Measures as Graph Limits

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Winter School in Abstract Analysis
Sněžné, 11 January 2022

Motivation

Let $\{G_i\}_{i=1}^{\infty}$ be a sequence of graphs.

- What does it mean that $\{G_i\}_{i=1}^{\infty}$ is convergent?
- What is the limit of the sequence?
- Can we always find a convergent subsequence?

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A graph is a pair (V, E) where V is a finite set and E is a family of two-element subsets of V.

Graph convergence

- Benjamini-Schramm convergence

  (I. Benjamini, O. Schramm, 2001)
- Local-global convergence

  (B. Bollobás, O. Riordan, 2011;

  H. Hatami, L. Lovász, B. Szegedy, 2014)
- Convergence of subgraph densities

  (L. Lovász, B. Szegedy, 2006;

  C. Borgs, J. T. Chayes, L. Lovász, V. T. Sós, K. Vesztergombi, 2008)
- Action convergence

  (A. Backhausz, B. Szegedy, 2018)
- s-convergence

  (D. Kunszenti-Kovács, L. Lovász, B. Szegedy, 2019)
- X-convergence

  (J. Nešetřil, P. Ossona de Mendez, 2020)

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sparse graph sequences dense graph sequences

arbitrary sequences

s-convergence

The limit objects of s-convergent sequences of graphs:
s-graphons = symmetric Borel probability measures on $[0,1]^2$

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- Let G_i be the random graph on i vertices with edge density $\frac{1}{2}$. Then, with probability 1, $\text{s-lim}_{i\to\infty} G_i = \lambda^2$.

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- Let G_i be the random graph on i vertices with edge density $\frac{1}{3}$. Then, with probability 1, $\text{s-lim}_{i\to\infty} G_i = \lambda^2$.

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  Then, with probability 1, $\text{s-lim}_{i\to\infty} G_i = \lambda^2$.
- Let $G_i = C_i$ be the cycle of length i.
  Then $\text{s-lim}_{i\to\infty} G_i = \mu_\alpha$ (for some $\alpha \in \mathbb{R} \setminus \mathbb{Q}$), where

$$\mu_\alpha(Z) = \frac{1}{2} \int_{x\in[0,1]} \left(\mathbb{1}_Z(x, x+\alpha \bmod 1) + \mathbb{1}_Z(x, x-\alpha \bmod 1)\right) d\lambda(x),$$

$$Z \subseteq [0,1]^2.$$

s-convergence

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Let G be a graph (with non empty edge set) on n vertices.
Let A_G be the adjacency matrix of G.
For every $k \in \mathbb{N}$ we define the *k-shape* $C(G, k)$ of G by

$$C(G, k) = \left\{ \frac{1}{\|A_G\|_1} \cdot M A_G M^T : M \in K(k, n) \right\} \subseteq \mathbb{R}^{k \times k}.$$

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Definition

A graph sequence $\{G_i\}_{i=1}^{\infty}$ is s-convergent if, for every $k \in \mathbb{N}$, the sequence $\{C(G_i,k)\}_{i=1}^{\infty}$ is convergent in the Vietoris topology of $\mathcal{K}(\mathbb{R}^{k \times k})$.

s-convergence

Recall: $K(k,n)$ is the set of all nonnegative $k \times n$ matrices with

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Let $f_1, f_2, \dots, f_k$ be nonnegative continuous functions on $[0,1]$ with

- $\sum_{j=1}^{k} f_j \equiv 1$,
- $\int_{[0,1]} f_j \, d\lambda = \frac{1}{k}$ for every j.

We define the k-shape $C(\mu, k)$ of an s-graphon μ by

$$C(\mu,k) = \overline{\{M(f_1,f_2,\dots,f_k) : f_1,f_2,\dots,f_k\}} \subseteq \mathbb{R}^{k\times k},$$

where $M(f_1,f_2,\dots,f_k)(i,j) = \int_{(x,y)\in[0,1]^2} f_i(x)f_j(y)\, d\mu(x,y)$.

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A graph sequence $\{G_i\}_{i=1}^{\infty}$ is s-convergent to an s-graphon μ if, for every $k \in \mathbb{N}$, the sequence $\{C(G_i, k)\}_{i=1}^{\infty}$ is convergent to $C(\mu, k)$ in the Vietoris topology of $\mathcal{K}(\mathbb{R}^{k \times k})$.

Basic facts

Fact

Every sequence of graphs has an s-convergent subsequence.

Theorem (Kunszenti-Kovács, Lovász, Szegedy, 2019)

If $\{G_i\}_{i=1}^{\infty}$ is an s-convergent sequence of graphs then there is an s-graphon μ such that $\{G_i\}_{i=1}^{\infty}$ s-converges to μ.

Theorem (Kunszenti-Kovács, Lovász, Szegedy, 2019)

For every s-graphon μ there is a sequence of graphs which s-converges to μ.

