Some tools in mathematical analysis of compressible fluids

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1. To quickly overfly the existence proof of weak solutions to CNSE.
2. To detect the points in the proof, where the properties of transport and continuity equations play essential role.
3. To recall a part of the theory of renormalized solutions to the transport equations needed in the proofs.
4. To show some of its generalizations with the goal to target some applications of potentially physical interest (non zero inflow-outflow problems, some simple bi-fluid models).

Barotropic Navier-Stokes equations

$\Omega \subset \mathbb{R}^3$ is a bounded (Lipschitz) domain, $I = (0, T)$.

$$\partial_t \varrho + \text{div}(\varrho \mathbf{u}) = 0, \quad (1)$$

$$\partial_t(\rho \mathbf{u}) + \text{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla P(\varrho) = \text{div}\mathbb{S}(\nabla_x \mathbf{u}) \quad (2)$$

$$P(\varrho) \approx \varrho^\gamma,\ \gamma > 1,\ \mathbb{S}(\mathbb{Z}) = \mu(\mathbb{Z} + \mathbb{Z}^T) + \lambda \text{Tr}(\mathbb{Z})\mathbb{I},\ \mu > 0,\ \lambda + \frac{2}{3}\mu \geq 0$$

Initial and boundary conditions :

$$\varrho(0) = \varrho_0,\ \varrho \mathbf{u}(0) = \mathbf{m}_0,\ \mathbf{u}|_{\partial\Omega} = 0.$$

Energy (in)equality :

$$\int_\Omega \Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)\mathrm{d}x\Big|_0^\tau + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \mathrm{d}x\mathrm{d}t = 0$$

$$H(\varrho) = \varrho \int_1^\varrho \frac{P(s)}{s^2}\mathrm{d}s \approx \varrho^\gamma$$

CNS :Weak solutions

1. Classical solutions versus weak solutions
2. General requirements on the definition of weak solutions :
   1. *Existence*
   2. *Compatibility* with classical solutions
   3. *Weak strong uniqueness property*
3. Weak formulation :
   1. Rewriting of equations in the integral form by using convenient "test functions"
   2. Postulating total energy balance as an inequality

CNS : bounded energy weak solutions

Functional spaces :

$$\varrho \geq 0,\ \ \varrho \in C_{\rm weak}(\bar{I}; L^\gamma(\Omega)),\ \ \mathbf{u} \in L^2(I; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\varrho|\mathbf{u}|^2 \in L^\infty(I; L^1(\Omega)),\ \ \ \varrho\mathbf{u} \in C_{\rm weak}(\bar{I}; L^q(\Omega)),\ \ q > 1.$$

Continuity equation :

$$\int_\Omega \varrho(\tau,\cdot)\varphi(\tau,\cdot)\ {\rm d}x - \int_\Omega \varrho_0(\cdot)\varphi(0,\cdot)\ {\rm d}x = \int_0^\tau \int_\Omega \big(\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\big){\rm d}x{\rm d}t$$

for all $\tau \in \bar{I}$, $\forall \varphi \in C_c^1(\bar{I}\times\overline{\Omega})$

CNS weak solutions : continued

Momentum equation :

$$\int_\Omega \varrho\mathbf{u}\cdot\varphi(\tau,\cdot)\mathrm{d}x - \int_\Omega \varrho_0\mathbf{u}_0\cdot\varphi(0,\cdot)\mathrm{d}x = \int_0^\tau\int_\Omega \Big(\varrho\mathbf{u}\cdot\partial_t\varphi$$

$$+\ \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + P(\varrho)\mathrm{div}_x\varphi - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi\Big)\mathrm{d}x\mathrm{d}t,$$

$$\forall\tau\in\bar{I},\quad \varphi\in C^1_c([0,T)\times\Omega;\mathbb{R}^3).$$

Energy inequality :

$$\int_\Omega\Big(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho)\Big)(\tau,\cdot)\mathrm{d}x\Big|_0^\tau + \int_0^\tau\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\mathrm{d}x\mathrm{d}t \le 0$$

for a.a. $\tau\in I$.

