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Abstract

The main objects of the present work are the quantum Navier–Stokes and quantum Euler systems; for the first one, in particular, we will consider constant viscosity coefficients. We deal with the concept of dissipative solutions, for which we will first prove the weak-strong uniqueness principle and afterwards, we will show the global existence for any finite energy initial data. Finally, we will prove that both systems admit a semiflow selection in the class of dissipative solutions.

Mathematics Subject Classification: 35A01, 35Q35, 76N10

Keywords: quantum fluid systems; dissipative solutions; weak–strong uniqueness; existence; semiflow selection

1 Introduction

At temperatures close to absolute zero, quantum effects appear relevant in the motion of some fluids: instead of individual atoms bouncing around, the particles move like one single body and, as a consequence of the vanishing viscosity, the fluid start to "creep" along the surfaces of its container, coming out of it if the latter is not properly sealed. This bizarre phenomena is just one of the many applications that motivate the study of *quantum fluid dynamics*: it provides

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useful tools for understanding not only the behaviour of atomic Bose–Einstein condensates and the transition of the aforementioned fluids into zero-viscosity ones (*superfluids*) [39], but also the mechanics of quantum semiconductors [26] and the trajectories arising from the de Broglie–Bohm theory [45].

1.1 The system

Motivated by the Thomas–Fermi–Dirac–Weizsäcker density functional theory [46], the motion of a quantum fluid can be modelled starting from the classical systems describing viscous or inviscid fluids and adding an extra term containing the Bohm quantum potential [41]

$$Q(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho) = \frac{\hbar}{2} \frac{\Delta_x \sqrt{\varrho}}{\sqrt{\varrho}}, \tag{1.1}$$

where ϱ denotes the density of the fluid. More precisely, we are going to consider the following two models.

- The compressible *quantum Navier–Stokes* system, whenever we are dealing with viscous fluids:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.2}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \nabla_x Q(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho). \tag{1.3}$$

- The compressible *quantum Euler* system, in case of inviscid fluids:

$$\partial_t \varrho + \operatorname{div}_x \mathbf{J} = 0, \tag{1.4}$$

$$\partial_t \mathbf{J} + \operatorname{div}_x \left(\frac{\mathbf{J} \otimes \mathbf{J}}{\varrho} \right) + \nabla_x p(\varrho) = \varrho \nabla_x Q(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho). \tag{1.5}$$

In both systems, the unknown variables are the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$ and the momentum $\mathbf{J} = (\varrho \mathbf{u})(t,x)$ of the fluid, while $p = p(\varrho)$ denotes the barotropic pressure, $\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ the viscous stress tensor and $Q = Q(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho)$ the quantum potential defined in (1.1). More precisely, we will consider the standard isentropic pressure

$$p(\varrho) = a \varrho^\gamma \tag{1.6}$$

with a a positive constant and $\gamma > \frac{d}{2}$ the adiabatic exponent, while the viscous stress tensor will be a linear function of the velocity gradient

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^\top \mathbf{u} - \frac{2}{d} (\operatorname{div}_x \mathbf{u}) \mathbb{I} \right) + \lambda (\operatorname{div}_x \mathbf{u}) \mathbb{I} \tag{1.7}$$

where $\mu > 0$ and $\lambda \geq 0$ denote the shear and bulk viscosities, respectively. Notice that we can write

$$\varrho \nabla_x Q(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho) = \operatorname{div}_x \mathbb{K}(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho)$$

with

$$\mathbb{K}(\varrho, \nabla_x \varrho, \nabla_x^2 \varrho) = \frac{\hbar}{4} \left(\nabla_x^2 \varrho - 4\, \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho} \right).$$

We will study both systems on the set $(0,\infty)\times\Omega$, where $\Omega\subset\mathbb{R}^d$, $d=2,3$, is a bounded domain of class C^2, on the boundary of which we impose the homogeneous Neumann condition for the density and the no-slip condition for the velocity

$$\nabla_x\varrho\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \mathbf{u}|_{\partial\Omega}=0, \tag{1.8}$$

when considering system (1.2)–(1.3), while we replace the boundary condition for the velocity with the one for the momentum

$$\nabla_x\varrho\cdot\mathbf{n}|_{\partial\Omega}=0,\quad \mathbf{J}\cdot\mathbf{n}|_{\partial\Omega}=0, \tag{1.9}$$

when considering system (1.4)–(1.5).

The last ingredient we need to formally close the systems is the energy. Introducing the *pressure potential* $P=P(\varrho)$ as a solution of

$$\varrho P'(\varrho)-P(\varrho)=p(\varrho), \tag{1.10}$$

which we will consider as

$$P(\varrho)=\frac{a}{\gamma-1}\varrho^\gamma,$$

the *total energy balance* associated to the quantum Navier–Stokes system (1.2)–(1.3) with the boundary conditions (1.8) is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(t)\,\mathrm{d}x+\int_\Omega\mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\,\mathrm{d}x=0\quad\text{with}\quad E(t)=\frac{1}{2}\varrho|\mathbf{u}|^2+P(\varrho)+\frac{\hbar}{2}|\nabla_x\sqrt{\varrho}|^2, \tag{1.11}$$

and similarly, the total energy balance associated to the quantum Euler system (1.4)–(1.5) with the boundary conditions (1.9) reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(t)\,\mathrm{d}x=0\quad\text{with}\quad E(t)=\frac{1}{2}\frac{|\mathbf{J}|^2}{\varrho}+P(\varrho)+\frac{\hbar}{2}|\nabla_x\sqrt{\varrho}|^2. \tag{1.12}$$

See Section A.2 for more details.

We finally point out that the quantum potential $Q=Q(\varrho,\nabla_x\varrho,\nabla_x^2\varrho)$ can be rewritten as

$$Q(\varrho,\nabla_x\varrho,\nabla_x^2\varrho)=K(\varrho)\Delta_x\varrho+\frac{1}{2}K'(\varrho)|\nabla_x\varrho|^2, \tag{1.13}$$

choosing the function $K=K(\varrho)$ such that

$$K(\varrho)=\frac{\hbar}{4\varrho}.$$

Systems (1.2), (1.3) and (1.4), (1.5) with the quantum potential replaced by the more general expression appearing on the right-hand side of (1.13) are called *Navier–Stokes–Korteweg* and *Euler–Korteweg* systems, respectively; usually, $K:(0,\infty)\to(0,\infty)$ is a smooth function.

1.2 State of the art

Given their importance in many applications, quantum fluid systems were widely studied in the last years. However, in literature we may typically encounter density-dependent instead of constant viscosity coefficients in the definition of the viscous stress tensor (1.7), leading to more mathematical difficulties due to the possible presence of vacuum. This alternative formulation is a consequence of a different derivation of the model, based on a Chapman–Enskog expansion of the Wigner function [11]. For the quantum Navier–Stokes system with non-constant viscosity coefficients, the existence of global-in-time weak solutions with special test function $\rho\phi$ instead of classical test function ϕ on the d-dimensional torus, $d = 2, 3$, was shown by Jüngel [34] with the constraint $\gamma > 3$ for $d = 3$ and the viscosity constant smaller than the scaled Plank constant; his result was later improved by Dong [19] and by Jiang [32], including the cases when the viscosity constant is equal and bigger, respectively, to the scaled Plank constant. Subsequently, the existence of global-in-time weak solutions with the standard test function ϕ was achieved with the help of extra terms in the equations that could guarantee the velocity to be well-defined even in the vacuum region: for instance, Gisclon and Lacroix-Violet [30] considered a cold pressure term, while Vasseur and Yu [43] added a damping term in the balance of momentum. Inspired by Li and Xin [38], Antonelli and Spirito [3] proved the global-in-time existence result for weak solutions without any extra terms, but requiring the viscosity and capillarity constants to be comparable. Recently, this assumption was removed by the same authors in [4], and by Lacroix-Violet and Vasseur [36]. Stability, i.e. the continuous dependence of solutions on initial data, was studied by Giesselmann, Lattanzio and Tzavaras [29] via a relative energy approach. Recently, Bresch, Gisclon and Lacroix-Violet [10] proved the existence of global-in-time dissipative solutions on the d-dimensional torus, $d = 2, 3$, of the quantum Navier–Stokes system with a linear density-dependent shear viscosity and zero bulk viscosity. Moreover, taking the vanishing viscosity limit, they obtained the existence of global-in-time dissipative solutions to the quantum Euler system (1.4), (1.5). For the latter, there are several results concerning well-posedness in the class of weak solutions. Donatelli, Feireisl and Marcati [16] showed that the system is ill-posed as uniqueness fails to be verified: for sufficiently smooth initial data, the system admits infinitely many weak solutions, even considering only the class of those satisfying the energy inequality. Later on, Antonelli and Marcati [2] proved the existence of global-in-time irrotational weak solutions by converting the Euler system into the non-linear Schrödinger one, while Audiard and Haspot [6] showed global well-posedness for small irrotational data in dimension $d \geq 3$. Last but not least, let us stress that important progresses have been made on singular limits and other topics for quantum fluid models [5, 12, 15, 17, 18, 33, 37].

Even though there is a wide range of significant results concerning well-posedness of quantum systems, we emphasize that there aren't any regarding the existence of global-in-time weak solutions for the quantum Navier–Stokes system (1.2), (1.3) with constant viscosity coefficients, even in dimension $d = 2$, and for the quantum Euler system (1.4), (1.5) for large initial data, as pointed out by Bresch et al. [10]. Therefore, the latter are important and interesting issues.

1.3 Structure of the paper

In the present study, we are interested in well-posedness of the aforementioned quantum systems; specifically, we are concerned with existence and uniqueness of global-in-time solutions for any finite energy initial data. Inspired by the work of Abbatiello, Feireisl and Novotný [1], we will consider *dissipative solutions*, i.e. solutions satisfying the equations and the energy inequality in the distributional sense but with extra "defect terms", which we may call *Reynolds stresses*, collecting the possible oscillations and/or concentrations arising from the convective, pressure and quantum terms, cf. Definitions 2.1 and 2.2. This notion of solution can be seen as a generalization of the concept of dissipative measure–valued solution, developed by Feireisl, Gwiazda, Świerczewska-Gwiazda and Wiedemann [22], implying in particular that they can be taken into account in the analysis of convergence of certain numerical schemes and, therefore, they can be identified as strong limits of finite element–finite volume schemes in the spirit of Feireisl and Lukáčová–Medviďová [24]. We point out that our definition of dissipative solution differs from the one considered in [10], as the latter is based on a relative energy inequality. A natural question is whether strong solutions are uniquely determined in the class of dissipative solutions; in order to give a positive answer, we will prove the *weak-strong uniqueness principle*: if the system admits a sufficiently regular solution in the classical sense then it must coincide with the dissipative solution emanating from the same initial data, cf. Theorems 3.1 and 3.2. As the name suggests, this technique was first developed by Prodi [40] considering weak/strong solutions for the incompressible Navier–Stokes equations, and later adapted for compressible systems (see e.g. [21], [23], [25], [28], [31], [44]). Our next goal is the existence of dissipative solutions. More precisely, we will first prove the existence result for the quantum Navier–Stokes system (1.2), (1.3) applying the classical fixed point argument in the spirit of [20], cf. Theorem 4.1, and afterwards, we will obtain the existence result for the quantum Euler system (1.4), (1.5) as a vanishing viscosity limit of the Navier–Stokes equations, cf. Theorem 4.2. Finally, to handle the problem of uniqueness, especially in view of the "negative" result stated in [16] for the quantum Euler system, we may look for that particular dissipative solution in the class of the ones emanating from the same initial data satisfying the *semigroup* or *semiflow property*: if we let the system run from time 0 to time t_1, we restart it and let it run for a time interval of amplitude t_2, the trajectory described by the selected solution will be the same as we have run the system directly from time 0 to time $t_1 + t_2$. We will refer to the process of finding such particular solution as *semiflow selection*, cf. Definition 5.1. Clearly, if uniqueness holds, the semigroup property is verified by any solution and the semiflow selection is simply the map associating to any admissible data that one unique solution emanating from it. The construction of a semiflow selection was originally a stochastic tool, first developed by Krylov [35] to study well-posedness of certain systems and later adapted by Flandoli and Romito [27], Breit, Feireisl and Hofmanová [8] for the incompressible and compressible, respectively, Navier–Stokes systems. Inspired by deterministic adaptation of Cardona and Kapitanski [13], we will prove the existence of a semiflow selection for the quantum Navier–Stokes and quantum Euler systems in the class of dissipative solutions, cf. Theorems 5.2 and 5.3. We will essentially follow the same strategy developed by Breit, Feireisl and Hofmanová [9] for the compressible Euler system in the class of measure–valued solutions. However, there will be a slightly difference in the choice of the trajectory space: instead of the space of continuous functions as in [13] or the space of integrable functions as in [9], we will work with the Skorokhod space of càglàd (a French acronym for "left-continuous and having right-hand limits") functions. The advantages of this choice is that on the one hand we are able to consider the energy, which is typically a non–increasing quantity with possible jumps, as a third state variable, while on the other hand we will get the existence of well-defined semiflow selections at any time. We point out that, thanks to the weak-strong uniqueness principle, solutions in the classical sense are always contained in the selected semiflow as long as they exist.

2 Dissipative solutions

In this section, we provide the definition of dissipative solution for both the quantum Navier–Stokes and quantum Euler systems. We will refer to the measure $\mathfrak{R}$ appearing in the weak formulations of the balance of momentum and energy inequality as *Reynolds stress*. For the definition of all the involved spaces see Section A.1.

Definition 2.1 (Dissipative solution of the quantum Navier–Stokes system)**.** The pair of functions $[\varrho, \mathbf{u}]$ with total energy E constitutes a *dissipative solution* to problem (1.2)–(1.3) with the isentropic pressure (1.6), the viscous stress tensor (1.7), the boundary conditions (1.8) and the initial data

$$[\varrho(0,\cdot), (\varrho\mathbf{u})(0,\cdot), E(0-)] = [\varrho_0, \mathbf{J}_0, E_0] \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d) \times [0,\infty)$$

if the following holds:

(i) *regularity class*: $\varrho > 0$ in $(0,\infty)\times\Omega$ and

$$\varrho \in C_{\text{weak,loc}}([0,\infty); L^\gamma(\Omega)) \cap L^\infty(0,\infty; W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)) \tag{2.1}$$

$$\varrho\mathbf{u} \in C_{\text{weak,loc}}([0,\infty); L^q(\Omega;\mathbb{R}^d)), \quad q = \max\left\{\frac{2\gamma}{\gamma+1}, \frac{4\gamma d}{(3d-2)\gamma+d}\right\}, \tag{2.2}$$

$$\mathbf{u} \in L^2_{\text{loc}}(0,\infty; W^{1,2}_0(\Omega;\mathbb{R}^d)), \tag{2.3}$$

$$E \in \mathfrak{D}([0,\infty)); \tag{2.4}$$

(ii) *weak formulation of the continuity equation*: the integral identity

$$\left[\int_\Omega \varrho\varphi(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi]\,\mathrm{d}x\mathrm{d}t \tag{2.5}$$

holds for any $\tau > 0$ and any $\varphi \in C^1_c([0,\infty)\times\overline{\Omega})$;

(iii) *weak formulation of the balance of momentum*: there exists

$$\mathfrak{R} \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}}))$$

such that the integral identity

$$
\begin{aligned}
\left[\int_\Omega \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + (\varrho\mathbf{u}\otimes\mathbf{u}):\nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\
&+ \frac{\hbar}{4}\int_0^\tau\int_\Omega \left[\nabla_x\varrho\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\varphi} + 4(\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}):\nabla_x\boldsymbol{\varphi}\right]\,\mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t
\end{aligned}
\tag{2.6}
$$

holds for any $\tau > 0$ and any $\boldsymbol{\varphi}\in C_c^1([0,\infty);C^2(\overline{\Omega};\mathbb{R}^d))$, $\boldsymbol{\varphi}|_{\partial\Omega}=0$;

(iv) *energy inequality:* there exists a constant $\lambda > 0$ such that

$$
\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + \frac{\hbar}{2}|\nabla_x\sqrt{\varrho}|^2\right](\tau,\cdot)\ \mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau) = E(\tau)
$$

for a.e. $\tau > 0$, and the energy inequality

$$
\left[E(t)\psi(t)\right]_{t=\tau_1^-}^{t=\tau_2^+} - \int_{\tau_1}^{\tau_2} E\psi'\ \mathrm{d}t + \int_{\tau_1}^{\tau_2}\psi\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\mathbf{u}\ \mathrm{d}x\mathrm{d}t \le 0 \tag{2.7}
$$

holds for any $0\le\tau_1\le\tau_2$ and any $\psi\in C_c^1([0,\infty))$, $\psi\ge 0$.

