Poroplasticity

Tomáš Ligurský

The Czech Academy of Sciences, Institute of Geonics

The model

- non-stationary isothermal saturated water flow in a deformable porous medium
- isotropic elastoplastic skeleton
- negligible inertial effects
- the assumption of small perturbations (small transformations, small displacements, small variations of the porosity and of the water mass density) + the assumption of small deformation velocity
- continuum approach, continuity assumption
- compressive-positive pore pressures, tensile-positive stresses
- a summary from [Cou04] + the Eulerian approach based on [LS98] but extended from poroelasticity by myself in the case of compressible solid matrix (Biot's coefficient $\alpha < 1$)

Notation

t — the time

$\boldsymbol{u}$ — the displacement vector of the skeleton

$\boldsymbol{id} + \boldsymbol{u}$ — the deformation of the skeleton

$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{\nabla u}$ — the deformation gradient

$\boldsymbol{\varepsilon} \equiv \frac{1}{2}\big(\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^\top\big)$ — the linear strain tensor

$\varepsilon_v \equiv \operatorname{tr} \boldsymbol{\varepsilon} = \operatorname{div} \boldsymbol{u}$ — the volumetric strain

$J = \det(\boldsymbol{I} + \boldsymbol{\nabla u})$ — the Jacobian of the deformation
($\approx 1 + \varepsilon_v$ under the assumption of small transformations)

n — the Eulerian porosity

$\phi = Jn$ — the Lagrangian porosity

Balance equations

Water mass balance

The Eulerian form (in the current configuration):

$$\frac{\mathrm{D}_s(\rho_w n)}{\mathrm{D}t} + \rho_w n \operatorname{div} \boldsymbol{v}_s + \operatorname{div}(\rho_w \boldsymbol{q}_{rw}) = 0 \tag{1}$$

ρ_w — the water mass density

$\boldsymbol{v}_s$ — the solid velocity

$\boldsymbol{v}_w$ — the water velocity

$\boldsymbol{q}_{rw} \equiv n(\boldsymbol{v}_w - \boldsymbol{v}_s)$ — the water specific discharge relative to the solid (or Darcy velocity or filtration vector)

$\dfrac{\mathrm{D}_s}{\mathrm{D}t} = \dfrac{\partial}{\partial t} + \boldsymbol{v}_s \cdot \nabla$ — the total time derivative with respect to the solid

The Lagrangian form (in the initial configuration):

$$\frac{\mathrm{d}(\rho_w \phi)}{\mathrm{d}t} + \operatorname{div} \boldsymbol{M} = 0 \tag{2}$$

$\boldsymbol{M} \equiv J\boldsymbol{F}^{-1}(\rho_w \boldsymbol{q}_{rw})$ — the Lagrangian relative flow vector of water mass

Solid mass balance
The Eulerian form:

$$\frac{\mathrm{D}_s(\rho_s(1-n))}{\mathrm{D}t} + \rho_s(1-n)\,\mathrm{div}\,\boldsymbol{v}_s = 0 \tag{3}$$

ρ_s — the solid mass density

The Lagrangian alternative:

$$\rho_s(1-n)J = \rho_{s0}(1-\phi_0) \tag{4}$$

ρ_{s0} — the initial solid mass density
$\phi_0 (= n_0)$ — the initial Lagrangian (= initial Eulerian) porosity

Equilibrium equation
The Eulerian form:

$$\mathbf{div}\,\boldsymbol{\sigma} + (\rho_s(1-n) + \rho_w n)\boldsymbol{f} = \mathbf{0} \tag{5}$$

$\boldsymbol{\sigma}$ — the Cauchy stress tensor $\boldsymbol{f}$ — a body force density

The Lagrangian counterpart:

$$\mathbf{div}(\boldsymbol{F}\boldsymbol{\Pi}) + (\rho_{s0}(1-\phi_0) + \rho_w\phi)\boldsymbol{f} = \mathbf{0} \tag{6}$$

