29], using integration by parts, we have

$$\begin{aligned}
IV_2=&\frac{d}{dt}\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]\theta-\mu\int\left[\nabla u+(\nabla u)'\right]:\left[\nabla\dot u+(\nabla\dot u)'\right]\theta\\
&+\mu\int\left[\nabla u+(\nabla u)'\right]:\left[\nabla u\cdot\nabla u+(\nabla u\cdot\nabla u)'\right]\theta-2\lambda\int\mathrm{div}u\,\mathrm{div}\dot u\theta\\
&+2\lambda\int\mathrm{div}u(\nabla u)':\nabla u\theta-\mu\int\frac{|\nabla u+(\nabla u)'|^2}{2}\mathrm{div}u\,\theta-\lambda\int(\mathrm{div}u)^3\theta\\
&-\mu\int\frac{|\nabla u+(\nabla u)'|^2}{2}u\cdot\nabla\theta-\lambda\int|\mathrm{div}u|^2u\cdot\nabla\theta.
\end{aligned}$$

This together with Hölder inequality, Sobolev inequality, and Corollary 4.5 yields

$$\begin{aligned}
IV_2+IV_3\le&\frac{d}{dt}\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]\theta+C\int|\nabla u|\,|\nabla\dot u|\,|\theta|\\
&+C\int|\nabla u|^3\,|\theta|\\
\le&\frac{d}{dt}\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]\theta+C\|\nabla u\|_{L^3}\|\nabla\dot u\|_{L^2}\|\theta-\tilde\theta\|_{L^6}\\
&+C\|\nabla u\|_{L^2}\|\nabla\dot u\|_{L^2}+C\|\nabla u\|_{L^{\frac{18}{5}}}^3\|\theta-\tilde\theta\|_{L^6}+C\|\nabla u\|_{L^3}^3\\
\le&\frac{d}{dt}\int\left[\frac{\mu}{2}\left|\nabla u+(\nabla u)'\right|^2+\lambda(\mathrm{div}u)^2\right]\theta+C\|\nabla u\|_{L^3}\|\nabla\dot u\|_{L^2}\|\nabla\theta\|_{L^2}\\
&+C\|\nabla\dot u\|_{L^2}+C\|\nabla u\|_{L^{\frac{18}{5}}}^3\|\nabla\theta\|_{L^2}+C\|\nabla u\|_{L^3}^3.
\end{aligned}$$

Using the interpolation inequality, and (4.16), we have

$$\|\nabla u\|_{L^{\frac{18}{5}}} \le C\|\nabla u\|_{L^2}^{\frac{1}{9}}\|\nabla u\|_{L^4}^{\frac{8}{9}} \le C\|\nabla u\|_{L^4}^{\frac{8}{9}}. \tag{5.14}$$

Inserting (5.14) to the estimate of $IV_2 + IV_3$, and using Young inequality, we have

$$\begin{aligned} IV_2 + IV_3 \le & \frac{d}{dt}\int \Big[\frac{\mu}{2}\big|\nabla u + (\nabla u)'\big|^2 + \lambda(\mathrm{div}u)^2\Big]\theta + C\|\nabla u\|_{L^3}\|\nabla\dot{u}\|_{L^2}\|\nabla\theta\|_{L^2} \\ & + C\|\nabla\dot{u}\|_{L^2} + C\|\nabla u\|_{L^4}^{\frac{8}{3}}\|\nabla\theta\|_{L^2} + C\|\nabla u\|_{L^3}^3 \\ \le & \frac{d}{dt}\int \Big[\frac{\mu}{2}\big|\nabla u + (\nabla u)'\big|^2 + \lambda(\mathrm{div}u)^2\Big]\theta + \delta\|\nabla\dot{u}\|_{L^2}^2 + C_\delta\|\nabla u\|_{L^3}^2\|\nabla\theta\|_{L^2}^2 \\ & + C_\delta + C\|\nabla u\|_{L^4}^4 + C\|\nabla\theta\|_{L^2}^4 + C\|\nabla u\|_{L^3}^3. \end{aligned}$$