Renormalized solutions of the continuity equation

1. $$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \quad \times B'(\varrho)$$
   implies
   $$\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\mathbf{u}) + (\varrho B'(\varrho) - B(\varrho))\mathrm{div}\mathbf{u} = 0$$
   with any $B \in C^1[0,\infty)$.
2. We say that ϱ is *renormalized* solution of the continuity equation iff
   $$\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\mathbf{u}) + (\varrho B'(\varrho) - B(\varrho))\mathrm{div}\mathbf{u} = 0,$$
   in the weak sense for all B Lipschitz on $[0,\infty)$.
3. DiPerna-Lions : If $\varrho \in L^2(I; L^2(\Omega))$ is a weak solution of c.e. with $\mathbf{u} \in L^2(I; W^{1,2}(\Omega))$ then it is a renormalized solution.

Existence result, $\mathbf{u}|_{\partial\Omega} = 0$

Theorem [Lions (1998) $\gamma \geq 9/5$, Feireisl(2000) $\gamma > 3/2$, Feireisl, Petzeltová, N. (2001)]
The compressible Navier-Stokes equations in barotropic regime admit at least one weak solution with finite energy initial data.

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1. $(\varrho_n, \mathbf{u}_n)$ is sequence of approximations : Main goal is to show that $\varrho_n \to \varrho$ a.e. in $Q_T \Rightarrow P(\varrho_n) \to P(\varrho)$.
2. Improved estimates of density ($\gamma \geq 9/5 \Rightarrow \varrho \in L^\beta(Q_T)$, $\beta \geq 2$)
3. Effective viscous flux identity
$$0 \leq \overline{P(\varrho)\varrho} - \overline{P(\varrho)}\,\varrho = (2\mu + \lambda)\Big(\overline{\varrho \mathrm{div}\mathbf{u}} - \varrho \mathrm{div}\mathbf{u}\Big).$$
4. If $\gamma \geq 9/5$ solutions of the continuity equation are renormalized solutions. RCE+ EVF $\Rightarrow$ strong convergence of density.
5. Oscillations defect measure is bounded $\Rightarrow$ Solutions of the limiting continuity equation are renormalized. RCE+EVF $\Rightarrow$ strong convergence of density.

Alternative approach by D. Bresch and P.E. Jabin for $\gamma \geq 9/5$ (2020).

Interaction of EVF with renormalization

1. Effective viscous flux identity :

$$\overline{\varrho \mathrm{div}\mathbf{u}} - \varrho \mathrm{div}\mathbf{u} \geq 0$$

2. RCE : $\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\mathbf{u}) + (\varrho B'(\varrho) - B(\varrho))\mathrm{div}\mathbf{u} = 0.$:
3. RCE with $B(\varrho) = \varrho \log \varrho$ (i.e. $\varrho B'(\varrho) - B(\varrho) = \varrho$)
   1. At level n : $\partial_t(\varrho_n \log \varrho_n) + \mathrm{div}(\varrho_n \log \varrho_n \mathbf{u}_n) = -\varrho_n \mathrm{div}\mathbf{u}_n$
   2. After limit $n \to \infty$ with test function 1 :

$$\int_\Omega \overline{\varrho \log \varrho(\tau)}\, \mathrm{d}x - \int_\Omega \varrho_0 \log \varrho_0\, \mathrm{d}x = -\int_0^\tau \int_\Omega \overline{\varrho \mathrm{div}\mathbf{u}}\, \mathrm{d}x\mathrm{d}t$$

   3. At the limiting level with test function 1 :

$$\int_\Omega \varrho \log \varrho(\tau)\, \mathrm{d}x - \int_\Omega \varrho_0 \log \varrho_0\, \mathrm{d}x = -\int_0^\tau \int_\Omega \varrho \mathrm{div}\mathbf{u}\, \mathrm{d}x\mathrm{d}t$$

4. Conclusion :

$$\int_\Omega \Big(\overline{\varrho \log \varrho} - \varrho \log \varrho\Big)\, \mathrm{d}x = \int_0^\tau \int_\Omega \Big(\varrho \mathrm{div}\mathbf{u} - \overline{\varrho \mathrm{div}\mathbf{u}}\Big)\, \mathrm{d}x\mathrm{d}t \leq 0$$