$\boldsymbol{\Pi} \equiv J\boldsymbol{F}^{-1}\boldsymbol{\sigma}\boldsymbol{F}^{-\top}$ — the Piola-Kirchhoff stress tensor

Constitutive relationships

Water density

$$\frac{\mathrm{d}\rho_w}{\rho_w} = \frac{\mathrm{d}p_w}{K_w} \tag{7}$$

p_w — the water pressure K_w — the water bulk modulus

Considering K_w constant (over some range of pressures), one can integrate (7) into the form:

$$\rho_w = \rho_{w0}e^{(p_w - p_{w0})/K_w}$$

ρ_{w0}, p_{w0} — initial values of the water density and pressure

Darcy's law

$$\boldsymbol{q}_{rw} = \frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f}) \tag{8}$$

$\boldsymbol{k}$ — the (intrinsic) permeability tensor of the porous medium
μ_w — the dynamic viscosity of water

Plastic strain and plastic porosity
Poroplasticity is the ability of porous materials to undergo permanent strains and permanent changes in porosity. In the context of small transformations, the incremental strain $\mathrm{d}\boldsymbol{\varepsilon}$ and the incremental Lagrangian porosity $\mathrm{d}\phi$ can be decomposed into their reversible (elastic) and irreversible (plastic) parts as follows:

$$\mathrm{d}\boldsymbol{\varepsilon} = \mathrm{d}\boldsymbol{\varepsilon}^{el} + \mathrm{d}\boldsymbol{\varepsilon}^{p} \qquad \mathrm{d}\phi = \mathrm{d}\phi^{el} + \mathrm{d}\phi^{p}$$

$\mathrm{d}\boldsymbol{\varepsilon}^{el}$ — the incremental elastic strain $\mathrm{d}\phi^{el}$ — the incremental elastic Lagrangian porosity
$\mathrm{d}\boldsymbol{\varepsilon}^{p}$ — the incremental plastic strain $\mathrm{d}\phi^{p}$ — the incremental plastic Lagrangian porosity

Elastic and plastic strains and elastic and plastic Lagrangian porosities are defined as the integrals of the increments from an initial reference state to the current one so that:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{el} + \boldsymbol{\varepsilon}^{p} \qquad \phi - \phi_0 = \phi^{el} + \phi^{p}$$

One can say that the poroplastic constitutive relationships for the skeleton are obtained from the poroelastic ones by replacing the (total) strain $\boldsymbol{\varepsilon}$ with the reversible strain $\boldsymbol{\varepsilon}^{el} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{p}$ and additionally also the porosity ϕ with the reversible part of porosity $\phi - \phi^{p}$ in the Lagrangian approach.

Stress tensor

$$\mathrm{d}\boldsymbol{\sigma}' = \mathrm{d}\boldsymbol{\sigma} + \mathrm{d}p_w \boldsymbol{I} = \boldsymbol{D}(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}^{p} - \mathrm{d}\boldsymbol{\varepsilon}_s^{p_w}) \tag{9}$$

$\boldsymbol{\sigma}' \equiv \boldsymbol{\sigma} + p_w \boldsymbol{I}$ — Terzaghi's effective stress

$\boldsymbol{D}$ — a tangent elastic stiffness tensor of the solid skeleton

$$\mathrm{d}\boldsymbol{\varepsilon}_s^{p_w} = -\frac{1}{3K_s}\mathrm{d}p_w \boldsymbol{I} \text{ — the incremental strain of the solid matrix (grains) produced by an incremental water pressure } \mathrm{d}p_w \tag{10}$$

