Very Weak Solutions and Convergence of Numerical Schemes

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Prologue - Lax equivalence principle

Formulation for LINEAR problems

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- **Stability** - uniform bounds of approximate solutions
- **Consistency** - vanishing approximation error

$$\Longrightarrow$$

- **Convergence** - approximate solutions converge to exact solution

Example - compressible viscous fluid

Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

Isentropic pressure

$$p = p(\varrho) = a\varrho^\gamma, \ a > 0, \ \gamma > 1.$$

Periodic boundary conditions

$$t \in [0, T], \ x \in \mathbb{T}^d, \ d = 2, 3.$$

State–of–the art

- **Global existence.** Global-in-time existence of weak solutions $\gamma > \frac{d}{2}$, uniqueness open
- **Local existence.** Local existence of smooth solutions, global existence of smooth solutions for the data close to equilibrium
- **Finite time blow up.** Solutions on R^3 may develop finite time blow up.
  F. Merle, P. Raphael, I. Rodnianski, and J. Szeftel [2020]: Blow up for certain γ but not for $\gamma = \frac{5}{3}$
  T. Buckmaster, G. Cao-Labora, and J. Gomez-Serrano [2022]: Blow up for $\gamma = \frac{7}{5}$
- **Conditional regularity.** Sun, Wang, and Zhang [2011]: If the maximal existence time $T_{\max}$ is finite, then

$$\|(\varrho, \mathbf{u})\|_{L^\infty} \to \infty \text{ as } t \to T_{\max}$$

FV numerical scheme

$$\int_{\mathbb{T}^d} D_t \varrho_h \varphi_h \, \mathrm{d}x - \sum_{\sigma \in \Sigma} \int_\sigma F_h(\varrho_h, \mathbf{u}_h) \, [[\varphi_h]] \, \mathrm{d}\sigma = 0 \quad \text{for all } \varphi_h \in Q_h$$

$$\int_{\mathbb{T}^d} D_t(\varrho_h \mathbf{u}_h) \cdot \boldsymbol{\varphi}_h \, \mathrm{d}x - \sum_{\sigma \in \Sigma} \int_\sigma \mathbf{F}_h(\varrho_h \mathbf{u}_h, \mathbf{u}_h) \cdot [[\boldsymbol{\varphi}_h]] \, \mathrm{d}\sigma - \sum_{\sigma \in \Sigma} \int_\sigma \{\!\{ p(\varrho_h) \}\!\} \, \mathbf{n} \cdot [[\boldsymbol{\varphi}_h]] \, \mathrm{d}\sigma$$

$$= -\mu \frac{1}{h} \sum_{\sigma \in \Sigma} \int_\sigma [[\mathbf{u}_h]] \cdot [[\boldsymbol{\varphi}_h]] \, \mathrm{d}\sigma - \lambda \int_{\mathbb{T}^d} \mathrm{div}_h \mathbf{u}_h \mathrm{div}_h \boldsymbol{\varphi}_h \, \mathrm{d}x \quad \text{for all } \boldsymbol{\varphi}_h \in \mathbf{Q}_h$$

$$\lambda = \frac{1}{d} \mu + \eta$$

Discrete time derivative

$$D_t r_K^k = \frac{r_K^k - r_K^{k-1}}{\Delta t}$$

Upwind, fluxes

$$\mathrm{Up}[r, \mathbf{v}] = \overline{r} \; \overline{\mathbf{v}} \cdot \mathbf{n} - \frac{1}{2} |\overline{\mathbf{v}} \cdot \mathbf{n}| \, [[r]]$$

$$F_h(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\alpha \, [[r]]$$

Convergence of the numerical method

Hypothesis: FV scheme produces a family of numerical solutions $(\varrho_h, \mathbf{u}_h)$ such that

$$\|\varrho_h, \mathbf{u}_h\|_{L^\infty((0,T)\times\mathbb{T}^d;R^{d+1})} \leq C \text{ for } h \searrow 0$$

Conclusion:

- $$\varrho_h \to \varrho \ \boxed{\textit{strongly}} \text{ in } L^q((0,T)\times\mathbb{T}^d)$$
  $$\mathbf{u}_h \to \mathbf{u} \ \boxed{\textit{strongly}} \text{ in } L^q((0,T)\times\mathbb{T}^d; R^d)$$
  for any $1 \leq q < \infty$
- The functions $(\varrho, \mathbf{u})$ are classical solution of the Navier–Stokes system

Proof, step I

Weak convergence:

$$\varrho_h \to \varrho \text{ weakly-(*) in } L^\infty((0,T)\times \mathbb{T}^d)$$
$$\mathbf{u}_h \to \mathbf{u} \text{ weakly-(*) in } L^\infty((0,T)\times \mathbb{T}^d; R^d)$$

Limit system:

$$\partial_t \varrho + \mathrm{div}_x \overline{\varrho \mathbf{u}} = 0$$
$$\partial_t \overline{\varrho \mathbf{u}} + \mathrm{div}_x \overline{\varrho \mathbf{u} \otimes \mathbf{u}} + \nabla_x \overline{p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{g}$$

Energy inequality:

$$\int_{\mathbb{T}^d} \overline{\frac{1}{2}\varrho |\mathbf{u}|^2 + P(\varrho)}(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_{\mathbb{T}^d} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x$$
$$\leq \int_{\mathbb{T}^d} \frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0)\ \mathrm{d}x$$

Step II, weak–strong uniqueness

Measure–valued solutions:

$$\overline{B(\varrho, \mathbf{u})}(t,x) = \langle \nu_{t,x}; B(\varrho, \mathbf{u}) \rangle$$

ν – a Young measure associated to the sequence $(\varrho_h, \mathbf{u}_h)_{h>0}$

MV – strong uniqueness EF, P. Gwiazda, A.Swierczewska–Gwiazda, E. Wiedemann [2016]

- The MV–solution coincides with the strong solution as long as the latter exists.
- The Young measure reduces to a Dirac mass.
- The convergence is strong.