A bi-fluid system

1. $$\partial_t\alpha + (\mathbf{u}\cdot\nabla)\alpha = 0, \quad 0 \le \alpha \le 1,\ \alpha(0) = \alpha_0 \in (0,1),$$
   $$\partial_t z + \mathrm{div}(z\mathbf{u}) = 0,\ z(0) = z_0$$
   $$\partial_t \varrho + \mathrm{div}(\varrho\mathbf{u}) = 0,\ \varrho(0) = \varrho_0$$
   $$\partial_t((\varrho+z)\mathbf{u}) + \mathrm{div}((\varrho+z)\mathbf{u}\otimes\mathbf{u}) + \nabla P(f(\alpha)\varrho, g(\alpha)z) = \mathrm{div}\mathbb{S}(\nabla_x\mathbf{u}),\ \mathbf{m}_0,$$
2. Transformed system : $R = f(\alpha)\varrho,\ Z = g(\alpha)z$
   $$\partial_t Z + \mathrm{div}(Z\mathbf{u}) = 0,\ Z_0 = g(\alpha_0)z_0,$$
   $$\partial_t R + \mathrm{div}(R\mathbf{u}) = 0,\ R_0 = f(\alpha_0)\varrho_0$$
   $$\partial_t z + \mathrm{div}(z\mathbf{u}) = 0,\ z(0) = z_0,$$
   $$\partial_t \varrho + \mathrm{div}(\varrho\mathbf{u}) = 0,\ \varrho(0) = \varrho_0$$
   $$\partial_t((\varrho+z)\mathbf{u}) + \mathrm{div}((\varrho+z)\mathbf{u}\otimes\mathbf{u}) + \nabla P(R,Z) = \mathrm{div}\mathbb{S}(\nabla_x\mathbf{u}),$$
   $$(\varrho+z)\mathbf{u}(0) = \mathbf{m}_0$$

Principal difficulties

1. Apparent difficulty is limit in $P(R_n, Z_n) = P(R_n, s_nR_n)$, $s_n = Z_n/R_n$.

$$P(R_n, s_nR_n) = P(R_n, sR_n) + [P(R_n, s_nR_n) - P(R_n, s_nR_n)P(R_n, sR_n)]$$

$$= \Pi(R_n, t, x) + \partial_Z P(R_n, \zeta_n)R_n(s_n - s)$$

A sort of compactness of $s_n = Z_n/R_n$ is needed

2. Passage from the "transformed system" (with continuity equations) to the original system (with one pure transport and one continuity equation). Formally α, $\tilde{\alpha}$,

$$f(\alpha) = \frac{R}{\varrho} \text{ and } g(\tilde{\alpha}) = \frac{Z}{z},$$

verify the pure transport equation with the same initial data.
A sort of uniqueness for the pure transport equation is needed

1. Vasseur, Wen, Yu (2019) :

$$\partial_t Z + \mathrm{div}(Z\mathbf{u}) = 0, \quad Z(0) = Z_0,$$

$$\partial_t R + \mathrm{div}(R\mathbf{u}) = 0, \quad R(0) = R_0$$

$$\partial_t((R+Z)\mathbf{u}) + \mathrm{div}((R+Z)\mathbf{u}\otimes\mathbf{u}) + \nabla P(R,Z) = \mathrm{div}\mathbb{S}(\nabla_x\mathbf{u}),$$

$$P(R,Z) = R^\gamma + Z^\beta$$

2. "Almost compactness" : *Let* $(R_n, \mathbf{u}_n)$, $(Z_n, \mathbf{u}_n)$ *satisfy continuity equation and let* $0 \le R_n \rightharpoonup_* R$, $0 \le Z_n \rightharpoonup_* Z$ *in* $L^\infty(I; L^2(\Omega))$, $\mathbf{u}_n \rightharpoonup \mathbf{u}$ *in* $L^2(I; W_0^{1,2}(\Omega))$. *Let* $0 \le s_n, s \le C$, *be functions such that* $s_n R_n = Z_n$, $sR = Z$, *then*

$$\int_0^T \int_\Omega R_n (s_n - s)^2 \,\mathrm{d}x\mathrm{d}t \to 0.$$

3. Pokorny, N. (2020) - Revisiting Vasseur, Wen, Yu from the point of view of the theory of renormalized solutions to the transport equations gives a slightly different formulation of *Almost compactness* by Vasseur and collaborators and the property of *Almost uniqueness*.