Definition 2.2 (Dissipative solution of the quantum Euler system)**.** The pair of functions $[\varrho,\mathbf{J}]$ with total energy E constitutes a *dissipative solution* to problem (1.4)–(1.5) with the isentropic pressure (1.6), the boundary conditions (1.9) and the initial data

$$
[\varrho(0,\cdot),\mathbf{J}(0,\cdot),E(0-)] = [\varrho_0,\mathbf{J}_0,E_0]\in L^\gamma(\Omega)\times L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d)\times[0,\infty)
$$

if the following holds:

(i) *regularity class*: $\varrho > 0$ in $(0,\infty)\times\Omega$ and

$$
\begin{aligned}
&\varrho\in C_{\mathrm{weak,loc}}([0,\infty);L^\gamma\cap W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)) \\
&\mathbf{J}\in C_{\mathrm{weak,loc}}([0,\infty);L^q(\Omega;\mathbb{R}^d)),\quad q=\max\left\{\frac{2\gamma}{\gamma+1},\frac{4\gamma d}{(3d-2)\gamma+d}\right\}, \\
&E\in\mathfrak{D}([0,\infty));
\end{aligned}
$$

(ii) *weak formulation of the continuity equation*: the integral identity

$$
\left[\int_\Omega \varrho\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega [\varrho\partial_t\varphi + \mathbf{J}\cdot\nabla_x\varphi]\ \mathrm{d}x\mathrm{d}t \tag{2.8}
$$

holds for any $\tau > 0$ and any $\varphi\in C_c^1([0,\infty)\times\overline{\Omega})$;

(iii) *weak formulation of the balance of momentum*: there exists

$$
\mathfrak{R}\in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}}))
$$

such that the integral identity

$$
\begin{aligned}
\left[\int_\Omega \mathbf{J}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[\mathbf{J}\cdot\partial_t\boldsymbol{\varphi} + \frac{\mathbf{J}\otimes\mathbf{J}}{\varrho} : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t \\
&\quad + \frac{\hbar}{4}\int_0^\tau\int_\Omega \left[\nabla_x\varrho\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\varphi} + 4(\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}) : \nabla_x\boldsymbol{\varphi}\right]\ \mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t
\end{aligned}
\tag{2.9}
$$

holds for any $\tau > 0$ and any $\boldsymbol{\varphi}\in C_c^1([0,\infty);C^2(\overline{\Omega};\mathbb{R}^d)),\ \boldsymbol{\varphi}|_{\partial\Omega} = 0,;$

(iv) *energy inequality:* there exists a constant $\lambda > 0$ such that

$$
\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}|^2}{\varrho} + P(\varrho) + \frac{\hbar}{2}|\nabla_x\sqrt{\varrho}|^2\right](\tau,\cdot)\ \mathrm{d}x + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau) = E(\tau)
$$

for a.e. $\tau > 0$, and the energy inequality

$$
\big[E(t)\psi(t)\big]_{t=\tau_1^-}^{t=\tau_2^+} - \int_{\tau_1}^{\tau_2} E\psi'\ \mathrm{d}t \le 0 \tag{2.10}
$$

holds for any $0\le\tau_1\le\tau_2$ and any $\psi\in C_c^1([0,\infty))$, $\psi\ge 0$.

3 Weak–strong uniqueness

In this section, our goal is to prove the *weak–strong uniqueness principle*: if the quantum Navier–Stokes (or quantum Euler) system admits a sufficiently regular classical solution, then it must coincide with the dissipative solution emanating from the same initial data. Hereafter, let

$$
\begin{aligned}
p_1 &:= \min\left\{\frac{\gamma}{\gamma-1}, \frac{2d\gamma}{(d+2)\gamma-d}\right\}, \\
p_2 &:= \min\left\{\frac{2d\gamma}{(d+2)\gamma-2d}, \frac{2d\gamma}{4\gamma-d}\right\}, \\
p_3 &:= \min\left\{\frac{d\gamma}{2\gamma-d}, \frac{2d\gamma}{(6-d)\gamma-d}\right\}.
\end{aligned}
$$

Theorem 3.1 (Weak–strong uniqueness for the quantum Navier–Stokes system)**.** *Let* $[\widetilde{\varrho},\widetilde{\mathbf{u}}]$ *with* $\widetilde{\varrho} > 0$ *and*

$$
\begin{aligned}
&\widetilde{\varrho}\in L^\infty(0,\infty;L^\gamma\cap W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)), \\
&\widetilde{\mathbf{u}}\in L^\infty(0,\infty;L^{2p_1}(\Omega;\mathbb{R}^d)) + L^2_{\mathrm{loc}}(0,\infty;W_0^{1,2}(\Omega;\mathbb{R}^d)),
\end{aligned}
\tag{3.1}
$$

be a strong solution of system (1.2)–(1.3)*, satisfying the constitutive relations* (1.6)–(1.7) *and the boundary conditions* (1.8)*, where in addition the density* $\widetilde{\varrho}$ *is such that*

$$
\begin{aligned}
\partial_t P'(\widetilde{\varrho}) &\in L^1_{\rm loc}(0,\infty;L^{p_1}(\Omega)),\\
\nabla_x P'(\widetilde{\varrho}) &\in L^2_{\rm loc}(0,\infty;L^{p_2}(\Omega;\mathbb{R}^d)) + L^1_{\rm loc}(0,\infty;L^{2p_1}(\Omega;\mathbb{R}^d)),\\
\partial_t \nabla_x \log\widetilde{\varrho} &\in L^1_{\rm loc}(0,\infty;L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^d)),\\
\nabla_x^2 \log\widetilde{\varrho} &\in L^2_{\rm loc}(0,\infty;L^{\frac{2d\gamma}{2\gamma-d}}(\Omega;\mathbb{R}^{d\times d})) + L^1_{\rm loc}(0,\infty;L^{\infty}(\Omega;\mathbb{R}^{d\times d})),
\end{aligned}
\tag{3.2}
$$

the velocity $\widetilde{\mathbf{u}}$ *is such that*

$$
\begin{aligned}
\partial_t \widetilde{\mathbf{u}} &\in L^2_{\rm loc}(0,\infty;L^{p_2}(\Omega;\mathbb{R}^d)) + L^1_{\rm loc}(0,\infty;L^{2p_1}(\Omega;\mathbb{R}^d)),\\
\nabla_x \widetilde{\mathbf{u}} &\in L^\infty(0,\infty;L^{p_3}(\Omega;\mathbb{R}^{d\times d})) + L^2_{\rm loc}(0,\infty;L^{2p_3}(\Omega;\mathbb{R}^{d\times d}))\\
&\quad + L^1_{\rm loc}(0,\infty;L^{\infty}(\Omega;\mathbb{R}^{d\times d})),\\
\mathrm{div}_x \widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{\infty}(\Omega)),\\
\mathrm{div}_x \nabla_x^\top \widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^d)),
\end{aligned}
\tag{3.3}
$$

and

$$
\frac{\mathbb{S}(\nabla_x \widetilde{\mathbf{u}})}{\widetilde{\varrho}} \in L^2_{\rm loc}(0,\infty;L^{\frac{2d\gamma}{2\gamma-d}}(\Omega;\mathbb{R}^{d\times d})).
$$

Let $[\varrho,\mathbf{u}]$ *be a dissipative solution of the same system with dissipation defect* $\mathfrak{R}$ *in the sense of Definition 2.1. If*

$$
[\widetilde{\varrho}(0,x),(\widetilde{\varrho}\widetilde{\mathbf{u}})(0,x)] = [\varrho(0,x),(\varrho\mathbf{u})(0,x)] \quad \textit{for a.e. } x\in\Omega \tag{3.4}
$$

then $\mathfrak{R}\equiv 0$ *and*

$$
[\widetilde{\varrho}(t,x),\widetilde{\mathbf{u}}(t,x)] = [\varrho(t,x),\mathbf{u}(t,x)] \quad \textit{for a.e. } (t,x)\in(0,\infty)\times\Omega. \tag{3.5}
$$

Theorem 3.2 (Weak–strong uniqueness for the quantum Euler system)**.** *Let* $[\widetilde{\varrho},\widetilde{\mathbf{u}}]$ *with*

$$
\begin{aligned}
&\widetilde{\varrho} \in L^\infty(0,\infty;L^\gamma\cap W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)),\\
&\widetilde{\mathbf{u}} \in L^\infty(0,\infty;L^{2p_1}(\Omega;\mathbb{R}^d)),
\end{aligned}
\tag{3.6}
$$

be a strong solution of system (1.4)–(1.5) *satisfying the constitutive relation* (1.6)*, where in addition the density* $\widetilde{\varrho}>0$ *and the velocity* $\widetilde{\mathbf{u}}$ *are such that* $\widetilde{\mathbf{u}}\cdot\mathbf{n}|_{\partial\Omega}=0$ *and*

$$
\begin{aligned}
\partial_t P'(\widetilde{\varrho}) &\in L^1_{\rm loc}(0,\infty;L^{p_1}(\Omega)),\\
\mathrm{div}_x \widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{\infty}(\Omega)),\\
\nabla_x P'(\widetilde{\varrho}),\ \nabla_x\Delta_x\log\widetilde{\varrho},\ \partial_t\widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{2p_1}(\Omega;\mathbb{R}^d)),\\
\partial_t\nabla_x\log\widetilde{\varrho},\ \mathrm{div}_x\nabla_x^\top\widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{\frac{2\gamma}{\gamma-1}}(\Omega;\mathbb{R}^d)),\\
\nabla_x^2\log\widetilde{\varrho},\ \nabla_x\widetilde{\mathbf{u}} &\in L^1_{\rm loc}(0,\infty;L^{\infty}(\Omega;\mathbb{R}^{d\times d})).
\end{aligned}
\tag{3.7}
$$

Let $[\varrho, \mathbf{J}]$ *be a dissipative solution of the same system with dissipation defect* $\mathfrak{R}$ *in the sense of Definition 2.2. If*

$$[\widetilde{\varrho}(0,x), (\widetilde{\varrho}\widetilde{\mathbf{u}})(0,x)] = [\varrho(0,x), \mathbf{J}(0,x)] \quad \textit{for a.e. } x \in \Omega$$

then $\mathfrak{R} \equiv 0$ *and*

$$[\widetilde{\varrho}(t,x), (\widetilde{\varrho}\widetilde{\mathbf{u}})(t,x)] = [\varrho(t,x), \mathbf{J}(t,x)] \quad \textit{for a.e. } (t,x) \in (0,\infty) \times \Omega.$$

The proofs are based on showing that a slightly modified version of the energy, known as relative energy, and the Reynolds stress vanish almost everywhere.

3.1 Proof of Theorem 3.1

We introduce the *relative energy functional*:

$$E(\varrho, \nabla_x \varrho, \mathbf{u} \mid \widetilde{\varrho}, \nabla_x \widetilde{\varrho}, \widetilde{\mathbf{u}}) = \frac{1}{2}\varrho|\mathbf{u} - \widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) + \frac{\hbar}{2}\left|\nabla_x \sqrt{\varrho} - \sqrt{\frac{\varrho}{\widetilde{\varrho}}}\, \nabla_x \sqrt{\widetilde{\varrho}}\right|^2.$$

To simplify notation, we introduce the *drift velocities*

$$\mathbf{v} = \frac{\nabla_x \sqrt{\varrho}}{\sqrt{\varrho}}, \quad \widetilde{\mathbf{v}} = \frac{\nabla_x \sqrt{\widetilde{\varrho}}}{\sqrt{\widetilde{\varrho}}}, \tag{3.8}$$

and therefore the relative energy functional can be rewritten as

$$E(\varrho, \mathbf{u}, \mathbf{v} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) = \frac{1}{2}\varrho|\mathbf{u} - \widetilde{\mathbf{u}}|^2 + P(\varrho) - P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - P(\widetilde{\varrho}) + \frac{\hbar}{2}\varrho\,|\mathbf{v} - \widetilde{\mathbf{v}}|^2.$$

Step 1. First of all, proving Theorem 3.1 is equivalent in showing that

$$\mathfrak{R} \equiv 0, \quad E(\varrho, \mathbf{u}, \mathbf{v} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) \equiv 0 \quad \text{a.e. in } (0,\infty) \times \Omega. \tag{3.9}$$

Indeed, since the pressure $\varrho \mapsto p(\varrho)$ is strictly increasing in $(0,\infty)$, the pressure potential $\varrho \mapsto P(\varrho)$ is strictly convex. For a differentiable function, this is equivalent in saying that the function lies above all of its tangents,

$$P(\varrho) \geq P'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) + P(\widetilde{\varrho}) \tag{3.10}$$

for all $\varrho, \widetilde{\varrho} \in (0,\infty)$. Therefore, we can deduce that

$$E(\varrho, \mathbf{u}, \mathbf{v} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{v}}) \geq 0. \tag{3.11}$$

Moreover, the equality in (3.10) holds if and only if $\varrho = \widetilde{\varrho}$ and consequently the equality in (3.11) holds if and only if (3.5) holds.

Step 2. We will now show that any dissipative solution satisfies an extended version of the energy inequality, whenever $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ are smooth and compactly supported functions. Let us suppose that

$$\widetilde{\varrho} \in C_c^\infty([0,\infty) \times \overline{\Omega}),$$
$$\widetilde{\mathbf{u}} \in C_c^\infty([0,\infty) \times \overline{\Omega}; \mathbb{R}^d);$$

then, we can take $\varphi = \frac{1}{2}|\widetilde{\mathbf{u}}|^2,\ P'(\widetilde{\varrho}),\ \frac{\hbar}{2}|\widetilde{\mathbf{v}}|^2,\ \hbar \operatorname{div}_x \widetilde{\mathbf{v}}$ as test functions in the weak formulation of the continuity equation (2.5) to get

$$\frac{1}{2}\left[\int_\Omega \varrho|\widetilde{\mathbf{u}}|^2(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \varrho\widetilde{\mathbf{u}} \cdot \big(\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \mathbf{u}\big)\ \mathrm{d}x\mathrm{d}t, \tag{3.12}$$

$$\left[\int_\Omega \varrho P'(\widetilde{\varrho})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega [\varrho\partial_t P'(\widetilde{\varrho}) + \varrho\mathbf{u} \cdot \nabla_x P'(\widetilde{\varrho})]\ \mathrm{d}x\mathrm{d}t, \tag{3.13}$$

$$\frac{\hbar}{2}\left[\int_\Omega \varrho|\widetilde{\mathbf{v}}|^2(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \hbar\int_0^\tau \int_\Omega \varrho\widetilde{\mathbf{v}} \cdot \big(\partial_t \widetilde{\mathbf{v}} + \nabla_x \widetilde{\mathbf{v}} \cdot \mathbf{u}\big)\ \mathrm{d}x\mathrm{d}t, \tag{3.14}$$

$$\hbar\left[\int_\Omega \varrho\mathbf{v} \cdot \widetilde{\mathbf{v}}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \hbar\int_0^\tau \int_\Omega \big[\varrho\mathbf{v} \cdot \big(\partial_t \widetilde{\mathbf{v}} + \nabla_x \widetilde{\mathbf{v}} \cdot \mathbf{u}\big) + \varrho\nabla_x \mathbf{u} : \nabla_x \widetilde{\mathbf{v}}\big]\ \mathrm{d}x\mathrm{d}t, \tag{3.15}$$

where we recall identity (A.6), and $\boldsymbol{\varphi} = \widetilde{\mathbf{u}}$ as test function in the weak formulation of the balance of momentum (2.6) to get

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u} \cdot \widetilde{\mathbf{u}}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau \int_\Omega \big[\varrho\mathbf{u} \cdot \big(\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \mathbf{u}\big) + p(\varrho) \operatorname{div}_x \widetilde{\mathbf{u}}\big]\ \mathrm{d}x\mathrm{d}t \\ &\quad + \hbar\int_0^\tau \int_\Omega \left[\frac{1}{2}\varrho\mathbf{v} \cdot \operatorname{div}_x \nabla_x^\top \widetilde{\mathbf{u}} + \varrho\mathbf{v} \cdot \nabla_x \widetilde{\mathbf{u}} \cdot \mathbf{v}\right] \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau \int_{\overline{\Omega}} \nabla_x \widetilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t.\end{aligned} \tag{3.16}$$