Since the estimates of $\|\nabla u\|_{L^3}$ and $\|\nabla u\|_{L^4}$ have already been obtained in (5.6) and (5.9), respectively, we proceed to evaluate $IV_2 + IV_3$ and get

$$\begin{aligned} IV_2 + IV_3 \le & \frac{d}{dt}\int \Big[\frac{\mu}{2}\big|\nabla u + (\nabla u)'\big|^2 + \lambda(\mathrm{div}u)^2\Big]\theta + \delta\|\nabla\dot{u}\|_{L^2}^2 \\ & + C_\delta + C\|\sqrt{\rho}\dot{u}\|_{L^2}^4 + C_\delta\|\nabla\theta\|_{L^2}^4. \end{aligned} \tag{5.15}$$

For IV_4, using the interpolation inequality, the standard elliptic estimate for $(1.6)_3$, (3.1), (4.16), and Sobolev inequality, we have

$$\begin{aligned} \|\nabla\theta\|_{L^3} \le & \|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}} \\ \le & C\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\Big[\|\sqrt{\rho}\dot{\theta}\|_{L^2} + \|\theta - \tilde{\theta}\|_{L^6}\|\mathrm{div}u\|_{L^3} + \|\nabla u\|_{L^4}^2 + 1\Big]^{\frac{1}{2}} \\ \le & C\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\Big[\|\sqrt{\rho}\dot{\theta}\|_{L^2} + \|\nabla\theta\|_{L^2}\|\mathrm{div}u\|_{L^3} + \|\nabla u\|_{L^4}^2 + 1\Big]^{\frac{1}{2}}, \end{aligned} \tag{5.16}$$

and thus

$$\begin{aligned} IV_4 \le & C\|\Delta\theta\|_{L^2}\|u\|_{L^6}\|\nabla\theta\|_{L^3} \\ \le & C\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\Big[\|\sqrt{\rho}\dot{\theta}\|_{L^2} + \|\nabla\theta\|_{L^2}\|\mathrm{div}u\|_{L^3} + \|\nabla u\|_{L^4}^2 + 1\Big]^{\frac{3}{2}} \\ \le & \frac{C_v}{4}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2 + C\|\nabla\theta\|_{L^2}^2\|\mathrm{div}u\|_{L^3}^2 + C\|\nabla u\|_{L^4}^4 + C \\ \le & \frac{C_v}{4}\|\sqrt{\rho}\dot{\theta}\|_{L^2}^2 + C\|\sqrt{\rho}\dot{u}\|_{L^2}^4 + C\|\nabla\theta\|_{L^2}^4 + C, \end{aligned} \tag{5.17}$$

where we have used (5.6), (5.9), and Young inequality.

Plugging (5.13), (5.15), and (5.17) into (5.12), we have

$$\begin{aligned} & \int C_v\rho|\dot{\theta}|^2 + \frac{d}{dt}\int\Big[\kappa|\nabla\theta|^2 - \Big(\mu\big|\nabla u + (\nabla u)'\big|^2 + 2\lambda(\mathrm{div}u)^2\Big)\theta\Big] \\ \le & C(\|\sqrt{\rho}\dot{u}\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2)^2 + 2\delta\|\nabla\dot{u}\|_{L^2}^2 + C_\delta. \end{aligned} \tag{5.18}$$

Multiplying (5.18) by $\frac{2C}{C_v}$, and plugging the result in (5.11), we have

$$\begin{aligned}
&\frac{d}{dt}\int\Big[\rho|\dot u|^2+\frac{2C\kappa}{C_v}|\nabla\theta|^2-\frac{2C}{C_v}\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)\theta\Big]\\
&+\int\Big(C\rho|\dot\theta|^2+\mu|\nabla\dot u|^2+(\mu+\lambda)|\mathrm{div}\,\dot u|^2\Big)\\
&\le C(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)^2+4\delta C\|\nabla\dot u\|_{L^2}^2+C_\delta.
\end{aligned}\tag{5.19}$$