DiPerna-Lions ⇒ Renormalized solutions to CE

1. Let $0 \le \varrho \in L^2(I; L^2(\Omega))$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega))$ satisfy continuity equation in the weak sense :

$$\int_{Q_T} \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi \Big) \mathrm{d}x \mathrm{d}t = 0, \ \forall \varphi \in C_c^1(I \times \overline{\Omega}). \tag{3}$$

Then it satisfies the continuity equation in the renormalized sense

$$\int_{Q_T} \Big(B(\varrho)\partial_t \varphi + B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - (\varrho B'(\varrho) - B(\varrho)) \mathrm{div}\mathbf{u} \varphi \Big) \mathrm{d}x \mathrm{d}t = 0 \tag{4}$$

with any $\varphi \in C_c^1(I \times \overline{\Omega})$, $B \in C^1[0,\infty)$, $B' \in L^\infty(0,\infty)$.

2. If moreover $\varrho \in L^\infty(I, L^\gamma(\Omega))$, $\gamma > 1$ then $\varrho \in C(\overline{I}; L^1(\Omega))$ and equations (3), (4) hold in the time integrated form (with test functions in $\varphi \in C^1(\overline{Q_T})$) :

$$\int_\Omega B(\varrho)\varphi(\tau, x) \mathrm{d}x - \int_\Omega B(\varrho(0,x))\varphi(0,x) \mathrm{d}x$$

$$= \int_0^\tau \int_\Omega \Big(B(\varrho)\partial_t \varphi + B(\varrho)\mathbf{u} \cdot \nabla \varphi - (\varrho B'(\varrho) - B(\varrho)) \mathrm{div}\mathbf{u} \varphi \Big) \mathrm{d}x \mathrm{d}t$$

Regularization procedure

We extend $(\varrho, \mathbf{u})$ *by* $(0,0)$ *outside* Ω. The extended couple verifies

$$\partial_t \varrho + \mathrm{div}(\varrho \mathbf{u}) = 0 \text{ in } \mathcal{D}'(I \times \mathbb{R}^3).$$

We regularize equation by using mollifiers :

$$\partial_t [\varrho]_\varepsilon + \mathrm{div}([\varrho]_\varepsilon \mathbf{u}) = \mathfrak{R}_\varepsilon := \mathrm{div}([\varrho]_\varepsilon \mathbf{u}) - \mathrm{div}([\varrho \mathbf{u}]_\varepsilon) \text{ a.e. in } Q_T.$$

This implies (multiplication by $B'([\varrho]_\varepsilon)$),

$$\partial_t B([\varrho]_\varepsilon) + \mathrm{div}(B([\varrho]_\varepsilon)\mathbf{u}) + ([\varrho]_\varepsilon B'([\varrho]_\varepsilon) - B([\varrho]_\varepsilon))\mathrm{div}\mathbf{u} = \mathfrak{R}_\varepsilon B'([\varrho]_\varepsilon).$$

and we get renormalized continuity equation as $\varepsilon \to 0$, provided $\mathfrak{R}_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(I \times \Omega)$:

$$\partial_t B(\varrho) + \mathrm{div}(B(\varrho)\mathbf{u}) + (\varrho B'(\varrho) - B(\varrho))\mathrm{div}\mathbf{u} = 0 \text{ in } \mathcal{D}'(I \times \mathbb{R}^3).$$

PTE : From weak to renormalized time integrated solutions

1. Let $0 \le s \in L^\infty(Q_T)$, $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega))$ satisfy the pure transport equation

$$\partial_t s + \mathbf{u} \cdot \nabla_x s = 0 \text{ weakly in } Q_T.$$

Then $s \in C(\bar{I}; L^1(\Omega))$ and it satisfies the time integrated transport equation in the renormalized sense up to the boundary :

$$\int_\Omega B(s)\varphi \, \mathrm{d}x \Big|_0^\tau = \int_0^\tau \int_\Omega \Big(B(s)\partial_t\varphi + B(s)\mathbf{u} \cdot \nabla_x \varphi + B(s)\mathrm{div}\mathbf{u}\varphi \Big) \, \mathrm{d}x\mathrm{d}t$$

for all $\tau \in \bar{I}$, for all $\varphi \in C_c^1(\bar{I} \times \overline{\Omega})$ with any $B \in C^1[0,\infty)$.