Next, if we sum the integral identities (3.12), (3.14) and subtract (3.13), (3.15), (3.16) from the energy inequality (2.7), keeping in mind that

$$\left[\int_\Omega \big[\widetilde{\varrho}P'(\widetilde{\varrho}) - P(\widetilde{\varrho})\big](t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega \frac{\partial}{\partial t}\big[\widetilde{\varrho}P'(\widetilde{\varrho}) - P(\widetilde{\varrho})\big]\ \mathrm{d}x\mathrm{d}t = \int_0^\tau \int_\Omega \widetilde{\varrho}P''(\widetilde{\varrho})\partial_t\widetilde{\varrho}\ \mathrm{d}x\mathrm{d}t,$$

we obtain

$$\begin{aligned}&\left[\int_\Omega E(\varrho,\mathbf{u},\mathbf{v} \mid \widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x(\mathbf{u} - \widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\ &\quad \le -\int_0^\tau \int_\Omega \varrho(\mathbf{u} - \widetilde{\mathbf{u}}) \cdot [\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot (\mathbf{u} - \widetilde{\mathbf{u}})]\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_\Omega p(\varrho) \operatorname{div}_x \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\ &\quad - \hbar\int_0^\tau \int_\Omega \varrho(\mathbf{v} - \widetilde{\mathbf{v}}) \cdot \big[\partial_t \widetilde{\mathbf{v}} + \nabla_x \widetilde{\mathbf{v}} \cdot \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{v}} \cdot (\mathbf{u} - \widetilde{\mathbf{u}})\big]\ \mathrm{d}x\mathrm{d}t \\ &\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \widetilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t - \int_0^\tau F_1(t)\ \mathrm{d}t - \hbar\int_0^\tau F_2(t)\ \mathrm{d}t,\end{aligned}$$

with

$$F_1(t) = \int_\Omega \frac{\varrho}{\widetilde{\varrho}}\, p'(\widetilde{\varrho})\big(\partial_t \widetilde{\varrho} + \mathbf{u}\cdot\nabla_x \widetilde{\varrho}\big)(t,\cdot)\ \mathrm{d}x - \int_\Omega p'(\widetilde{\varrho})\partial_t \widetilde{\varrho}(t,\cdot)\ \mathrm{d}x,$$

$$F_2(t) = \int_\Omega \varrho \left(\frac{1}{2}\ \mathbf{v}\cdot \mathrm{div}_x\, \nabla_x^\top \widetilde{\mathbf{u}} + \mathbf{v}\cdot\nabla_x \widetilde{\mathbf{u}}\cdot \mathbf{v} + \frac{1}{2}\nabla_x \mathbf{u} : \nabla_x \widetilde{\mathbf{v}}\right)(t,\cdot)\ \mathrm{d}x,$$

recalling that $p'(\widetilde{\varrho}) = \widetilde{\varrho} P''(\widetilde{\varrho})$. Now, we can sum and subtract the following integrals

$$\int_0^\tau \int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot \left[\nabla_x P'(\widetilde{\varrho}) - \frac{1}{\widetilde{\varrho}}\mathrm{div}_x\, \mathbb{S}(\nabla_x \widetilde{\mathbf{u}}) - \frac{1}{\widetilde{\varrho}}\mathrm{div}_x\, \mathbb{K}(\widetilde{\varrho}, \nabla_x \widetilde{\mathbf{v}})\right] \mathrm{d}x\mathrm{d}t, \tag{3.17}$$

$$\int_0^\tau \int_\Omega [p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) + p(\widetilde{\varrho})]\, \mathrm{div}_x\, \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t, \tag{3.18}$$

$$\frac{\hbar}{2}\int_0^\tau \int_\Omega \frac{\varrho}{\widetilde{\varrho}}\ (\mathbf{v}-\widetilde{\mathbf{v}})\cdot \mathrm{div}_x\left(\widetilde{\varrho}\nabla_x^\top \widetilde{\mathbf{u}}\right)\ \mathrm{d}x\mathrm{d}t \tag{3.19}$$

from the previous inequality to get

$$\begin{aligned}
&\left[\int_\Omega E(\varrho, \mathbf{u}, \mathbf{v}\mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{v}})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)\\
&\qquad + \int_0^\tau \int_\Omega \mathbb{S}\big(\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\big) : \nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t\\
&\quad \leq -\int_0^\tau \int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\left[\partial_t\widetilde{\mathbf{u}} + \nabla_x\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - \frac{1}{\widetilde{\varrho}}\mathrm{div}_x\, \mathbb{S}(\nabla_x\widetilde{\mathbf{u}}) - \frac{1}{\widetilde{\varrho}}\mathrm{div}_x\, \mathbb{K}(\widetilde{\varrho}, \nabla_x\widetilde{\mathbf{v}})\right] \mathrm{d}x\mathrm{d}t\\
&\qquad - \int_0^\tau \int_\Omega \varrho(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\nabla_x\widetilde{\mathbf{u}}\cdot(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t\\
&\qquad - \int_0^\tau \int_\Omega [p(\varrho) - p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho}) - p(\widetilde{\varrho})]\, \mathrm{div}_x\, \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t,\\
&\qquad - \int_0^\tau \int_\Omega \left(\frac{\varrho}{\widetilde{\varrho}} - 1\right)(\mathbf{u}-\widetilde{\mathbf{u}})\cdot \mathrm{div}_x\, \mathbb{S}(\nabla_x\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t\\
&\qquad - \hbar\int_0^\tau \int_\Omega \varrho(\mathbf{v}-\widetilde{\mathbf{v}})\cdot\left[\partial_t\widetilde{\mathbf{v}} + \nabla_x\widetilde{\mathbf{v}}\cdot\widetilde{\mathbf{u}} + \frac{1}{2\widetilde{\varrho}}\mathrm{div}_x\left(\widetilde{\varrho}\nabla_x^\top\widetilde{\mathbf{u}}\right)\right] \mathrm{d}x\mathrm{d}t\\
&\qquad - \hbar\int_0^\tau \int_\Omega \varrho(\mathbf{v}-\widetilde{\mathbf{v}})\cdot\nabla_x\widetilde{\mathbf{v}}\cdot(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t\\
&\qquad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x\widetilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t - \int_0^\tau \widetilde{F}_1(t)\ \mathrm{d}t - \hbar\int_0^\tau \widetilde{F}_2(t)\ \mathrm{d}t,
\end{aligned}$$

with

$$\begin{aligned}
\widetilde{F}_1(t) &= F_1(t) - \int_\Omega \frac{\varrho}{\widetilde{\varrho}}\, p'(\widetilde{\varrho})\big[\mathbf{u}\cdot\nabla_x\widetilde{\varrho} - \mathrm{div}_x\left(\widetilde{\varrho}\widetilde{\mathbf{u}}\right)\big](t,\cdot)\ \mathrm{d}x - \int_\Omega p'(\widetilde{\varrho})\, \mathrm{div}_x\left(\widetilde{\varrho}\widetilde{\mathbf{u}}\right)(t,\cdot)\ \mathrm{d}x\\
&= \int_\Omega p'(\widetilde{\varrho})\left(\frac{\varrho}{\widetilde{\varrho}} - 1\right)[\partial_t\widetilde{\varrho} + \mathrm{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})](t,\cdot)\ \mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{F}_2(t) &= F_2(t) + \frac{1}{\hbar}\int_\Omega \frac{\varrho}{\widetilde{\varrho}}\ (\mathbf{u}-\widetilde{\mathbf{u}})\cdot \mathrm{div}_x\, \mathbb{K}(\widetilde{\varrho}, \nabla_x\widetilde{\mathbf{v}})\ \mathrm{d}x - \frac{1}{2}\int_\Omega \frac{\varrho}{\widetilde{\varrho}}\ (\mathbf{v}-\widetilde{\mathbf{v}})\cdot\mathrm{div}_x\left(\widetilde{\varrho}\nabla_x^\top\widetilde{\mathbf{u}}\right)\ \mathrm{d}x\\
&= -\int_\Omega \varrho(\mathbf{v}-\widetilde{\mathbf{v}})\cdot\nabla_x\widetilde{\mathbf{v}}\cdot(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x + \int_\Omega \varrho(\mathbf{v}-\widetilde{\mathbf{v}})\cdot\nabla_x\widetilde{\mathbf{u}}\cdot(\mathbf{v}-\widetilde{\mathbf{v}})\ \mathrm{d}x
\end{aligned}$$

We have finally obtained the *relative energy inequality*:

$$
\begin{aligned}
&\left[\int_{\Omega} E(\varrho, \mathbf{u}, \mathbf{v} \mid \widetilde{\varrho}, \widetilde{\mathbf{u}}, \widetilde{\mathbf{v}})(t, \cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} + \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau) \\
&\quad + \int_0^\tau \int_{\Omega} \mathbb{S}\big(\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\big) : \nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\
&\leq -\int_0^\tau \int_{\Omega} \varrho(\mathbf{u}-\widetilde{\mathbf{u}}) \cdot \left[\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) - \frac{1}{\widetilde{\varrho}} \operatorname{div}_x \mathbb{S}(\nabla_x \widetilde{\mathbf{u}}) - \frac{1}{\widetilde{\varrho}} \operatorname{div}_x \mathbb{K}(\widetilde{\varrho}, \nabla_x \widetilde{\mathbf{v}})\right] \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\Omega} \varrho\big[(\mathbf{u}-\widetilde{\mathbf{u}}) \otimes (\mathbf{u}-\widetilde{\mathbf{u}})\big] : \nabla_x \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\Omega} [p(\varrho) - p'(\widetilde{\varrho})(\varrho - \widetilde{\varrho}) - p(\widetilde{\varrho})] \operatorname{div}_x \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\Omega} p'(\widetilde{\varrho}) \left(\frac{\varrho}{\widetilde{\varrho}} - 1\right) [\partial_t \widetilde{\varrho} + \operatorname{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})]\ \mathrm{d}x\mathrm{d}t, \\
&\quad - \int_0^\tau \int_{\Omega} \left(\frac{\varrho}{\widetilde{\varrho}} - 1\right) (\mathbf{u}-\widetilde{\mathbf{u}}) \cdot \operatorname{div}_x \mathbb{S}(\nabla_x \widetilde{\mathbf{u}})\ \mathrm{d}x\mathrm{d}t \\
&\quad - \hbar \int_0^\tau \int_{\Omega} \varrho(\mathbf{v}-\widetilde{\mathbf{v}}) \cdot \left[\partial_t \widetilde{\mathbf{v}} + \nabla_x \widetilde{\mathbf{v}} \cdot \widetilde{\mathbf{u}} + \frac{1}{2\widetilde{\varrho}} \operatorname{div}_x \left(\widetilde{\varrho} \nabla_x^\top \widetilde{\mathbf{u}}\right)\right] \mathrm{d}x\mathrm{d}t \\
&\quad - \hbar \int_0^\tau \int_{\Omega} \varrho\big[(\mathbf{v}-\widetilde{\mathbf{v}}) \otimes (\mathbf{v}-\widetilde{\mathbf{v}})\big] : \nabla_x \widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t \\
&\quad - \int_0^\tau \int_{\overline{\Omega}} \nabla_x \widetilde{\mathbf{u}} : \mathrm{d}\mathfrak{R}\ \mathrm{d}t.
\end{aligned} \tag{3.20}
$$

Step 3. The class of functions $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ satisfying the relative energy inequality (3.20) can be extended by a density argument as long as integrals (3.12)–(3.19) remain well-defined. After a careful analysis, we recover the regularity class given by (3.1)–(3.3). Notice that we have used the Sobolev embedding

$$
W^{1,\frac{2\gamma}{\gamma+1}} \hookrightarrow L^{\gamma^*}(\Omega) \quad \text{with} \quad \gamma^* := \frac{2\gamma d}{(d-2)\gamma + d} \tag{3.21}
$$

implying in particular that

$$
\varrho \in C_{\mathrm{weak,loc}}([0,\infty); L^p(\Omega)), \quad p := \max\{\gamma, \gamma^*\} \tag{3.22}
$$

and the fact that $\gamma^* > \gamma$ as long as $\{d = 2\}$ or $\{d = 3,\ d/2 < \gamma < d\}$ to get the optimal regularity for the density ϱ and the momentum $\varrho\mathbf{u}$.

Step 4. Let us now suppose that the couple $[\widetilde{\varrho}, \widetilde{\mathbf{u}}]$ is a strong solution of problem (1.2)–(1.3), meaning that

$$
\begin{aligned}
\partial_t \widetilde{\varrho} + \operatorname{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}}) &= 0, \\
\partial_t \widetilde{\mathbf{u}} + \nabla_x \widetilde{\mathbf{u}} \cdot \widetilde{\mathbf{u}} + \nabla_x P'(\widetilde{\varrho}) &= \frac{1}{\widetilde{\varrho}} \operatorname{div}_x \big[\mathbb{S}(\nabla_x \widetilde{\mathbf{u}}) + \mathbb{K}(\widetilde{\varrho}, \nabla_x \widetilde{\mathbf{v}})\big], \\
\partial_t \widetilde{\mathbf{v}} + \nabla_x \widetilde{\mathbf{v}} \cdot \widetilde{\mathbf{u}} &= -\frac{1}{2\widetilde{\varrho}} \operatorname{div}_x \big(\widetilde{\varrho} \nabla_x^\top \widetilde{\mathbf{u}}\big),
\end{aligned}
$$

where the last one was deduced taking the gradient in the continuity equation (1.2). Then, the relative energy inequality (3.20) reduces to

$$\begin{aligned}
&\left[\int_\Omega E(\varrho,\mathbf{u},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(t,\cdot)\,\mathrm{d}x\right]_{t=0}^{t=\tau}+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)+\int_0^\tau\int_\Omega\mathbb{S}\big(\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\big):\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\,\mathrm{d}x\mathrm{d}t\\
&\qquad\le-\int_0^\tau\int_\Omega\varrho\left[(\mathbf{u}-\widetilde{\mathbf{u}})\otimes(\mathbf{u}-\widetilde{\mathbf{u}})+\hbar(\mathbf{v}-\widetilde{\mathbf{v}})\otimes(\mathbf{v}-\widetilde{\mathbf{v}})\right]:\nabla_x\widetilde{\mathbf{u}}\,\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_\Omega[p(\varrho)-p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-p(\widetilde{\varrho})]\,\mathrm{div}_x\,\widetilde{\mathbf{u}}\,\mathrm{d}x\mathrm{d}t,\\
&\qquad-\int_0^\tau\int_\Omega\left(\frac{\varrho}{\widetilde{\varrho}}-1\right)(\mathbf{u}-\widetilde{\mathbf{u}})\cdot\mathrm{div}_x\,\mathbb{S}(\nabla_x\widetilde{\mathbf{u}})\,\mathrm{d}x\mathrm{d}t-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\widetilde{\mathbf{u}}:\mathrm{d}\mathfrak{R}\,\mathrm{d}t.
\end{aligned}\tag{3.23}$$

On the one hand, we have

$$\int_0^\tau\int_\Omega\mathbb{S}\big(\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\big):\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})\,\mathrm{d}x\mathrm{d}t\ge\mu\int_0^\tau\int_\Omega|\nabla_x(\mathbf{u}-\widetilde{\mathbf{u}})|^2\,\mathrm{d}x\mathrm{d}t;$$

on the other hand, we have

$$\begin{aligned}
|\varrho(\mathbf{u}-\widetilde{\mathbf{u}})\otimes(\mathbf{u}-\widetilde{\mathbf{u}})|&\le c_1\frac{1}{2}\,\mathrm{Tr}[\varrho(\mathbf{u}-\widetilde{\mathbf{u}})\otimes(\mathbf{u}-\widetilde{\mathbf{u}})]=c_1\frac{1}{2}\varrho|\mathbf{u}-\widetilde{\mathbf{u}}|^2,\\
\hbar|\varrho(\mathbf{v}-\widetilde{\mathbf{v}})\otimes(\mathbf{v}-\widetilde{\mathbf{v}})|&\le c_1\frac{\hbar}{2}\,\mathrm{Tr}[\varrho(\mathbf{v}-\widetilde{\mathbf{v}})\otimes(\mathbf{v}-\widetilde{\mathbf{v}})]=c_1\frac{\hbar}{2}\varrho|\mathbf{v}-\widetilde{\mathbf{v}}|^2,\\
p(\varrho)-p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-p(\widetilde{\varrho})&=(\gamma-1)\big[P(\varrho)-P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-P(\widetilde{\varrho})\big],\\
|\mathfrak{R}|&\le\frac{c_2}{\lambda}\,\mathrm{Tr}[\mathfrak{R}].
\end{aligned}$$

Moreover, it is easy to see that

$$P(\varrho)-P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-P(\widetilde{\varrho})\ge c(\widetilde{\varrho})\begin{cases}(\varrho-\widetilde{\varrho})^2&\text{if }\varrho\in\left[\frac{\widetilde{\varrho}}{2},2\widetilde{\varrho}\right]\\(1+\varrho^\gamma)&\text{otherwise,}\end{cases}$$

and therefore, it is possible to show that

$$\int_0^\tau\int_\Omega\left|\frac{\varrho}{\widetilde{\varrho}}-1\right|\,|\mathbf{u}-\widetilde{\mathbf{u}}|\,\mathrm{d}x\mathrm{d}t\le c(\delta)\int_0^\tau\int_\Omega E(\varrho,\mathbf{u}\mid\widetilde{\varrho},\widetilde{\mathbf{u}})\,\mathrm{d}x\mathrm{d}t+\delta\int_0^\tau\int_\Omega|\mathbf{u}-\widetilde{\mathbf{u}}|^2\,\mathrm{d}x\mathrm{d}t$$

for any $\delta>0$ (see for instance [23], Section 4.1.1). Therefore, from the Poincaré inequality and hypothesis (3.4), we can rewrite (3.23) as

$$\begin{aligned}
&\int_\Omega E(\varrho,\mathbf{u},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(\tau,\cdot)\,\mathrm{d}x+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)+(1-\delta)\int_0^\tau\int_\Omega|\mathbf{u}-\widetilde{\mathbf{u}}|^2\,\mathrm{d}x\mathrm{d}t\\
&\qquad\le c(\delta,\widetilde{\varrho},\nabla_x\widetilde{\mathbf{u}},\mathrm{div}_x\,\mathbb{S}(\nabla_x\widetilde{\mathbf{u}}))\int_0^\tau\left(\int_\Omega E(\varrho,\mathbf{u},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(t,\cdot)\,\mathrm{d}x+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](t)\right)\mathrm{d}t.
\end{aligned}$$

Applying Gronwall's lemma, we can recover that

$$\int_\Omega E(\varrho,\mathbf{u},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(\tau,\cdot)\,\mathrm{d}x+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)\le0,$$

but since the quantity on the left-hand side is non-negative, this is possible if and only if (3.9) holds.