For $\delta=\frac{\mu}{8C}$, we have

$$\begin{aligned}
&\frac{d}{dt}\int\Big[\rho|\dot u|^2+\frac{2C\kappa}{C_v}|\nabla\theta|^2-\frac{2C}{C_v}\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)\theta\Big]\\
&+\int\Big(C\rho|\dot\theta|^2+\frac{\mu}{2}|\nabla\dot u|^2\Big)\le C(\|\sqrt{\rho}\dot u\|_{L^2}^2+\|\nabla\theta\|_{L^2}^2)^2+C.
\end{aligned}\tag{5.20}$$

Denoting

$$\bar G_2(\rho,u,\theta)=\int\rho|\dot u|^2+\frac{2C\kappa}{C_v}\int|\nabla\theta|^2-\frac{2C}{C_v}\int\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)\theta,$$

and noticing

$$\begin{aligned}
\|\nabla u\|_{L^{\frac{12}{5}}}^2\le&C(\|\mathrm{curl}u\|_{L^{\frac{12}{5}}}^2+\|\mathrm{div}u\|_{L^{\frac{12}{5}}}^2)\\
\le&C\|\mathrm{curl}u\|_{L^2}^{\frac32}\|\nabla\mathrm{curl}u\|_{L^2}^{\frac12}+C\|F\|_{L^2}^{\frac32}\|\nabla F\|_{L^2}^{\frac12}+C\|\rho(\theta-\tilde\theta)\|_{L^2}^{\frac32}\|\rho(\theta-\tilde\theta)\|_{L^6}^{\frac12}\\
&+C\|\tilde\theta(\rho-\tilde\rho)\|_{L^2}^{\frac32}\|\tilde\theta(\rho-\tilde\rho)\|_{L^6}^{\frac12}\\
\le&C\|\sqrt{\rho}\dot u\|_{L^2}^{\frac12}+C\|\nabla\theta\|_{L^2}^{\frac12}+C,
\end{aligned}$$

we have

$$\begin{aligned}
&\int\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)\theta\\
=&\int\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)(\theta-\tilde\theta)+\tilde\theta\int\Big(\mu\big|\nabla u+(\nabla u)'\big|^2+2\lambda(\mathrm{div}u)^2\Big)\\
\le&C\|\nabla u\|_{L^{\frac{12}{5}}}^2\|\theta-\tilde\theta\|_{L^6}+C\\
\le&C\|\sqrt{\rho}\dot u\|_{L^2}^{\frac12}\|\nabla\theta\|_{L^2}+C\|\nabla\theta\|_{L^2}^{\frac12}\|\nabla\theta\|_{L^2}+C\|\nabla\theta\|_{L^2}+C.
\end{aligned}$$

This implies

$$\frac{1}{\bar M_3}\int(\rho|\dot u|^2+|\nabla\theta|^2)\le\bar G_2(\rho,u,\theta)+\bar M_2\le\bar M_3+\bar M_3\int(\rho|\dot u|^2+|\nabla\theta|^2),$$

for some positive constants $\bar M_2$ and $\bar M_3$. This relation, together with (5.20) and Gronwall inequality, yields (5.1). □

Corollary 5.2. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\begin{aligned}&\sup_{0\le t\le T}(\|\nabla F\|_{L^2}+\|\nabla \mathrm{curl} u\|_{L^2}+\|\nabla u\|_{L^6}+\|u\|_{L^\infty})\\ &+\int_0^T(\|\mathrm{div} u\|_{L^\infty}^2+\|\nabla^2\theta\|_{L^2}^2)\le C.\end{aligned}\tag{5.21}$$