2. Holds also for $B(s_1, s_2) \ldots$

Some formal calculations

$$[\partial_t R + \mathbf{u}\cdot\nabla_x R + R\mathrm{div}\mathbf{u} = 0] \times (-\frac{Z}{R^2}) \ \Rightarrow \ Z\partial_t(\frac{1}{R}) + Z\mathbf{u}\cdot\nabla_x(\frac{1}{R}) - \frac{Z}{R}\mathrm{div}\mathbf{u} = 0$$

$$[\partial_t Z + \mathbf{u}\cdot\nabla_x Z + Z\mathrm{div}\mathbf{u} = 0] \times (\frac{1}{R}) \ \Rightarrow \ \frac{1}{R}\partial_t Z + \frac{1}{R}\mathbf{u}\cdot\nabla_x Z + \frac{Z}{R}\mathrm{div}\mathbf{u} = 0$$

$$\partial_t(\frac{Z}{R}) + \mathbf{u}\cdot\nabla_x(\frac{Z}{R}) = 0$$

What we are doing is :

1. Take $B(R,Z) = Z/R$.
2. Multiply continuity equation for R and multiply by $\partial_R B(R,Z)$.
3. Multiply continuity equation for Z by $\partial_Z B(R,Z)$.
4. $B(R,Z)$ is not good renormalizing function (we have to take $B_\delta(R,Z) = Z/(R+\delta)$ and then let $\delta \to 0$ - by Lebesgeue dominated convergence theorem)
5. For the Lebesgue dominated convergence one needs the domination condition $0 \le Z \le \overline{a}R$

Lemma 1 : From CE to PTE

Let

$$R \in L^2(Q_T) \cap L^\infty(I; L^\gamma(\Omega)),\ \gamma > 1,$$

$$\forall t \in \bar{I},\ 0 \le Z \le \bar{a}R,\ \mathbf{u} \in L^2(I, W_0^{1,2}(\Omega)),$$

satisfy

$$\partial_t R + \mathrm{div}(R\mathbf{u}) = 0,\ \partial_t Z + \mathrm{div}(Z\mathbf{u}) = 0 \text{ in the weak sense in } Q_T. \quad (5)$$

Then, in particular, $R, Z \in C(\bar{I}, L^1(\Omega))$ and we can define

$$\forall t \in \bar{I},\ s(t,x) := \frac{Z(t,x)}{R(t,x)} \text{ if } R(t,x) > 0,\ s(t,x) := a \in \mathbb{R} \text{ otherwise.} \quad (6)$$

Then $s \in C(\bar{I}; L^1(\Omega))$ and

$$\partial_t B(s) + \mathbf{u} \cdot \nabla_x B(s) = 0 \quad (7)$$

holds with any $B \in C^1[0, \infty)$ in the time integrated form and up to the boundary.

Some formal calculation

$$[\partial_t B(s) + \mathbf{u} \cdot \nabla_x B(s) = 0] \times R \;\Rightarrow\; R\partial_t B(s) + R\mathbf{u} \cdot \nabla_x B(s) = 0$$

$$[\partial_t R + \mathbf{u} \cdot \nabla_x R + R\mathrm{div}\mathbf{u} = 0] \times B(s) \;\Rightarrow\; B(s)\partial_t R + B(s)\mathbf{u} \cdot \nabla_x R + RB(s)\mathrm{div}\mathbf{u} = 0$$

$$\partial_t (RB(s)) + \mathrm{div}(RB(s)\mathbf{u}) = 0$$

Lemma 2 : From PTE to CE

1. Let

$$0 \le R \in L^2(Q_T) \cap L^\infty(I; L^\gamma(\Omega)), \ \mathbf{u} \in L^2(I, W_0^{1,2}(\Omega)), \ 0 \le s \in L^\infty(Q_T)$$

and let couple $(R, \mathbf{u})$ satisfy the continuity equation and couple $(s, \mathbf{u})$ the pure transport equation in the weak sense. Then :

2. Then

$$s, R, RB(s) \in C(\bar{I}; L^1(\Omega))$$

and $RB(s)$ satisfies continuity equation in the time integrated form and up to the boundary.