K_s — the matrix bulk modulus

For an isotropic material

$$\boldsymbol{D}\boldsymbol{I} = 3K\boldsymbol{I} \qquad K \text{ — the skeleton bulk modulus}$$

and one obtains:

$$\mathrm{d}\boldsymbol{\sigma} + \alpha \mathrm{d}p_w \boldsymbol{I} = \boldsymbol{D}(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}^{p})$$

$$\alpha = 1 - \frac{K}{K_s} \text{ — Biot's coefficient} \tag{11}$$

Porosity
(for the Lagrangian approach solely)

$$\mathrm{d}\phi - \mathrm{d}\phi^{p} = \alpha(\mathrm{d}\varepsilon_v - \mathrm{d}\varepsilon_v^{p}) + \frac{\mathrm{d}p_w}{N} \tag{12}$$

ε_v^{p} — the volumetric plastic strain $\qquad N$ — Biot's modulus

Considering α and N constant (over some ranges of strains and pressures), one can integrate (12) into the form:

$$\phi - \phi^{p} - \phi_0 = \alpha(\varepsilon_v - \varepsilon_v^{p}) + \frac{p_w - p_{w0}}{N} \tag{13}$$

Moreover under the assumptions that K and K_s are constant as well and the variations of the porosity are small, the equality in (11) can be recovered and

$$\frac{1}{N} = \frac{\alpha - n_0}{K_s} \tag{14}$$

Solid density
(for the Eulerian approach solely)
By assuming $\rho_s = \rho_s(p_w, \mathrm{tr}\,\boldsymbol{\sigma}')$ one gets:

$$\frac{\mathrm{d}\rho_s}{\rho_s} = \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial p_w}\mathrm{d}p_w + \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial(\mathrm{tr}\,\boldsymbol{\sigma}')}\mathrm{d}(\mathrm{tr}\,\boldsymbol{\sigma}')$$

Employing

$$\frac{1}{\rho_s}\frac{\partial \rho_s}{\partial p_w} = \frac{1}{K_s} \qquad \frac{1}{\rho_s}\frac{\partial \rho_s}{\partial (\operatorname{tr}\boldsymbol{\sigma}')} = -\frac{1}{3(1-n)K_s}$$

$$\mathrm{d}(\operatorname{tr}\boldsymbol{\sigma}') \overset{(9)}{=} \operatorname{tr}\boldsymbol{D}(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}^p - \mathrm{d}\boldsymbol{\varepsilon}_s^{p_w}) \overset{(10)}{=} 3K\left(\mathrm{d}\varepsilon_v - \mathrm{d}\varepsilon_v^p + \frac{\mathrm{d}p_w}{K_s}\right) \tag{15}$$

and using the expression for α in (11), one arrives at:

$$\frac{\mathrm{d}\rho_s}{\rho_s} = \frac{1}{1-n}\left(\frac{\alpha - n}{K_s}\mathrm{d}p_w - (1-\alpha)(\mathrm{d}\varepsilon_v - \mathrm{d}\varepsilon_v^p)\right) \tag{16}$$

Complete equations

Lagrangian approach

When adopting the small perturbation assumption,

$$\begin{aligned}
\frac{\partial(\rho_w\phi)}{\partial t} &= \rho_w\frac{\partial \phi}{\partial t} + \phi\frac{\partial \rho_w}{\partial t} \overset{(12),(7)}{=} \rho_w\left(\alpha\left(\frac{\partial \varepsilon_v}{\partial t} - \frac{\partial \varepsilon_v^p}{\partial t}\right) + \frac{\partial \phi^p}{\partial t} + \frac{1}{N}\frac{\partial p_w}{\partial t}\right) + \phi\frac{\rho_w}{K_w}\frac{\partial p_w}{\partial t} \\
&\approx \rho_{w0}\left(\frac{1}{N} + \frac{\phi_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha\left(\frac{\partial \varepsilon_v}{\partial t} - \frac{\partial \varepsilon_v^p}{\partial t}\right) + \rho_{w0}\frac{\partial \phi^p}{\partial t} \\
\operatorname{div}\boldsymbol{M} \approx \operatorname{div}(\rho_w\boldsymbol{q}_{rw}) &\overset{(8)}{=} \operatorname{div}\left(\rho_w\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_w\boldsymbol{f})\right) \approx \operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right)
\end{aligned}$$

and the Lagrangian water mass balance equation (2) leads to:

$$\boxed{\rho_{w0}\left(\frac{1}{N} + \frac{\phi_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha\left(\frac{\partial \varepsilon_v}{\partial t} - \frac{\partial \varepsilon_v^p}{\partial t}\right) + \rho_{w0}\frac{\partial \phi^p}{\partial t} = -\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right)} \tag{17}$$

