INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Preprint No. 26-2022

PRAHA 2022

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Abstract. In a compactly generated triangulated category, we introduce a class of tilting objects satisfying certain purity condition. We call these the decent tilting objects and show that the tilting heart induced by any such object is equivalent to a category of contramodules over the endomorphism ring of the object endowed with a natural linear topology. This extends the recent result for n-tilting modules of Positselski and Šťovíček. In the setting of the derived category of a ring, we show that the decent tilting complexes are precisely the silting complexes such that their character dual is cotilting. We show that the hearts of cotilting complexes of finite type are equivalent to the category of discrete modules with respect to the same topological ring. Finally, we show that decent tilting complexes parametrize pairs consisting of a tilting and a cotilting derived equivalence as above together with a tensor compatibility condition.

Contents

1. Introduction

In his landmark result [Ric89], Rickard established a full Morita theory for derived categories of modules: There is a triangle equivalence $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S)$ between the bounded derived categories of right modules if and only if there is a compact tilting complex T in $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ with $S \cong \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$. In addition, the triangle equivalence can be represented by the derived functor $\mathbf{R}\mathsf{Hom}_R(T,-)$; this fact is best observed using the formal endomorphism dg-ring of T, as explained by Keller [Kel93]. The whole picture can be made more symmetric following the observation made in [Ric91]: T also induces a triangle equivalence $T \otimes_R^{\mathbf{L}} - : \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \cong \mathsf{D}^{\mathsf{b}}(S\text{-}\mathsf{Mod})$ on the side of left modules, and these equivalences are compatible with the tensor products in the sense that the following diagram commutes:

$$(1) \qquad \begin{array}{ccc} \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \times \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{-\otimes_R^{\mathbf{L}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \\ {\scriptstyle (\mathbf{R}\mathsf{Hom}_R(T,-),\, T\otimes_R^{\mathbf{L}}-)}\downarrow\cong & & \downarrow = \\ \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S) \times \mathsf{D}^{\mathsf{b}}(S\text{-}\mathsf{Mod}) & \xrightarrow{-\otimes_S^{\mathbf{L}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \end{array}$$

More recently, efforts were made to see to which extent the theory can be stated for tilting objects which are not necessarily compact. For a survey of both the history and the modern aspects of the theory of "large" silting and tilting objects we refer to [AH19]. When the tilting complex T is not

2020 *Mathematics Subject Classification.* Primary: 13D09, 14F08; Secondary: 16D90, 16S50.
The author was supported by the GAČR project 20-13778S and RVO: 67985840.

compact, its endomorphism ring cannot be derived equivalent to R. However, Bazzoni showed in [Baz10] that if T is a 1-tilting module which has the additional property of being good, then the derived category of R embeds via $\mathbf{R}\mathsf{Hom}_R(T,-)$ to the derived category of $\mathsf{End}_R(T)$, and in fact, this fully faithful functor is a part of a recollement of triangulated categories. The assumption of being good is very mild in the sense that every tilting module is additively equivalent to a good one. This was later extended to n-tilting modules by Bazzoni, Mantese, and Tonolo [BMT11], and to a general setting of dg categories by Nicolás and Saorín [NS18].

A very recent breakthrough came in the paper [PŠ21]. There, the endomorphism ring of a tilting module T is endowed with a linear topology such that the resulting topological ring $\mathfrak{S} = \mathsf{End}_R(T)$ is complete and separated. Such a structure comes associated with an abelian category of right $\mathfrak{S}$-contramodules. This category can be morally viewed as the well-behaved replacement of the ill-behaved category of complete and separated $\mathfrak{S}$-modules. The theory of contramodules, developed chiefly by Positselski, has quickly found many strong applications in algebra and algebraic geometry, see e.g. the survey [Pos21] and [Pos12]. Positselski and Šťovíček showed in [PŠ21] that the heart of the tilting t-structure induced by T is equivalent to the category $\mathsf{Ctra}\text{-}\mathfrak{S}$ of right $\mathfrak{S}$-contramodules. Assuming again the mild additional condition of T being good, the forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{S} \to \mathsf{Mod}\text{-}\mathfrak{S}$ is fully faithful, both on the abelian and the derived level. It follows that the image of the fully faithful functor $\mathbf{R}\mathsf{Hom}_R(T,-)$ in the setting of [BMT11] is now given an algebraic description as the derived category of contramodules.

In the present paper, we wish to extend the latter result from tilting modules to tilting complexes, or more generally, to silting objects in the sense of Psaroudakis and Vitória [PV18] and Nicolás, Saorín, and Zvonareva [NSZ19]. It turns out that this does not work without additional assumptions — there are tilting complexes whose tilting heart cannot be equivalent to a category of contramodules over any complete and separated topological ring (Example 3.12). Therefore, we are forced to find a condition on a tilting complex which guarantees a better behavior. The condition we consider comes from the purity theory of compactly generated triangulated categories as established by Krause [Kra00], and rely on the new techniques using Grothendieck derivators developed recently by Laking [Lak20]. In Section 2, we introduce the notion of a decent tilting object as a silting object such that its definable closure is contained in the silting heart. Such objects are automatically tilting in the sense of [PV18], and therefore are expected to provide derived equivalences. In the generality of a compactly generated triangulated category, we show that the heart of the t-structure induced by a decent tilting object is equivalent to a contramodule category of the endomorphism ring $\mathfrak{S}$ endowed with a suitable linear topology we call the compact topology. The way we prove this is by using the restricted Yoneda functor to reduce the problem from the triangulated setting to the abelian setting of modules over a ringoid, where the results of [PŠ21] apply directly.

In Section 3, we specialize to the setting of derived category of modules over a ring. There, it turns out that our condition has a very natural interpretation: For any ring R, a (bounded) silting complex T of right R-modules is decent if and only if its character dual is a cotilting complex of left R-modules (Theorem 3.4). It follows that our notion of a decent tilting complex includes all tilting modules and all compact tilting complexes, and the decent tilting complexes correspond bijectively to cotilting complexes of cofinite type via the character duality. Moreover, we discuss some recently studied sources of interesting examples coming from commutative algebra [PV21], [HNŠ22].

Motivated by the aforementioned results by Bazzoni et al., we consider in Section 4 an appropriate version of the good property of a silting complex T, and show that this allows to represent the derived equivalence $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$ induced by a good and decent tilting complex by the derived functor $\mathbf{R}\mathsf{Hom}_R(T,-)$. In Section 5 we focus on the cotilting setting and show that the cotilting heart associated to the character dual of a good and decent tilting complex is equivalent to another category induced by the topological ring — the category of left discrete $\mathfrak{S}$-modules. This gives a rather explicit description of the hearts induced by cotilting complexes of cofinite type (Corollary 5.3). Analogously to Rickard's result, the cotilting derived equivalence can be represented by $T \otimes_R^{\mathbf{L}} -$. In Theorem 5.5, we obtain a commutative square for a good and decent tilting complex similar to Eq. (1), which shows that the representable derived equivalences are compatible with the tensor and contratensor structures:

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \times \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{-\otimes_R^{\mathbf{L}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \\ {\scriptstyle (\mathbf{R}\mathsf{Hom}_R(T,-),T\otimes_R^{\mathbf{L}}-)}\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle =} \\ \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S}) \times \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr}) & \xrightarrow{-\odot_{\mathfrak{S}}^{\mathbf{L}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \end{array} \tag{2}$$

Note that in this picture, certain asymmetry appears between the tilting and cotilting equivalence which was not visible in the classical tilting situation Eq. (1). In Proposition 5.6, we also show that a kind of converse holds, characterizing the derived equivalences induced by decent tilting complexes: If there is a complete and separated topological ring $\mathfrak{S}$, and a couple of triangle equivalences $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \cong \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$ and $\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \cong \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr})$ which make the square as in Eq. (2) commute then there is a good and decent tilting complex in $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ such that its endomorphism ring endowed with the compact topology is topologically Morita equivalent to $\mathfrak{S}$.

In the final Section 6, we discuss an example arising from a codimension filtration of the Zariski spectrum of a one-dimensional commutative noetherian ring.

2. Add-closures in compactly generated triangulated categories

The goal of this section is to partially extend the techniques of [PŠ21, §6, §7] to triangulated context by giving a description of Add-closure of an object M in terms of a topological structure of the endomorphism ring of M.

2.1. Contramodules over topological rings. For a comprehensive resource about the theory of contramodules over complete and separated topological rings as developed by Positselski, we refer the reader to the survey [Pos21] and references therein, as well as to [PŠ21, §6, §7] where the setting is very close to ours. Here we recall the basic concept and notation. Let $\mathfrak{R}$ be a (unital, associative) ring. We say that $\mathfrak{R}$ is a (left) topological ring if it comes endowed with a linear topology of left ideals, that is, with a filter $(\mathfrak{I}_\alpha)_{\alpha \in A}$ of left ideals of $\mathfrak{R}$ such that for each $r \in \mathfrak{R}$ and $\alpha \in A$ there is $\alpha' \in A$ such that $\mathfrak{I}_{\alpha'} \cdot r \subseteq \mathfrak{I}_\alpha$. With such a filter fixed, we call the ideals it contains the open left ideals of the topological ring $\mathfrak{R}$. There is a natural map $\lambda : \mathfrak{R} \to \varprojlim_{\mathfrak{I} \subseteq \mathfrak{R} \text{ open}} \mathfrak{R}/\mathfrak{I}$ from $\mathfrak{R}$ to the completion with respect to the topology of open left ideals. We say that $\mathfrak{R}$ is complete if the map λ is surjective and separated if the map is injective.

Let $\mathfrak{R}$ be a complete and separated topological ring. Given a set X, let $\mathfrak{R}[[X]]$ denote the set of all (possibly infinite) formal linear combinations $\sum_{x \in X} x \cdot r_x$ of elements of the set X with coefficients $r_x \in \mathfrak{R}$ such that the family $(r_x)_{x \in X}$ converges to zero in the topology of $\mathfrak{R}$, that is, if for any open ideal $\mathfrak{I}$ we have $r_x \in \mathfrak{I}$ for all but finitely many $x \in X$. This assignment defines a functor $\mathfrak{R}[[-]] : \mathsf{Sets} \to \mathsf{Sets}$ on the category of all sets. Indeed, given a map $f : X \to Y$ of sets, the induced map $\mathfrak{R}[[f]] : \mathfrak{R}[[X]] \to \mathfrak{R}[[Y]]$ is defined by sending an element $\sum_{x \in X} x \cdot r_x$ to $\sum_{x \in X} f(x) \cdot r_x = \sum_{y \in Y} y \cdot s_y$, where the coefficient $s_y = \sum_{f(x)=y} r_x$ is well-defined using the fact that $\mathfrak{R}$ is complete and separated and the coefficients converge to zero. There is the "opening of parentheses" map $\mu_X : \mathfrak{R}[[\mathfrak{R}[[X]]]] \to \mathfrak{R}[[X]]$ (which is the obvious assignment of a formal linear combination to a formal linear combination of formal linear combinations) and the "trivial linear combination" map $\epsilon_X : X \to \mathfrak{R}[[X]]$, the well-defined-ness of μ_X is again ensured by the complete and separated assumption on the topology. Then the functor $\mathfrak{R}[[-]]$ on Sets together with the two natural transformation μ and ϵ form an additive monad on the category of sets, and one can therefore speak about modules (=algebras) over this monad — these are precisely the right $\mathfrak{R}$-contramodules. Explicitly, a right $\mathfrak{R}$-contramodule is a set $\mathfrak{M}$ together with a contraaction map $\pi : \mathfrak{R}[[\mathfrak{M}]] \to \mathfrak{M}$ satisfying two axioms: first we have two maps $\mu_{\mathfrak{M}}, \mathfrak{R}[[\pi]] : \mathfrak{R}[[\mathfrak{R}[[\mathfrak{M}]]]] \to \mathfrak{R}[[\mathfrak{M}]]$ and these need to equalize after composing with $\pi : \mathfrak{R}[[\mathfrak{M}]] \to \mathfrak{M}$ ("contra-associativity"), and secondly the composition of $\pi \circ \epsilon_{\mathfrak{M}} : \mathfrak{M} \to \mathfrak{R}[[\mathfrak{M}]] \to \mathfrak{M}$ needs to be the identity map ("contra-unitality").

We denote the category of all right $\mathfrak{R}$-contramodules by $\mathsf{Ctra}\text{-}\mathfrak{R}$. It turns out that $\mathsf{Ctra}\text{-}\mathfrak{R}$ is a complete and cocomplete locally presentable abelian category. For any set X, the map μ_X endows the set $\mathfrak{R}[[X]]$ with a natural structure of a right $\mathfrak{R}$-contramodule, and in fact, if $\star$ is a singleton set then $\mathfrak{R} = \mathfrak{R}[[\star]]$ is a projective generator of $\mathfrak{S}$, while $\mathfrak{R}[[X]]$ is the coproduct of X copies of $\mathfrak{R}$ in $\mathsf{Ctra}\text{-}\mathfrak{R}$. In particular, $\mathsf{Ctra}\text{-}\mathfrak{R}$ has enough projectives and the full subcategory of projective objects $\mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$ consists precisely of direct summands of objects of the form $\mathfrak{R}[[X]]$ for some set X. There is a natural forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{R} \to \mathsf{Mod}\text{-}\mathfrak{R}$ which simply restricts the contraaction just to finite linear combinations with coefficients in $\mathfrak{R}$. This forgetful functor is in general not fully faithful, and while it respects products, it usually does not preserve coproducts.

2.2. **Modules over ringoids.** By a ringoid we understand a skeletally small preadditive category. A ringoid $\mathcal{R}$ gives rise to the category $\mathsf{Mod}\text{-}\mathcal{R}$ of all contravariant additive functors $\mathcal{R} \to \mathsf{Mod}\text{-}\mathbb{Z}$, we call its objects the right $\mathcal{R}$-modules. Recall that $\mathsf{Mod}\text{-}\mathcal{R}$ is a Grothendieck category with a generating set of finitely presented projective objects given by the representable functors $\mathsf{Hom}_{\mathcal{R}}(-, R)$, where R runs over a skeleton of $\mathcal{R}$. The classical case of right modules over an associative unital ring is recovered by restricting to ringoids with precisely one object.