3. Holds also for $RB(s_1, s_2)$, $B \in C^1([0, \infty)^2)$.

Lemma 3 : Almost uniqueness for the pure transport equation

Let $\mathbf{u} \in L^2(I; W_0^{1,2}(\Omega; R^3))$. Let $0 \le s_i \in L^\infty(Q_T)$, $i = 1, 2$ be two weak solutions of the pure transport equation in the weak sense (up to the boundary). Then $s_i \in C(\bar{I}, L^1(\Omega))$. If moreover $s_1(0, \cdot) = s_2(0, \cdot)$ then

$$\text{for all } \tau \in \bar{I}\ s_1(\tau, \cdot) = s_2(\tau, \cdot) \text{ for a.a. } x \in \{\varrho(\tau, \cdot) > 0\}, \tag{8}$$

where ϱ is *any* time integrated weak solution to the continuity equation with the same transporting velocity in the class $0 \le \varrho \in C(\bar{I}, L^1(\Omega)) \cap L^2(Q_T) \cap L^\infty(I; L^p(\Omega))$, $p > 1$.

1. Lemma 3 can be viewed as extension of the results of Di Perna -Lions (1989) and Bianchini-Bonicato (2018) in the following sense :
2. It yields uniqueness under assumption $\mathrm{div}\mathbf{u} \in L^1(I; L^\infty))$ (which is classical result of DL, 1989)
3. It yields uniqueness under weaker assumption than DL namely that ”continuity equation with transporting velocity $\mathbf{u}$ admits a strictly positive and bounded distributional solution” (which is what can be deduced from BB, 2018).

Sketch of proof

- $s_i \in C(\bar{I}; L^1(\Omega))$ is time integrated weak solution of the PTE.
- $(s_1 - s_2)^2$ is also time integrated weak solution of the PTE.
- $\varrho \in C(\bar{I}; L^1(\Omega))$ is time integrated weak solution to the continuity equation.
- $\varrho(s_1 - s_2)^2$ is time integrated weak solution of the continuity equation.
- Take in the latter $\varphi = 1$:

$$\forall \tau \in \bar{I}, \int_\Omega \varrho(s_1 - s_2)^2(\tau)\, \mathrm{d}x = \int_\Omega \varrho(s_1 - s_2)^2(0)\, \mathrm{d}x.$$

Lemma 4 : Convergence induced by Lemmas 1 and 2

Let

1. $$\mathbf{u}_n \in_b L^2(I, W_0^{1,2}(\Omega)), \ \varrho_n \in_b L^2(Q_T) \cap L^\infty(I; L^q(\Omega)), \ 0 \le Z_n \le \overline{a}\varrho_n$$

   be bounded sequences.
2. Suppose that both couples $(\varrho_n, \mathbf{u}_n)$, $(Z_n, \mathbf{u}_n)$ satisfy continuity equation (3) with initial data ϱ_0 resp. Z_0.

Convergence induced by L1 and L2 continued

Then we have :

1. Up to a subsequence (not relabeled)

$$(\varrho_n, Z_n) \to (\varrho, Z) \text{ in } C_{\rm weak}(\bar I; L^q(\Omega)),\ \mathbf{u}_n \rightharpoonup \mathbf{u} \text{ in } L^2(I; W^{1,2}(\Omega)),$$

   where $(\varrho, \mathbf{u})$ as well as (Z,**u**) verify continuity equation in the renormalized sense.

2. Define sequence $s_n(t,x)$ and function $s(t,x)$ as in (6). Then $s_n, s \in C(\bar I; L^q(\Omega))$, $1 \le q < \infty$ and for all $t \in \bar I$, $0 \le s_n(t,x) \le \bar a$, $0 \le s(t,x) \le \bar a$ for a. a. $x \in \Omega$. Moreover, both $(s_n, \mathbf{u}_n)$ and $(s, \mathbf{u})$ satisfy transport equation up to the boundary.

3. Finally,

$$\int_\Omega \varrho_n |s_n - s|^2(\tau, x) {\rm d}x \to 0 \text{ for all } \tau \in \bar I. \tag{9}$$