Furthermore

$$\mathbf{div}(\boldsymbol{F\Pi}) \approx \mathbf{div}\,\boldsymbol{\sigma} \qquad \rho_w\phi \approx \rho_{w0}\phi_0$$

and the Lagrangian equilibrium equation (6) becomes:

$$\boxed{\mathbf{div}\,\boldsymbol{\sigma} + \big(\rho_{s0}(1-\phi_0) + \rho_{w0}\phi_0\big)\boldsymbol{f} = \mathbf{0}} \tag{18}$$

Eulerian approach

When adopting the assumptions of small perturbations and small deformation velocity,

$$\rho_w \approx \rho_{w0} \qquad n \approx n_0 \qquad \rho_s \approx \rho_{s0}$$

$$\frac{\mathrm{D}_s}{\mathrm{D}t} = \frac{\partial}{\partial t} + \boldsymbol{v}_s\cdot\nabla \approx \frac{\partial}{\partial t} \qquad \operatorname{div}\boldsymbol{v}_s = \operatorname{div}\frac{\mathrm{D}_s\boldsymbol{u}}{\mathrm{D}t} \approx \operatorname{div}\frac{\partial \boldsymbol{u}}{\partial t} = \frac{\partial \varepsilon_v}{\partial t}$$

and the Eulerian mass balance equations (1) and (3) and the constitutive equation for ρ_s (16) can be written as:

$$\rho_{w0}\frac{\partial n}{\partial t} + n_0\frac{\partial \rho_w}{\partial t} + \rho_{w0}n_0\frac{\partial \varepsilon_v}{\partial t} = -\operatorname{div}(\rho_{w0}\boldsymbol{q}_{rw}) \tag{19}$$

$$\frac{\partial(1-n)}{\partial t} + \frac{1-n_0}{\rho_{s0}}\frac{\partial \rho_s}{\partial t} + (1-n_0)\frac{\partial \varepsilon_v}{\partial t} = 0 \tag{20}$$

$$\frac{1-n_0}{\rho_{s0}}\frac{\partial \rho_s}{\partial t} = \frac{\alpha - n_0}{K_s}\frac{\partial p_w}{\partial t} - (1-\alpha)\left(\frac{\partial \varepsilon_v}{\partial t} - \frac{\partial \varepsilon_v^p}{\partial t}\right) \tag{21}$$

Elimination of $\partial\rho_s/\partial t$ from (20) by (21) gives:

$$\frac{\partial n}{\partial t} = \frac{\alpha - n_0}{K_s}\frac{\partial p_w}{\partial t} + (\alpha - n_0)\frac{\partial \varepsilon_v}{\partial t} + (1-\alpha)\frac{\partial \varepsilon_v^p}{\partial t} \tag{22}$$

which inserted together with (7) and (8) into (19) yields:

$$\boxed{\rho_{w0}\left(\frac{\alpha - n_0}{K_s} + \frac{n_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha\frac{\partial \varepsilon_v}{\partial t} + \rho_{w0}(1-\alpha)\frac{\partial \varepsilon_v^p}{\partial t} = -\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right)} \tag{23}$$