Proof. It follows from (4.11) and (5.4) that

$$\|\nabla F\|_{L^2}\le C\|\rho\dot u\|_{L^2}\le C,$$

$$\|\nabla \mathrm{curl} u\|_{L^2}\le C\|\rho\dot u\|_{L^2}\le C,$$

and

$$\begin{aligned}&\int_0^T\|\mathrm{div} u\|_{L^\infty}^2\\ \le& C\int_0^T\|F\|_{L^\infty}^2+C\int_0^T\|\theta-\tilde\theta\|_{L^\infty}^2+C\\ \le& C\int_0^T\|F\|_{L^6}^2+\int_0^T\|\nabla F\|_{L^6}^2+C\int_0^T\|\theta-\tilde\theta\|_{L^6}^2+\int_0^T\|\nabla\theta\|_{L^6}^2+C\\ \le& C\int_0^T\|\nabla F\|_{L^2}^2+\int_0^T\|\rho\dot u\|_{L^6}^2+C\int_0^T\|\nabla\theta\|_{L^2}^2+\int_0^T\|\nabla^2\theta\|_{L^2}^2+C\\ \le& C\int_0^T\|\sqrt{\rho}\dot u\|_{L^2}^2+C\int_0^T\|\nabla\dot u\|_{L^2}^2+C\int_0^T\|\nabla^2\theta\|_{L^2}^2+C\le C,\end{aligned}$$

where we have used (3.1), (5.1), Sobolev inequality, and the following elliptic estimate:

$$\begin{aligned}\|\nabla^2\theta\|_{L^2}\le& C\|\sqrt{\rho}\dot\theta\|_{L^2}+C\|\theta-\tilde\theta\|_{L^6}\|\mathrm{div} u\|_{L^3}+C\|\nabla u\|_{L^4}^2+C\\ \le& C\|\sqrt{\rho}\dot\theta\|_{L^2}+C\|\nabla\theta\|_{L^2}\|\mathrm{div} u\|_{L^3}+C\|\nabla u\|_{L^4}^2+C\\ \le& C\|\sqrt{\rho}\dot\theta\|_{L^2}+C.\end{aligned}\tag{5.22}$$

Therefore we have

$$\int_0^T\|\nabla^2\theta\|_{L^2}^2\le C.$$

By (5.5), we have

$$\begin{aligned}\|\nabla u\|_{L^6}\le& C\|\mathrm{div} u\|_{L^6}+C\|\mathrm{curl} u\|_{L^6}\\ \le& C\|F\|_{L^6}+C\|\mathrm{curl} u\|_{L^6}+C\|\theta-\tilde\theta\|_{L^6}+C\\ \le& C\|\nabla F\|_{L^2}+C\|\nabla\mathrm{curl} u\|_{L^2}+C\|\nabla\theta\|_{L^2}+C\\ \le& C.\end{aligned}\tag{5.23}$$

By (4.16), (5.23), and Sobolev inequality, we have

$$\|u\|_{L^\infty}\le C\|u\|_{L^6}+C\|\nabla u\|_{L^6}\le C\|\nabla u\|_{L^2}+C\|\nabla u\|_{L^6}\le C.$$

□

Lemma 5.3. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\sup_{0\le t\le T}\int\rho|\theta_t|^2+\int_0^T\int|\nabla\theta_t|^2\le C. \tag{5.24}$$

Proof. As in [29], differentiating $(1.6)_3$ with respect to t, multiplying the result by θ_t, and using integration by parts, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int C_v\rho|\theta_t|^2+\kappa\int|\nabla\theta_t|^2\\
&=-\int\rho_t\left(C_v\theta_t+C_vu\cdot\nabla\theta+\theta\mathrm{div}u\right)\theta_t-\int\rho(C_vu_t\cdot\nabla\theta+\theta_t\,\mathrm{div}u)\,\theta_t\\
&\quad-\int\rho\theta\,\mathrm{div}u_t\,\theta_t+\mu\int\left(\nabla u+(\nabla u)'\right):\left(\nabla u_t+(\nabla u_t)'\right)\theta_t\\
&\quad+2\lambda\int\mathrm{div}u\,\mathrm{div}u_t\,\theta_t=\sum_{i=1}^5V_i.
\end{aligned} \tag{5.25}$$