Given an object X in an additive category $\mathcal{C}$ with arbitrary coproducts, we let $\mathsf{Add}_{\mathcal{C}}(X)$ denote the full subcategory consisting of direct summands of coproducts of copies of X, dually we also define $\mathsf{Prod}_{\mathcal{C}}(X)$ using products. We drop the subscript if the ambient category is clear from the context. Given a ringoid $\mathcal{R}$ and a module $M \in \mathsf{Mod}\text{-}\mathcal{R}$, consider the endomorphism ring $\mathfrak{S} = \mathsf{End}_{\mathsf{Mod}\text{-}\mathcal{R}}(M)$. Following [PŠ21, §7.1] and the references therein, we endow $\mathfrak{S}$ with the finite topology, in which the filter basis of open left ideals consists of ideals of the form $\mathfrak{I}_F = \{g \in \mathsf{End}_{\mathsf{Mod}\text{-}\mathcal{R}}(M) \mid g_{\restriction F} = 0\}$ where F runs through finitely generated subobjects F of M. Equivalently, the basis consists of ideals of the form $\mathfrak{I}_f = \{g \in \mathsf{End}_{\mathsf{Mod}\text{-}\mathcal{R}}(M) \mid f \circ g = 0\}$ where f runs through morphisms $f : P \to M$ where P is a finitely generated projective object of $\mathsf{Mod}\text{-}\mathcal{R}$. The following result of Positselski and Šťovíček is the initial point of our study.

Theorem 2.1. [PŠ21, Theorem 7.1] *Let $\mathcal{R}$ be a ringoid and $M \in \mathsf{Mod}\text{-}\mathcal{R}$. Consider the endomorphism ring $\mathfrak{S} = \mathsf{End}_{\mathsf{Mod}\text{-}\mathcal{R}}(M)$ endowed with the finite topology. Then $\mathfrak{S}$ is a complete and separated (left) topological ring and the functor $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{R}}(M, -) : \mathsf{Mod}\text{-}\mathcal{R} \to \mathsf{Mod}\text{-}\mathfrak{S}$ factors through the forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{S} \to \mathsf{Mod}\text{-}\mathfrak{S}$ and induces an equivalence $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{R}}(M, -) : \mathsf{Add}(M) \xrightarrow{\cong} \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$.*

Remark 2.2. [PŠ21, Theorem 7.1] shows in particular that the two additive monads on the category of sets defined by the rules $X \mapsto \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{R}}(M, M^{(X)})$ and $X \mapsto \mathfrak{S}[[X]]$ (see [PŠ21, §6.3]) are isomorphic, which is the reason why the right $\mathfrak{S}$-module structure on $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{R}}(M, M^{(X)})$ extends to a right $\mathfrak{S}$-module structure.

If $M \in \mathsf{Mod}\text{-}\mathcal{R}$ is a finitely generated module then the finite topology on $\mathfrak{S}$ becomes discrete, in which case $\mathsf{Ctra}\text{-}\mathfrak{S} = \mathsf{Mod}\text{-}\mathfrak{S}$. In this case, Theorem 2.1 recovers the classical equivalence $\mathsf{Add}(M) \cong \mathsf{Mod}\text{-}\mathfrak{S}_{\mathsf{proj}}$ of [Dre69].

2.3. **Compactly generated triangulated categories.** Let $\mathcal{T}$ be a triangulated category with suspension functor $-[1]$ and assume that $\mathcal{T}$ has arbitrary coproducts. Recall that an object $F \in \mathcal{T}$ is compact if the functor $\mathsf{Hom}_{\mathcal{T}}(F, -) : \mathcal{T} \to \mathsf{Mod}\text{-}\mathbb{Z}$ preserves coproducts, and denote by $\mathcal{T}^{\mathsf{c}}$ the full subcategory of compact objects. Unless specified otherwise, we assume that the triangulated category $\mathcal{T}$ is compactly generated, which means that $\mathcal{T}^{\mathsf{c}}$ is skeletally small and any object $X \in \mathcal{T}$ such that $\mathsf{Hom}_{\mathcal{T}}(F, X) = 0$ for all $F \in \mathcal{T}^{\mathsf{c}}$ has to be zero; this assumption also implies that $\mathcal{T}$ has arbitrary products.

Considering $\mathcal{T}^{\mathsf{c}}$ to be a ringoid yields a theory of purity in $\mathcal{T}$ as developed by Krause [Kra00]. The main ingredient is the restricted Yoneda functor $\mathbf{y} : \mathcal{T} \to \mathsf{Mod}\text{-}\mathcal{T}^{\mathsf{c}}$ defined by the assignment $X \mapsto \mathsf{Hom}_{\mathcal{T}}(-, X)_{\restriction \mathcal{T}^{\mathsf{c}}}$. This is a conservative cohomological functor (however, it is usually not fully faithful). A triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \to X[1]$ in $\mathcal{T}$ is pure provided that it is sent to a short exact sequence by $\mathbf{y}$. If this is the case, we say that f (resp. g) is a pure monomorphism (resp. pure epimorphism) in $\mathcal{T}$. An object $E \in \mathcal{T}$ is pure-injective provided that $\mathbf{y}E$ is an injective object in $\mathsf{Mod}\text{-}\mathcal{T}^c$. A pure-injective envelope of object X is an object $PE(X)$ together with a map $e : X \to PE(X)$ such that $\mathbf{y}e : \mathbf{y}X \to \mathbf{y}PE(X)$ identifies with the injective envelope map in $\mathsf{Mod}\text{-}\mathcal{T}^c$. The pure-injective envelope exists for any object $X \in \mathcal{T}$ and $PE(X)$ is uniquely determined up to isomorphism. Pure-projective objects in $\mathcal{T}$ are defined similarly. A morphism f in $\mathcal{T}$ is called phantom provided that $\mathbf{y}f = 0$. For $X, Y \in \mathcal{T}$, the kernel of $\mathsf{Hom}_{\mathcal{T}}(X, Y) \to \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}X, \mathbf{y}Y)$ consists precisely of the phantom morphisms $X \to Y$. If X is pure-projective (in particular, if X is compact) or if Y is pure-injective, any phantom map $X \to Y$ is zero in $\mathcal{T}$. See [Kra00, §1, §2] for details.

Let $M \in \mathcal{T}$ and denote its endomorphism ring by $\mathfrak{S} = \mathsf{End}_{\mathcal{T}}(M)$. In complete analogy with the finite topology, we define a natural linear topology on the ring $\mathfrak{S}$ which we call the compact topology. The basis of open left ideals of $\mathfrak{S}$ is given by ideals of the form $\mathfrak{I}_f = \{g \in \mathfrak{S} \mid g \circ f = 0\}$ for all maps $f : F \to M$ with $F \in \mathcal{T}^c$. The compact topology makes $\mathfrak{S}$ into a left topological ring. Indeed, for any $f : F \to M$ with $F \in \mathcal{T}^c$ and any $s \in \mathfrak{S}$ we have $\mathfrak{I}_{sf} \cdot s = \{gs \in \mathfrak{S} \mid gsf = 0\} \subseteq \mathfrak{I}_f = \{g \in \mathfrak{S} \mid gf = 0\}$.

Proposition 2.3. *Let $M \in \mathcal{T}$ be such that the restriction $\mathbf{y}_{\restriction \mathsf{Add}(M)} : \mathsf{Add}(M) \to \mathsf{Mod}\text{-}\mathcal{T}^c$ of the restricted Yoneda functor to $\mathsf{Add}(M)$ is fully faithful. Then $\mathfrak{S}$ endowed with the compact topology is a complete and separated topological ring and there is an equivalence $\mathsf{Hom}_{\mathcal{T}}(M, -) : \mathsf{Add}(M) \xrightarrow{\cong} \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$.*

Proof. The assumption yields that $\mathbf{y}$ induces an isomorphism $\mathfrak{S} \cong \mathsf{End}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}M)$. Since $\mathbf{y}$ preserves coproducts, we also have that $\mathsf{Add}(M) = \mathsf{Add}_{\mathcal{T}}(M) \cong \mathsf{Add}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}M)$. Then it follows directly from Theorem 2.1 that $\mathfrak{S}$ is complete and separated and $\mathsf{Add}(M)$ is equivalent to $\mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$, with the caveat that here $\mathfrak{S}$ is endowed with the finite topology induced on $\mathbf{y}M \in \mathsf{Mod}\text{-}\mathcal{T}^c$. The equivalence is induced by the functor $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}M, -) : \mathsf{Mod}\text{-}\mathcal{T}^c \to \mathsf{Ctra}\text{-}\mathfrak{S}$, and the restriction of this functor to $\mathsf{Add}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}M)$ is identified with the functor $\mathsf{Hom}_{\mathcal{T}}(M, -) : \mathsf{Add}(M) \to \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$.

It remains to show that the compact topology induced in $\mathcal{T}$ and the finite topology induced in $\mathsf{Mod}\text{-}\mathcal{T}^c$ on $\mathfrak{S}$ coincide. But to see that, it is enough to recall that finitely generated subobjects $F \hookrightarrow \mathbf{y}M$ are precisely the images of morphisms $\mathbf{y}C \to \mathbf{y}M$ with $C \in \mathcal{T}^c$, because the objects of the form $\mathbf{y}C$ with $C \in \mathcal{T}^c$ are precisely the finitely generated projective objects of $\mathsf{Mod}\text{-}\mathcal{T}^c$. But any such morphism is of the form $\mathbf{y}(f)$ for a morphism $f : C \to M$ in $\mathcal{T}$, see [Kra00, Theorem 1.8]. □

Corollary 2.4. *Let M be a pure-projective object of $\mathcal{T}$. Then $\mathsf{Add}(M) \cong \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$. In particular, if M is a compact object then $\mathsf{Add}(M) \cong \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}} = \mathsf{Mod}\text{-}\mathfrak{S}_{\mathsf{proj}}$.*

Proof. If M is pure-projective then the isomorphism follows directly from Proposition 2.3 and [Kra00, Theorem 1.8]. If M is compact then M is pure-projective and the compact topology on $\mathfrak{S}$ becomes discrete, so $\mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}} = \mathsf{Mod}\text{-}\mathfrak{S}_{\mathsf{proj}}$. □

2.4. **Hearts of t-structures.** For a short moment, let $\mathcal{T}$ be any triangulated category. Given a full subcategory $\mathcal{C} \subseteq \mathcal{T}$, we define full subcategories $\mathcal{C}^{\perp \bigcirc} = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(C, X[i]) \ \forall C \in \mathcal{C}, i \bigcirc\}$ of $\mathcal{T}$ where $\bigcirc$ denotes a symbol such as $= 0, \leq 0, > 0, \neq 0$ and others which specify a set of integers. Analogously, we define full subcategories ${}^{\perp \bigcirc}\mathcal{C} = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(X, C[i]) \ \forall C \in \mathcal{C}, i \bigcirc\}$ and if $\mathcal{C} = \{C\}$, we write just $C^{\perp \bigcirc}$ and ${}^{\perp \bigcirc}C$. Recall that a t-structure $\mathbb{T}$ in $\mathcal{T}$ is a pair $\mathbb{T} = (\mathcal{U}, \mathcal{V})$ of full subcategories such that:

(i) $\mathcal{V} = \mathcal{U}^{\perp_0}$,
(ii) $\mathcal{U}$ is closed under suspension, that is, $\mathcal{U}[1] \subseteq \mathcal{U}$,
(iii) each object $X \in \mathcal{T}$ fits into a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

Following Beilinson, Bernstein, and Deligne [BBD82], any t-structure gives rise to an abelian category $\mathcal{H}_{\mathbb{T}} = \mathcal{U} \cap \mathcal{V}[1]$ called the heart of the t-structure. The exact sequences in $\mathcal{H}_{\mathbb{T}}$ are induced by the triangles of $\mathcal{T}$ with all three components lying in the heart. In particular, a morphism $f : X \to Y$ in $\mathcal{H}$ is a monomorphism if and only if $\mathsf{Cone}(f) \in \mathcal{H}$.

Let now $\mathcal{T}$ be again a compactly generated triangulated category. A subcategory $\mathcal{C}$ of $\mathcal{T}$ is called <u>definable</u> provided that there is a set Φ of maps in $\mathcal{T}^c$ such that

$$\mathcal{C} = \{X \in \mathcal{T} \mid \mathsf{Hom}_{\mathcal{T}}(f, X) \text{ is the zero map } \ \forall f \in \Phi\}.$$

Given an object $X \in \mathcal{T}$, let $\mathsf{Def}_{\mathcal{T}}(X)$ (or just $\mathsf{Def}(X)$ if $\mathcal{T}$ is clear from context) denote the smallest definable subcategory of $\mathcal{T}$ which contains X. Note that $\mathsf{Def}(X)$ exists, as it can be defined using the set Φ of all maps f in $\mathcal{T}^c$ such that $\mathsf{Hom}_{\mathcal{T}}(f, X)$ is the zero map. It is clear from the definition that any definable subcategory is closed under coproducts, products, pure subobjects, and pure quotients. Here, an object $Y \in \mathcal{T}$ is a <u>pure subobject</u> of $X \in \mathcal{T}$ if there is a pure monomorphism $Y \to X$, the <u>pure quotients</u> are defined similarly. By [AHMV17, Corollary 4.4], definable subcategories are also closed under taking pure-injective envelopes, meaning that $PE(X)$ belongs to a definable subcategory whenever X does.

Proposition 2.5. *Let $\mathbb{T}$ be a t-structure in $\mathcal{T}$ with heart $\mathcal{H}$ and consider an object $M \in \mathcal{H}$. Assume the following condition: $\mathsf{Def}_{\mathcal{T}}(M) \subseteq \mathcal{H}$.*

Then the restriction $\mathbf{y}_{\restriction \mathsf{Def}_{\mathcal{T}}(M)} : \mathsf{Def}_{\mathcal{T}}(M) \to \mathsf{Mod}\text{-}\mathcal{T}^c$ is fully faithful. As a consequence, $\mathsf{Add}_{\mathcal{T}}(M) \cong \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$, where $\mathfrak{S} = \mathsf{End}_{\mathcal{T}}(M)$ endowed with the compact topology is complete and separated.