Further

$$\rho_s(1-n) \approx \rho_{s0}(1-n_0) \qquad \rho_w n \approx \rho_{w0} n_0$$

and the Eulerian equilibrium equation (5) leads to:

$$\boxed{\operatorname{\mathbf{div}}\boldsymbol{\sigma} + (\rho_{s0}(1-n_0) + \rho_{w0}n_0)\boldsymbol{f} = \mathbf{0}} \tag{24}$$

Additionally, taking α and K_s constant one can integrate (21) and (22) into:

$$\begin{aligned} \rho_s &= \rho_{s0}\left(1 + \frac{1}{1-n_0}\left(\frac{\alpha - n_0}{K_s}(p_w - p_{w0}) - (1-\alpha)(\varepsilon_v - \varepsilon_v^p)\right)\right) \\ n &= n_0 + \frac{\alpha - n_0}{K_s}(p_w - p_{w0}) + (\alpha - n_0)\varepsilon_v + (1-\alpha)\varepsilon_v^p \end{aligned} \tag{25}$$

Remark. If $\boldsymbol{D}^{ep}$ is the tangent elastoplastic tensor such that the constitutive equation for the stress tensor (9) can be written as

$$\mathrm{d}\boldsymbol{\sigma}' = \boldsymbol{D}^{ep}(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}_s^{p_w}) \tag{26}$$

then one can proceed as previously and reformulate the state equations (16) and (22) for ρ_s and n and the flow equation (23) in terms of $\boldsymbol{D}^{ep}$ instead of Biot's coefficient α including the skeleton (elastic) bulk modulus K.
In particular, if one can take K^{ep} such that (15) can be replaced by:

$$\mathrm{d}(\operatorname{tr}\boldsymbol{\sigma}') \overset{(26)}{=} \operatorname{tr}\boldsymbol{D}^{ep}(\mathrm{d}\boldsymbol{\varepsilon} - \mathrm{d}\boldsymbol{\varepsilon}_s^{p_w}) = 3K^{ep}\left(\mathrm{d}\varepsilon_v + \frac{\mathrm{d}p_w}{K_s}\right)$$

then the equations (16), (22) and (23) can be rewritten in the same form as for an isotropic *elastic* skeleton with $\alpha^{ep} = 1 - K^{ep}/K_s$ instead of α:

$$\begin{aligned} \frac{\mathrm{d}\rho_s}{\rho_s} &= \frac{1}{1-n}\left(\frac{\alpha^{ep} - n}{K_s}\mathrm{d}p_w - (1-\alpha^{ep})\mathrm{d}\varepsilon_v\right) \\ \frac{\partial n}{\partial t} &= \frac{\alpha^{ep} - n_0}{K_s}\frac{\partial p_w}{\partial t} + (\alpha^{ep} - n_0)\frac{\partial \varepsilon_v}{\partial t} \\ \rho_{w0}\left(\frac{\alpha^{ep} - n_0}{K_s} + \frac{n_0}{K_w}\right)\frac{\partial p_w}{\partial t} + \rho_{w0}\alpha^{ep}\frac{\partial \varepsilon_v}{\partial t} &= -\operatorname{div}\left(\rho_{w0}\frac{\boldsymbol{k}}{\mu_w}(-\nabla p_w + \rho_{w0}\boldsymbol{f})\right) \end{aligned}$$

Models of poroplasticity

An ideal poroplastic model

A *poroelasticity domain* C_E — a domain in the loading space $\{\boldsymbol{\sigma} \times p_w\}$ such that the strain and the change in porosity remain reversible along any loading path lying entirely within this domain. It can be defined as:

$$C_E = \{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w) < 0\} \qquad f \text{ — a loading function}$$

A yield surface — the boundary of C_E, where plastic evolutions may occur:

$$\{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w) = 0\}$$

A set of plastically admissible loadings — the closure of C_E:

$$\{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w) \leq 0\}$$

For an *ideal poroplastic* material, the poroelasticity domain C_E is not altered by plastic evolutions.
A *(plastic) flow rule* — it specifies how the plastic evolutions occur:

$$\mathrm{d}\boldsymbol{\varepsilon}^p = \mathrm{d}\lambda h_{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}, p_w) \qquad \mathrm{d}\phi^p = \mathrm{d}\lambda h_\phi(\boldsymbol{\sigma}, p_w)$$