3.2 Proof of Theorem 3.2

We repeat the same passages performed before. Notice that in this case the relative energy functional is

$$E(\varrho,\mathbf{J},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})=\frac{1}{2\varrho}|\mathbf{J}-\varrho\widetilde{\mathbf{u}}|^2+P(\varrho)-P'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-P(\widetilde{\varrho})+\frac{\hbar}{2}\varrho\,|\mathbf{v}-\widetilde{\mathbf{v}}|^2\,,$$

and therefore the relative energy associated to the Euelr-Korteweg system (1.4)–(1.5) reads

$$\begin{aligned}
&\left[\int_\Omega E(\varrho,\mathbf{J},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)\\
&\quad\le-\int_0^\tau\int_\Omega(\mathbf{J}-\varrho\widetilde{\mathbf{u}})\cdot\left[\partial_t\widetilde{\mathbf{u}}+\nabla_x\widetilde{\mathbf{u}}\cdot\widetilde{\mathbf{u}}+\nabla_x P'(\widetilde{\varrho})-\frac{1}{\widetilde{\varrho}}\mathrm{div}_x\,\mathbb{K}(\widetilde{\varrho},\nabla_x\widetilde{\mathbf{v}})\right]\mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_\Omega\frac{1}{\varrho}\big[(\mathbf{J}-\varrho\widetilde{\mathbf{u}})\otimes(\mathbf{J}-\varrho\widetilde{\mathbf{u}})\big]:\nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_\Omega[p(\varrho)-p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-p(\widetilde{\varrho})]\,\mathrm{div}_x\,\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_\Omega p'(\widetilde{\varrho})\left(\frac{\varrho}{\widetilde{\varrho}}-1\right)[\partial_t\widetilde{\varrho}+\mathrm{div}_x(\widetilde{\varrho}\widetilde{\mathbf{u}})]\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\hbar\int_0^\tau\int_\Omega\varrho(\mathbf{v}-\widetilde{\mathbf{v}})\cdot\left[\partial_t\widetilde{\mathbf{v}}+\nabla_x\widetilde{\mathbf{v}}\cdot\widetilde{\mathbf{u}}+\frac{1}{\widetilde{\varrho}}\mathrm{div}_x\left(\widetilde{\varrho}\nabla_x^\top\widetilde{\mathbf{u}}\right)\right]\mathrm{d}x\mathrm{d}t\\
&\qquad-\hbar\int_0^\tau\int_\Omega\varrho\big[(\mathbf{v}-\widetilde{\mathbf{v}})\otimes(\mathbf{v}-\widetilde{\mathbf{v}})\big]:\nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\widetilde{\mathbf{u}}:\mathrm{d}\mathfrak{R}\ \mathrm{d}t.
\end{aligned}$$

If we suppose that the couple $[\widetilde{\varrho},\widetilde{\mathbf{u}}]$ is a strong solution then the previous inequality simplifies as

$$\begin{aligned}
&\left[\int_\Omega E(\varrho,\mathbf{J},\mathbf{v}\mid\widetilde{\varrho},\widetilde{\mathbf{u}},\widetilde{\mathbf{v}})(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}+\frac{1}{\lambda}\int_{\overline{\Omega}}\mathrm{d}\,\mathrm{Tr}[\mathfrak{R}](\tau)\\
&\qquad-\int_0^\tau\int_\Omega\frac{1}{\varrho}\big[(\mathbf{J}-\varrho\widetilde{\mathbf{u}})\otimes(\mathbf{J}-\varrho\widetilde{\mathbf{u}})\big]:\nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_\Omega[p(\varrho)-p'(\widetilde{\varrho})(\varrho-\widetilde{\varrho})-p(\widetilde{\varrho})]\,\mathrm{div}_x\,\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t,\\
&\qquad-\hbar\int_0^\tau\int_\Omega\varrho\big[(\mathbf{v}-\widetilde{\mathbf{v}})\otimes(\mathbf{v}-\widetilde{\mathbf{v}})\big]:\nabla_x\widetilde{\mathbf{u}}\ \mathrm{d}x\mathrm{d}t\\
&\qquad-\int_0^\tau\int_{\overline{\Omega}}\nabla_x\widetilde{\mathbf{u}}:\mathrm{d}\mathfrak{R}\ \mathrm{d}t,
\end{aligned}$$

and therefore it is enough to proceed as before.

4 Existence

In this section, we aim to prove the existence of dissipative solutions for both the quantum Navier–Stokes and quantum Euler systems. More precisely, we will focus on the following two results.

Theorem 4.1 (Existence of dissipative solutions for the quatum Navier–Stokes system). *For any arbitrarily large $T > 0$ and any fixed initial data*

$$[\varrho_0, \mathbf{J}_0, E_0] \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d) \times [0,\infty)$$

such that

$$\varrho_0 > 0, \qquad \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_0} + P(\varrho_0) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_0}|^2 \right] \mathrm{d}x \le E_0,$$

the quantum Navier–Stokes system (1.2)–(1.3) *with constitutive relations* (1.6)–(1.7) *and boundary conditions* (1.8) *admits a dissipative solution in the sense of Definition 2.1.*

Theorem 4.2 (Existence of dissipative solutions for the quantum Euler system). *For any arbitrarily large $T > 0$ and any fixed initial data*

$$[\varrho_0, \mathbf{J}_0, E_0] \in L^\gamma(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}(\Omega;\mathbb{R}^d) \times [0,\infty)$$

such that

$$\varrho_0 > 0, \qquad \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_0} + P(\varrho_0) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_0}|^2 \right] \mathrm{d}x \le E_0,$$

the quantum Euler system (1.4)–(1.5) *with the isentropic pressure* (1.6) *and boundary conditions* (1.9) *admits a dissipative solution in the sense of Definition 2.2.*

The first theorem will be proved by employing a two-level approximation scheme based on addition of artificial viscosity terms, in order to convert the hyperbolic system into a parabolic one, and approximation via the Faedo-Galerkin technique. The second theorem will be obtained by letting the viscosity to go to zero in the quantum Navier–Stokes equations.

4.1 Proof of Theorem 4.1

From now one, let the time $T > 0$ be fixed arbitrarily large. We start by choosing a family $\{X_n\}_{n\in\mathbb{N}}$ of finite-dimensional spaces $X_n \subset L^2(\Omega;\mathbb{R}^d)$, such that

$$X_n := \mathrm{span}\{\mathbf{w}_i | \ \mathbf{w}_i \in C_c^\infty(\Omega;\mathbb{R}^d), \ i = 1, \dots, n\},$$

where $\mathbf{w}_i$ are orthonormal with respect to the standard scalar product in $L^2(\Omega;\mathbb{R}^d)$. Now, for each $\varepsilon > 0$ and $n \in \mathbb{N}$ fixed, we consider the following system

$$\partial_t \varrho_{\varepsilon,n} + \mathrm{div}_x(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}) = \varepsilon \Delta_x \varrho_{\varepsilon,n}, \tag{4.1}$$

$$\begin{aligned} \partial_t(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}) + \mathrm{div}_x(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n} \otimes \mathbf{u}_{\varepsilon,n}) + \nabla_x p(\varrho_{\varepsilon,n}) + \varepsilon \nabla_x \mathbf{u}_{\varepsilon,n} \cdot \nabla_x \varrho_{\varepsilon,n} \\ = \mathrm{div}_x \left(\mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) + \mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) \right) \end{aligned} \tag{4.2}$$

on $(0,T) \times \Omega$, where we look for approximated velocities

$$\mathbf{u}_{\varepsilon,n} \in C([0,T]; X_n).$$

Moreover, we impose the homogeneous Neumann and no-slip boundary conditions for the density and velocity, respectively

$$\nabla_x \varrho_{\varepsilon,n} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbf{u}_{\varepsilon,n}|_{\partial\Omega} = 0, \tag{4.3}$$

and we fix the initial conditions

$$\varrho_{\varepsilon,n}(0,\cdot) = \varrho_{0,n}, \quad (\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n})(0,\cdot) = \mathbf{J}_0 \quad \text{on } \Omega,$$

where the initial densities $\{\varrho_{0,n}\}_{n\in\mathbb{N}} \subset W^{1,2}(\Omega)$, $0 < \underline{\varrho}_n \leq \varrho_{0,n} \leq \overline{\varrho}_n < \infty$, are chosen in such a way that

$$\varrho_{0,n} \to \varrho_0 \text{ in } L^1(\Omega) \text{ as } n \to \infty,$$

with the couple $(\varrho_0, \mathbf{J}_0)$ as in the hypotheses of Theorem 4.1. Solvability of the approximated problem will be discussed in the following sections.

4.1.1 On the approximated continuity equation

For any fixed $\varepsilon > 0$, $n \in \mathbb{N}$ and given $\mathbf{u}_{\varepsilon,n} \in C([0,T]; X_n)$, let us focus on finding that unique weak solution $\varrho_{\varepsilon,n} = \varrho[\mathbf{u}_{\varepsilon,n}]$ of equation (4.1). Before stating the existence result for the approximated continuity equation, notice that since X_n is finite-dimensional, all the norms on X_n induced by $W^{k,p}$-norms, with $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, are equivalent; thus, we deduce that

$$\mathbf{u}_{\varepsilon,n} \in L^\infty(0,T; W^{1,\infty}(\Omega; \mathbb{R}^d)),$$

and there exist two constants $0 < \underline{n} < \overline{n} < \infty$, depending solely on the dimension n of X_n, such that for any $t \in [0,T]$

$$\underline{n}\|\mathbf{u}_{\varepsilon,n}(t,\cdot)\|_{W^{1,\infty}(\Omega)} \leq \|\mathbf{u}_{\varepsilon,n}(t,\cdot)\|_{X_n} \leq \overline{n}\|\mathbf{u}_{\varepsilon,n}(t,\cdot)\|_{W^{1,\infty}(\Omega)}. \tag{4.4}$$

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain of class C^2 and let $\varepsilon > 0$, $n \in \mathbb{N}$ be fixed. For any given $\mathbf{u}_{\varepsilon,n} \in C([0,T]; X_n)$, there exists a unique solution*

$$\varrho_{\varepsilon,n} \in L^2((0,T); W^{2,2}(\Omega)) \cap W^{1,2}(0,T; L^2(\Omega))$$

of equation (4.1) *with $\varrho_{\varepsilon,n}(0,\cdot) = \varrho_{0,n}$. Moreover,*

(i) (bound from above - maximum principle) the weak solution $\varrho_{\varepsilon,n}$ satisfies

$$\|\varrho_{\varepsilon,n}\|_{L^\infty((0,\tau)\times\Omega)} \leq \overline{\varrho}_n \exp\left(\tau\|\operatorname{div}_x \mathbf{u}_{\varepsilon,n}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{4.5}$$

for any $\tau \in [0,T]$, with

$$\overline{\varrho}_n := \max_\Omega \varrho_{0,n}; \tag{4.6}$$

(ii) (bound from below) the weak solution $\varrho_{\varepsilon,n}$ satisfies

$$\operatorname*{ess\,inf}_{(0,\tau)\times\Omega} \varrho_{\varepsilon,n}(t,x) \geq \underline{\varrho}_n \exp\left(-\tau\|\operatorname{div}_x \mathbf{u}_{\varepsilon,n}\|_{L^\infty((0,T)\times\Omega)}\right), \tag{4.7}$$

for any $\tau \in [0,T]$, with

$$\underline{\varrho}_n := \min_\Omega \varrho_{0,n}; \tag{4.8}$$

(iii) let $\mathbf{u}_1, \mathbf{u}_2 \in C([0,T]; X_n)$ *be such that*

$$\max_{i=1,2} \|\mathbf{u}_i\|_{L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d))} \leq K$$

with $K \in (0,\infty)$*, and let* $\varrho_i = \varrho[\mathbf{u}_i]$*,* $i = 1,2$ *be the weak solutions of the approximated continuity equation* (4.1)*sharing the same initial data* $\varrho_{0,n}$*. Then, for any* $\tau \in [0,T]$*,*

$$\|(\varrho_1 - \varrho_2)(\tau,\cdot)\|_{L^2(\Omega)} \leq c_1 \|\mathbf{u}_1 - \mathbf{u}_2\|_{L^\infty(0,\tau;W^{1,\infty}(\Omega;\mathbb{R}^d))} \tag{4.9}$$

with $c_1 = c_1(\varepsilon, \varrho_0, T, K)$*.*

Proof. The proof is a straightforward consequence of Lemma 4.3 in [14]. □

4.1.2 On the approximated balance of momentum

Let us now turn our attention to the approximated balance of momentum (4.2). The approximate velocities $\mathbf{u}_{\varepsilon,n} \in C([0,T]; X_n)$ are looked for to satisfy the integral identity

$$\begin{aligned}
\left[\int_\Omega \varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}(t,\cdot)\cdot\boldsymbol{\psi}\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega \left[(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\otimes\mathbf{u}_{\varepsilon,n}) : \nabla_x\boldsymbol{\psi} + p(\varrho_{\varepsilon,n})\,\mathrm{div}_x\,\boldsymbol{\psi}\right]\mathrm{d}x\mathrm{d}t \\
&+ \frac{\hbar}{4}\int_0^\tau\int_\Omega \left[\nabla_x\varrho_{\varepsilon,n}\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\psi} + 4(\nabla_x\sqrt{\varrho_{\varepsilon,n}}\otimes\nabla_x\sqrt{\varrho_{\varepsilon,n}}) : \nabla_x\boldsymbol{\psi}\right]\ \mathrm{d}x\mathrm{d}t \\
&- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_{\varepsilon,n}) : \nabla_x\boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega \nabla_x\varrho_{\varepsilon,n}\cdot\nabla_x\mathbf{u}_{\varepsilon,n}\cdot\boldsymbol{\psi}\ \mathrm{d}x\mathrm{d}t
\end{aligned} \tag{4.10}$$

for any test function $\boldsymbol{\psi} \in X_n$ and all $\tau \in [0,T]$, with $(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n})(0,\cdot) = \mathbf{J}_0$. Now, the integral identity (4.10) can be rephrased for any $\tau \in [0,T]$ as

$$\langle \mathcal{M}[\varrho_{\varepsilon,n}(\tau,\cdot)](\mathbf{u}_{\varepsilon,n}(\tau,\cdot)), \boldsymbol{\psi}\rangle = \langle \mathbf{J}_0^*, \boldsymbol{\psi}\rangle + \langle \int_0^\tau \mathcal{N}[\varrho_{\varepsilon,n}(s,\cdot), \mathbf{u}_{\varepsilon,n}(s,\cdot)]\ \mathrm{d}s, \boldsymbol{\psi}\rangle$$

with

$$\begin{aligned}
\mathcal{M}[\varrho] : X_n \to X_n^*, \quad & \langle \mathcal{M}[\varrho]\mathbf{v}, \mathbf{w}\rangle := \int_\Omega \varrho\mathbf{v}\cdot\mathbf{w}\ \mathrm{d}x, \\
\mathbf{J}_0^* \in X_n^*, \quad & \langle \mathbf{J}_0^*, \boldsymbol{\psi}\rangle := \int_\Omega \mathbf{J}_0\cdot\boldsymbol{\psi}\ \mathrm{d}x, \\
\mathcal{N}[\varrho,\mathbf{u}] \in X_n^*, \quad & \langle \mathcal{N}[\varrho,\mathbf{u}], \boldsymbol{\psi}\rangle := \int_\Omega \left[(\varrho\mathbf{u}\otimes\mathbf{u}) : \nabla_x\boldsymbol{\psi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\psi}\right]\mathrm{d}x \\
& \qquad + \frac{\hbar}{4}\int_\Omega \left[\nabla_x\varrho\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\psi} + 4(\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}) : \nabla_x\boldsymbol{\psi}\right]\ \mathrm{d}x \\
& \qquad - \int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\psi}\ \mathrm{d}x - \varepsilon\int_\Omega \nabla_x\varrho\cdot\nabla_x\mathbf{u}\cdot\boldsymbol{\psi}\ \mathrm{d}x.
\end{aligned}$$

We are now ready to apply the following lemma.

Lemma 4.4. *Let*

$$\mathcal{B}(0, \underline{n}K) := \left\{\mathbf{v} \in C([0,T(n)]; X_n) \Big| \sup_{t\in[0,T(n)]} \|\mathbf{v}(t,\cdot)\|_{X_n} \leq \underline{n}K\right\},$$

with $\underline{n}$ defined as in (4.4)*. For $K > 0$ sufficiently large and $T(n)$ sufficiently small, the map*

$$\mathcal{F} : \mathcal{B}(0, \underline{n}K) \to C([0, T(n)]; X_n)$$

such that

$$\mathcal{F}[\mathbf{u}_{\varepsilon,n}](\tau, \cdot) := \mathcal{M}^{-1}[\varrho_{\varepsilon,n}(\tau, \cdot)] \left(\mathbf{J}_0^* + \int_0^\tau \mathcal{N}[\varrho_{\varepsilon,n}(s, \cdot), \mathbf{u}_{\varepsilon,n}(s, \cdot)] \, \mathrm{d}s \right),$$

is a contraction mapping from the closed ball $\mathcal{B}(0, \underline{n}K)$ onto itself and therefore it admits a unique fixed point $\mathbf{u}_{\varepsilon,n} \in C([0, T(n)]; X_n)$.