Proof. Recall from the discussion above that $\mathsf{Def}_{\mathcal{T}}(M)$ is closed under pure subobjects, pure quotients, and pure-injective envelopes. Then the condition $\mathsf{Def}_{\mathcal{T}}(M) \subseteq \mathcal{H}$ implies that for any $Y \in \mathsf{Def}_{\mathcal{T}}(M)$, all three components of the triangle $Y \to PE(Y) \to L \xrightarrow{+}$ belong to $\mathsf{Def}_{\mathcal{T}}(M)$, and thus also to $\mathcal{H}$. It follows that the pure-injective envelope map $i : Y \to PE(Y)$ is a monomorphism in $\mathcal{H}$. If $X \in \mathcal{H}$ and $f : X \to Y$ is a phantom map in $\mathcal{T}$, then $i \circ f$ is zero (both in $\mathcal{T}$ and $\mathcal{H}$). Since i is a monomorphism in $\mathcal{H}$, it follows that $f = 0$. We showed that $\mathbf{y} : \mathsf{Hom}_{\mathcal{T}}(X, Y) \to \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}X, \mathbf{y}Y)$ is a monomorphism for any $X, Y \in \mathcal{H}$ and $Y \in \mathsf{Def}_{\mathcal{T}}(M)$.

Furthermore, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathsf{Hom}_{\mathcal{T}}(X, Y[-1]) & \longrightarrow & \mathsf{Hom}_{\mathcal{T}}(X, Y) & \longrightarrow & \mathsf{Hom}_{\mathcal{T}}(X, PE(Y)) & \longrightarrow & \mathsf{Hom}_{\mathcal{T}}(X, L) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}X, \mathbf{y}Y) & \to & \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}X, \mathbf{y}PE(Y)) & \to & \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathcal{T}^c}(\mathbf{y}X, \mathbf{y}L)
\end{array}$$

The three rightmost vertical arrows are monomorphisms by the previous discussion, while the third arrow from the left is an isomorphism since $PE(Y)$ is pure-injective [Kra00, Theorem 1.8]. It follows by Four Lemma that the second vertical arrow from the left is an isomorphism and we are done.

Finally, since $\mathsf{Add}_{\mathcal{T}}(M) \subseteq \mathsf{Def}_{\mathcal{T}}(M)$, we get the last sentence of the statement from Proposition 2.3. ☐

Proposition 2.6. *Let $\mathbb{T}$ be a t-structure in $\mathcal{T}$ with heart $\mathcal{H}$ and a projective generator P of $\mathcal{H}$. Assume that $\mathsf{Def}_{\mathcal{T}}(P) \subseteq \mathcal{H}$ is satisfied for P and let $\mathfrak{S} = \mathsf{End}_{\mathcal{T}}(P)$ be endowed with the compact topology. Then $\mathfrak{S}$ is complete and separated and there is an equivalence of abelian categories $\mathsf{Hom}_{\mathcal{H}}(P, -) : \mathcal{H} \xrightarrow{\cong} \mathsf{Ctra}\text{-}\mathfrak{S}$.*

Proof. By Proposition 2.5, $\mathsf{Hom}_{\mathcal{T}}(P, -)$ restricts to an equivalence $\mathsf{Add}_{\mathcal{T}}(P) \xrightarrow{\cong} \mathsf{Ctra}\text{-}\mathfrak{S}_{\mathsf{proj}}$. Note that the assumption on P ensures that $\mathsf{Add}_{\mathcal{T}}(P) = \mathsf{Add}_{\mathcal{H}}(P) = \mathcal{H}_{\mathsf{proj}}$.

Clearly, $\mathsf{Hom}_{\mathcal{H}}(P, -)$ is well-defined as a functor $\mathcal{H} \to \mathsf{Mod}\text{-}\mathfrak{S}$. Following [PŠ21, Corollary 6.3] (also in view of [PŠ21, Remark 6.4 and §6.3]), this functor actually factorizes as $\mathsf{Hom}_{\mathcal{H}}(P, -) : \mathcal{H} \to \mathsf{Ctra}\text{-}\mathfrak{S} \to \mathsf{Mod}\text{-}\mathfrak{S}$, where the latter is the forgetful functor. Since $\mathsf{Hom}_{\mathcal{T}}(P, -) = \mathsf{Hom}_{\mathcal{H}}(P, -) : \mathcal{H} \to \mathsf{Ctra}\text{-}\mathfrak{S}$ is an exact functor between two abelian categories with enough projectives which restricts to an equivalence between the respective categories of projectives, a standard argument yields that we have the desired equivalence. ☐

2.5. **Silting objects.** In what follows we will focus on t-structures coming from the theory of (large) silting and cosilting objects, we refer the reader to the survey [AH19, §5, §6] for details. An object T of $\mathcal{T}$ is silting if the pair $(T^{\perp_{>0}}, T^{\perp_{\leq 0}})$ constitutes a t-structure in $\mathcal{T}$, called the silting t-structure. Any object $T' \in \mathcal{T}$ satisfying $\mathsf{Add}(T) = \mathsf{Add}(T')$ is also a silting object inducing the same silting t-structure. In this situation, we say that T and T' are equivalent silting objects. Denote the associated heart by $\mathcal{H}_T$ and the induced cohomological functor as $H^0_T : \mathcal{T} \to \mathcal{H}_T$. Then $\mathcal{H}_T$ is an abelian category with exact products such that $H^0_T(T)$ is a projective generator ([AHMV17, Lemma 2.7]). Note also that in this situation we have $\mathcal{H}_T = T^{\perp_{\neq 0}}$. If in addition $\mathsf{Add}(T) \subseteq T^{\perp_{<0}}$, or equivalently $\mathsf{Add}(T) \subseteq \mathcal{H}_T$, we call T a tilting object.

Definition 2.7. A silting object $T \in \mathcal{T}$ is decent provided that $\mathsf{Def}(T) \subseteq \mathcal{H}_T$. Note that since $\mathsf{Add}(T) \subseteq \mathsf{Def}(T)$, this implies in particular that T is a tilting object. Also, clearly the property of being decent is invariant under equivalence of silting objects, and therefore can be viewed as a property of the silting t-structure.

For the next part it will be important to use the fact that definable subcategories are characterized by their closure properties in case $\mathcal{T}$ admits a good enough enhancement. This was developed by Laking [Lak20], where the enhancement comes in the form of a compactly generated derivator and using the notion of a directed homotopy colimit and homotopy coherent reduced product we refer to [Lak20, §2, Appendix] and references therein. For what comes in the next sections, we recall that the unbounded derived category $\mathsf{D}(\mathsf{Mod}\text{-}R)$ of a module category over a ring R underlies a standard compactly generated derivator, the directed homotopy colimits are in this case computed as ordinary direct limits of diagrams in the category of cochain complexes of R-modules, see e.g. [HN21, Appendix].

Theorem 2.8 ([Lak20]). *Let $\mathcal{C}$ be a full subcategory of triangulated category $\mathcal{T}$ which is the base of a compactly generated derivator. Then:*

(1) $\mathcal{C}$ is definable if and only if it is closed under products, pure subobjects, and directed homotopy colimits.
(2) For any $Y \in \mathcal{T}$, the definable closure $\mathsf{Def}_{\mathcal{T}}(Y)$ consists precisely of pure subobjects of directed homotopy colimits of objects from $\mathsf{Prod}_{\mathcal{T}}(Y)$.

Proof. Both the claims are proved in [Lak20, Theorem 3.11] and [Lak20, Corollary 3.12]. □

Lemma 2.9. *Let $\mathcal{T}$ be a triangulated category which is the base of a compactly generated derivator and $T \in \mathcal{T}$ a pure-projective tilting object. Then $\mathcal{H}_T$ is a definable subcategory of $\mathcal{T}$. In particular, T is decent.*

Proof. Since T is pure-projective, both the subcategories $T^{\perp_{>0}}$ and $T^{\perp_{<0}}$ are closed under pure monomorphisms, pure epimorphisms, and products. It follows that both the subcategories are closed under directed homotopy colimits and so are definable, this follows by a similar argument as in the proof of [Lak20, Lemma 4.5]. Then their intersection $\mathcal{H}_T$ is definable as well, and the condition $\mathsf{Def}(T) \subseteq \mathcal{H}_T$ clearly holds and T is decent. □

An object C of $\mathcal{T}$ is cosilting if the pair $({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ constitutes a t-structure in $\mathcal{T}$. In order to adhere to the standard notation, it is convenient to consider the associated heart shifted by degree -1 so that the equality $\mathcal{H}_C = {}^{\perp_{\neq 0}}C$ holds. We denote the induced (again, shifted) cohomological functor as $H^0_C : \mathcal{T} \to \mathcal{H}_C$. The heart $\mathcal{H}_C$ is an abelian category with exact coproducts such that $H^0_C(C)$ is an injective cogenerator. If in addition $\mathsf{Prod}(C) \subseteq {}^{\perp_{<0}}C$, or equivalently $\mathsf{Prod}(C) \subseteq \mathcal{H}_C$, we call C a cotilting object. Similarly to the cosilting situation, we say that two cosilting objects C and C' are equivalent if they induced the same t-structure, which amounts to $\mathsf{Prod}(C) = \mathsf{Prod}(C')$.

Lemma 2.10. *Let $\mathcal{T}$ be a triangulated category which is the base of a compactly generated derivator and $C \in \mathcal{T}$ a pure-injective cotilting object. Then $\mathsf{Def}(C) \subseteq \mathcal{H}_C$.*

Proof. Since C is pure-injective, both the categories ${}^{\perp_{>0}}C$ and ${}^{\perp_{<0}}C$ are closed under pure monomorphisms, pure epimorphisms, and coproducts. It follows that their intersection $\mathcal{H}_C$ is closed under pure monomorphisms and directed homotopy colimits (see the proof of [Lak20, Lemma 4.5]). Since C is cotilting, we have $\mathsf{Prod} C \subseteq \mathcal{H}_C$. Now it is enough to recall from Theorem 2.8(2) that $\mathsf{Def}(C)$ consists precisely of all pure subobjects of directed homotopy colimits of objects in $\mathsf{Prod}(C)$. □

Remark 2.11. Certain asymmetry is now apparent from Lemmas 2.9 and 2.10. First, unlike for pure-projective tilting objects, it is not always the case that the heart $\mathcal{H}_C$ of a pure-injective cotilting object is a definable subcategory.

Secondly, one encounters far fewer pure-projective tilting objects in practice than the pure-injective cotilting counterparts. Of course, any compact tilting object is pure-projective. There exist pure-projective tilting complexes which are not equivalent to a compact one but these are rather exotic, see [BHP$^+$20]. On the other hand, any (bounded) cotilting complex in a derived category of a ring is known to be pure-injective [MV18, Proposition 3.10]. In fact, the author is not aware of any example of a cosilting object of a compactly generated triangulated category which is not pure-injective.

In what follows, we will show that the condition of being decent for a tilting complex T in a derived category of a ring holds much more generally then under the pure-projective assumption, but it does not hold always.

3. Tilting complexes and hearts in the derived category of a ring

If $\mathcal{A}$ is an abelian category, we let $\mathsf{D}(\mathcal{A})$ denote the (unbounded) derived category and $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ the bounded derived category of cochain complexes over $\mathcal{A}$. The abelian categories we consider never encounter issues with the existence of their derived categories.

Unless said otherwise, R will always denote an arbitrary (associative, unital) ring. An object $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ of the derived category of right R-modules is a (bounded) silting complex[1] if T is a silting object which is quasi-isomorphic to a bounded complex of right projective R-modules. A silting complex T is called a tilting complex in case it is a tilting object in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. Finally, a silting complex is decent if it is decent as a silting object, that is, if $\mathsf{Def}(T) \subseteq \mathcal{H}_T$. We recall that any decent silting complex is automatically tilting. Note that since $T^{\perp_{>0}}$ is always a definable subcategory of $\mathsf{D}(\mathsf{Mod}\text{-}R)$ [MV18, Theorem 3.6], a silting complex T is decent if and only if $\mathsf{Def}(T) \subseteq T^{\perp_{<0}}$.

Dually, an object $C \in \mathsf{D}(R\text{-}\mathsf{Mod})$ of the derived category of left R-modules is a cosilting complex if C is isomorphic in $\mathsf{D}(R\text{-}\mathsf{Mod})$ to a bounded complex of injective R-modules and it is a cosilting object in this triangulated category. Any (bounded) cosilting complex is pure-injective in $\mathsf{D}(R\text{-}\mathsf{Mod})$ and $\mathcal{H}_C$ is a Grothendieck category with an injective cogenerator $H^0_C(C)$ [MV18, Proposition 3.10], [AHMV17, Lemma 2.7, Theorem 3.6]. If C is in addition a cotilting object in $\mathsf{D}(R\text{-}\mathsf{Mod})$, we call it a cotilting complex. Finally, we say that a cosilting complex is of cofinite type if the associated cosilting t-structure is compactly generated.

Consider the character duality functor $(-)^+ := \mathsf{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ acting as a functor $\mathsf{Mod}\text{-}R \to R\text{-}\mathsf{Mod}$. Since $(-)^+$ is exact, it naturally extends to a functor $(-)^+ : \mathsf{D}(\mathsf{Mod}\text{-}R) \to \mathsf{D}(R\text{-}\mathsf{Mod})$. Note that this is a conservative functor, a property we will utilize throughout the rest of the paper. We will also need another duality functor $(-)^* := \mathbf{R}\mathsf{Hom}_R(-, R)$ which induces a contravariant equivalence $\mathsf{D}(\mathsf{Mod}\text{-}R)^{\mathsf{c}} \xrightarrow{\cong} \mathsf{D}(R\text{-}\mathsf{Mod})^{\mathsf{c}}$ on the categories of compact objects. See [AHH21, §2.2] for details about these dualities, and also the recent preprint [BW22] where the triangulated character duality is developed in larger generality. By symmetry, we also freely consider both functors going in the opposite direction.