Sketch of proof

1. Let $s_n(t) := Z_n(t)/\varrho_n(t)$. Then $s_n \in C(\bar{I}; L^1(\Omega))$ and $(s_n, \mathbf{u}_n)$ satisfies time integrated weak formulation of PTE.
2. $\varrho_n s_n^2 \in C(\bar{I}; L^1(\Omega)$ and $(\varrho_n s_n^2, \mathbf{u}_n)$ satisfies time integrated weak formulation of CE.
3. $Z, \varrho \in C(\bar{I}; L^1(\Omega))$ and $(Z, \mathbf{u})$, $(\varrho, \mathbf{u})$ satisfy time integrated weak formulation of CE
4. $s = Z/\varrho \in C(\bar{I}; L^1(\Omega))$ and $(s, \mathbf{u})$ satisfies time integrated weak formulation of PTE.
5. $\varrho s^2 \in C(\bar{I}; L^1(\Omega))$ and $(\varrho s^2, \mathbf{u})$ satisfies time integrated weak formulation of CE.
6. $\int_\Omega \varrho s^2(\tau)\,\mathrm{d}x = \int_\Omega \varrho_0 s_0^2\,\mathrm{d}x$
7. $\int_\Omega \varrho_n s_n^2(\tau)\,\mathrm{d}x = \int_\Omega \varrho_0 s_0^2\,\mathrm{d}x$
8. $\lim_{n\to\infty} \int_\Omega \varrho_n s_n s(\tau)\,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega Z_n s(\tau)\,\mathrm{d}x = \int_\Omega Z s(\tau)\,\mathrm{d}x = \int_\Omega \varrho s^2(\tau)\,\mathrm{d}x$

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CNSE with non zero inflo-outflow

1. Boundary data :

$$0 \le \varrho_B \in C(\mathbb{R}^3),\ \mathbf{u}_B \in C^1_c(\mathbb{R}^3),\ \mathbf{u}_B = 0 \text{ on } \mathfrak{g}$$

2. Weak formulation of the continuity equation : There is

$$0 \le \varrho \in C_{\rm weak}(\bar I; L^\gamma(\Omega)),\ \varrho \in L^\gamma(I; L^\gamma(\Gamma^{\rm out}; |\mathbf{u}_B \cdot \mathbf{n}| {\rm d}S_x)),$$

$$\mathbf{u} - \mathbf{u}_B \in L^2(I; W^{1,2}_0(\Omega))$$

such that

$$\int_\Omega \varrho\varphi(\cdot, x)\ {\rm d}x\Big|_0^\tau + \int_0^\tau \int_{\Gamma^{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n}\varphi {\rm d}S_x {\rm d}t + \int_0^\tau \int_{\Gamma^{\rm out}} \varrho \mathbf{u}_B \cdot \mathbf{n}\varphi {\rm d}S_x {\rm d}t,$$

$$= \int_0^\tau \int_\Omega \Big(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi\Big) {\rm d}x {\rm d}t$$

for all $\tau \in \bar I$ and $\varphi \in C^1(\bar I \times \overline{\Omega})$.

Non-zero inflow/outflow b.c. : Extension lemma

Suppose that $(\varrho, \mathbf{u} - \mathbf{u}_B) \in [L^2(I \times \Omega) \cap L^\gamma(I; L^\gamma(\Gamma^{\rm out}))] \times L^2(I; W_0^{1,2}(\Omega))$ satisfies continuity equation in the weak sense :

$$\int_I \int_\Omega \left(\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi\right) {\rm d}x{\rm d}t$$

$$= \int_I \int_{\Gamma^{\rm in}} \varrho_B \mathbf{u}_B \cdot \mathbf{n} \varphi \ {\rm d}S_x {\rm d}t + \int_I \int_{\Gamma^{\rm out}} \varrho \mathbf{u}_B \cdot \mathbf{n} \varphi \ {\rm d}S_x {\rm d}t,$$

$\forall \varphi \in C^1_c(I \times \overline{\Omega})$, then it satisfies the renormalized continuity equation

$$\int_0^T \int_\Omega \Big(B(\varrho)\partial_t \varphi + B(\varrho)\mathbf{u} \cdot \nabla_x \varphi - \varphi(\varrho B'(\varrho) - B(\varrho){\rm div}\mathbf{u}\Big){\rm d}x{\rm d}t =$$

$$\int_0^T \int_{\Gamma^{\rm in}} B(\varrho_B)\mathbf{u}_B \cdot \mathbf{n}\varphi \ {\rm d}S_x {\rm d}t \ + \int_0^T \int_{\Gamma^{\rm out}} B(\varrho)\mathbf{u}_B \cdot \mathbf{n}\varphi \ {\rm d}S_x {\rm d}t$$

Extension lemma : Sketch of the proof

Extension outside $\Gamma = \Gamma^{\rm in}$ (recall Γ is C^2 parametrized surface).