Proof. The lemma is a straightforward consequence of the Banach-Cacciopoli fixed point theorem. Notice in particular that $\varrho_{\varepsilon,n} = \varrho[\mathbf{u}_{\varepsilon,n}]$ is the weak solution of equation (4.1) uniquely determined by $\mathbf{u}_{\varepsilon,n}$ and thus by Lemma 4.3 we can deduce that $0 < \underline{\varrho}_n e^{-Kt} \leq \varrho_{\varepsilon,n}(t, x) \leq \overline{\varrho}_n e^{Kt}$ for any $t \in [0, T(n)]$ whenever $\mathbf{u}_{\varepsilon,n} \in \mathcal{B}(0, \underline{n}K)$, where $\overline{\varrho}_n$, $\underline{\varrho}_n$ are defined as in (4.6), (4.8) respectively. Therefore, the operator $\mathcal{M}$ is invertible and from (4.9) it is also easy to show that $\mathcal{F}$ is a contraction mapping, see e.g. [14], Section 4.3.2 for more details. □

So far, we have found the velocity $\mathbf{u}_{\varepsilon,n}$ solving the integral identity (4.10) on the time interval $[0, T(n)]$. However, the previous procedure can be repeated a finite number of times until we reach $T = T(n)$, as long as we have a bound on $\mathbf{u}_{\varepsilon,n}$ independent of $T(n)$; in other words, we need some *energy estimates*. We have that

$$\begin{aligned}
\int_\Omega \partial_t(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}) \cdot \boldsymbol{\psi} \, \mathrm{d}x &= \int_\Omega [(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n} \otimes \mathbf{u}_{\varepsilon,n}) : \nabla_x \boldsymbol{\psi} + p(\varrho_{\varepsilon,n}) \operatorname{div}_x \boldsymbol{\psi}] \, \mathrm{d}x \\
&\quad - \int_\Omega \left[\mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) + \mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) \right] : \nabla_x \boldsymbol{\psi} \, \mathrm{d}x \\
&\quad - \varepsilon \int_\Omega \nabla_x \varrho_{\varepsilon,n} \cdot \nabla_x \mathbf{u}_{\varepsilon,n} \cdot \boldsymbol{\psi} \, \mathrm{d}x
\end{aligned} \tag{4.11}$$

holds on $(0, T(n))$ for any $\boldsymbol{\psi} \in X_n$, with $\varrho_{\varepsilon,n} = \varrho[\mathbf{u}_{\varepsilon,n}]$. We can then take $\boldsymbol{\psi} = \mathbf{u}_{\varepsilon,n}(t, \cdot)$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho_{\varepsilon,n} |\mathbf{u}_{\varepsilon,n}|^2 \, \mathrm{d}x = &- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x + \int_\Omega p(\varrho_{\varepsilon,n}) \operatorname{div}_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x \\
&- \int_\Omega \mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x \\
&- \int_\Omega \frac{1}{2} |\mathbf{u}_{\varepsilon,n}|^2 \big(\partial_t \varrho_{\varepsilon,n} + \operatorname{div}_x(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}) - \varepsilon \Delta_x \varrho_{\varepsilon,n} \big) \mathrm{d}x,
\end{aligned}$$

where the last line vanishes due to (4.1). Multiplying (4.1) by $P'(\varrho)$ we recover that in this context the pressure potential $P = P(\varrho)$ satisfies the following identity

$$p(\varrho_{\varepsilon,n}) \operatorname{div}_x \mathbf{u}_{\varepsilon,n} = -\partial_t P(\varrho_{\varepsilon,n}) - \operatorname{div}_x (P(\varrho_{\varepsilon,n})\mathbf{u}_{\varepsilon,n}) + \varepsilon P'(\varrho_{\varepsilon,n}) \Delta_x \varrho_{\varepsilon,n}.$$

Therefore, the previous integral identity can be rewritten as

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho_{\varepsilon,n} |\mathbf{u}_{\varepsilon,n}|^2 + P(\varrho_{\varepsilon,n}) \right) \, \mathrm{d}x = &- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x - \varepsilon \int_\Omega P''(\varrho_{\varepsilon,n}) |\nabla_x \varrho_{\varepsilon,n}|^2 \, \mathrm{d}x \\
&- \int_\Omega \mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x.
\end{aligned} \tag{4.12}$$

Moreover, we have

$$
\begin{aligned}
-\int_\Omega \mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x &= \int_\Omega \mathrm{div}_x \left[\mathbb{K}(\varrho_{\varepsilon,n}, \nabla_x \varrho_{\varepsilon,n}, \nabla_x^2 \varrho_{\varepsilon,n}) \right] \cdot \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x \\
&= \frac{\hbar}{2} \int_\Omega \varrho_{\varepsilon,n} \nabla_x \left(\frac{\Delta_x \sqrt{\varrho_{\varepsilon,n}}}{\sqrt{\varrho_{\varepsilon,n}}} \right) \cdot \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x = \frac{\hbar}{2} \int_\Omega \frac{\Delta_x \sqrt{\varrho_{\varepsilon,n}}}{\sqrt{\varrho_{\varepsilon,n}}} \, \mathrm{div}_x (\varrho_{\varepsilon,n} \mathbf{u}_{\varepsilon,n}) \, \mathrm{d}x \\
&= \frac{\hbar}{4} \int_\Omega \left(\frac{1}{\varrho_{\varepsilon,n}} \Delta_x \varrho_{\varepsilon,n} - \frac{1}{2 \varrho_{\varepsilon,n}^2} |\nabla_x \varrho_{\varepsilon,n}|^2 \right) \, (\partial_t \varrho_{\varepsilon,n} - \varepsilon \Delta_x \varrho_{\varepsilon,n}) \, \mathrm{d}x \\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\hbar}{2} |\nabla_x \sqrt{\varrho_{\varepsilon,n}}|^2 \, \mathrm{d}x - \varepsilon \frac{\hbar}{4} \int_\Omega \varrho_{\varepsilon,n} \, |\nabla_x^2 (\log \varrho_{\varepsilon,n})|^2 \, \mathrm{d}x,
\end{aligned}
$$

where we used formulas

$$
\begin{aligned}
\Delta_x f(\varrho) &= f'(\varrho) \Delta_x \varrho + f''(\varrho) |\nabla_x \varrho|^2, \\
|\nabla_x^2 f(\varrho)|^2 &= \frac{1}{2} \Delta_x |\nabla_x f(\varrho)|^2 - \nabla_x f(\varrho) \cdot \nabla_x \Delta_x f(\varrho),
\end{aligned}
$$

to write

$$
\begin{aligned}
\int_\Omega \left(\frac{1}{\varrho} \Delta_x \varrho - \frac{1}{2\varrho^2} |\nabla_x \varrho|^2 \right) \Delta_x \varrho \, \mathrm{d}x &= \int_\Omega \Delta_x (\log \varrho) \, \Delta_x \varrho \, \mathrm{d}x + \frac{1}{2} \int_\Omega |\nabla_x (\log \varrho)|^2 \, \Delta_x \varrho \, \mathrm{d}x \\
&= \int_\Omega \varrho \left(-\nabla_x \Delta_x (\log \varrho) \cdot \nabla_x (\log \varrho) + \frac{1}{2} \Delta_x |\nabla_x (\log \varrho)|^2 \right) \, \mathrm{d}x \\
&= \int_\Omega \varrho \, |\nabla_x^2 (\log \varrho)|^2 \, \mathrm{d}x.
\end{aligned}
$$

We have finally obtained

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left(\frac{1}{2} \varrho_{\varepsilon,n} |\mathbf{u}_{\varepsilon,n}|^2 + P(\varrho_{\varepsilon,n}) + \frac{\hbar}{2} |\nabla_x \sqrt{\varrho_{\varepsilon,n}}|^2 \right) \, \mathrm{d}x \\
&\quad + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x \\
&\quad + \varepsilon \int_\Omega \left(P''(\varrho_{\varepsilon,n}) |\varrho_{\varepsilon,n} \mathbf{v}_{\varepsilon,n}|^2 + \frac{\hbar}{4} \, \varrho_{\varepsilon,n} |\nabla_x^2 (\log \varrho_{\varepsilon,n})|^2 \right) \, \mathrm{d}x = 0.
\end{aligned}
$$

Integrating the previous expression over $(0, \tau)$, we get the following energy inequality

$$
\begin{aligned}
&\int_\Omega \left[\frac{1}{2} \varrho_{\varepsilon,n} |\mathbf{u}_{\varepsilon,n}|^2 + P(\varrho_{\varepsilon,n}) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_{\varepsilon,n}}|^2 \right] (\tau, \cdot) \, \mathrm{d}x \\
&\quad + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_{\varepsilon,n}) : \nabla_x \mathbf{u}_{\varepsilon,n} \, \mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon \int_0^\tau \int_\Omega \left(P''(\varrho_{\varepsilon,n}) |\nabla_x \varrho_{\varepsilon,n}|^2 + \frac{\hbar}{4} \varrho_{\varepsilon,n} \, |\nabla_x^2 (\log \varrho_{\varepsilon,n})|^2 \right) \mathrm{d}x\mathrm{d}t \\
&\quad \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_{0,n}}|^2 \right] \mathrm{d}x,
\end{aligned}
\tag{4.13}
$$

for any time $\tau \in [0, T(n)]$. In particular, if we suppose that

$$
\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_{0,n}}|^2 \right] \mathrm{d}x \leq E_0 \tag{4.14}
$$

where the constant E_0 is independent of $n > 0$, the term on the left-hand side of (4.13) is bounded. Consequently, it is not difficult to show that the functions $\mathbf{u}_{\varepsilon,n}(t,\cdot)$ remain bounded in X_n for any t independently of $T(n) \leq T$. Thus we are allowed to iterate the previous local existence result to construct a solution defined on the whole time interval $[0,T]$, see e.g. the last part of Section 7.3.4 in [20] for more details.

Summarizing, so far we proved the following result.

Lemma 4.5. *For every fixed $\varepsilon > 0$, $n \in \mathbb{N}$, and any $\varrho_{0,n} \in W^{1,2}(\Omega)$ such that*

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2}\ |\nabla_x \sqrt{\varrho_{0,n}}|^2\right] \mathrm{d}x \leq E_0,$$

where the constant E_0 is independent of n, there exist

$$\begin{aligned}&\varrho_{\varepsilon,n} \in L^2((0,T);W^{2,2}(\Omega)) \cap W^{1,2}(0,T;L^2(\Omega)),\\ &\mathbf{u}_{\varepsilon,n} \in C([0,T];X_n),\end{aligned}$$

such that

(i) the integral identity

$$\left[\int_\Omega \varrho_{\varepsilon,n}\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau\int_\Omega (\varrho_{\varepsilon,n}\partial_t\varphi + \varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\cdot\nabla_x\varphi - \varepsilon\nabla_x\varrho_{\varepsilon,n}\cdot\nabla_x\varphi)\ \mathrm{d}x\mathrm{d}t \tag{4.15}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho_{\varepsilon,n}(0,\cdot) = \varrho_{0,n}$;

(ii) the integral identity

$$\begin{aligned}&\left[\int_\Omega \varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau}\\ &= \int_0^\tau\int_\Omega [\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\cdot\partial_t\boldsymbol{\varphi} + (\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\otimes\mathbf{u}_{\varepsilon,n}) : \nabla_x\boldsymbol{\varphi} + p(\varrho_{\varepsilon,n})\,\mathrm{div}_x\,\boldsymbol{\varphi}]\ \mathrm{d}x\mathrm{d}t\\ &+ \frac{\hbar}{4}\int_0^\tau\int_\Omega \left[\nabla_x\varrho_{\varepsilon,n}\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\varphi} + 4(\nabla_x\sqrt{\varrho_{\varepsilon,n}}\otimes\nabla_x\sqrt{\varrho_{\varepsilon,n}}) : \nabla_x\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t\\ &- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_{\varepsilon,n}) : \nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t - \varepsilon\int_0^\tau\int_\Omega \nabla_x\varrho_{\varepsilon,n}\cdot\nabla_x\mathbf{u}_{\varepsilon,n}\cdot\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t\end{aligned} \tag{4.16}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T];X_n)$, with $(\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n})(0,\cdot) = \mathbf{J}_0$;

(iii) the integral inequality

$$\begin{aligned}&\int_\Omega \left[\frac{1}{2}\varrho_{\varepsilon,n}|\mathbf{u}_{\varepsilon,n}|^2 + P(\varrho_{\varepsilon,n}) + \frac{\hbar}{2}\ |\nabla_x\sqrt{\varrho_{\varepsilon,n}}|^2\right](\tau,\cdot)\ \mathrm{d}x\\ &+ \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_{\varepsilon,n}) : \nabla_x\mathbf{u}_{\varepsilon,n}\ \mathrm{d}x\mathrm{d}t\\ &+ \varepsilon\int_0^\tau\int_\Omega \left(P''(\varrho_{\varepsilon,n})|\nabla_x\varrho_{\varepsilon,n}|^2 + \frac{\hbar}{4}\varrho_{\varepsilon,n}\ |\nabla_x^2(\log\varrho_{\varepsilon,n})|^2\right)\mathrm{d}x\mathrm{d}t\\ &\leq \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2}\ |\nabla_x\sqrt{\varrho_{0,n}}|^2\right]\mathrm{d}x,\end{aligned} \tag{4.17}$$

holds for any time $\tau \in [0,T]$.

4.1.3 Limit $\varepsilon \to 0$

In order to perform the limit $\varepsilon \to 0$, we need the following result.

Lemma 4.6. *Let $n \in \mathbb{N}$ be fixed and let $\{\varrho_{\varepsilon,n}, \mathbf{u}_{\varepsilon,n}\}_{\varepsilon>0}$ be as in Lemma 4.5. Then, passing to suitable subsequences as the case may be, the following convergences hold as $\varepsilon \to 0$.*

$$\varrho_{\varepsilon,n} \overset{*}{\rightharpoonup} \varrho_n \quad \text{in } L^\infty((0,T)\times\Omega) \tag{4.18}$$

$$\mathbf{u}_{\varepsilon,n} \overset{*}{\rightharpoonup} \mathbf{u}_n \quad \text{in } L^\infty(0,T;W^{1,\infty}(\Omega;\mathbb{R}^d)), \tag{4.19}$$

$$\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n} \overset{*}{\rightharpoonup} \varrho_n\mathbf{u}_n \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d), \tag{4.20}$$

$$\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n}\otimes\mathbf{u}_{\varepsilon,n} \overset{*}{\rightharpoonup} \varrho_n\mathbf{u}_n\otimes\mathbf{u}_n \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^{d\times d}), \tag{4.21}$$

$$\nabla_x\varrho_{\varepsilon,n} \overset{*}{\rightharpoonup} \nabla_x\varrho_n \quad \text{in } L^\infty(0,T;L^2(\Omega)), \tag{4.22}$$

$$p(\varrho_{\varepsilon,n}) \overset{*}{\rightharpoonup} \overline{p(\varrho_n)} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})), \tag{4.23}$$

$$\nabla_x\sqrt{\varrho_{\varepsilon,n}}\otimes\nabla_x\sqrt{\varrho_{\varepsilon,n}} \overset{*}{\rightharpoonup} \overline{\nabla_x\sqrt{\varrho_n}\otimes\nabla_x\sqrt{\varrho_n}} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d})), \tag{4.24}$$

$$\sqrt{\varepsilon}\ \nabla_x\varrho_{\varepsilon,n} \rightharpoonup \sqrt{\varepsilon}\ \nabla_x\varrho_n \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d), \tag{4.25}$$

$$\sqrt{\varepsilon}\ \nabla_x\varrho_{\varepsilon,n}\cdot\nabla_x\mathbf{u}_{\varepsilon,n} \rightharpoonup \sqrt{\varepsilon}\ \overline{\nabla_x\varrho_n\cdot\nabla_x\mathbf{u}}_n \quad \text{in } L^2((0,T)\times\Omega;\mathbb{R}^d). \tag{4.26}$$

Proof. From (4.17) it is easy to deduce the following uniform bounds

$$\|P(\varrho_{\varepsilon,n})\|_{L^\infty(0,T;L^1(\Omega))} \le c(\overline{E}) \tag{4.27}$$

$$\|\nabla_x\sqrt{\varrho_{\varepsilon,n}}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le c(\overline{E}) \tag{4.28}$$

$$\|\nabla_x\mathbf{u}_{\varepsilon,n}\|_{L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})} \le c(\overline{E}). \tag{4.29}$$

Estimate (4.29) combined with the Poincaré inequality provides

$$\|\mathbf{u}_{\varepsilon,n}\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^d))} \le c_1$$

for some positive constant c_1 independent of $\varepsilon > 0$. Applying Lemma 4.3, we get

$$e^{-c_1T}\underline{\varrho}_n \le \varrho_{\varepsilon,n}(t,x) \le e^{c_1T}\overline{\varrho}_n, \quad \text{for all } (t,x)\in[0,T]\times\overline{\Omega}. \tag{4.30}$$

which yields to convergence (4.18). From the fact that $\mathbf{u}_{\varepsilon,n}$ belongs to $C([0,T];X_n)$, it is easy to deduce

$$\sup_{t\in[0,T]} \|\mathbf{u}_{\varepsilon,n}(t,\cdot)\|_{W^{1,\infty}(\Omega;\mathbb{R}^d)} \le c_2, \tag{4.31}$$

from which convergence (4.19) follows. Combining (4.30) and (4.31), we can recover

$$\varrho_{\varepsilon,n}\mathbf{u}_{\varepsilon,n} \overset{*}{\rightharpoonup} \overline{\varrho_n\mathbf{u}_n} \quad \text{in } L^\infty((0,T)\times\Omega;\mathbb{R}^d).$$