The cosilting complexes of cofinite type are known to be precisely the character duals of silting complexes, up to equivalence.

Proposition 3.1. [AHH21, Theorem 3.3] *If $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ is a silting complex then $T^+ \in \mathsf{D}(R\text{-}\mathsf{Mod})$ is a cosilting complex. This assignment yields a bijection between equivalence classes*

[1]Silting complexes not necessarily isomorphic to a bounded complex of projectives are also considered in the literature. In this paper, we focus on the bounded versions of silting and cosilting complexes.

of silting complexes in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ *and equivalence classes of cosilting complexes in* $\mathsf{D}(R\text{-}\mathsf{Mod})$ *of cofinite type.*

The following lemma shows that applying the character duality induces a pair of definable categories which are in the usual terminology from the model theory of modules called dual definable.

Lemma 3.2. *Let* $Y \in \mathsf{D}(\mathsf{Mod}\text{-}R)$. *Then:*

(i) For any $X \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ *we have* $X \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(Y) \iff X^+ \in \mathsf{Def}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(Y^+)$,

(ii) For any $Z \in \mathsf{D}(R\text{-}\mathsf{Mod})$ *we have* $Z \in \mathsf{Def}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(Y^+) \iff Z^+ \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(Y)$.

Proof. Let Φ be the set of all maps in f in $\mathsf{D}(\mathsf{Mod}\text{-}R)^{\mathsf{c}}$ such that $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(f, Y)$ is zero so that $\mathsf{Def}(Y) = \{X \in \mathsf{D}(\mathsf{Mod}\text{-}R) \mid \mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(f, X) \text{ is zero } \ \forall f \in \Phi\}$. Put $\Phi^* = \{f^* \mid f \in \Phi\}$. We recall that $f^* = \mathbf{R}\mathsf{Hom}_R(f, R)$ and Φ^* is a set of maps between compact objects of $\mathsf{D}(R\text{-}\mathsf{Mod})$. For any map f in $\mathsf{D}(\mathsf{Mod}\text{-}R)^{\mathsf{c}}$ and any $X \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ we have an equivalence

$$\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(f, X) \text{ is zero } \iff \mathsf{Hom}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(f^*, X^+) \text{ is zero},$$

see the proof of [AHH21, Lemma 2.3]. Plugging $X = Y$, we obtain that the set Φ^* of maps in $\mathsf{D}(R\text{-}\mathsf{Mod})^{\mathsf{c}}$ defines $\mathsf{Def}(Y^+)$, and then using it for general $X \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ we obtain (i). Since Φ^{**} identifies with Φ, we also obtain (ii) by the same argument. □

Lemma 3.3. *Let* T *be a tilting complex in* $\mathsf{D}(\mathsf{Mod}\text{-}R)$ *and put* $C = T^+$. *For any* $X \in \mathsf{D}(R\text{-}\mathsf{Mod})$ *we have* $X \in \mathcal{H}_C$ *if and only if* $X^+ \in \mathcal{H}_T$.

Proof. Note that $\mathcal{H}_T = T^{\perp_{\neq 0}}$ and our convention also ensures that $\mathcal{H}_C = {}^{\perp_{\neq 0}}C$. Using adjunction, there are isomorphisms for any $X \in \mathsf{D}(R\text{-}\mathsf{Mod})$ and $i \in \mathbb{Z}$

$$\mathsf{Hom}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(X, T^+[i]) \cong (T \otimes_R^{\mathbf{L}} X[-i])^+ \cong \mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, X^+[i]),$$

which is enough to conclude the proof. □

We follow with the main result of this section which provides a more intuitive and useful characterization of the decent property of a silting complex.

Theorem 3.4. *Let* $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ *be a silting complex and put* $C = T^+$. *The following are equivalent:*

(i) T *is decent,*

(ii) the cosilting complex T^+ *is cotilting.*

Proof. $(i) \implies (ii)$: Since $\mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ is closed under taking the double character dual $(-)^{++}$ (this follows e.g. by Lemma 3.2), condition (i) ensures that for any $X \in \mathsf{Add}(T)$ we have $X^{++} \in \mathcal{H}_T$. This in turn means $X^+ \in \mathcal{H}_C$ by Lemma 3.3. This already shows that $C^{\varkappa} \in \mathcal{H}_C$ for any cardinal $\varkappa$. Since $\mathcal{H}_C$ is closed under direct summands, we have $\mathsf{Prod}(C) \subseteq \mathcal{H}_C$ as desired.

$(ii) \implies (i)$: By Lemma 3.2, the definable closures of T and $C = T^+$ satisfy that $X \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ if and only if $X^+ \in \mathsf{Def}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(C)$. By Lemmas 2.10 and 3.3, we have for any $X \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ that $X^{++} \in \mathcal{H}_T$. As a consequence, any pure-injective object from $\mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ belongs to $\mathcal{H}_T$, see [AHH21, Corollary 2.8].

Now let $X \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ be any object. Since $T^{\perp_{>0}}$ is definable, it is enough to show $X \in T^{\perp_{<0}}$ to demonstrate that $X \in \mathcal{H}_T$. There is a sequence of triangles $X_i \to P_i \to X_{i+1} \to X_i[1]$ indexed by $i \geq 0$ determined by the following properties: The map $X_i \to P_i$ is the pure-injective envelope for all $i \geq 0$ and $X_0 = X$. It follows that $X_i \in \mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ for all $i \geq 0$. By the previous paragraph, $P_i \in \mathcal{H}_T$ for all $i \geq 0$. Therefore, we have for any $l > 0$ an isomorphism $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, X_0[-l]) \cong \mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, X_i[-l-i])$.

For any set of integers I, the full subcategory of $\mathsf{D}(\mathsf{Mod}\text{-}R)$ determined by vanishing of cohomology in degrees outside of I is easily seen to be definable. Then since T is a cohomologically bounded object, there are integers $a < b$ such that any object from $\mathsf{Def}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ has cohomology vanishing outside of degrees in the interval $[a, b]$. But then $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, X_i[-l-i])$ has to vanish whenever $-l-i < -n$ for $n \gg 0$. Using the above isomorphism, we infer that $\mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, X_0[-l]) = 0$ for all $l > 0$, meaning that $X_0 = X \in T^{\perp_{<0}}$ as desired. □

The following is an easy consequence of Theorem 3.4. We will show in Example 3.12 that the converse implication is not true in general.

Corollary 3.5. *Let T be a silting complex in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ such that T^+ is a cotilting complex in $\mathsf{D}(R\text{-}\mathsf{Mod})$. Then T is tilting.*

Corollary 3.6. *The assignment $T \mapsto T^+$ of Proposition 3.1 restricts to a bijection*

$$\left\{ \begin{array}{c} \textit{Decent tilting complexes} \\ \textit{in } \mathsf{D}(\mathsf{Mod}\text{-}R) \\ \textit{up to equivalence} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Cotilting complexes of cofinite type} \\ \textit{in } \mathsf{D}(R\text{-}\mathsf{Mod}) \\ \textit{up to equivalence} \end{array} \right\}.$$

Proof. Follows directly from Theorem 3.4, as a silting complex T is decent (and thus in particular, tilting) if and only if T^+ is a cotilting complex. □

Corollary 3.7. *Let R be a commutative noetherian ring. The assignment $T \mapsto T^+$ of Proposition 3.1 restricts to a bijection*

$$\left\{ \begin{array}{c} \textit{Decent tilting complexes} \\ \textit{in } \mathsf{D}(\mathsf{Mod}\text{-}R) \\ \textit{up to equivalence} \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Cotilting complexes} \\ \textit{in } \mathsf{D}(R\text{-}\mathsf{Mod}) \\ \textit{up to equivalence} \end{array} \right\}.$$

Proof. In this situation, any cosilting complex is automatically of cofinite type by [HN21, Corollary 2.14] and [MV18, Proposition 3.10]. □

Example 3.8. Let $M \in \mathsf{Mod}\text{-}R$. Then M is a silting complex in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ if and only if M is an n-tilting module for some $n \geq 0$ (see e.g. [PŠ21, §2]). Then any such n-tilting module is well-known to satisfy condition (ii) of Theorem 3.4. Together with Lemma 2.9, we see that decent tilting complexes generalize both n-tilting modules and compact tilting complexes.

3.1. **Examples over commutative rings.** A silting complex is <u>2-term</u> if it is isomorphic in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ to a complex of projectives concentrated in degrees 0 and -1. The main result of [AHH17] says that 2-term silting complexes T in $\mathsf{D}(R)$ over a commutative ring R correspond up to an equivalence to hereditary torsion pairs $(\mathcal{T}_\mathcal{G}, \mathcal{F}_\mathcal{G})$ of finite type in $\mathsf{Mod}\text{-}R$, which in turn correspond to Gabriel filters $\mathcal{G}$ of finite type. A Gabriel filter of finite type is a linear topology of open ideals of R with a basis of finitely generated ideals closed under ideal products, see [Hrb16, Lemma 2.3]. Then $\mathcal{T}_\mathcal{G} = \{M \in \mathsf{Mod}\text{-}R \mid \mathsf{Ann}(m) \in \mathcal{G} \ \forall m \in M\}$, where $\mathsf{Ann}(m)$ is the annihilator ideal of m. Furthermore, given such torsion pair, the tilting heart $\mathcal{H}_T$ can be described just using cohomology as

$$X \in \mathcal{H}_T \iff H^i(X) \begin{cases} \in \mathcal{E}_\mathcal{G} & i = -1 \\ \in \mathcal{D}_\mathcal{G} & i = 0 \\ = 0 & \text{else} \end{cases},$$

where $\mathcal{D}_\mathcal{G} = \{M \in \mathsf{Mod}\text{-}R \mid M = IM \ \forall I \in \mathcal{G}\}$ is the torsion class of all $\mathcal{G}$-divisible modules and $\mathcal{E}_\mathcal{G} = \mathcal{D}_\mathcal{G}{}^{\perp_0}$ is the corresponding torsion-free class of $\mathcal{G}$-reduced modules, i.e. modules with no non-zero $\mathcal{G}$-divisible submodule. In other words, $\mathcal{H}_T$ is obtained as the Happel-Reiten-Smalø tilt with respect to the torsion pair $(\mathcal{D}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ of $\mathsf{Mod}\text{-}R$.

Denote by $t_\mathcal{G} : \mathsf{Mod}\text{-}R \to \mathsf{Mod}\text{-}R$ the torsion radical corresponding to the hereditary torsion class $\mathcal{T}_\mathcal{G}$. We say that a $\mathcal{G}$-torsion module $M \in \mathcal{T}_\mathcal{G}$ is <u>$\mathcal{G}$-bounded</u> if there is $I \in \mathcal{G}$ such that $IM = 0$.

Proposition 3.9. *Let R be commutative noetherian ring. Then any 2-term silting complex $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ is decent tilting.*

Proof. This follows from [PV21, Corollary 5.12] and Theorem 3.4. □

Remark 3.10. Much more is proved in [PV21]: Any intermediate and compactly generated t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ which restricts to a t-structure in the bounded derived category $\mathsf{D}^\mathsf{b}(\mathsf{mod}\text{-}R)$ of finitely generated module is induced by a cotilting complex. Such t-structures are abundant [ATJLS10, §6] and correspond to decent tilting complexes via Corollary 3.7.

Another source of decent tilting complexes over commutative noetherian rings is provided in [HNŠ22, §6, §7]. There, it is shown that any codimension function $\mathsf{d} : \mathsf{Spec}(R) \to \mathbb{Z}$ on the Zariski spectrum of a commutative noetherian ring R of finite Krull dimension gives rise to a silting complex T_{d} in the derived category. If R is in addition a homomorphic image of a Cohen-Macaulay ring, then T_{d} is in fact tilting and its character dual is cotilting. Unless R is itself Cohen-Macaulay, T_{d} is not quasi-isomorphic to a stalk complex, [HNŠ22, Remark 5.9].

Lemma 3.11. *Let R be a commutative ring and T be a 2-term silting complex in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ corresponding to a Gabriel filter $\mathcal{G}$. Then:*

(1) If $t_{\mathcal{G}}(R)$ is $\mathcal{G}$-reduced then T is a tilting complex.
(2) $t_{\mathcal{G}}(R)$ is $\mathcal{G}$-bounded if and only if T^+ is a cotilting complex.

Proof. There is an exact sequence

$$0 \to t_{\mathcal{G}}(R) \to R \to D_0 \to D_1 \to 0$$

where $D_0, D_1 \in \mathcal{D}_{\mathcal{G}}$. Indeed, this follows since $R/t_{\mathcal{G}}(R)$ admits a monomorphic $\mathcal{D}_{\mathcal{G}}$-preenvelope. Then T is tilting by [CHZ19, Theorem A] (cf. Section 4.1).

Now we prove the second statement. By [PV21, Theorem 5.6], T^+ is cotilting if and only if $J \in \mathcal{G}$, where $J = \mathsf{tr}_{R/t_{\mathcal{G}}(R)} R$ is the trace ideal of the cyclic module $R/t_{\mathcal{G}}(R)$. If the $\mathcal{G}$-torsion in R is bounded there is $I \in \mathcal{G}$ such that $It_{\mathcal{G}}(R) = 0$. This implies $I \subseteq J$ and so $J \in \mathcal{G}$. On the other hand, clearly $Jt_{\mathcal{G}}(R) = 0$, and so $J \in \mathcal{G}$ implies that the $\mathcal{G}$-torsion of R is bounded. □

We are ready to provide an example of an indecent tilting complex.

Example 3.12. There is a commutative ring R and a 2-term tilting complex T in $\mathsf{D}(R)$ such the cosilting complex T^+ is not cotilting. Furthermore, we show that the natural map $T^{(\omega)} \to T^\omega$ is not a monomorphism in $\mathcal{H}_T$. As a consequence $\mathcal{H}_T$ cannot be equivalent to a category of contramodules over any complete and separated topological ring, see [PŠ21, last paragraph of §6.2].