For V_1, using $(1.6)_1$ and integration by parts, we have

$$\begin{aligned}
V_1=&-\int\rho u\cdot\nabla\theta_t\left(2C_v\theta_t+C_vu\cdot\nabla\theta+\theta\mathrm{div}u\right)\\
&-\int C_v\rho u\cdot\left(\nabla u\cdot\nabla\theta+u\cdot\nabla\nabla\theta\right)\theta_t\\
&-\int\rho u\cdot\left(\nabla\theta\mathrm{div}u+\theta\nabla\mathrm{div}u\right)\theta_t\\
=&\sum_{i=1}^3V_{1,i}.
\end{aligned} \tag{5.26}$$

For $V_{1,1}$, using Cauchy inequality, (3.1), (4.16), (5.1), (5.6), and (5.21), we have

$$\begin{aligned}
V_{1,1}\le&\frac{\kappa}{12}\int|\nabla\theta_t|^2+C\int\rho|\theta_t|^2+C\int\rho^2|u|^2|\theta-\tilde\theta|^2|\mathrm{div}u|^2\\
&+C\int\rho^2|u|^2|\mathrm{div}u|^2+C\\
\le&\frac{\kappa}{12}\int|\nabla\theta_t|^2+C\int\rho|\theta_t|^2+C\|\theta-\tilde\theta\|_{L^6}^2\|\mathrm{div}u\|_{L^3}^2+C\\
\le&\frac{\kappa}{12}\int|\nabla\theta_t|^2+C\int\rho|\theta_t|^2+C.
\end{aligned} \tag{5.27}$$

For $V_{1,2}$ and $V_{1,3}$, similarly to [29], we have

$$V_{1,2}\le C\int\rho|\theta_t|^2+C\int|\nabla^2\theta|^2+C. \tag{5.28}$$

and

$$
\begin{aligned}
V_{1,3} \leq & C\int \rho|\theta_t|^2 + C\|\nabla\theta\|_{L^3}^2\|\mathrm{div}u\|_{L^6}^2 - \frac{1}{2\mu+\lambda}C\int \rho(\theta-\tilde{\theta})u\cdot\nabla F\theta_t \\
& - \frac{1}{2\mu+\lambda}C\tilde{\theta}\int \rho u\cdot\nabla F\theta_t + \frac{1}{2(2\mu+\lambda)}C\int \rho^2\theta^2 u\cdot\nabla\theta_t \\
& + \frac{1}{2(2\mu+\lambda)}C\int \rho^2\theta^2\mathrm{div}u\,\theta_t \\
\leq & C\int \rho|\theta_t|^2 + C\|\nabla^2\theta\|_{L^2} + C\|\theta-\tilde{\theta}\|_{L^6}\|u\|_{L^6}\|\theta_t\|_{L^6}\|\nabla F\|_{L^2} + C \\
& + C\|\theta-\tilde{\theta}\|_{L^6}^2\|u\|_{L^6}\|\nabla\theta_t\|_{L^2} + C\|\rho u\|_{L^2}\|\nabla\theta_t\|_{L^2} \\
& + C\|\theta-\tilde{\theta}\|_{L^6}^2\|\mathrm{div}u\|_{L^2}\|\theta_t\|_{L^6} \\
\leq & \frac{\kappa}{12}\int |\nabla\theta_t|^2 + C\int \rho|\theta_t|^2 + C\|\nabla^2\theta\|_{L^2} + C, \qquad (5.29)
\end{aligned}
$$

where we have used Cauchy inequality, Hölder inequality, the interpolation inequality, integration by parts, and the relations (3.1), (4.16), (5.1), and (5.21).