Now, notice that (4.18) can be strengthened to

$$\varrho_{\varepsilon,n} \to \varrho_n \quad \text{in } C_{\text{weak}}([0,T];L^p(\Omega)) \quad \text{for all } 1<p<\infty$$

as $\varepsilon \to 0$, so that, relaying on the compact Sobolev embedding

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-1,1}(\Omega) \quad \text{for all } p \geq 1,$$

we obtain

$$\varrho_{\varepsilon,n} \to \varrho_n \quad \text{in } C([0,T]; W^{-1,1}(\Omega))$$

as $\varepsilon \to 0$. The last convergence combined with (4.19), implies

$$\overline{\varrho_n \mathbf{u}_n} = \varrho_n \mathbf{u}_n \quad \text{a.e. in } (0,T) \times \Omega,$$

and thus, we get (4.20). Similarly, from (4.19) and (4.20) we can deduce (4.21). Noticing that

$$\nabla_x \varrho_{\varepsilon,n} = 2\sqrt{\varrho_{\varepsilon,n}}\, \nabla_x \sqrt{\varrho_{\varepsilon,n}}, \tag{4.32}$$

convergence (4.22) can be deduced combining (4.28) and (4.30). From (4.27) and (4.28), we can deduce that the sequences $\{p(\varrho_{\varepsilon,n})\}_{\varepsilon>0}$, $\{\nabla_x \sqrt{\varrho_{\varepsilon,n}} \otimes \nabla_x \sqrt{\varrho_{\varepsilon,n}}\}_{\varepsilon>0}$ are uniformly bounded in $L^\infty(0,T; L^1(\Omega))$. However, since the L^1-space cannot be identified as the dual space of any separable space and therefore it is not possible to apply the Banach-Alaoglu theorem, a suitable idea consists in the embedding of $L^1(\Omega)$ in the space of measures $\mathcal{M}(\overline{\Omega})$, which, on the contrary, is the dual space of $C(\overline{\Omega})$; we get convergences (4.23), (4.24). Finally, from (4.30) and the energy inequality (4.17), we have

$$\varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_{\varepsilon,n}|^2 \,\mathrm{d}x\mathrm{d}t \leq \varepsilon\, c(\underline{\varrho}_n) \int_0^\tau \int_\Omega P''(\varrho_{\varepsilon,n}) |\nabla_x \varrho_{\varepsilon,n}|^2 \,\mathrm{d}x\mathrm{d}t \leq c(\underline{\varrho}_n, T).$$

In this way we get (4.25) and, in view of (4.31), (4.26). $\square$

We are now ready to let $\varepsilon \to 0$ in the weak formulations (4.15), (4.16); notice in particular that, in view of (4.25) and (4.26), for any $\tau \in [0,T]$, any $\varphi \in C^1([0,T] \times \overline{\Omega})$ and any $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$

$$\varepsilon \int_0^\tau \int_\Omega \nabla_x \varrho_{\varepsilon,n} \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t = \sqrt{\varepsilon} \int_0^\tau \int_\Omega \sqrt{\varepsilon}\, \nabla_x \varrho_{\varepsilon,n} \cdot \nabla_x \varphi \,\mathrm{d}x\mathrm{d}t \to 0,$$
$$\varepsilon \int_0^\tau \int_\Omega \nabla_x \varrho_{\varepsilon,n} \cdot \nabla_x \mathbf{u}_{\varepsilon,n} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t = \sqrt{\varepsilon} \int_0^\tau \int_\Omega \sqrt{\varepsilon}\, \nabla_x \varrho_{\varepsilon,n} \cdot \nabla_x \mathbf{u}_{\varepsilon,n} \cdot \boldsymbol{\varphi} \,\mathrm{d}x\mathrm{d}t \to 0$$

as $\varepsilon \to 0$. We therefore obtain that the weak formulations of the continuity equation (2.5) and balance of momentum (1.3) hold for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T] \times \overline{\Omega})$, $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$, respectively, with the Reynolds stresses

$$\mathfrak{R}_n \in L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

such that

$$\mathrm{d}\mathfrak{R}_n := (\overline{p(\varrho_n)} - p(\varrho_n))\mathbb{I}\,\mathrm{d}x + \hbar\, (\overline{\nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}} - \nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n})\,\mathrm{d}x.$$

We claim that $\mathfrak{R}_n$ are positive measure, meaning that

$$\mathfrak{R}_n \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\mathrm{sym}})); \tag{4.33}$$

more precisely, we have to show that for any $\boldsymbol{\xi} \in \mathbb{R}^d$ and any bounded open set $\mathcal{B} \subset \Omega$

$$\mathfrak{R}_n : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \geq 0 \quad \text{in } \mathcal{D}'((0,T) \times \mathcal{B}).$$

We have

$$\mathfrak{R}_n : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) = (\overline{p(\varrho_n)} - p(\varrho_n))|\boldsymbol{\xi}|^2 + \hbar \, (\overline{\nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}} - \nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi});$$

on the one hand, the first term on the right-hand side is non-negative due to the convexity of the function $\varrho \mapsto p(\varrho)$ and therefore $p(\varrho_n) \leq \overline{p(\varrho_n)}$ (see e.g. [21], Theorem 2.1.1); on the other hand, we have

$$\begin{aligned}
&(\overline{\nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}} - \nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \\
&\quad = \lim_{\varepsilon \to 0} \left[(\nabla_x \sqrt{\varrho_{\varepsilon,n}} \otimes \nabla_x \sqrt{\varrho_{\varepsilon,n}}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \right] - (\nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}) : (\boldsymbol{\xi} \otimes \boldsymbol{\xi}) \\
&\quad = \lim_{\varepsilon \to 0} |\nabla_x \sqrt{\varrho_{\varepsilon,n}} \cdot \boldsymbol{\xi}|^2 - |\nabla_x \sqrt{\varrho_n} \cdot \boldsymbol{\xi}|^2 = \overline{|\nabla_x \sqrt{\varrho_n} \cdot \boldsymbol{\xi}|^2} - |\nabla_x \sqrt{\varrho_n} \cdot \boldsymbol{\xi}|^2
\end{aligned}$$

in $\mathcal{D}'((0,T) \times \mathcal{B})$, where it is interesting to notice that the derivatives of the function $\varrho \mapsto f_{\boldsymbol{\xi}}(\varrho) = |\nabla_x \sqrt{\varrho} \cdot \boldsymbol{\xi}|^2$ are such that

$$f_{\boldsymbol{\xi}}^{(k)}(\varrho) = (-1)^k \, \frac{k!}{\varrho^k} \, f_{\boldsymbol{\xi}}(\varrho) \quad \text{for any } k \in \mathbb{N};$$

in particular, $f_{\boldsymbol{\xi}}$ is convex for any fixed $\boldsymbol{\xi} \in \mathbb{R}^d$ and therefore $f_{\boldsymbol{\xi}}(\varrho_n) \leq \overline{f_{\boldsymbol{\xi}}(\varrho_n)}$. We get (4.33).

Similarly, we can pass to the limit in (4.17) to get

$$\begin{aligned}
\int_\Omega \left[\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + P(\varrho_n) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_n}|^2 \right] (\tau, \cdot) \, \mathrm{d}x& \\
+ \int_{\overline{\Omega}} \mathrm{d}\mathfrak{E}_n(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \, \mathrm{d}x\mathrm{d}t& \\
\leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_{0,n}}|^2 \right] \mathrm{d}x,&
\end{aligned} \tag{4.34}$$

with

$$\mathfrak{E}_n \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}))$$

such that

$$\mathrm{d}\mathfrak{E}_n = \left(\overline{P(\varrho_n)} - P(\varrho_n) \right) \mathrm{d}x + \frac{\hbar}{2} \left(\overline{|\nabla_x \sqrt{\varrho_n}|^2} - |\nabla_x \sqrt{\varrho_n}|^2 \right) \mathrm{d}x.$$

Furthermore, introducing $\lambda = \lambda(d, \gamma) = \max\{d(\gamma - 1), 2\}$, we obtain

$$\begin{aligned}
\mathrm{Tr}[\mathfrak{R}_n] &= d \left(\overline{p(\varrho_n)} - p(\varrho_n) \right) + \hbar \left(\lim_{\varepsilon \to 0} \mathrm{Tr}[\nabla_x \sqrt{\varrho_{\varepsilon,n}} \otimes \nabla_x \sqrt{\varrho_{\varepsilon,n}}] - \mathrm{Tr}[\nabla_x \sqrt{\varrho_n} \otimes \nabla_x \sqrt{\varrho_n}] \right) \\
&= d(\gamma - 1) \left(\overline{P(\varrho_n)} - P(\varrho_n) \right) + \hbar \left(\overline{|\nabla_x \sqrt{\varrho_n}|^2} - |\nabla_x \sqrt{\varrho_n}|^2 \right) \leq \lambda \mathfrak{E}_n
\end{aligned}$$

and therefore, the energy inequality (4.34) will still hold replacing $\mathfrak{E}_n$ with $\lambda^{-1} \, \mathrm{Tr}[\mathfrak{R}_n]$.

Lemma 4.7. *For every fixed $n \in \mathbb{N}$, and any $\varrho_{0,n} \in C(\overline{\Omega})$ such that*

$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2}\ |\nabla_x\sqrt{\varrho_{0,n}}|^2\right] \mathrm{d}x \le E_0,$$

where the constant E_0 is independent of n, there exist

$$\varrho_n \in L^\infty(0,T;L^\infty(\Omega)),$$
$$\mathbf{u}_n \in C([0,T];X_n),$$

with

$$e^{-cT}\underline{\varrho}_n \le \varrho_n(t,x) \le e^{cT}\overline{\varrho}_n, \quad \textit{for all } (t,x) \in [0,T]\times\overline{\Omega},$$

for a positive constant c, such that

(i) the integral identity

$$\left[\int_\Omega \varrho_n\varphi(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_\Omega (\varrho_n\partial_t\varphi + \varrho_n\mathbf{u}_n\cdot\nabla_x\varphi)\ \mathrm{d}x \tag{4.35}$$

holds for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T]\times\overline{\Omega})$, with $\varrho_n(0,\cdot) = \varrho_{0,n}$;

(ii) there exists

$$\mathfrak{R}_n \in L^\infty(0,T;\mathcal{M}^+(\overline{\Omega};\mathbb{R}^{d\times d}_{\mathrm{sym}}))$$

such that the integral identity

$$\begin{aligned}\left[\int_\Omega \varrho_n\mathbf{u}_n\cdot\boldsymbol{\varphi}(t,\cdot)\ \mathrm{d}x\right]_{t=0}^{t=\tau} &= \int_0^\tau\int_\Omega [\varrho_n\mathbf{u}_n\cdot\partial_t\boldsymbol{\varphi} + (\varrho_n\mathbf{u}_n\otimes\mathbf{u}_n):\nabla_x\boldsymbol{\varphi} + p(\varrho_n)\,\mathrm{div}_x\,\boldsymbol{\varphi}]\,\mathrm{d}x\mathrm{d}t \\ &+ \frac{\hbar}{4}\int_0^\tau\int_\Omega \left[\nabla_x\varrho_n\cdot\mathrm{div}_x\,\nabla_x^\top\boldsymbol{\varphi} + 4(\nabla_x\sqrt{\varrho_n}\otimes\nabla_x\sqrt{\varrho_n}):\nabla_x\boldsymbol{\varphi}\right]\mathrm{d}x\mathrm{d}t \\ &- \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_n):\nabla_x\boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t + \int_0^\tau\int_{\overline{\Omega}} \nabla_x\boldsymbol{\varphi}:\mathrm{d}\mathfrak{R}_n\ \mathrm{d}t\end{aligned} \tag{4.36}$$

holds for any $\tau \in [0,T]$ and any $\boldsymbol{\varphi} \in C^1([0,T];X_n)$, with $(\varrho_n\mathbf{u}_n)(0,\cdot) = \mathbf{J}_0$;

(iii) there exists a positive constant $\lambda = \lambda(d,\gamma)$ such that the integral inequality

$$\begin{aligned}\int_\Omega &\left[\frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + P(\varrho_n) + \frac{\hbar}{2}\ |\nabla_x\sqrt{\varrho_n}|^2\right](\tau,\cdot)\ \mathrm{d}x \\ &+ \frac{1}{\lambda}\int_{\overline{\Omega}} \mathrm{d}\,\mathrm{Tr}[\mathfrak{R}_n](\tau) + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}_n):\nabla_x\mathbf{u}_n\ \mathrm{d}x\mathrm{d}t \\ &\le \int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{\varrho_{0,n}} + P(\varrho_{0,n}) + \frac{\hbar}{2}\ |\nabla_x\sqrt{\varrho_{0,n}}|^2\right] \mathrm{d}x,\end{aligned} \tag{4.37}$$

holds for a.e. $\tau \in (0,T)$.

4.1.4 Limit $n \to \infty$

In order to perform the last limit, we need the following result.

Lemma 4.8. *Let $\{\varrho_n, \mathbf{u}_n, \mathfrak{R}_n\}_{n\in\mathbb{N}}$ be as in Lemma 4.7. Then, passing to suitable subsequences as the case may be, the following convergences hold as $n \to \infty$.*

$$\varrho_n \to \varrho \quad \text{in } C_{\text{weak}}([0,T]; L^\gamma(\Omega)) \tag{4.38}$$

$$\varrho_n \mathbf{u}_n \to \varrho\mathbf{u} \quad \text{in } C_{\text{weak}}([0,T]; L^q(\Omega;\mathbb{R}^d)), \tag{4.39}$$

$$\mathbf{u}_n \rightharpoonup \mathbf{u} \quad \text{in } L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^d)), \tag{4.40}$$

$$\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n \rightharpoonup \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{in } L^2(0,T; L^r(\Omega;\mathbb{R}^{d\times d})),\ r>1, \tag{4.41}$$

$$\varrho_n \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,\infty; W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)), \tag{4.42}$$

$$p(\varrho_n) \overset{*}{\rightharpoonup} \overline{p(\varrho)} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})), \tag{4.43}$$

$$\nabla_x\sqrt{\varrho_n}\otimes\nabla_x\sqrt{\varrho_n} \overset{*}{\rightharpoonup} \overline{\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega})), \tag{4.44}$$

$$\mathfrak{R}_n \overset{*}{\rightharpoonup} \widetilde{\mathfrak{R}} \quad \text{in } L^\infty(0,T;\mathcal{M}(\overline{\Omega};\mathbb{R}^{d\times d}_{\text{sym}})), \tag{4.45}$$

where the exponent q is defined as in (2.2).

Proof. From the energy inequality (4.37) we can recover the following uniform bounds:

$$\|\sqrt{\varrho_n}\mathbf{u}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le c(\overline{E}), \tag{4.46}$$

$$\|\nabla_x\sqrt{\varrho_n}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^d))} \le c(\overline{E}) \tag{4.47}$$

$$\|P(\varrho_n)\|_{L^\infty(0,T;L^1(\Omega))} \le c(\overline{E}), \tag{4.48}$$

$$\|\operatorname{Tr}[\mathfrak{R}_n]\|_{L^\infty(0,T;L^1(\Omega))} \le c(\overline{E}), \tag{4.49}$$

$$\|\nabla_x\mathbf{u}_n\|_{L^2((0,T)\times\Omega;\mathbb{R}^{d\times d})} \le c(\overline{E}). \tag{4.50}$$

From (4.48), it is easy to deduce that, passing to a suitable subsequence,

$$\varrho_n \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,T;L^\gamma(\Omega)); \tag{4.51}$$

this convergence can be strengthened to (4.38) as a consequence of the Arzelà-Ascoli theorem. Convergence (4.38) combined with identity (4.32), the uniform bound (4.47) and the fact that $\gamma > \frac{2\gamma}{\gamma+1}$ imply (4.42). Convergence (4.40) can be recovered from (4.50), while from (4.46), (4.51), (4.42), the Sobolev embedding (3.21) and the fact that for a.e. $t\in[0,T]$, as a consequence of Hölder inequality,

$$\|(\varrho_n\mathbf{u}_n)(t,\cdot)\|_{L^q(\Omega;\mathbb{R}^d)} \le \|(\sqrt{\varrho_n}\mathbf{u})(t,\cdot)\|_{L^2(\Omega;\mathbb{R}^d)}\|\sqrt{\varrho_n}(t,\cdot)\|_{L^{2p}(\Omega)},$$

with q and p defined as in (2.2) and (3.22), respectively, we get

$$\varrho_n\mathbf{u}_n \overset{*}{\rightharpoonup} \overline{\varrho\mathbf{u}} \quad \text{in } L^\infty(0,T;L^q(\Omega;\mathbb{R}^d)). \tag{4.52}$$

Now, from the compact Sobolev embedding $L^\gamma(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$, we get the strong convergence of the densities in $C([0,T]; W^{-1,2}(\Omega))$ and therefore

$$\overline{\varrho \mathbf{u}} = \varrho \mathbf{u} \quad \text{a.e. on } (0,T) \times \Omega.$$

Once again, convergence (4.52) can be strengthened to (4.39). Next, convergences (4.39), (4.40) combined with the Sobolev embedding $L^q(\Omega) \hookrightarrow\hookrightarrow W^{-1,2}(\Omega)$ imply (4.41), where the exponent r must satisfy

$$\frac{1}{r} = \frac{1}{q} + \frac{d-2}{2d}.$$

Finally, convergences (4.43) and (4.44) can be deduced from (4.47) and (4.48) respectively, repeating the same passages performed in the proof of Lemma 4.6, while convergence (4.45) follows from (4.49). $\square$