This example follows [Pos16, Example 2.6]. Let k be a field and R be the commutative k-algebra generated by the infinite sequence $x_1, x_2, x_3, \ldots$ and another generator y subject to relations $x_i x_j = 0$ and $y^i x_i = 0$ for all $i, j > 0$. Consider the Gabriel filter $\mathcal{G}$ generated by the principal ideal (y). Then it is easy to see that the $\mathcal{G}$-torsion $t_{\mathcal{G}}(R)$ is not $\mathcal{G}$-bounded, but it is $\mathcal{G}$-reduced, so the induced silting complex T is tilting, but T^+ is not cotilting by Lemma 3.11.

Furthermore, let $f : R \to R[y^{-1}]$ be the localization map. Recall from [AHMV19, Proposition 1.3] and [AHH21, Example 4.14(5)] that we can chose $T = R[y^{-1}] \oplus \mathsf{Cone}(f)$ (see also [AHH21, Proposition 5.15]). We claim that the map $g : T^{(\omega)} \to T^\omega$ is not a monomorphism in $\mathcal{H}_T$, or equivalently, that $\mathsf{Cone}(g) \notin \mathcal{H}_T$. Clearly, $H^{-1}\mathsf{Cone}(g) \cong \mathsf{Coker}(H^{-1}g)$. Furthermore, $H^{-1}g$ is just the map $t_{\mathcal{G}}(R)^{(\omega)} \to t_{\mathcal{G}}(R)^\omega$. and so $\mathsf{Coker}(H^{-1}g) \cong t_{\mathcal{G}}(R)^\omega / t_{\mathcal{G}}(R)^{(\omega)}$. Since $t_{\mathcal{G}}(R)^\omega / t_{\mathcal{G}}(R)^{(\omega)}$ is precisely the ω-reduced product, any ω-directed limit of copies of $t_{\mathcal{G}}(R)$ can be embedded into it [Pre09, Theorem 3.3.2].

It remains to show that there is a ω-directed limit of copies of $t_{\mathcal{G}}(R)$ which is not $\mathcal{G}$-reduced. Note that $t_{\mathcal{G}}(R) \cong \bigoplus_{n>0} (x_n)$, and $(x_n) \cong k[y]/(y^n)$ with the obvious R-action in which $x_i = 0$ for $i > 0$. Then there is an monic endomorphism h of $t_{\mathcal{G}}(R)$ which sends x_n to yx_{n+1} for each $n > 0$. The direct limit of the system $t_{\mathcal{G}}(R) \xrightarrow{h} t_{\mathcal{G}}(R) \xrightarrow{h} t_{\mathcal{G}}(R) \xrightarrow{h} t_{\mathcal{G}}(R) \xrightarrow{h} \cdots$ is not $\mathcal{G}$-reduced because multiplication by y acts surjectively on the direct limit (and it is non-zero).

4. Good tilting complexes

Let T be a silting complex in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ and let $\mathfrak{S} = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ be the endomorphism ring which we again consider as a topological ring by endowing it with the compact topology. We also let $A = \mathsf{dgEnd}_R(T) \cong \mathbf{R}\mathsf{Hom}_R(T, T)$ be the endomorphism dg-ring of T; this is a weakly non-positive dg-ring as $H^i(A) = \mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T, T[i]) = 0$ for $i > 0$. We have $\mathfrak{S} = H^0(A)$ and the weak non-positivity ensures that there is the standard t-structure $(\mathsf{D}^{\leq 0}, \mathsf{D}^{>0})$ in the derived category of right dg-modules $\mathsf{D}(\mathsf{dgMod}\text{-}A)$ defined by vanishing of cohomology; the heart of this t-structure is $\mathsf{Mod}\text{-}\mathfrak{S}$.

The following definition was introduced for 1-tilting modules by [Baz10], and then generalized to n-tilting modules by [BMT11]. Our version for silting complexes is akin to the condition used in a more general dg setting in [NS18]. Recall that a full triangulated subcategory of a triangulated category is thick if it is closed under direct summands, and we denote the thick closure operator as $\mathsf{thick}(-)$.

Definition 4.1. A silting complex $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ is good provided that $R \in \mathsf{thick}(T)$.

The assumption of being good is satisfied for any compact silting object. For non-compact ones being good is not automatic, but it is still a rather mild assumption thanks to the following well-known observation, which we reprove here in our setting.

Lemma 4.2. *For any silting complex T there is a good silting complex T' equivalent to T.*

Proof. Since T is a silting complex we have $R \in \mathsf{thick}(\mathsf{Add}(T))$, see [AH19, Proposition 5.3]. In particular, there are $T_0, \dots, T_{n-1} \in \mathsf{Add}(T)$ such that $R \in \mathsf{thick}(T_0, \dots, T_{n-1})$. Put $T' = T \oplus \bigoplus_{i=0}^{n-1} T_i$, then clearly $R \in \mathsf{thick}(T')$ and $\mathsf{Add}(T) = \mathsf{Add}(T')$. The last equality implies that T' is a silting complex and that it is equivalent to T as such. □

We record some adjunction formulas for dg-modules, well-known to experts, for further use.

Lemma 4.3. *Let A and B be dg-rings.*

There is an evaluation morphism

$$\gamma_{X,Y,Z} : \mathbf{R}\mathsf{Hom}_A(X, Y) \otimes^{\mathbf{L}}_B Z \to \mathbf{R}\mathsf{Hom}_A(X, Y \otimes^{\mathbf{L}}_B Z)$$

which is natural in $X \in \mathsf{D}(A\text{-}\mathsf{dgMod}), Y \in \mathsf{D}((A \otimes_{\mathbb{Z}} B^{\mathsf{op}})\text{-}\mathsf{Mod}), Z \in \mathsf{D}(B\text{-}\mathsf{dgMod})$.

There is an evaluation morphism

$$\delta_{X,Y,Z} : \mathbf{R}\mathsf{Hom}_A(Y, X) \otimes^{\mathbf{L}}_B Z \to \mathbf{R}\mathsf{Hom}_A(\mathbf{R}\mathsf{Hom}_B(Z, Y), X)$$

natural in $Y \in \mathsf{D}(\mathsf{Mod}\text{-}(A \otimes_{\mathbb{Z}} B^{\mathsf{op}})), X \in \mathsf{D}(\mathsf{dgMod}\text{-}A), Z \in \mathsf{D}(B\text{-}\mathsf{dgMod})$.

If Z is a compact object in $\mathsf{D}(B\text{-}\mathsf{dgMod})$ then both $\gamma_{X,Y,Z}$ and $\delta_{X,Y,Z}$ are isomorphisms (in $\mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z})$).

Proof. For the first morphism we refer to [Yek20, Theorem 12.9.10,Theorem 14.1.22]. The existence of the second morphism is covered e.g. in [BM17, Lemma 1.3]. The fact that $\delta_{X,Y,Z}$ is an isomorphism if Z is compact follows by a standard argument. Indeed, this map is easily checked to be an isomorphism if $Z = B$, and therefore it is an isomorphism also if $Z \in \mathsf{thick}_{\mathsf{D}(B\text{-}\mathsf{dgMod})}(B) = \mathsf{D}(B\text{-}\mathsf{dgMod})^{\mathsf{c}}$. □

The following results are available in [NS18] (see also [BM17]), but we gather the relevant parts here in a form directly applicable for our purposes.

Theorem 4.4. *Let $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ be a good silting complex. Then:*

- *(i) T is compact as an object of $\mathsf{D}(A\text{-}\mathsf{dgMod})$,*
- *(ii) the canonical morphism $R \to \mathbf{R}\mathsf{Hom}_A(T, T)$ is a quasi-isomorphism.*
- *(iii) both the functors $\mathbf{R}\mathsf{Hom}_R(T, -) : \mathsf{D}(\mathsf{Mod}\text{-}R) \to \mathsf{D}(\mathsf{dgMod}\text{-}A)$ and $T \otimes^{\mathbf{L}}_R - : \mathsf{D}(R\text{-}\mathsf{Mod}) \to \mathsf{D}(A\text{-}\mathsf{dgMod})$ are fully faithful.*
- *(iv) the essential images from both the functors from (iii) are thick subcategories.*

Proof. (i) : There is a finite sequence of triangles witnessing that $R \in \mathsf{thick}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$, and by applying $\mathbf{R}\mathsf{Hom}_R(-, T)$ on these triangles we obtain that $T \in \mathsf{thick}_{\mathsf{D}(A\text{-}\mathsf{dgMod})}(A) = \mathsf{D}(A\text{-}\mathsf{dgMod})^{\mathsf{c}}$.

(ii) : Follows as in [BM17, Theorem 1.4]. Indeed, consider the unit morphism $\alpha_X : X \to \mathbf{R}\mathsf{Hom}_A(\mathbf{R}\mathsf{Hom}_R(X, T), T)$ of the adjoint pair $\mathbf{R}\mathsf{Hom}_R(-, T) : \mathsf{D}(\mathsf{Mod}\text{-}R)^{\mathsf{op}} \rightleftarrows \mathsf{D}(A\text{-}\mathsf{dgMod})^{\mathsf{op}} : \mathbf{R}\mathsf{Hom}_A(-, T)$. Clearly,

$$\alpha_T : T \to \mathbf{R}\mathsf{Hom}_A(\mathbf{R}\mathsf{Hom}_R(T, T), T) \cong \mathbf{R}\mathsf{Hom}_A(A, T) \cong T$$

is an isomorphism. But since $R \in \mathsf{thick}_{\mathsf{D}(\mathsf{Mod}\text{-}R)} T$, also

$$\alpha_R : R \to \mathbf{R}\mathsf{Hom}_A(\mathbf{R}\mathsf{Hom}_R(R, T), T) \cong \mathbf{R}\mathsf{Hom}_A(T, T)$$

is an isomorphism, and this is easily checked to coincide with the canonical isomorphism.

(iii) : That $\mathbf{R}\mathsf{Hom}_R(T,-)$ is fully faithful follows from [NS18, Theorem 6.4] or also [BM17, Theorem 1.4]. The second statement follows similarly and is also contained in [NS18, Theorem 6.4], but we sketch it here for convenience. It is enough to show that the unit morphism $X \to \mathbf{R}\mathsf{Hom}_A(T, T \otimes^{\mathbf{L}}_R X)$ is an isomorphism for any $X \in \mathsf{D}(R\text{-}\mathsf{Mod})$. By Lemma 4.3, the natural morphism $\gamma_{X,T,T} : \mathbf{R}\mathsf{Hom}_A(T,T) \otimes^{\mathbf{L}}_R X \to \mathbf{R}\mathsf{Hom}_A(T, T \otimes^{\mathbf{L}}_R X)$ is an isomorphism. Then the above unit morphism factorizes as

$$X \xrightarrow{\cong} R \otimes^{\mathbf{L}}_R X \xrightarrow{\cong} \mathbf{R}\mathsf{Hom}_A(T,T) \otimes^{\mathbf{L}}_R X \xrightarrow{\gamma_{X,T,T}} \mathbf{R}\mathsf{Hom}_A(T, T \otimes^{\mathbf{L}}_R X),$$

and therefore it is an isomorphism.

(iv): The functors in question are fully faithful by (iii) and admit a left or right adjoint, respectively. □

Corollary 4.5. *Let $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ be a good and decent tilting object. Then the forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{S} \to \mathsf{Mod}\text{-}\mathfrak{S}$ is fully faithful.*

Proof. The functor $\mathsf{Hom}_{\mathcal{H}_T}(T,-) : \mathcal{H}_T \to \mathsf{Mod}\text{-}\mathfrak{S}$ factorizes into the composition of the equivalence $\mathsf{Hom}_{\mathcal{H}_T}(T,-) : \mathcal{H}_T \xrightarrow{\cong} \mathsf{Ctra}\text{-}\mathfrak{S}$ and the forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{S} \to \mathsf{Mod}\text{-}\mathfrak{S}$. On the other hand, the functor $\mathbf{R}\mathsf{Hom}_R(T,-) : \mathsf{D}(\mathsf{Mod}\text{-}R) \to \mathsf{D}(\mathsf{dgMod}\text{-}A)$ is fully faithful by Theorem 4.4. Then also the restriction of $\mathbf{R}\mathsf{Hom}_R(T,-)$ to the heart $\mathcal{H}_T$ is fully faithful, and this functor naturally identifies with $\mathsf{Hom}_{\mathcal{H}_T}(T,-) : \mathcal{H}_T \to \mathsf{Mod}\text{-}\mathfrak{S}$ (recall that $\mathsf{Mod}\text{-}\mathfrak{S}$ can be identified with the heart of the standard t-structure in $\mathsf{D}(\mathsf{dgMod}\text{-}A)$). □

4.1. **Derived equivalence and realization functors.** Given an abelian category $\mathcal{A}$, a t-structure $(\mathcal{U}, \mathcal{V})$ in $\mathsf{D}(\mathcal{A})$ (or in $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$) is intermediate if there are integers $n < m$ such that $\mathsf{D}^{\leq n} \subseteq \mathcal{U} \subseteq \mathsf{D}^{\leq m}$, where again $\mathsf{D}^{\leq n} = \{X \in \mathsf{D}(\mathcal{A}) \mid H^i(X) = 0\ \forall i > n\}$. It is easy to check that any intermediate t-structure in the unbounded derived category restricts to a t-structure in the bounded derived category and that any t-structure induced by a silting complex in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ is intermediate, and similarly for cosilting complexes in $\mathsf{D}(R\text{-}\mathsf{Mod})$.

Following [BBD82], as explained in a more detail in [PV18, §2, §3, §4], for any intermediate t-structure $\mathbb{T}$ in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ there is a triangle functor $\mathsf{real}_{\mathbb{T}} : \mathsf{D}^{\mathsf{b}}(\mathcal{H}_{\mathbb{T}}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ which extends the inclusion $\mathcal{H}_{\mathbb{T}} \subseteq \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$. This realization functor may in principle be non-unique[2] and in fact it is constructed using, and determined by, a suitable enhancement of $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ called the f-enhancement, see [PV18, §3]. An example of an f-enhancement is the structure of a filtered (bounded) derived category, which is always available for $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$.

Let $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ be a silting complex. Then the realization functor $\mathsf{real}_T : \mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ is a triangle equivalence if and only if T is tilting [PV18, Corollary 5.2]. The analogous result is also true for bounded cosilting objects in $\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod})$.