$(h_{\boldsymbol{\varepsilon}}, h_\phi)$ — a couple of functions defining the directions of plastic increments
$\mathrm{d}\lambda$ — a plastic multiplier scaling the intensity of the plastic increments

The complementarity conditions:

$$\mathrm{d}\lambda \geq 0 \qquad f \leq 0 \qquad \mathrm{d}\lambda \cdot f = 0 \tag{27}$$

The consistency condition:

$$\mathrm{d}\lambda \cdot \mathrm{d}f = 0 \tag{28}$$

Non-negativeness of dissipated energy (plastic work):

$$\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^p + p\mathrm{d}\phi^p \geq 0$$

Owing to the non-negativeness of $\mathrm{d}\lambda$, this inequality requires $(h_{\boldsymbol{\varepsilon}}, h_\phi)$ to satisfy:

$$\boldsymbol{\sigma} : h_{\boldsymbol{\varepsilon}} + ph_\phi \geq 0$$

A hardening poroplastic model
For a *hardening poroplastic* material, the poroelasticity domain C_E is generally altered by plastic evolutions:

$$C_E = \{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta}) < 0\}$$

$\boldsymbol{\zeta} = \{\zeta_J\},\ \zeta_J = -\dfrac{\partial U}{\partial \chi_J}(\boldsymbol{\chi})$ — evolutionary hardening forces accounting for the current hardening state
U — a trapped energy
$\boldsymbol{\chi} = \{\chi_J\}$ — a set of hardening state variables

The yield surface:

$$\{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta}) = 0\}$$

The set of plastically admissible loadings:

$$\{(\boldsymbol{\sigma}, p_w) \mid f(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta}) \leq 0\}$$

Flow rule:

$$\mathrm{d}\boldsymbol{\varepsilon}^p = \mathrm{d}\lambda h_{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta}) \qquad \mathrm{d}\phi^p = \mathrm{d}\lambda h_\phi(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta}) \qquad \mathrm{d}\chi_J = \mathrm{d}\lambda h_J(\boldsymbol{\sigma}, p_w, \boldsymbol{\zeta})$$

The plastic multiplier $\mathrm{d}\lambda$ still obeys the complementarity conditions (27) and the consistency condition (28).
Non-negativeness of dissipated energy:

$$\boldsymbol{\sigma} : \mathrm{d}\boldsymbol{\varepsilon}^p + p\mathrm{d}\phi^p + \zeta_J \mathrm{d}\chi_J \geq 0$$

Owing to the non-negativeness of $\mathrm{d}\lambda$, this inequality requires $(h_{\boldsymbol{\varepsilon}}, h_\phi, h_J)$ to satisfy:

$$\boldsymbol{\sigma} : h_{\boldsymbol{\varepsilon}} + p h_\phi + \zeta_J h_J \geq 0$$

Standard material — the loading function f is a so-called associated potential:

$$h_{\boldsymbol{\varepsilon}} = \frac{\partial f}{\partial \boldsymbol{\sigma}} \qquad h_\phi = \frac{\partial f}{\partial p_w} \qquad h_J = \frac{\partial f}{\partial \zeta_J}$$

Non-standard material — there exists a non-associated potential $g \neq f$:

$$h_{\boldsymbol{\varepsilon}} = \frac{\partial g}{\partial \boldsymbol{\sigma}} \qquad h_\phi = \frac{\partial g}{\partial p_w} \qquad h_J = \frac{\partial g}{\partial \zeta_J}$$