We are now ready to let $n \to \infty$. Once again, we get that the weak formulations of the continuity equation (2.5) and balance of momentum (1.3) hold for any $\tau \in [0,T]$ and any $\varphi \in C^1([0,T] \times \overline{\Omega})$, $\boldsymbol{\varphi} \in C^1([0,T]; X_n)$, respectively, with the Reynolds stress

$$\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\Omega; \mathbb{R}^{d\times d}_{\rm sym}))$$

such that

$$\mathrm{d}\mathfrak{R} = \mathrm{d}\widetilde{\mathfrak{R}} \, + \left(\overline{p(\varrho)} - p(\varrho)\right) \mathbb{I} \, \mathrm{d}x + \hbar \left(\overline{\nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}} - \nabla_x \sqrt{\varrho} \otimes \nabla_x \sqrt{\varrho}\right) \mathrm{d}x.$$

Choosing a càglàd function $E = E(\tau)$ such that

$$E(\tau) = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho}|^2\right] (\tau, \cdot) \, \mathrm{d}x + \frac{1}{\lambda} \int_{\overline{\Omega}} \mathrm{d} \, \mathrm{Tr}[\mathfrak{R}](\tau)$$

for a.e. $\tau \in (0,T)$, the integral inequality

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x\mathrm{d}t \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{J}_0|^2}{\varrho_0} + P(\varrho_0) + \frac{\hbar}{2} \, |\nabla_x \sqrt{\varrho_0}|^2\right] \mathrm{d}x,$$

holds for a.e. $\tau \in (0,T)$. Finally, notice that the spaces X_n can be chosen in such a way that the validity of (2.6) can be extended to any $\boldsymbol{\varphi} \in C^1([0,T]; C^2_c(\Omega; \mathbb{R}^d))$ by a density argument. Given

$$\boldsymbol{\varphi} \in C^1([0,T]; C^2(\overline{\Omega}; \mathbb{R}^d)), \;\; \boldsymbol{\varphi}|_{\partial\Omega} = 0,$$

we can construct a sequence $\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}} \subset C^1([0,T]; C^2_c(\Omega; \mathbb{R}^d))$ such that

$$\{\boldsymbol{\varphi}_n\}_{n\in\mathbb{N}} \text{ is uniformly bounded in } W^{1,\infty}(0,T; W^{2,\infty}(\Omega; \mathbb{R}^d))$$

and, for any $(t,x) \in (0,T) \times \Omega$

$$\begin{gathered} \boldsymbol{\varphi}_n(t,x) \to \boldsymbol{\varphi}(t,x), \quad \partial_t \boldsymbol{\varphi}_n(t,x) \to \partial_t \boldsymbol{\varphi}(t,x), \\ \nabla_x \boldsymbol{\varphi}_n(t,x) \to \nabla_x \boldsymbol{\varphi}(t,x), \quad \nabla_x^2 \boldsymbol{\varphi}_n(t,x) \to \nabla_x^2 \boldsymbol{\varphi}(t,x). \end{gathered}$$

This concludes the proof of Theorem 4.1.

4.2 Proof of Theorem 4.2

Let $\{[\varrho_\delta, \mathbf{u}_\delta]\}_{\delta>0}$ be a family of dissipative solutions of the quantum Navier–Stokes system

$$\partial_t \varrho_\delta + \mathrm{div}_x(\varrho_\delta \mathbf{u}_\delta) = 0, \tag{4.53}$$

$$\partial_t(\varrho_\delta \mathbf{u}_\delta) + \mathrm{div}_x\left(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta\right) + \nabla_x p(\varrho_\delta) = \delta\, \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathbf{u}_\delta) + \mathrm{div}_x\, \mathbb{K}(\varrho_\delta, \nabla_x \varrho_\delta, \nabla_x^2 \varrho_\delta), \tag{4.54}$$

with correspondent Reynolds stress $\mathfrak{R}_\delta$, pressure (1.6), viscous stress tensor (1.7), boundary conditions (1.8) and initial conditions $[\varrho_0, \mathbf{J}_0]$ as in the hypotheses of Theorem (4.2). For each fixed $\delta > 0$, the existence of a dissipative solution $[\varrho_\delta, \mathbf{u}_\delta]$ in the sense of Definition 2.1 was proven in Theorem (4.1). Similarly to what was done in the previous section, passing to suitable subsequences as the case may be, we have the following convergences as $\delta \to 0$.

$$\varrho_\delta \to \varrho \quad \text{in } C_{\text{weak}}([0,T]; L^\gamma(\Omega)) \tag{4.55}$$

$$\varrho_\delta \mathbf{u}_\delta \to \mathbf{J} \quad \text{in } C_{\text{weak}}([0,T]; L^q(\Omega; \mathbb{R}^d)), \tag{4.56}$$

$$\varrho_\delta \overset{*}{\rightharpoonup} \varrho \quad \text{in } L^\infty(0,T; W^{1,\frac{2\gamma}{\gamma+1}}(\Omega)), \tag{4.57}$$

$$p(\varrho_\delta) \overset{*}{\rightharpoonup} \overline{p(\varrho)} \quad \text{in } L^\infty(0,T; \mathcal{M}(\overline{\Omega})), \tag{4.58}$$

$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \overset{*}{\rightharpoonup} \overline{\frac{\mathbf{J}\otimes\mathbf{J}}{\varrho}} \quad \text{in } L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \tag{4.59}$$

$$\nabla_x\sqrt{\varrho_\delta} \otimes \nabla_x\sqrt{\varrho_\delta} \overset{*}{\rightharpoonup} \overline{\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}} \quad \text{in } L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \tag{4.60}$$

$$\mathfrak{R}_\delta \overset{*}{\rightharpoonup} \widetilde{\mathfrak{R}} \quad \text{in } L^\infty(0,T; \mathcal{M}(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}})), \tag{4.61}$$

$$\sqrt{\delta}\, \mathbb{S}(\nabla_x \mathbf{u}_\delta) \rightharpoonup \sqrt{\delta}\, \widetilde{\mathbb{S}} \quad \text{in } L^2((0,T)\times\Omega; \mathbb{R}^{d\times d}), \tag{4.62}$$

with q defined as in (2.2).

We are now ready to let $\delta \to 0$ in (2.5)–(2.7). Notice that the term with the δ-dependent viscous stress tensor vanishes due to convergence (4.62); indeed,

$$\delta \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t = \sqrt{\delta} \int_0^\tau \int_\Omega \sqrt{\delta}\, \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \boldsymbol{\varphi}\ \mathrm{d}x\mathrm{d}t \to 0.$$

We get the weak formulations of the continuity equation (2.8), of the balance of momentum (2.9) and of the energy inequality (2.10) for the quantum Euler system, with

$$\mathfrak{R} \in L^\infty(0,T; \mathcal{M}^+(\overline{\Omega}; \mathbb{R}^{d\times d}_{\text{sym}}))$$

such that

$$\begin{aligned}\mathrm{d}\mathfrak{R} = \mathrm{d}\widetilde{\mathfrak{R}} + \left(\overline{p(\varrho)} - p(\varrho)\right)\mathbb{I}\ \mathrm{d}x + \left(\overline{\frac{\mathbf{J}\otimes\mathbf{J}}{\varrho}} - \frac{\mathbf{J}\otimes\mathbf{J}}{\varrho}\right)\mathrm{d}x \\ + \hbar\left(\overline{\nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}} - \nabla_x\sqrt{\varrho}\otimes\nabla_x\sqrt{\varrho}\right)\mathrm{d}x.\end{aligned}$$

Indeed, proceeding as in the previous section, we can write for any $\boldsymbol{\xi} \in \mathbb{R}^d$ and any bounded open set $\mathcal{B} \subset \Omega$

$$\left(\overline{\frac{\mathbf{J}\otimes\mathbf{J}}{\varrho}} - \frac{\mathbf{J}\otimes\mathbf{J}}{\varrho}\right) : (\boldsymbol{\xi}\otimes\boldsymbol{\xi}) = \overline{\left|\frac{\mathbf{J}\cdot\boldsymbol{\xi}}{\varrho}\right|^2} - \left|\frac{\mathbf{J}\cdot\boldsymbol{\xi}}{\varrho}\right|^2 \quad \text{in } \mathcal{D}'((0,T)\times\mathcal{B}),$$

where the non-negativity of the right-hand side quantity will follow from the convexity of the lower semi-continuous function $[\varrho, \mathbf{J}] \mapsto \left|\frac{\mathbf{J}\cdot\boldsymbol{\xi}}{\varrho}\right|^2$. This concludes the proof of Theorem 4.2.

5 Semiflow selection

We start by fixing a proper setting. We let

- $H := W^{-k,2}(\Omega) \times W^{-k,2}(\Omega;\mathbb{R}^d) \times \mathbb{R}$ with $k > \frac{d}{2}+1$ fixed; notice that with this particular choice of the constant k we can guarantee

$$L^p(\Omega) \hookrightarrow\hookrightarrow W^{-k,2}(\Omega) \quad \text{for any } p \geq 1; \tag{5.1}$$

- $\mathcal{D}$ denote the space of initial data associated to the quantum Navier–Stokes or quantum Euler systems; in both cases, it can be chosen as

$$\mathcal{D} := \left\{ [\varrho_0, \mathbf{J}_0, E_0] \in H: \ \varrho_0 \in L^1(\Omega), \ \varrho_0 \geq 0, \ \mathbf{J}_0 \in L^1(\Omega;\mathbb{R}^d) \text{ satisfying (5.2)} \right\}$$

where
$$\int_\Omega \left[\frac{1}{2}\frac{|\mathbf{J}_0|^2}{\varrho_0} + P(\varrho_0) + \frac{\hbar}{2}|\nabla_x\sqrt{\varrho_0}|^2 \right] \mathrm{d}x \leq E_0; \tag{5.2}$$

- $\mathcal{T} = \mathfrak{D}([0,\infty); H)$ represents the trajectory space;

- $\mathcal{U}: \mathcal{D} \to 2^{\mathcal{T}}$ represents the set–valued mapping that associate to every $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$ the family of dissipative solutions in the sense of Definition 2.1 or 2.2 if we are considering the quantum Navier–Stokes or quantum Euler system, respectively, arising from the initial data $[\varrho_0, \mathbf{J}_0, E_0]$. More precisely, for every $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$

$$\begin{aligned}&\mathcal{U}[\varrho_0, \mathbf{J}_0, E_0] = \\ &\quad \{[\varrho, \mathbf{J}, E] \in \mathcal{T}: \ [\varrho, \mathbf{J}, E] \text{ is a dissipative solution with initial data } [\varrho_0, \mathbf{J}_0, E_0]\}.\end{aligned}$$

  Notice that also in the context of the quantum Navier–Stokes system, we consider the momentum $\mathbf{J} = \varrho\mathbf{u}$ as a state variable along with the density ϱ instead of the velocity $\mathbf{u}$ because it is at least weakly continuous in time.

We are now ready to give the following definition.

Definition 5.1 (Semiflow selection)**.** A *semiflow selection* in the class of dissipative solutions is a Borel measurable map $U: \mathcal{D} \to \mathcal{T}$ such that

$$U[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{J}_0, E_0] \text{ for every } [\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$$

satisfying the semigroup property: for any $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$ and any $t_1, t_2 \geq 0$

$$U[\varrho_0, \mathbf{J}_0, E_0](t_1 + t_2) = U[\varrho(t_1), \mathbf{J}(t_1), E(t_1)](t_2)$$

where $[\varrho, \mathbf{J}, E] = U[\varrho_0, \mathbf{J}_0, E_0]$.

The goal of this section is to prove the following two results.

Theorem 5.2 (Semiflow selection for the quantum Navier–Stokes system)**.** *The quantum Navier–Stokes system* (1.2)–(1.3) *with constitutive relations* (1.6)–(1.7) *and boundary conditions* (1.8) *admits a semiflow selection in the sense of Definition 5.1.*

Theorem 5.3 (Semiflow selection for the quantum Euler system)**.** *The quantum Euler system* (1.4)–(1.5) *with the isentropic pressure* (1.6) *and boundary conditions* (1.9) *admits a semiflow selection in the sense of Definition 5.1.*

Both Theorems 5.2 and (5.3) are a direct consequence of Theorem 3.2 in [7] once we have verified that the set–valued map $\mathcal{U}$ verifies the following five properties.

(**P1**) *Non-emptiness*: $\mathcal{U}[\varrho_0, \mathbf{J}_0, E_0]$ is a non-empty subset of $\mathcal{T}$ for any $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$.

(**P2**) *Compactness*: $\mathcal{U}[\varrho_0, \mathbf{J}_0, E_0]$ is a compact subset of $\mathcal{T}$ for every $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$.

(**P3**) *Measurability*: $\mathcal{U} : \mathcal{D} \to 2^{\mathcal{T}}$ is Borel measurable.

(**P4**) *Shift invariance*: introducing the positive shift operator $S_T \circ \Phi$ for every $T > 0$ and $\Phi \in \mathcal{T}$ as

$$S_T \circ \Phi(t) = \Phi(T + t), \quad \text{for all } t \geq 0,$$

then, for any $T > 0$, $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$ and $[\varrho, \mathbf{J}, E] \in \mathcal{U}[\varrho_0, \mathbf{J}_0, E_0]$, we have

$$S_T \circ [\varrho, \mathbf{J}, E] \in \mathcal{U}([\varrho(T), \mathbf{J}(T), E(T-)]).$$

(**P5**) *Continuation*: introducing the continuation operator $\Phi_1 \cup_T \Phi_2$ for every $T > 0$ and $\Phi_1, \Phi_2 \in \mathcal{T}$ as

$$\Phi_1 \cup_T \Phi_2(t) = \begin{cases} \Phi_1(t) & \text{for } 0 \leq t \leq T, \\ \Phi_2(t - T) & \text{for } t > T, \end{cases} \quad \text{for all } t \geq 0,$$

then, for any $T > 0$, $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$,

$$[\varrho_1, \mathbf{J}_1, E_1] \in \mathcal{U}[\varrho_0, \mathbf{J}_0, E_0],$$
$$[\varrho_2, \mathbf{J}_2, E_2] \in \mathcal{U}[\varrho_1(T), \mathbf{J}_1(T), E_1(T-)],$$

we have

$$[\varrho_1, \mathbf{J}_1, E_1] \cup_T [\varrho_2, \mathbf{J}_2, E_2] \in \mathcal{U}[\varrho_0, \mathbf{J}_0, E_0].$$

To this end, we have the following facts.

- Property (**P1**) is equivalent in showing the *existence* of a dissipative solution in the sense of Definitions 2.1 and 2.2 for any fixed initial data $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$. This has already been achieved in Theorems 4.1 and 4.2.

- Properties (**P2**) and (**P3**) hold true if we manage to prove the *weak sequential stability* of the solution set $\mathcal{U}[\varrho_0, \mathbf{J}_0, E_0]$ for every $[\varrho_0, \mathbf{J}_0, E_0] \in \mathcal{D}$ fixed, since it will in particular imply compactness and the closed-graph property of the mapping

$$\mathcal{D} \ni [\varrho_0, \mathbf{J}_0, E_0] \to \mathcal{U}[\varrho_0, \mathbf{J}_0, E_0] \in 2^{\mathcal{T}},$$

and thus the Borel–measurality of $\mathcal{U}$, cf. Lemma 12.1.8 in [42].

- Properties (**P4**) and (**P5**) can be easily checked for both systems following the same arguments done in [9], Lemma 4.2 and 4.3.

Therefore, the proofs of Theorems 5.2 and 5.3 reduce to the proof of the weak sequential stability results.

Proposition 5.4 (Weak sequential stability for the quantum Navier–Stokes system)**.** *Let* $\{[\varrho_n, \mathbf{u}_n]\}_{n\in\mathbb{N}}$ *be a family of dissipative solutions of the quantum Navier–Stokes system* (1.2), (1.3) *with the corresponding total energies* $\{E_n\}_{n\in\mathbb{N}}$ *and initial data* $\{[\varrho_{0,n}, \mathbf{J}_{0,n}, E_{0,n}]\}_{n\in\mathbb{N}}$ *in the sense of Definition 2.1. If*

$$[\varrho_{0,n}, \mathbf{J}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{J}_0, E_0] \quad \textit{in } H,$$

then, at least for suitable subsequences,

$$[\varrho_n, \mathbf{J}_n = \varrho_n \mathbf{u}_n, E_n] \to [\varrho, \mathbf{J} = \varrho \mathbf{u}, E] \quad \textit{in } \ \mathfrak{D}([0,\infty); H), \tag{5.3}$$

where $[\varrho, \mathbf{u}]$ *is another dissipative solution of the same problem with total energy* E*.*

Proposition 5.5 (Weak sequential stability for the quantum system)**.** *Let* $\{[\varrho_n, \mathbf{J}_n]\}_{n\in\mathbb{N}}$ *be a family of dissipative solutions of the quantum Euler system* (1.4), (1.5) *with the corresponding total energies* $\{E_n\}_{n\in\mathbb{N}}$ *and initial data* $\{[\varrho_{0,n}, \mathbf{J}_{0,n}, E_{0,n}]\}_{n\in\mathbb{N}}$ *in the sense of Definition 2.2. If*

$$[\varrho_{0,n}, \mathbf{J}_{0,n}, E_{0,n}] \to [\varrho_0, \mathbf{J}_0, E_0] \quad \textit{in } H,$$

then, at least for suitable subsequences,

$$[\varrho_n, \mathbf{J}_n, E_n] \to [\varrho, \mathbf{J}, E] \quad \textit{in } \ \mathfrak{D}([0,\infty); H), \tag{5.4}$$

where $[\varrho, \mathbf{J}]$ *is another dissipative solution of the same problem with total energy* E*.*

We are not going to show the two aforementioned propositions in details since the proofs would be essentially a repetition of what was done in Sections 4.1.4 and 4.2. We just point out that the convergences

$$[\varrho_n, \mathbf{J}_n] \to [\varrho, \mathbf{J}] \quad \text{in } C_{\text{weak,loc}}([0,\infty); L^p(\Omega) \times L^q(\Omega; \mathbb{R}^d))$$

can be strengthened to

$$[\varrho_n, \mathbf{J}_n] \to [\varrho, \mathbf{J}] \quad \text{in } C_{\text{loc}}([0,\infty); W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; \mathbb{R}^d))$$

thanks to the compact embedding (5.1), implying in particular that,

$$[\varrho_n, \mathbf{J}_n] \to [\varrho, \mathbf{J}] \quad \text{in } \mathfrak{D}([0,\infty); W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; \mathbb{R}^d)),$$

as for continuous functions, the convergence in the Skorokhod space coincides with the uniform one, cf. condition (ii) of Proposition 2.1 in [7]. Moreover, the energies $\{E_n\}_{n\in\mathbb{N}}$ are non-increasing functions, locally of bounded variation; therefore, from Helly's selection theorem, there exists a subsequence converging pointwise

$$E_n(t) \to E(t) \quad \text{for all } t \in [0,\infty),$$

implying in particular that

$$E_n \to E \quad \text{in } \mathfrak{D}([0,\infty)),$$

as for monotone functions, the convergence in the Skorokhod space coincides with the almost everywhere one, cf. condition (i) of Proposition 2.1 in [7].