Proposition 4.6. *Let R, S be rings and let $\mathbb{R}$ and $\mathbb{S}$ be two t-structures in $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ and $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S)$ respectively. Let $F : \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S)$ be a triangle functor which satisfies the following conditions:*

(i) F is t-exact with respect to the t-structures $\mathbb{R}$ and $\mathbb{S}$, that is, F preserves both the left and the right constituents of the t-structures,

(ii) F is fully faithful and its essential image is a thick subcategory of $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S)$.

By (i), F restricts to an exact functor $F_0 : \mathcal{H}_{\mathbb{R}} \to \mathcal{H}_{\mathbb{S}}$ between the hearts of the two t-structures. Then the following diagram commutes (up to natural equivalence):

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}(\mathcal{H}_{\mathbb{R}}) & \xrightarrow{F_0} & \mathsf{D}^{\mathsf{b}}(\mathcal{H}_{\mathbb{S}}) \\ \downarrow{\scriptstyle \mathsf{real}_{\mathbb{R}}} & & \downarrow{\scriptstyle \mathsf{real}_{\mathbb{S}}} \\ \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) & \xrightarrow{F} & \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S) \end{array}$$

[2] This pathology seems to disappear once we switch to a strong enough enhancement of the derived category, cf. [Lur17, Proposition 1.3.3.7].

where in the upper row F_0 is naturally extended to the bounded derived categories and the realization functors are taken with respect to suitable f-enhancements.

Proof. This follows by combining [PV18, Theorem 3.13] and [PV18, Corollary 3.9]. Indeed, by [PV18, Example 3.2], the bounded derived category $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}S)$ admits an f-enhancement in the form of a filtered (bounded) derived category, see [PV18, §3.1] for the definitions. By (ii) and [PV18, Corollary 3.9], there is an induced f-enhancement on $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ such that F admits an f-lifting with respect to the two f-enhancements. Then [PV18, Theorem 3.13] applies. □

For any morphism $f : A \to B$ of dg-rings such that f is a quasi-isomorphism of the underlying complexes, the forgetful functor $U_f : \mathsf{D}(\mathsf{dgMod}\text{-}B) \to \mathsf{D}(\mathsf{dgMod}\text{-}A)$ is a triangle equivalence with $I_f := - \otimes_A^{\mathbf{L}} B \cong \mathbf{R}\mathsf{Hom}_A(B, -) : \mathsf{D}(\mathsf{dgMod}\text{-}A) \to \mathsf{D}(\mathsf{dgMod}\text{-}B)$, the inverse equivalence. Moreover, both the functors U_f and I_f preserve the cohomology of the objects. For details, see [Yek20, Theorem 12.7.2, Lemma 12.7.3].

Now let R be a ring, $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ a tilting complex, $\mathfrak{S} = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ be the endomorphism ring endowed with the compact topology, and let $A = \mathsf{dgEnd}_R(T)$ be the endomorphism dg-ring of T. Since T is tilting, we have $H^0(A) = \mathfrak{S}$ and $H^i(A) = 0$ for all $i \neq 0$. Then the zig-zag of quasi-isomorphisms of dg-rings $\mathfrak{S} = \tau^{\geq 0}\tau^{\leq 0}A \xleftarrow{r} \tau^{\leq 0}A \xrightarrow{l} A$ induces a triangle equivalence $\epsilon : \mathsf{D}(\mathsf{dgMod}\text{-}A) \to \mathsf{D}(\mathsf{Mod}\text{-}\mathfrak{S})$, where $\epsilon = I_r \circ U_l$. Clearly, ϵ restricts to an equivalence $\mathsf{D}^{\mathsf{b}}(\mathsf{dgMod}\text{-}A) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$ of the corresponding bounded derived categories. By abuse of notation, we will denote by ϵ also the analogous equivalence $\mathsf{D}(A\text{-}\mathsf{dgMod}) \xrightarrow{\cong} \mathsf{D}(\mathfrak{S}\text{-}\mathsf{Mod})$ on the side of left (dg-)modules.

Note that T being isomorphic in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ to a bounded complex of projectives implies that both the functors $\mathbf{R}\mathsf{Hom}_R(T, -)$ and $T \otimes_R^{\mathbf{L}} -$ restrict to functors between the respective bounded derived categories.

Theorem 4.7. *Assume that T is good and decent. Then the forgetful functor $\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$ is fully faithful, and the functor $G = \epsilon \circ \mathbf{R}\mathsf{Hom}_R(T, -)$ induces a triangle equivalence $G : \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$ with inverse equivalence $(- \otimes_A^{\mathbf{L}} T) \circ (\epsilon^{-1})_{\restriction \mathsf{D}(\mathsf{Ctra}\text{-}\mathfrak{S})}$.*

Proof. Observe that the functor G restricted to $\mathcal{H}_T$ is equivalent to the functor $\mathsf{Hom}_{\mathcal{H}_T}(T, -) : \mathcal{H}_T \to \mathsf{Ctra}\text{-}\mathfrak{S} \subseteq \mathsf{Mod}\text{-}\mathfrak{S}$, the last inclusion being fully faithful is provided by Corollary 4.5. Since T is a tilting complex, the realization functor $\mathsf{real}_T : \mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ is a triangle equivalence [PV18, Corollary 5.2]. The functor G is clearly t-exact with respect to the tilting t-structure $(T^{\perp_{>0}}, T^{\perp_{\leq 0}})$ in $\mathsf{D}(\mathsf{Mod}\text{-}R)$ and the standard t-structure in $\mathsf{D}(\mathsf{Mod}\text{-}\mathfrak{S})$. By Theorem 4.4, G is a fully faithful functor realizing $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ as a thick subcategory of $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$. Then Proposition 4.6 applies and yields a commutative diagram as follows:

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) & \xrightarrow{\mathsf{Hom}_{\mathcal{H}_T}(T,-)} & \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S}) \\
{\scriptstyle \cong}\big\downarrow{\scriptstyle \mathsf{real}_T} & & \big\downarrow{\scriptstyle =} \\
\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) & \xrightarrow{G} & \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})
\end{array}$$

Since G is fully faithful, and since $\mathsf{Hom}_{\mathcal{H}_T}(T, -)$ factorizes through $\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$, it follows from the upper row that the forgetful functor $\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S}) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$ is fully faithful. Then the essential image of $\mathsf{Hom}_{\mathcal{H}_T}(T, -) : \mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) \to \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$ is precisely the full subcategory $\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$ and the commutative square above yields another commutative square

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(\mathcal{H}_T) & \xrightarrow[\cong]{\mathsf{Hom}_{\mathcal{H}_T}(T,-)} & \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S}) \\
{\scriptstyle \cong}\big\downarrow{\scriptstyle \mathsf{real}_T} & & \big\downarrow{\scriptstyle =} \\
\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) & \xrightarrow{G} & \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})
\end{array}$$

with the upper functor being an equivalence. It follows that G is a triangle equivalence.

The inverse equivalence is $(- \otimes_A^{\mathbf{L}} T) \circ (\epsilon^{-1})_{\restriction \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})}$ since this is the left adjoint to G and $\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S})$ is a full subcategory of $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}\mathfrak{S})$. □

Corollary 4.8. *In the setting as above, the equivalence of Theorem 4.7 restricts to an equivalence* $H^0\,\mathbf{R}\mathsf{Hom}_R(T,-) : \mathcal{H}_T \cong \mathsf{Ctra}\text{-}\mathfrak{S} : - \otimes^{\mathbf{L}}_A T$*, when we consider* $\mathsf{Ctra}\text{-}\mathfrak{S}$ *as a full subcategory of the standard heart of* $\mathsf{D}(\mathsf{dgMod}\text{-}A)$.

5. Cotilting hearts and discrete modules

5.1. **Discrete modules and contratensor product.** Let $\mathfrak{R}$ be a (left) topological ring. A left $\mathfrak{R}$-module N is called discrete if for any element $n \in N$ its annihilator $\mathsf{Ann}_{\mathfrak{R}}(n) = \{r \in \mathfrak{R} \mid rn = 0\}$ is open. The full subcategory $\mathfrak{R}\text{-}\mathsf{Discr}$ of $\mathfrak{R}\text{-}\mathsf{Mod}$ consisting of all discrete left $\mathfrak{R}$-modules is a locally finitely generated Grothendieck category with an injective cogenerator obtained by taking the maximal discrete submodule of an injective cogenerator of $\mathfrak{R}\text{-}\mathsf{Mod}$.

Now assume that $\mathfrak{R}$ is complete and separated. For any $N \in \mathfrak{R}\text{-}\mathsf{Discr}$ and any abelian group V, the right $\mathfrak{R}$-module $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathbb{Z}}(N,V)$ structure naturally extends to a structure of a right $\mathfrak{R}$-contramodule. There is the contratensor product functor $- \odot_{\mathfrak{R}} - : \mathsf{Ctra}\text{-}\mathfrak{R} \times \mathfrak{R}\text{-}\mathsf{Discr} \to \mathsf{Mod}\text{-}\mathbb{Z}$, which is defined as a suitable quotient of the ordinary tensor product. The defining property of this functor is the adjunction isomorphism $\mathsf{Hom}_{\mathsf{Mod}\text{-}\mathbb{Z}}(\mathfrak{M} \odot_{\mathfrak{R}} N, V) \cong \mathsf{Hom}_{\mathsf{Ctra}\text{-}\mathfrak{R}}(\mathfrak{M}, \mathsf{Hom}_{\mathsf{Mod}\text{-}\mathbb{Z}}(N,V))$. We refer to [PŠ21, §7.2] for more details. Finally, we record the following observation.

Lemma 5.1. *Let T be a good and decent tilting complex in* $\mathsf{D}(\mathsf{Mod}\text{-}R)$ *and* $\mathfrak{S}$ *as before. Then the contratensor product* $-\odot_{\mathfrak{S}} - : \mathsf{Ctra}\text{-}\mathfrak{S} \times \mathfrak{S}\text{-}\mathsf{Discr} \to \mathsf{Mod}\text{-}\mathbb{Z}$ *is naturally equivalent to (the restriction of) the ordinary tensor product* $- \otimes_{\mathfrak{S}} -$.

Proof. By Corollary 4.5, the forgetful functor $\mathsf{Ctra}\text{-}\mathfrak{R} \to \mathsf{Mod}\text{-}\mathfrak{R}$ is fully faithful and this implies our claim, see [PŠ21, Lemma 7.11]. □

5.2. **Cotilting hearts.** We are ready to characterize cotilting hearts obtained from character duals of decent tilting complexes as categories of discrete modules.

Theorem 5.2. *Let* $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ *be a good and decent tilting complex and* $C = T^+$*. The functor* $H^0(T \otimes^{\mathbf{L}}_R -)$ *induces an equivalence* $\mathcal{H}_C \to \mathfrak{S}\text{-}\mathsf{Discr}$ *with the inverse equivalence* $\mathbf{R}\mathsf{Hom}_A(T,-)$*, where we consider* $\mathfrak{S}\text{-}\mathsf{Discr}$ *as a full subcategory of the heart* $\mathfrak{S}\text{-}\mathsf{Mod}$ *of the standard t-structure in* $\mathsf{D}(A\text{-}\mathsf{dgMod})$.

Proof. We start by checking that the two functors are well-defined. First, the functor $H^0(T \otimes^{\mathbf{L}}_R -)$ constitutes a well-defined functor $\mathsf{D}(R\text{-}\mathsf{Mod}) \to \mathfrak{S}\text{-}\mathsf{Mod}$. This is because the functor $T \otimes^{\mathbf{L}}_R -$ takes values in $\mathsf{D}(A\text{-}\mathsf{dgMod})$, which is sent to $\mathfrak{S}\text{-}\mathsf{Mod}$ by H^0. We show that this functor lands in the full subcategory $\mathfrak{S}\text{-}\mathsf{Discr}$; in fact, we show that $H^0(T \otimes^{\mathbf{L}}_R X)$ is a discrete left $\mathfrak{S}$-module for any $X \in \mathsf{D}(R\text{-}\mathsf{Mod})$. First, assume that X is a compact object of $\mathsf{D}(R\text{-}\mathsf{Mod})$. The compactness of X yields an isomorphism $H^0(T \otimes^{\mathbf{L}}_R X) \cong H^0\,\mathbf{R}\mathsf{Hom}_R(X^*,T)$ where $X^* = \mathbf{R}\mathsf{Hom}_R(X,R)$ is a compact object in $\mathsf{D}(\mathsf{Mod}\text{-}R)$, see §3 or Lemma 4.3. But then $H^0\,\mathbf{R}\mathsf{Hom}_R(X^*,T) = \mathsf{Hom}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(X^*,T)$ is clearly a discrete left $\mathfrak{S}$-module by the definition of the compact topology on $\mathfrak{S}$. Now consider a general object $X \in \mathsf{D}(R\text{-}\mathsf{Mod})$, we may and will assume that X is a dg-flat complex. By [CH15, Theorem], we can write X as a direct limit $X = \varinjlim_{i\in I} X_i$ of complexes X_i which are compact. Then $H^0(T \otimes^{\mathbf{L}}_R X) = H^0(T \otimes^{\mathbf{L}}_R \varinjlim_{i\in I} X_i) \cong \varinjlim_{i\in I} H^0(T \otimes^{\mathbf{L}}_R X_i)$. Since a direct limit of discrete modules is discrete, this argument is finished.