Step 1. A lemma of differential geometry (Foote) : There are open sets $T^+ \subset \mathbb{R}^3 \setminus \overline{\Omega}$, $T^- \subset \Omega$, $T := T^+ \cup T^- \cup \Gamma$ open, such that

1. $\forall x \in T$, $\exists ! P(x) \in \Gamma$, $|x - P(x)| = d_\Gamma(x)$.
2. $P \in C^1(\overline{T})$, $d_\Gamma \in C^2(\overline{T^\pm})$

Step 2. We examine the flow of $-\mathbf{u}_B$:

$$\frac{d}{dt}\mathfrak{X}(s;x) = -\mathbf{u}_B(\mathfrak{X}),\ \mathfrak{X}(0,x) = x.$$

1. $\mathfrak{X} \in C^1(\mathbb{R} \times \mathbb{R}^3)$, $\mathfrak{X}(s,\cdot)$ is a diffeomorphism $\mathbb{R}^3 \mapsto \mathbb{R}^3$.
2. The map

$$\Phi : \mathbb{R} \times \Gamma \to \mathfrak{X}(\mathbb{R},\Gamma) \subset \mathbb{R}^3 : \Phi(s,x) = \mathfrak{X}(s,x)$$

is a local diffeomorphism with the determinant of the Jacobi matrix > 0 (proportional to $\mathbf{u}_B \cdot \mathbf{n}$).

Step 3.

1. There is an open set $\{0\} \times \Gamma \subset V \subset \mathbb{R} \times \Gamma$ such that $\Phi|_V$ is a diffeomorphism of V onto (open set) $U = \Phi(V) \subset T$. Moreover, if $V^{\pm} = V \cap \mathbb{R}^*_{\pm} \times \Gamma$, then $U^{\pm} := \Phi(V^{\pm}) \subset T^{\pm}$.
2. Thus : for all $\xi \in U$ there exists a unique $(s, x_B) \in V$ such that $\xi = \mathfrak{X}(s; x_B)$.
3. We set $\tilde{\Omega} = U^+ \cup \Gamma \cup \Omega$ and

$$\tilde{\mathbf{u}}(t,x) = \left\{ \begin{array}{c} \mathbf{u}(t,x), \ x \in \Omega \\ \mathbf{u}_B(x), \ x \in \overline{U}^+ \end{array} \right\}$$

$$\tilde{\varrho}(t,x) = \left\{ \begin{array}{c} \varrho(t,x), \ x \in \Omega \\ \varrho_B(x_B)\exp\Big(\int_0^s \mathrm{div}\mathbf{u}_B(\mathfrak{X}(z;x_B))\mathrm{d}z\Big), \ x = \mathfrak{X}(s,x_B) \in \overline{U}^+ \end{array} \right\}.$$

Step 4 :
We have $(\tilde{\varrho}, \tilde{\mathbf{u}}) \in C^1(\bar{I} \times \overline{U^+})$ and

$$\partial_t \tilde{\varrho} + \mathrm{div}(\tilde{\varrho}\tilde{\mathbf{u}}) = 0 \text{ in } \bar{I} \times \overline{U^+} \;\Rightarrow$$

$$\partial_t \tilde{\varrho} + \mathrm{div}(\tilde{\varrho}\tilde{\mathbf{u}}) = 0 \text{ in } \mathcal{D}'(I \times \tilde{\Omega})$$

to which we can apply DiPerna-Lions' regularization technique :

$$\int_I \int_{\tilde{\Omega}} \Big(B(\tilde{\varrho})\partial_t \varphi + B(\tilde{\varrho})\mathbf{u} \cdot \nabla_x \varphi - (\tilde{\varrho}B'(\tilde{\varrho}) - B(\tilde{\varrho}))\mathrm{div}\mathbf{u}\varphi \Big) \mathrm{d}x\mathrm{d}t = 0$$

Seeing that

$$\partial_t B(\tilde{\varrho}) + \mathrm{div}(B(\tilde{\varrho}\tilde{\mathbf{u}})) + (\tilde{\varrho}B'(\tilde{\varrho}) - B(\tilde{\varrho}))\mathrm{div}\tilde{\mathbf{u}} = 0 \;\text{ in } \bar{I} \times \overline{U^+}.$$

we obtain the result.