Substituting (5.27), (5.28), and (5.29) into (5.26), we have

$$
V_1 \leq \frac{\kappa}{6}\int |\nabla\theta_t|^2 + C\int \rho|\theta_t|^2 + C\int |\nabla^2\theta|^2 + C. \qquad (5.30)
$$

For V_2 and V_3, using Cauchy inequality, Hölder inequality, (3.1), (4.16), (5.1), and (5.21) again, we have

$$
\begin{aligned}
V_2 \leq & C\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\dot{u}\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}} + C\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla\theta\|_{L^2}^{\frac{1}{2}}\|\nabla^2\theta\|_{L^2}^{\frac{1}{2}} \\
& + C\|\mathrm{div}u\|_{L^\infty}\int \rho|\theta_t|^2 \\
\leq & C\left(\|\mathrm{div}u\|_{L^\infty} + \|\nabla\dot{u}\|_{L^2}^2 + 1\right)\int \rho|\theta_t|^2 + C\|\nabla^2\theta\|_{L^2}^2 + C, \qquad (5.31)
\end{aligned}
$$

and

$$
\begin{aligned}
|V_3| \leq & C\|\theta-\tilde{\theta}\|_{L^6}\|\mathrm{div}\dot{u}\|_{L^2}\|\rho\theta_t\|_{L^3} + C\|\mathrm{div}\dot{u}\|_{L^2}\|\rho\theta_t\|_{L^2} \\
& + C\int \rho|\theta||\nabla u|^2|\theta_t| + |\int \rho\theta\theta_t u\cdot\nabla\mathrm{div}u| \\
\leq & C\int \rho|\theta_t|^2 + C\int |\mathrm{div}\dot{u}|^2 + \frac{\kappa}{64}\int |\nabla\theta_t|^2 + C\|\theta-\tilde{\theta}\|_{L^6}\|\rho\theta_t\|_{L^2}\|\nabla u\|_{L^6}^2 \\
& + C\int |\nabla u|^4 + \frac{1}{2\mu+\lambda}|\int \rho\theta\theta_t u\cdot\nabla F| + \frac{1}{2(2\mu+\lambda)}|\int \theta_t u\cdot\nabla(\rho\theta)^2| \\
\leq & C\int \rho|\theta_t|^2 + C\int |\mathrm{div}\dot{u}|^2 + \frac{\kappa}{64}\int |\nabla\theta_t|^2 + C\|\theta-\tilde{\theta}\|_{L^6}\|\theta_t\|_{L^6}\|u\|_{L^6}\|\nabla F\|_{L^2} \\
& + C + \frac{1}{2(2\mu+\lambda)}|\int \theta_t\mathrm{div}u\,(\rho\theta)^2| + \frac{1}{2(2\mu+\lambda)}|\int (\rho\theta)^2 u\cdot\nabla\theta_t|.
\end{aligned}
$$

For the last two terms on the right-hand side of V_3, we have

$$\begin{aligned}
&\frac{1}{2(2\mu+\lambda)}\Big|\int \theta_t \mathrm{div}u\,(\rho\theta)^2\Big| + \frac{1}{2(2\mu+\lambda)}\Big|\int (\rho\theta)^2 u\cdot\nabla\theta_t\Big| \\
\le& C\|\theta_t\|_{L^6}\|\mathrm{div}u\|_{L^2}\|\theta-\tilde\theta\|_{L^6}^2 + C\|\rho\theta_t\|_{L^2}\|\mathrm{div}u\|_{L^2} \\
&+ C\|\theta-\tilde\theta\|_{L^6}^2\|u\|_{L^6}\|\nabla\theta_t\|_{L^2} + C\|\rho u\|_{L^2}\|\nabla\theta_t\|_{L^2} \\
\le& \frac{\kappa}{64}\int|\nabla\theta_t|^2 + C\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C,
\end{aligned}$$

which yields

$$|V_3| \le C\int\rho|\theta_t|^2 + C\int|\mathrm{div}\dot u|^2 + \frac{\kappa}{24}\int|\nabla\theta_t|^2 + C. \tag{5.32}$$