A Appendix

A.1 Function spaces

Let $Q \subseteq \mathbb{R}^N$, $N \geq 1$, be an open set, X a Banach space and $M \geq 1$. We denote with

- $C_{\text{weak}}(Q; X)$ the space of functions defined on Q and ranging in X which are continuous with respect to the weak topology. If Q is bounded, we say that $f_n \to f$ in $C_{\text{weak}}(\overline{Q}; X)$ as $n \to \infty$ if for all $g \in X^*$
$$\sup_{y\in\overline{Q}} |\langle g; f_n(y) - f(y)\rangle_{X^*,X}| \to 0 \quad \text{as } n \to \infty;$$
- $C^k(Q; X)$, with k a non-negative integer, the space of k-times continuously differentiable functions on Q and $C^\infty(Q; X) = \bigcap_{k=0}^\infty C^k(Q; X)$;
- $\mathcal{D}(Q; X) = C_c^\infty(Q; X)$ the space of functions belonging to $C^\infty(Q; X)$ and having compact support in Q;
- $\mathcal{D}'(Q; \mathbb{R}^M) = [C_c^\infty(Q; \mathbb{R}^M)]^*$ the space of distributions;
- $\mathcal{M}(Q; \mathbb{R}^M) = \left[\overline{C_c(Q; \mathbb{R}^M)}^{\|\cdot\|_\infty}\right]^*$ the space of vector-valued Radon measures. If $\Omega \subset \mathbb{R}^N$ is a bounded domain, then $\mathcal{M}(\overline{\Omega}) = [C(\overline{\Omega})]^*$.
- $\mathcal{M}^+(Q)$ the space of positive Radon measures;
- $\mathcal{M}^+(Q; \mathbb{R}^{N\times N}_{\text{sym}})$ the space of tensor–valued Radon measures $\mathfrak{R}$ such that $\mathfrak{R} : (\xi \otimes \xi) \in \mathcal{M}^+(Q)$ for all $\xi \in \mathbb{R}^d$, and with components $\mathfrak{R}_{i,j} = \mathfrak{R}_{j,i}$;
- $L^p(Q; X)$, with $1 \leq p \leq \infty$, the Lebesgue space defined on Q and ranging in X;
- $W^{k,p}(Q; \mathbb{R}^M)$, with $1 \leq p \leq \infty$ and k a positive integer, the Sobolev space defined on Q;

- $W^{-k,p'}(Q;\mathbb{R}^m)$, with p' the conjugate exponent of $1 \leq p < \infty$ and k a positive integer, the dual space of $W_0^{k,p}(Q;\mathbb{R}^m) = \left[\overline{C_c(Q;\mathbb{R}^M)}^{\|\cdot\|_{W^{k,p}(Q;\mathbb{R}^M)}}\right]^*$;

- $\mathfrak{D}([0,\infty);H)$ the Skorokhod space of càglàd functions defined on $[0,\infty)$ taking values in a Hilbert space H. More precisely, Φ belongs to the space $\mathfrak{D}([0,\infty);H)$ if it is left–continuous and has right–hand limits:

  (i) for $t>0$, $\Phi(t-) = \lim_{s\uparrow t}\Phi(s)$ exists and $\Phi(t-) = \Phi(t)$;

  (ii) for $t \geq 0$, $\Phi(t+) = \lim_{s\downarrow t}\Phi(s)$ exists.

A.2 Energy

In this section, we will show how to deduce the total energy balances (1.11) and (1.12). First of all, introducing the *drift velocity* $\mathbf{v} = \mathbf{v}(\varrho, \nabla_x\varrho)$ such that

$$\mathbf{v} = \frac{\nabla_x\sqrt{\varrho}}{\sqrt{\varrho}}, \tag{A.1}$$

and taking the gradient in the continuity equations (1.2), (1.4), we get extra equations for $\mathbf{v}$:

$$\partial_t(\varrho\mathbf{v}) + \mathrm{div}_x(\varrho\mathbf{v}\otimes\mathbf{u}) + \frac{1}{2}\,\mathrm{div}_x\left(\varrho\nabla_x^\top\mathbf{u}\right) = 0, \tag{A.2}$$

when considering system (1.2)–(1.3), and

$$\partial_t(\varrho\mathbf{v}) + \frac{1}{2}\,\mathrm{div}_x\,\nabla_x^\top\mathbf{J} = 0, \tag{A.3}$$

when considering system (1.4)–(1.5). Furthermore, notice that we can write

$$\mathbb{K}(\varrho, \nabla_x\mathbf{v}) = \frac{\hbar}{2}\varrho\nabla_x\mathbf{v}.$$

Supposing that all the quantities in question are smooth, we can multiply the balance of momentum (1.3) of the quantum Navier–Stokes system by $\mathbf{u}$ and, using the continuity equation (1.2), we can deduce

$$\begin{aligned}\partial_t\left(\frac{1}{2}\varrho|\mathbf{u}|^2\right) + \mathrm{div}_x\left(\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + p(\varrho)\right]\mathbf{u}\right) - p(\varrho)\,\mathrm{div}_x\,\mathbf{u} + \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} + \frac{\hbar}{2}\varrho\nabla_x\mathbf{v} : \nabla_x\mathbf{u} \\ = \mathrm{div}_x\left(\mathbb{S}(\nabla_x\mathbf{u})\cdot\mathbf{u} + \mathbb{K}(\varrho, \nabla_x\mathbf{v})\cdot\mathbf{u}\right).\end{aligned} \tag{A.4}$$

Similarly, we multiply (A.2) by $\mathbf{v}$ to get

$$\partial_t\left(\frac{1}{2}\varrho|\mathbf{v}|^2\right) + \mathrm{div}_x\left(\frac{1}{2}\varrho|\mathbf{v}|^2\mathbf{u}\right) - \frac{1}{2}\varrho\nabla_x\mathbf{u} : \nabla_x\mathbf{v} = -\frac{1}{2}\,\mathrm{div}_x\left(\varrho\nabla_x^\top\mathbf{u}\cdot\mathbf{v}\right), \tag{A.5}$$

where we used the fact that

$$\nabla_x^\top\mathbf{u} : \nabla_x\mathbf{v} = \nabla_x\mathbf{u} : \nabla_x^\top\mathbf{v} = \nabla_x\mathbf{u} : \nabla_x\mathbf{v} \tag{A.6}$$

since $\nabla_x \mathbf{v}$ is symmetric. Multiplying (A.5) by $\hbar$ and summing the obtained identity to (A.4) we get

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{\hbar}{2}\varrho|\mathbf{v}|^2 \right) + \mathrm{div}_x \left(\left[\frac{1}{2}\varrho|\mathbf{u}|^2 + p(\varrho) + \frac{\hbar}{2}|\mathbf{v}|^2 \right] \mathbf{u} \right) - p(\varrho)\,\mathrm{div}_x\,\mathbf{u} + \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$$
$$= \mathrm{div}_x \left(\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} + \mathbb{K}(\varrho, \nabla_x \mathbf{v}) \cdot \mathbf{u} - \frac{\hbar}{2}\varrho \nabla_x^\top \mathbf{u} \cdot \mathbf{v} \right).$$

Recalling that the pressure potential $P = P(\varrho)$ is characterized by (1.10), from the continuity equation (1.2) we obtain the following identity

$$-p(\varrho)\,\mathrm{div}_x\,\mathbf{u} = \partial_t P(\varrho) + \mathrm{div}_x[P(\varrho)\mathbf{u}].$$

Now, it is enough to integrate over Ω and use the boundary conditions to get the desired expressions.

References

[1] A. Abbatiello, E. Feireisl and A. Novotný, *Generalized solutions to mathematical models of compressible viscous fluids*, Discrete & Continuous Dynamical Systems **41**(1): 1–28; 2021

[2] P. Antonelli and P. Marcati, *On the finite energy weak solutions to a system in quantum fluid dynamics*, Commun. Math. Phys. **287**: 657–686; 2009

[3] P. Antonelli and S. Spirito, *Global existence of finite energy weak solutions of quantum Navier–Stokes equations*, Arch. Rational Mech. Anal. **225**: 1161–1199; 2017

[4] P. Antonelli and S. Spirito, *On the compactness of weak solutions to the Navier-Stokes-Korteweg equations for capillary fluids*, Nonlinear Anal. **187**: 110–124; 2019.

[5] P. Antonelli, C. G. Cianfarani, C. Lattanzio and S. Spirito, *Relaxation limit from the quantum Navier-Stokes equations to the quantum drift-diffusion equation*, J. Nonlinear Sci., **31**(71); 2021

[6] C. Audiard and B. Haspot, *Global well-posedness of the Euler Korteweg system for small irrotational data*, Commun. Math. Phys. **315**: 201–247; 2017

[7] D. Basarić, *Semiflow selection to models of general compressible viscous fluids*, J. Math. Fluid Mech. **23**(2); 2021

[8] D. Breit, E. Feireisl and M. Hofmanová, *Markov selection for the stochastic compressible Navier–Stokes system*, Ann. Appl. Probab. **30**(6): 2547–2572; 2020

[9] D. Breit, E. Feireisl and M. Hofmanová, *Solution semiflow to the isentropic Euler system*, Arch. Rational Mech. Anal. **235**: 167–194; 2020

[10] D. Bresch, M. Gisclon and I. Lacroix-Violet, *On the Navier–Stokes–Korteweg and Euler–Korteweg system*, Arch. Rational Mech. Anal. **223**: 975–1025; 2019

[11] S. Brull and F. Méhats, *Derivation of viscous correction terms for the isothermal quantum Euler model*, ZAMM Z. Angew. Math. Mech. **90**: 219–230; 2010

[12] M. Caggio and D. Donatelli, *High Mach number limit for Korteweg fluids with density dependent viscosity*, J. Differential Equations **277**: 1–37; 2021.

[13] J. E. Cardona and L. Kapitanski, *Semiflow selection and Markov selection theorems*, Topol. Methods Nonlinear Anal. **56**(1): 197–227; 2020

[14] T. Chang, B. J. Jin and A. Novotný, *Compressible Navier-Stokes system with inflow-outflow boundary data*, SIAM J. Math. Anal. **51**(2): 1238–1278; 2019

[15] C. G. Cianfarani and C. Lattanzio, *High friction limit for Euler-Korteweg and Navier-Stokes-Korteweg models via relative entropy approach*, J. Differential Equations, **269**: 10495-10526; 2020.

[16] D. Donatelli, E. Feireisl and P. Marcati, *Well/ill posedness for the Euler–Korteweg–Poisson system and related problems*, Commun. Partial Differ. Equ. **40**: 1314–1335; 2015

[17] D. Donatelli and P. Marcati, *Quasi-neutral limit, dispersion, and oscillations for Korteweg-type fluids*, SIAM J. Math. Anal., **47**:2265–2282; 2015.

[18] D. Donatelli and P. Marcati, *Low Mach number limit for the quantum hydrodynamics system*, Res. Math. Sci., **3**(13); 2016

[19] J. Dong, *A note on barotropic compressible quantum Navier–Stokes equations*, Nonlinear Anal. Real World Appl. **73**: 854–856; 2010

[20] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford University Press, Oxford; 2003

[21] E. Feireisl, *On weak–strong uniqueness for the compressible Navier–Stokes system with non-monotone pressure law*, Commun. Partial Differ. Equ. **44**(3): 271–278; 2019

[22] E. Feireisl, P. Gwiazda, A. Świerczewska-Gwiazda and E. Wiedemann, *Dissipative measure-valued solutions to the compressible Navier–Stokes system*, Calc. Var. Partial Differ. Equ. **55** (6): 55–141; 2016

[23] E. Feireisl, B. J. Jin and A. Novotný, *Relative Entropies, Suitable Weak Solutions, and Weak–Strong Uniqueness for the Compressible Navier–Stokes System*, Journal of Mathematical Fluid Dynamics **14**: 717–730; 2012

[24] E. Feireisl and M. Lukáčová-Medviďová, *Convergence of a mixed finite element–finite volume scheme for the isentropic Navier–Stokes system via dissipative measure–valued solutions*, Found. Comput. Math. **18**: 703–730; 2018

[25] E. Feireisl and A. Novotný, *Weak–strong uniqueness property for models of compressible viscous fluids near vacuum*, Nonlinearity **34**(9); 2021

[26] D. K. Ferry and J.-R. Zhou, *Form of the quantum potential for use in hydrodynamic equations for semiconductor device modeling*, Phys. Rev. B **48**: 7944–7950; 1993

[27] F. Flandoli and M. Romito, *Markov selections for the 3D stochastics Navier–Stokes equations*, Probab. Theory Related Fields, **140**(3-4): 407–458; 2008

[28] P. Germain, *Weak–strong uniqueness for the isentropic compressible Navier–Stokes system*, J. Math. Fluid Mech. **13**(1): 137–146; 2011

[29] J. Giesselmann, C. Lattanzio and A. E. Tzavaras, *Relative energy for the Korteweg theory and related Hamiltonian flows in gas dynamics*, Arch. Rational Mech. Anal. **223**: 1427–1484; 2017

[30] M. Gisclon and I. Lacroix-Violet, *About the barotropic compressible quantum Navier–Stokes equations*, Nonlinear Anal. Theory Methods Appl. **128**: 106–121; 2015

[31] P. Gwiazda, A. Świerczewska-Gwiazda and E. Wiedemann, *Weak-strong uniqueness for measure-valued solutions of some compressible fluid models*, Nonlinearity **28**: 3873–3890; 2015

[32] F. Jiang, *A remark on weak solutions to the barotropic compressible quantum Navier–Stokes equations*, Nonlinear Anal. Real World Appl. **12**: 1733–1735; 2011

[33] A. Jüngel and H. L. Li, *Quantum Euler-Poisson systems: global existence and exponential decay*, Quart. Appl. Math., **62**: 569–600; 2004.

[34] A. Jüngel, *Global weak solutions to compressible Navier- Stokes equations for quantum fluids*, SIAM J. Math. Anal. **42**: 1025–1045; 2010

[35] N. V. Krylov, *The selection of a Markov process from a Markov system of processes, and the construction of quasidiffusion processes*, Izv. Akad. Nauk SSSR Ser. Mat., **37**: 691–708; 1973

[36] I. Lacroix-Violet and A. Vasseur, *Global weak solutions to the compressible quantum Navier–Stokes equations and its semi-classical limit*, J. Math. Pures Appl. **114**(9): 191–210; 2018

[37] H. L. Li and P. Marcati, *Existence and asymptotic behavior of multi-dimensional quantum hydrodynamic model for semiconductors*, Comm. Math. Phys., **245**: 215–247; 2004.

[38] J. Li and Z.P. Xin, *Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities*, arXiv:1504.06826; 2015

[39] M. I. Loffredo and L. M. Morato, *On the creation of quantized vortex lines in rotating He–2*, Nuovo Cim. B **108**: 205–216; 1993

[40] G. Prodi, *Un teorema di unicitá per le equazioni di Navier–Stokes*, Ann. Mat. Pura Appl. **48**(4): 173–182; 1959

[41] R. Slavchov and R. Tsekov, *Quantum hydrodynamics of electron gases*, J. Chem. Phys. **132**; 2010

[42] D. W. Stroock and S. R. S. Varadhan, *Multidimensional diffusion processes*, Classics in Mathematics, Springer–Verlag, Berlin; 2006

[43] A. Vasseur and C. Yu, *Global weak solutions to compressible Navier–Stokes equations with damping*, SIAM J. Math. Anal. **48**(2): 1489–1511; 2016

[44] E. Wiedemann, *Weak–strong uniqueness in fluid dynamics*, Partial Differential Equations in Fluid Dynamics: 289–326; 2018

[45] R. Wyatt, *Quantum dynamics with trajectories: introduction to quantum hydrodynamics*, Springer, New York; 2005

[46] E. Zaremba and H. C. Tso, *Thomas–Fermi–Dirac–von Weizsäcker hydrodynamics in parabolic wells*, Phys. Rev. B **49**; 1994