On the other hand, let $N \in \mathfrak{S}\text{-}\mathsf{Discr}$, then by compactness of $T \in \mathsf{D}(A\text{-}\mathsf{dgMod})$ we have that $\mathbf{R}\mathsf{Hom}_A(T,N)^+ \cong N^+ \otimes^{\mathbf{L}}_A T$ (see Lemma 4.3), and so $\mathbf{R}\mathsf{Hom}_A(T,N)^+ \in \mathcal{H}_T \subseteq \mathsf{D}(\mathsf{Mod}\text{-}R)$ by Corollary 4.8 since $N^+ \in \mathsf{Ctra}\text{-}\mathfrak{S}$. In view of Lemma 3.3, $\mathbf{R}\mathsf{Hom}_A(T,-)$ induces a well-defined funtor $\mathfrak{S}\text{-}\mathsf{Discr} \to \mathcal{H}_C$. Moreover, since $(T \otimes^{\mathbf{L}}_R X)^+ \cong \mathbf{R}\mathsf{Hom}_R(X,C)$ and $\mathcal{H}_C = {}^{\perp_{\neq 0}}C$, we have that $T \otimes^{\mathbf{L}}_R X$ has cohomology concentrated in degree 0 for any $X \in \mathcal{H}_C$. Therefore, $H^0(T \otimes^{\mathbf{L}}_R -)$ is naturally identified with $T \otimes^{\mathbf{L}}_R -$ as functors from the heart $\mathcal{H}_C$, and so the two functors from the statement are well-defined and also mutually adjoint. Therefore, to establish the equivalence it is enough to show that both the unit and the counit morphism of this adjunction are isomorphisms.

Let $N \in \mathfrak{S}\text{-}\mathsf{Discr}$, and consider the counit morphism $\nu : H^0(T \otimes^{\mathbf{L}}_R \mathbf{R}\mathsf{Hom}_A(T,N)) \to N$. Then we compute using a similar isomorphism utilizing Lemma 4.3 as above:

$$H^0(T \otimes^{\mathbf{L}}_R \mathbf{R}\mathsf{Hom}_A(T,N))^+ \cong H^0\,\mathbf{R}\mathsf{Hom}_R(T, \mathbf{R}\mathsf{Hom}_A(T,N)^+) \cong$$

$$\cong H^0\,\mathbf{R}\mathrm{Hom}_R(T, N^+ \otimes^{\mathbf{L}}_A T).$$

It follows that ν^+ is identified with the unit morphism of the equivalence of Corollary 4.8 evaluated at $N^+ \in \mathsf{Ctra}\text{-}\mathfrak{S}$, so ν^+ is an isomorphism, and therefore ν is an isomorphism.

Let $X \in \mathcal{H}_C$ and consider the unit morphism $\eta : X \to \mathbf{R}\mathrm{Hom}_A(T, T \otimes_R X)$. Then we compute:

$$\mathbf{R}\mathrm{Hom}_A(T, T \otimes^{\mathbf{L}}_R X)^+ \cong (T \otimes^{\mathbf{L}}_R X)^+ \otimes^{\mathbf{L}}_A T \cong$$

$$\cong \mathbf{R}\mathrm{Hom}_R(T, X^+) \otimes^{\mathbf{L}}_A T.$$

It follows that η^+ is the counit morphism of the equivalence of Corollary 4.8 evaluated at $X^+ \in \mathcal{H}_T$, and therefore η is an isomorphism. □

As a consequence, we obtain a description of the hearts induced by cotilting complexes of cofinite type. Recall that if R is commutative noetherian then any cotilting complex is of cofinite type [HN21, Corollary 2.14].

Corollary 5.3. *Let $C \in \mathsf{D}(R\text{-}\mathsf{Mod})$ be a cotilting complex of cofinite type. Then there is a complete and separated topological ring $\mathfrak{S}$ such that $\mathcal{H}_C \cong \mathfrak{S}\text{-}\mathsf{Discr}$.*

Proof. By Corollary 3.6, there is a decent tilting complex $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ such that C is equivalent to T^+ as cosilting complexes. By Lemma 4.2, there is a good and decent tilting complex T' which is equivalent to T. Putting $\mathfrak{S} = \mathrm{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T')$ and endowing $\mathfrak{S}$ with the compact topology, the proof is finished by noting that $\mathcal{H}_C \cong \mathcal{H}_{T'^+} \cong \mathfrak{S}\text{-}\mathsf{Discr}$ where the last equivalence follows from Theorem 5.2. □

5.3. **Cotilting derived equivalence and tensor compatibility.**

Theorem 5.4. *Assume that $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ is a good and decent tilting complex and let again $\mathfrak{S} = \mathrm{End}_{\mathsf{D}}(\mathsf{Mod}\text{-}R)(T)$. Then the forgetful functor $\mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr}) \to \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Mod})$ is fully faithful and the functor $H = \epsilon \circ (T \otimes^{\mathbf{L}}_R -)$ induces a triangle equivalence $H : \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \to \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr})$.*

Proof. This is proved similarly to Theorem 4.7. By Theorem 4.4, $(T \otimes^{\mathbf{L}}_R -) : \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \to \mathsf{D}^{\mathsf{b}}(A\text{-}\mathsf{dgMod})$ is fully faithful. Arguing as in Theorem 4.7, Proposition 4.6 yields a commutative square:

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(\mathcal{H}_C) & \xrightarrow{H^0(T \otimes^{\mathbf{L}}_R -)} & \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Mod}) \\
\cong \downarrow \mathsf{real}_C & & \downarrow = \\
\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{H} & \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Mod})
\end{array}$$

Since $H^0(T \otimes^{\mathbf{L}}_R -)$ factorizes as $\mathcal{H}_C \xrightarrow{\cong} \mathfrak{S}\text{-}\mathsf{Discr} \to \mathfrak{S}\text{-}\mathsf{Mod}$, we obtain that the forgetful functor $\mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr}) \to \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Mod})$ is fully faithful. Then the square above induces another commutative square

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(\mathcal{H}_C) & \xrightarrow{H^0(T \otimes^{\mathbf{L}}_R -)} & \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr}) \\
\cong \downarrow \mathsf{real}_C & & \downarrow = \\
\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{H} & \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr})
\end{array}$$

where the upper arrow is an equivalence. □

Theorem 5.5. *In the setting of Theorem 5.4, there is a commutative square as follows:*

$$\begin{array}{ccc}
\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \times \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{- \otimes^{\mathbf{L}}_R -} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \\
\cong \downarrow G \times H & & \downarrow = \\
\mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{S}) \times \mathsf{D}^{\mathsf{b}}(\mathfrak{S}\text{-}\mathsf{Discr}) & \xrightarrow{- \odot^{\mathbf{L}}_{\mathfrak{S}} -} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z})
\end{array}$$

Proof. Note first that here the contratensor product $- \odot_{\mathfrak{S}} -$ identifies with the ordinary tensor product $- \otimes_{\mathfrak{S}} -$ by Lemma 5.1. For $X \in \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ and $Y \in \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod})$, we have a sequence of natural isomorphisms in $\mathsf{D}(\mathbb{Z})$ as follows:

$$G(X) \otimes^{\mathbf{L}}_{\mathfrak{S}} H(Y) = \epsilon(\mathbf{R}\mathsf{Hom}_R(T, X)) \otimes^{\mathbf{L}}_{\mathfrak{S}} \epsilon(T \otimes^{\mathbf{L}}_R Y) \cong \mathbf{R}\mathsf{Hom}_R(T, X) \otimes^{\mathbf{L}}_A (T \otimes^{\mathbf{L}}_R Y) \cong$$
$$\cong (\mathbf{R}\mathsf{Hom}_R(T, X) \otimes^{\mathbf{L}}_A T) \otimes^{\mathbf{L}}_R Y \cong X \otimes^{\mathbf{L}}_R Y.$$

In the second isomorphism we use the fact that the equivalence $\epsilon : \mathsf{D}(\mathsf{dgMod}\text{-}A) \to \mathsf{D}(\mathsf{Mod}\text{-}\mathfrak{S})$ (resp. $\epsilon : \mathsf{D}(A\text{-}\mathsf{dgMod}) \to \mathsf{D}(\mathfrak{S}\text{-}\mathsf{Mod})$) preserves quasi-isomorphism and derived tensor products, see [Yek20, Theorem 12.7.2]. The last isomorphism follows from Theorem 4.4(iii). □

5.4. **A kind of a converse result.** There is a theory of Morita theory of complete and separated topological ring developed in [PŠ21] and [PŠ19b], which we recall now. Let $\mathfrak{R}$ be a complete and separated (left) topological ring and $\mathfrak{P} \in \mathsf{Ctra}\text{-}\mathfrak{R}$ a projective generator. Then the endomorphism ring $\mathfrak{R}' = \mathsf{End}_{\mathsf{Ctra}\text{-}\mathfrak{R}}(\mathfrak{P})$ admits a naturally induced linear topology of open left ideals such that $\mathfrak{R}'$ is complete and separated and there is an equivalence $\mathsf{Ctra}\text{-}\mathfrak{R} \cong \mathsf{Ctra}\text{-}\mathfrak{R}'$ which takes $\mathfrak{P}$ to $\mathfrak{R}'$. In this situation, we say that $\mathfrak{R}$ and $\mathfrak{R}'$ are topologically Morita equivalent. In particular, if $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ is a tilting complex with $\mathfrak{S} = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ endowed with the compact topology and T' is a tilting complex equivalent to T then $\mathfrak{S}' = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ admits a linear topology which makes it topologically Morita equivalent to $\mathfrak{S}$. Furthermore, one can check directly that this topology described in [PŠ21, Corollary 7.6] coincides with the compact topology defined on the endomorphism ring $\mathfrak{S}'$ (see also [PŠ19b, §5]). Finally, in this situation there is also an equivalence $\mathfrak{S}\text{-}\mathsf{Discr} \cong \mathfrak{S}'\text{-}\mathsf{Discr}$ of the discrete module categories [PŠ19b, Proposition 5.2].

If $\mathcal{A}$ is an abelian category with enough projectives, let $\mathsf{K}^{\mathsf{b}}(\mathcal{A}_{\mathsf{proj}})$ denote the homotopy category of bounded complexes of projectives objects of $\mathcal{A}$, considered as a full subcategory of $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$.

Proposition 5.6. *Assume that there is a complete, separated topological ring $\mathfrak{R}$ and a pair of triangle equivalences $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{R})$ and $\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathfrak{R}\text{-}\mathsf{Discr})$ which make the following diagram commute:*

$$\begin{array}{ccc} \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \times \mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) & \xrightarrow{-\otimes^{\mathbf{L}}_R -} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \\ \cong\downarrow & & \downarrow = \\ \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{R}) \times \mathsf{D}^{\mathsf{b}}(\mathfrak{R}\text{-}\mathsf{Discr}) & \xrightarrow{-\odot^{\mathbf{L}}_{\mathfrak{R}} -} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \end{array}$$

Then there is a good and decent tilting complex $T \in \mathsf{D}(\mathsf{Mod}\text{-}R)$ such that its endomorphism ring $\mathfrak{S} = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ endowed with compact topology is topologically Morita equivalent to $\mathfrak{R}$.

Proof. Under the equivalence $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{R})$, the projective generator $\mathfrak{R}$ of $\mathsf{Ctra}\text{-}\mathfrak{R}$ corresponds to an object $T \in \mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$, and similarly an injective cogenerator $W \in \mathfrak{R}\text{-}\mathsf{Discr}$ corresponds to an object C in $\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod})$. The equivalences transfer the standard t-structures to t-structures of the form $\mathbb{T}_T = (T^{\perp_{>0}}, T^{\perp_{\leq 0}})$ and $\mathbb{T}_C = ({}^{\perp_{\leq 0}}C, {}^{\perp_{>0}}C)$ in $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ and $\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod})$, respectively. We also have $\mathsf{Add}(T) \subseteq \mathcal{H}_T$ and $\mathsf{Prod}(C) \subseteq \mathcal{H}_C$, where $\mathcal{H}_T$ and $\mathcal{H}_C$ are the hearts of the two t-structures. Then $\mathsf{Add}(T) \subseteq T^{\perp_{\neq 0}}$ and $\mathsf{Prod}(C) \subseteq {}^{\perp_{\neq 0}}C$.

The equivalence $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{R})$ restricts to $\mathsf{K}^{\mathsf{b}}(\mathsf{Mod}\text{-}R_{\mathsf{proj}}) \xrightarrow{\cong} \mathsf{K}^{\mathsf{b}}(\mathsf{Ctra}\text{-}\mathfrak{R}_{\mathsf{proj}})$. Indeed, as in the proof of [PV18, Theorem 5.3] which refers to the argument [Ric89, Proposition 6.2], if $\mathcal{A}$ is a cocomplete abelian category with enough projectives then we can characterize $\mathsf{K}^{\mathsf{b}}(\mathcal{A}_{\mathsf{proj}})$ inside $\mathsf{D}^{\mathsf{b}}(\mathcal{A})$ internally as a full subcategory consisting of those objects X such that for any Y we have $\mathsf{Hom}_{\mathsf{D}^{\mathsf{b}}(\mathcal{A})}(X, Y[i])$ for $i \gg 0$. It follows in the same fashion as in [PV18, Theorem 5.3] that T is isomorphic in $\mathsf{D}^{\mathsf{b}}(\mathsf{Mod}\text{-}R)$ to a bounded complex of projective R-modules and furthermore, that $\mathsf{thick}(\mathsf{Add}(T)) = \mathsf{K}^{\mathsf{b}}(\mathsf{Mod}\text{-}R_{\mathsf{proj}})$. Then by [AHH21, Proposition 5.3], T is a bounded tilting object in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. An analogous argument using [AHH21, Proposition 6.8] shows that C is a bounded cotilting object in $\mathsf{D}(R\text{-}\mathsf{Mod})$.

Recall that by an adjunction argument we have $\mathsf{Hom}_{\mathsf{D}(R\text{-}\mathsf{Mod})}(X, T^+[i]) = 0$ if and only if $H^{-i}(T \otimes^{\mathbf{L}}_R X) = 0$. Via the identification of $T \otimes^{\mathbf{L}}_R -$ and $\mathfrak{R} \odot^{\mathbf{L}}_{\mathfrak{R}} -$, we see that the equivalence

$\mathsf{D}^{\mathsf{b}}(R\text{-}\mathsf{Mod}) \xrightarrow{\cong} \mathsf{D}^{\mathsf{b}}(\mathfrak{R}\text{-}\mathsf{Discr})$ identifies the standard t-structure in $\mathsf{D}^{\mathsf{b}}(\mathfrak{R}\text{-}\mathsf{Discr})$ with the cosilting t-structure induced by T^+. It follows that the cosilting complexes C and T^+ are equivalent, and so in particular, T^+ is cotilting and $\mathcal{H}_C = \mathcal{H}_{T^+}$. It follows from Theorem 3.4 that T is decent. Passing to a direct sum $T^{(X)}$ of copies of T for some set X, we can assume that T is also good by Lemma 4.2. As discussed above, this adjustment will replace the original topological ring $\mathfrak{R}$ by the endomorphism ring of $\mathfrak{R}^{(X)} \in \mathsf{Ctra}\text{-}\mathfrak{R}$, so it only changed up to topological Morita equivalence. Therefore, we may assume that T is good without loss of generality.