For V_4 and V_5, using integration by parts, we have

$$\begin{aligned}
V_4 =& \mu\int\big(\nabla u + (\nabla u)'\big) : \nabla u_t\,\theta_t + \mu\int\big(\nabla u + (\nabla u)'\big) : (\nabla u_t)'\,\theta_t \\
=& -\mu\int\big(\nabla u + (\nabla u)'\big) : \nabla\theta_t\otimes u_t - 2\mu\int(\Delta u + \nabla\mathrm{div}u)\cdot u_t\,\theta_t \\
&- \mu\int\big(\nabla u + (\nabla u)'\big) : u_t\otimes\nabla\theta_t,
\end{aligned}$$

and

$$V_5 = -2\lambda\int\nabla\mathrm{div}u\cdot u_t\,\theta_t - 2\lambda\int\mathrm{div}u\,u_t\cdot\nabla\theta_t.$$

Hence we have

$$\begin{aligned}
V_4+V_5 \le& C\int|\nabla u|\,|\nabla\theta_t|\,|u_t| + C\int\rho|\dot u|\,|u_t|\,|\theta_t| + 2|V_3| + 2\int\rho\theta u_t\cdot\nabla\theta_t \\
\le& C\|\nabla u\|_{L^3}\|\nabla\theta_t\|_{L^2}\|u_t\|_{L^6} + C\|\sqrt{\rho}\dot u\|_{L^2}\|u_t\|_{L^6}\|\sqrt{\rho}\theta_t\|_{L^3} + 2|V_3| \\
&+ C\|\rho(\theta-\tilde\theta)\|_{L^3}\|u_t\|_{L^6}\|\nabla\theta_t\|_{L^2} + C\|\rho u_t\|_{L^2}\|\nabla\theta_t\|_{L^2} \\
\le& C\int\rho|\theta_t|^2 + C\int|\nabla\dot u|^2 + \frac{\kappa}{6}\int|\nabla\theta_t|^2 + C,
\end{aligned}\tag{5.33}$$

where we have used (5.1), (5.21), and (5.32).

Putting (5.30), (5.31), (5.32) and (5.33) into (5.25), we have

$$\begin{aligned}
\frac{d}{dt}\int C_v\rho|\theta_t|^2 + \kappa\int|\nabla\theta_t|^2 \le& C\left(\|\mathrm{div}u\|_{L^\infty} + \|\nabla\dot u\|_{L^2}^2 + 1\right)\int\rho|\theta_t|^2 \\
&+ C\int(|\nabla\dot u|^2 + |\nabla^2\theta|^2) + C.
\end{aligned}\tag{5.34}$$

By virtue of (5.1), (5.21), (5.34) and Gronwall inequality, the proof of Lemma 5.3 is complete. □

Corollary 5.4. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\sup_{0\le t\le T}\int|\nabla^2\theta|^2 \le C. \tag{5.35}$$

Proof. Relation (5.22), together with (3.1), (5.1), (5.21), and (5.24), yields (5.35).

□

The remaining estimates in (4.1) can be obtained in the same way as in [29]:

Lemma 5.5. *Under the hypotheses of Theorem 1.4 and (3.1), there holds*

$$\sup_{0\le t\le T}\left(\|\nabla\rho\|_{L^l}+\|\rho_t\|_{L^l}+\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla^2 u\|_{L^2}\right)$$
$$+\int_0^T\left(\|u_t\|_{D^1}^2+\|(u,\theta)\|_{D^{2,q}}^2\right)\le C,$$

for $l=2,q$.

The proof of Theorem 1.4 is complete. □

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