Conclusion:
Numerical solutions converge strongly in L^p to the strong solution of the Navier–Stokes system on its life span (locally in time)

Step III, conditional regularity

boundedness of the numerical solutions $\Rightarrow$ the limit $(\varrho, \mathbf{u})$ is bounded

Global existence:

conditional regularity criterion of Sun, Wang, Zhang

$\Rightarrow$

convergence in $(0, T)$ to the classical solution

Error estimates

Relative energy:

$$E\left(\varrho_h, \mathbf{u}_h \middle| \varrho, \mathbf{u}\right) = \frac{1}{2}\varrho_h|\mathbf{u} - \mathbf{u}_h|^2 + P(\varrho_h) - P'(\varrho)(\varrho_h - \varrho) - P(\varrho)$$

Error estimates (M. Lukáčová, B. She):

$$\int_{\mathbb{T}^d} E\left(\varrho_h, \mathbf{u}_h \middle| \varrho, \mathbf{u}\right)(\tau, \cdot)\,\mathrm{d}x \le C(\Delta t + h)$$

$$0 \le \tau \le T$$

Statistical solutions

Data:

- initial data $[\varrho_0, \mathbf{u}_0]$
- viscosity coefficients μ, λ
- driving force $\mathbf{g}$
- EOS - the pressure law $p(\varrho) = a\varrho^\gamma$

uncertain data $\Rightarrow$ data considered as random variables

- **Weak stochastic approach:** Only distribution of the data is known. Monte–Carlo and related methods
- **Strong stochastic approach:** Data are known as random variables ranging in a suitable space. Stochastic Galerkin, stochastic collocation methods etc.

Numerical approximation

Step 1:
Choose regular (initial) data
Step 2: [Nonintrusive methods] Apply *deterministic* numerical method several times with (i) randomly generated data (weak approach) (ii) exact data at collocation points (strong approach)

Numerical solutions:

$$(\varrho_{h,n}, \mathbf{u}_{h,n}),\ n = 1, 2, \dots \text{ family of numerical solutions}$$

Empirical means:

$$(\varrho_{h,N}, \mathbf{u}_{h,N}) = \frac{1}{N} \sum_{n=1}^{N} (\varrho_{h,n}, \mathbf{u}_{h,n})$$

Boundedness in probability
Given $\varepsilon > 0$, there is $M(\varepsilon)$ such that:

$$\frac{1}{N} \# \Big\{ \|\varrho_{h,n}, \mathbf{u}_{h,n}\|_{L^\infty} < M(\varepsilon), n \leq N \Big\} > 1 - \varepsilon$$

Convergence analysis, I

Application of Skorokhod representation theorem:

- **Data:**

$$\left[\widetilde{\varrho}_{0,N}, \widetilde{\mathbf{u}}_{0,N}, \widetilde{\mu}_N, \widetilde{\eta}_N, \widetilde{\mathbf{g}}_N\right] \sim \left[\varrho_0, \mathbf{u}_0, \mu, \eta, \mathbf{g}\right]$$

$$\widetilde{\varrho}_{0,N} \to \widetilde{\varrho}_0 \sim \varrho_0$$
$$\widetilde{\mathbf{u}}_{0,N} \to \widetilde{\mathbf{u}}_0 \sim \mathbf{u}_0$$
$$\widetilde{\mu}_N \to \widetilde{\mu} \sim \mu, \widetilde{\eta}_N \to \widetilde{\eta} \sim \eta$$
$$\widetilde{\mathbf{g}}_N \to \widetilde{\mathbf{g}} \sim \mathbf{g}$$

- **Numerical solutions**

$$\mathcal{L}(\mathrm{law})(\widetilde{\varrho}_{h,N}, \widetilde{\mathbf{u}}_{h,N}) = \frac{1}{N} \sum_{n=1}^{N} \delta_{(\varrho_{n,h}, \mathbf{u}_{n,h})}$$

- **Boundedness a.s.**

$$(\widetilde{\varrho}_{h,N}, \widetilde{\mathbf{u}}_{h,N}) \leq C \text{ as } h \to 0,\ N \to \infty \text{ a.s.}$$

Convergence analysis, II

Application of Gyöngly–Krylov convergence criteria:

- The limit problem admits a regular solution $(\varrho, \mathbf{u})$ a.s.
- $$\|(\varrho_N, \mathbf{u}_N) - (\varrho, \mathbf{u})\|_{L^q} \to 0 \text{ in probability}, \ 1 \leq q < \infty$$

Possible applications to more complex systems

- **(Complete) Navier–Stokes–Fourier system.** The existence and weak strong uniqueness principle proved in a series of papers with J. Březina (Kyushu University) and A. Novotný (Toulon)
- **Regularity criterion by EF, Wen and Zhu [2022]**
  If $T_{\max} < \infty$ for the complete Navier–Stokes–Fourier system, then

$$\|\varrho(t,\cdot)\|_{L^\infty} + \|\vartheta(t,\cdot)\|_{L^\infty} \to \infty \text{ as } t \to T_{\max}$$