Matrix plastic incompressibility
The observable macroscopic volumetric plastic strain ε_v^p undergone by the skeleton is due to both the plastic change in porosity and the volumetric plastic strain ε_{sv}^p undergone by the solid matrix. One can show that in the framework of small transformations:

$$\varepsilon_v^p = (1 - n_0)\varepsilon_{sv}^p + \phi^p \tag{29}$$

In soil and rock mechanics the plastic evolutions are caused by irreversible relative sliding of the solid grains forming the matrix, whereas the volume change of the matrix due uniquely to plasticity turns out to be negligible. This entails $\varepsilon_{sv}^p = 0$ and (29) results in:

$$\phi^p = \varepsilon_v^p \tag{30}$$

In the case of an associated flow rule, this further yields:

$$\mathrm{d}\lambda \frac{\partial f}{\partial p_w} = \mathrm{d}\phi^p = \mathrm{d}\varepsilon_v^p = \mathrm{d}\lambda \,\mathrm{tr}\, \frac{\partial f}{\partial \boldsymbol{\sigma}}$$

which requires the loading function f to depend only upon Terzaghi's effective stress $\boldsymbol{\sigma}' = \boldsymbol{\sigma} + p_w \boldsymbol{I}$ instead of an arbitrary couple $(\boldsymbol{\sigma}, p_w)$. This holds even for Biot's coefficient $\alpha < 1$.

By extension, in the context of a non-associated flow rule both the loading function f and the non-associated potential g are expressed as functions only of $\boldsymbol{\sigma}'$ (and the set of hardening forces $\boldsymbol{\zeta}$ in the case of hardening). The elasticity and the plasticity criteria then read:

$$f(\boldsymbol{\sigma}') < 0, \quad f(\boldsymbol{\sigma}') = 0 \qquad \big(f(\boldsymbol{\sigma}', \boldsymbol{\zeta}) < 0, \quad f(\boldsymbol{\sigma}', \boldsymbol{\zeta}) = 0\big)$$

and the flow rule is written in the form:

$$\mathrm{d}\boldsymbol{\varepsilon}^p = \mathrm{d}\lambda \frac{\partial g}{\partial \boldsymbol{\sigma}'} \qquad \mathrm{d}\phi^p = \mathrm{d}\varepsilon_v^p \qquad \left(\mathrm{d}\chi_J = \mathrm{d}\lambda \frac{\partial g}{\partial \zeta_J}\right)$$

Isotropic hardening — the poroelasticity domain dilates (hardening) or contracts (softening) in an isotropic way around the origin of the effective loading space $\{\boldsymbol{\sigma}'\}$. A hardening scalar force ζ is introduced to define this homothetical transformation.
Kinematic hardening — the poroelasticity domain moves in a rigid way in the effective loading space $\{\boldsymbol{\sigma}'\}$. A second-order tensor $\boldsymbol{\zeta}$ is introduced to define this translation.

Concluding remarks

The Lagrangian approach seems to be more general than the Eulerian one since it enables to separate the plastic change in porosity from the volumetric plastic strain – recall the plastic strain partition (29). In the case of matrix plastic incompressibility with (30) and the expressions for α and $1/N$ from (11) and (14) (and with $\phi_0 = n_0$ as the initial configuration coincides with the current one at time $t = 0$):

- The Lagrangian system (17)&(18) reduces to the Eulerian one (23)&(24).
- The equation (13) for the Lagrangian porosity ϕ is related to the equation (25) for the Eulerian porosity n by

$$\phi = Jn \approx (1 + \varepsilon_v)n$$

  when ε_v, ε_v^p and $p_w - p_{w0}$ are small.

Further note that a plastic evolution appears in the Eulerian flow equation (23) only for Biot's coefficient $\alpha < 1$ (an elastically compressible matrix).

References

[Cou04] O. Coussy. *Poromechanics.* John Wiley & Sons, 2004.

[LS98] R. W. Lewis and B. A. Schrefler. *The Finite Element Method in the Static and Dynamic Deformation and Consolidation of Porous Media.* John Wiley, 2nd edition, 1998.