We have $\mathfrak{R} = \mathsf{End}_{\mathsf{D}(R)}(T)$ as ordinary rings. Let $\mathfrak{S}$ be the same ring as $\mathfrak{R}$ but endowed with the compact topology. Then we know from Corollary 4.8 and Theorem 5.2 that $\mathcal{H}_T \cong \mathsf{Ctra}\text{-}\mathfrak{S}$ and $\mathcal{H}_C \cong \mathfrak{S}\text{-}\mathsf{Discr}$. Using Theorem 5.5, we obtain a commutative square as follows

$$\begin{array}{ccc} \mathsf{Ctra}\text{-}\mathfrak{S} \times \mathfrak{S}\text{-}\mathsf{Discr} & \xrightarrow{-\otimes_{\mathfrak{S}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \\ {\scriptstyle\cong}\downarrow{\scriptstyle G' \times H'} & & \downarrow{\scriptstyle =} \\ \mathsf{Ctra}\text{-}\mathfrak{R} \times \mathfrak{R}\text{-}\mathsf{Discr} & \xrightarrow{-\odot_{\mathfrak{R}}-} & \mathsf{D}(\mathsf{Mod}\text{-}\mathbb{Z}) \end{array}$$

in which the equivalence G' is obtained as a composition of the equivalences $\mathcal{H}_T \cong \mathsf{Ctra}\text{-}\mathfrak{R}$ and $\mathcal{H}_T \cong \mathsf{Ctra}\text{-}\mathfrak{S}$, and similarly for H'. It follows from this square that for any $N \in \mathfrak{R}\text{-}\mathsf{Discr}$, we have $N \cong \mathfrak{R} \odot_{\mathfrak{R}} N \cong \mathfrak{S} \otimes_{\mathfrak{S}} H'(N) \cong H'(N)$, where the isomorphisms are natural and computed in $\mathsf{Mod}\text{-}\mathbb{Z}$. It follows that the equivalence H' induces an equivalence between the forgetful functors $\mathfrak{S}\text{-}\mathsf{Discr} \to \mathsf{Mod}\text{-}\mathbb{Z}$ and $\mathfrak{R}\text{-}\mathsf{Discr} \to \mathsf{Mod}\text{-}\mathbb{Z}$. Then it follows from [PŠ19b, Proposition 4.2] that $\mathfrak{R}$ and $\mathfrak{S}$ are isomorphic as topological rings. □

6. Example: Commutative noetherian rings of dimension one

In this section, R is a commutative noetherian ring of Krull dimension equal to one. Let $\mathsf{d} : \mathsf{Spec}(R) \to \mathbb{Z}$ be a codimension function on the Zariski spectrum (e.g., we can choose d to be the height function ht which assigns to a prime ideal $\mathfrak{p}$ its height $\mathsf{ht}(\mathfrak{p})$). Following [HNŠ22, Theorem 4.6], there is a silting complex of the form $T_{\mathsf{d}} = \bigoplus_{\mathfrak{p} \in \mathsf{Spec}(R)} \mathbf{R}\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}[\mathsf{d}(\mathfrak{p})]$, where $\mathbf{R}\Gamma_{\mathfrak{p}} R_{\mathfrak{p}}$ is the local cohomology of the local ring $R_{\mathfrak{p}}$ considered as an object in $\mathsf{D}(\mathsf{Mod}\text{-}R)$. It follows from [HNŠ22, Theorem 6.10] and Theorem 3.4 that T_{d} is a decent tilting complex. Since the tilting heart and the endomorphism ring of T_{d} does not depend on the choice of the codimension function d, see [HNŠ22, Remark 4.10], we assume $\mathsf{d} = \mathsf{ht}$ and denote simply $T = T_{\mathsf{d}}$ from now on. Let W_i be the set of all primes of height i for $i = 0, 1$; that is, W_0 consists of all minimal primes and W_1 of all maximal primes of $\mathsf{Spec}(R)$. Let $Q = \prod_{\mathfrak{p} \in W_0} R_{\mathfrak{p}}$ and $\widehat{R} = \prod_{\mathfrak{m} \in W_1} \widehat{R_{\mathfrak{m}}}$, where $\widehat{R_{\mathfrak{m}}}$ is the $\mathfrak{m}$-adic completion of the local ring $R_{\mathfrak{m}}$. Note that we have the canonical flat ring epimorphism $R \to Q$ and the faithfully flat map $R \to \widehat{R}$. We remark that if R is Cohen-Macaulay then $R \to Q$ is monic and Q is precisely the total ring of quotients of R. In any case, we can write the tilting complex as $T = Q \oplus K$, where $K = \mathsf{Cone}(R \to Q)$, see [HNŠ22, end of §4]. From this description, one can also see that T is good. Indeed, there is a triangle of the form $R \to Q \to K \to R[1]$, and so $R \in \mathsf{thick}(T)$.

The endomorphism ring $\mathfrak{S} = \mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(T)$ can be written explicitly, this is discussed already in [PŠ19a, Example 8.4] in the case when Q is the total ring of quotients; the non-Cohen-Macaulay situation is covered in [HNŠ22, Example 6.5]. The endomorphism ring has the following matrix presentation:

$$\mathfrak{S} = \begin{pmatrix} Q & 0 \\ Q \otimes_R \widehat{R} & \widehat{R} \end{pmatrix}.$$

Here, $\widehat{R}$ is identified with $\mathsf{End}_{\mathsf{D}(\mathsf{Mod}\text{-}R)}(K)$. In fact, we have the primary decomposition $K = \bigoplus_{\mathfrak{m} \in W_1} K_{\mathfrak{m}}$, and for each $\mathfrak{m}$ we can write $K = \mathsf{hocolim}_{n \geq 0} K(s_{\mathfrak{m}}^n)$ where $K(s_{\mathfrak{m}}^n)$ is the Koszul complex over any non-unit non-zerodivisor element $s_{\mathfrak{m}} \in R_{\mathfrak{m}}$, see e.g. [PŠ21, Example 5.7]. Since Koszul complexes are compact, this allows to describe the compact topology of left open ideals of $\mathfrak{S}$. The induced topology on the corner rings Q and $\widehat{R}$ makes them into topological rings. The ring Q is an artinian and as such its topology is discrete. From the presentation of K above, one can compute directly that $\widehat{R}$ carries the product topology of the $\mathfrak{m}$-adic topologies on $\widehat{R_\mathfrak{m}}$'s. Finally, $Q \otimes_R \widehat{R}$ is endowed with a topology of R-submodules induced as follows. First, note that $Q \otimes_R \widehat{R}$ is identified with the restricted product of $Q \otimes_R \widehat{R_\mathfrak{m}}$'s with respect to images of the maps $R_\mathfrak{m} \to Q \otimes_R \widehat{R_\mathfrak{m}}$ for each $\mathfrak{m} \in W_1$. In other words, this is a submodule of $\prod_{\mathfrak{m} \in W_1}(Q \otimes_R \widehat{R_\mathfrak{m}})$ consisting of those sequences $(q_\mathfrak{m} \otimes_R c_\mathfrak{m})$ with $q_\mathfrak{m} \in Q$ and $c_\mathfrak{m} \in R_\mathfrak{m}$ such that $q_\mathfrak{m} = 1$ for almost all $\mathfrak{m} \in W_1$. For each $\mathfrak{m} \in W_1$, the topology on $Q \otimes_R \widehat{R_\mathfrak{m}}$ has the base of open submodules of the form $s_\mathfrak{m}^{-k}(\mathfrak{m}^n \widehat{R_\mathfrak{m}})$ with $k, n \geq 0$. The topology on $\prod_{\mathfrak{m} \in W_1}(Q \otimes_R \widehat{R_\mathfrak{m}})$ is then the induced restricted product topology.

By a standard argument, the category $\mathfrak{S}$-Mod of left $\mathfrak{S}$-modules can be identified with the category in which the objects are R-module morphisms $V \xrightarrow{\varphi} M$ where $V \in$ Mod-Q and $M \in$ Mod-$\widehat{R}$ and the morphisms are commutative squares

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & M \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \gamma} \\ V' & \xrightarrow{\varphi'} & M' \end{array}$$

where η is a morphism of Q-modules and γ a morphism of $\widehat{R}$-modules. The action of an element $q \otimes_R c \in Q \otimes_R \widehat{R} \subseteq \mathfrak{S}$ on an object as above takes an element $v \in V$ to $c\varphi(qv) \in M$. The category Mod-$\mathfrak{S}$ of right $\mathfrak{S}$-modules has an analogous description with arrows $M \xrightarrow{\varphi} V$ going in the opposite direction and the right $\mathfrak{S}$-action of an element $q \otimes_R c \in Q \otimes_R \widehat{R}$ defined using the rule $m(q \otimes_R c) = \varphi(mc)q$ for $m \in M$.

Then the tilting heart $\mathcal{H}_T$ can be described explicitly as follows. By Corollary 4.8 and Corollary 4.5, we know that $\mathcal{H}_T \cong$ Ctra-$\mathfrak{S}$ and Ctra-$\mathfrak{S}$ is a full subcategory of Mod-$\mathfrak{S}$. Then $\mathcal{H}_T$ identifies with a full subcategory of the above described category of R-linear morphisms $M \xrightarrow{\varphi} V$ with $M \in$ Mod-$\widehat{R}$ and $V \in$ Mod-Q. Furthermore, it is clear from the contraaction of $\mathfrak{S}$ that the action of $\widehat{R}$ on M extends to the unique contraaction; note that Ctra-$\widehat{R} \subseteq$ Mod-$\widehat{R}$ is a full subcategory [Pos17, Corollary 13.13]. On the other hand, for any morphism $\mathfrak{M} \to V$ as above with $\mathfrak{M} \in$ Ctra-$\widehat{R}$ the right $\mathfrak{S}$-action on $\mathfrak{M} \oplus V$ extends to a right $\mathfrak{S}$-contraaction. The contraaction of the two corner rings of $\mathfrak{S}$ is clear: $\widehat{R}$ acts on the $\widehat{R}$-contramodule $\mathfrak{M}$ and Q acts as an ordinary ring on V. It remains to see how the contraaction is defined given a sequence $(q_\alpha \otimes_R c_\alpha)_{\alpha \in A}$ of elements of $Q \otimes_R \widehat{R}$ which converges to zero in the topology. By the description of the topology above, all but finitely many q_α's can be assumed to be 1's. Then for any collection $m_\alpha \in \mathfrak{M}, \alpha \in A$, the contramodule action is computed as follows: $\sum_{\alpha \in A}(m_\alpha)(q_\alpha \otimes_R c_\alpha) = \sum_{\alpha \in F}(m_\alpha)(q_\alpha \otimes_R c_\alpha) + \sum_{\alpha \in A \setminus F}(m_\alpha)(1 \otimes_R c_\alpha) = \sum_{\alpha \in F} \varphi(m_\alpha c_\alpha) q_\alpha + \varphi(\sum_{\alpha \in A \setminus F} m_\alpha c_\alpha)$, where F is a finite subset of A such that $q_\alpha = 1$ whenever $\alpha \in A \setminus F$. Note that $(c_\alpha)_{\alpha \in A}$ converges to zero in the topology of $\widehat{R}$, which ensures that $\sum_{\alpha \in A \setminus F} m_\alpha c_\alpha$ is well-defined using the $\widehat{R}$-contramodule structure of $\mathfrak{M}$. One can check directly that all this defines an assignment $\mathfrak{S}[[\mathfrak{M} \oplus V]] \to \mathfrak{M} \oplus V$ satisfying the conditions of contra-associativity and contra-unitality, and thus we obtain the contraaction. Since Ctra-$\mathfrak{S}$ is a full subcategory of Mod-$\mathfrak{S}$, this shows that $\mathcal{H}_T$ is equivalent to the full subcategory of Mod-$\mathfrak{S}$ consisting of R-linear morphisms $\mathfrak{M} \xrightarrow{\varphi} V$ with $\mathfrak{M} \in$ Ctra-$\mathfrak{S}$ and $V \in$ Mod-Q. We remark that $\mathfrak{M}$ decomposes into a product $\prod_{\mathfrak{m} \in W_1} \mathfrak{M}_\mathfrak{m}$ where $\mathfrak{M}_\mathfrak{m} \in$ Ctra-$\widehat{R_\mathfrak{m}}$; for a module-theoretic description of this category see [Pos17, Corollary 13.13].

We are left with the task of describing the cotilting heart induced by $C = T^+$. Using Theorem 5.4, $\mathcal{H}_C$ is equivalent to $\mathfrak{S}$-Discr. Arguing similarly (in fact, more easily) to above, we see that $\mathcal{H}_C$ is identified with the full subcategory of objects of the form $V \xrightarrow{\varphi} M$ where M is in $\widehat{R}$-Discr. Here, it is well-known that $\widehat{R}$-Discr is naturally identified with the full subcategory of R-Mod consisting of modules which are W_1-torsion, meaning that they are supported on the set $W_1 \subseteq \mathsf{Spec}(R)$. Yet another description is that these are those modules M such that $M \otimes_R Q = 0$. Finally, we remark that one can identify $\mathcal{H}_C$ with the data of the Zariski torsion model constructed in [BGPW20, Remark 8.8, see also Example 8.9, §9.3] by Balchin, Greenlees, Pol, and Williamson, while $\mathcal{H}_T$ is seemingly different from the complete model of Balchin and Greenlees in [BG22], see [BG22, §10] in particular.

Acknowledgement

The author would like to to thank Francesco Genovese, Leonid Positselski, and Jordan Williamson for helpful discussions. A special thanks is due to Leonid Positselski for giving a very useful feedback on an earlier version of the manuscript.

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