Alternative approach

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A non-negative continuous function $f\colon [0,1]^2 \to [0,\infty)$ is called fairly distributed (wrt λ) if for every $x, y \in [0,1]$ it holds

$$\int_{v\in[0,1]} f(x,v)\, d\lambda(v) = \int_{u\in[0,1]} f(u,y)\, d\lambda(u) = 1.$$

Let $\mathcal{FDC}$ denote the set of all fairly distributed functions on $[0,1]^2$.

Alternative approach

For every $\mu \in \mathfrak{s}\mathfrak{G}$ and every $f \in \mathcal{FDC}$ we define a function $\varphi(f, \mu) \in L^1([0,1]^2, \lambda^2)$ by

$$\varphi(f, \mu)(u, v) = \int_{(x,y)\in[0,1]^2} f(x, u) f(y, v)\, d\mu(x, y), \qquad u, v \in [0, 1].$$

Let $\Phi(f, \mu)$ be the absolutely continuous (wrt λ^2) measure on $[0, 1]^2$ with the Radon-Nikodym derivative $\varphi(f, \mu)$.

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Let $\Phi(f,\mu)$ be the absolutely continuous (wrt λ^2) measure on $[0,1]^2$ with the Radon-Nikodym derivative $\varphi(f,\mu)$.

For every $\mu \in \mathfrak{s}\mathfrak{G}$ we define the shape $C(\mu)$ of μ by

$$C(\mu) = \overline{\{\Phi(f,\mu) : f \in \mathcal{FDC}\}} \subseteq \mathfrak{s}\mathfrak{G}.$$

Main result

Theorem

Convergence of k-shapes is equivalent to convergence of shapes.

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That is, for s-graphons μ and μ_i, $i \in \mathbb{N}$, the following conditions are equivalent:

- $\forall_{k \in \mathbb{N}}\colon \lim_{i\to\infty} C(\mu_i, k) = C(\mu, k)$ *in the Vietoris topology of* $\mathcal{K}(\mathbb{R}^{k\times k})$,
- $\lim_{i\to\infty} C(\mu_i) = C(\mu)$ *in the Vietoris topology of* $\mathcal{K}(\mathfrak{s}\mathfrak{G})$.

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Similarly, for an s-graphon μ and graphs G_i, $i \in \mathbb{N}$, the following conditions are equivalent:

- $\forall_{k \in \mathbb{N}}\colon \lim_{i \to \infty} C(G_i, k) = C(\mu, k)$ *in the Vietoris topology of* $\mathcal{K}(\mathbb{R}^{k \times k})$,
- $\lim_{i \to \infty} C(G_i) = C(\mu)$ *in the Vietoris topology of* $\mathcal{K}(\mathfrak{s}\mathfrak{G})$.

Key steps of the proof

The k-shape $C(\mu, k)$ of an s-graphon μ is a subset of $\mathbb{R}^{k\times k}$.

But each $M \in C(\mu, k)$ can be naturally represented by an s-graphon μ_M.

So $C(\mu, k)$ can be represented by a subset $\widetilde{C}(\mu, k)$ of $\mathfrak{s}\mathfrak{G}$.

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Convergence of shapes $\Longrightarrow$ convergence of k-shapes:

Lemma

For every s-graphon μ and every $k \in \mathbb{N}$ we have

$$\widetilde{C}(\mu, k) = C(\mu) \cap \left\{ \mu_M : M \in \mathbb{R}^{k \times k} \right\}.$$

Key steps of the proof

Convergence of k-shapes $\Longrightarrow$ convergence of shapes:

Lemma

For every s-graphon μ we have

$$C(\mu) = \overline{\bigcup_{k \in \mathbb{N}} \widetilde{C}(\mu, k)}.$$

Lemma

Let ρ be an arbitrary compatible metric on $\mathfrak{s}\mathfrak{G}$. Then for every $\varepsilon > 0$ there is $K \in \mathbb{N}$ such that for every $\mu \in \mathfrak{s}\mathfrak{G}$ we have

$$d_H^{\rho}\Big(C(\mu), \widetilde{C}(\mu, K)\Big) \leq \varepsilon,$$

where d_H^{ρ} is the Hausdorff distance on $\mathfrak{s}\mathfrak{G}$ obtained from ρ.

Final remarks

Two s-graphons μ_1 and μ_2 are isomorphic if $C(\mu_1, k) = C(\mu_2, k)$ for every $k \in \mathbb{N}$.

Question (Kunszenti-Kovács, Lovász, Szegedy, 2019)

Is there a more simple analytic characterization of isomorphism between s-graphons?

Corollary

Two s-graphons μ_1 and μ_2 are isomorphic if and only if $C(\mu_1) = C(\mu_2)$.

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- Kunszenti-Kovács, Dávid; Lovász, László; Szegedy, Balázs. *Measures on the square as sparse graph limits*. J. Combin. Theory Ser. B 138 (2019), 1–40.