INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Motion of Several Rigid bodies in a Compressible Fluid: mixed case

Šárka Nečasová
Mythily Ramaswamy
Arnab Roy
Anja Schlömerkemper

Preprint No. 27-2022

PRAHA 2022

MOTION OF SEVERAL RIGID BODIES IN A COMPRESSIBLE FLUID: MIXED CASE

Š. NEČASOVÁ, M. RAMASWAMY, A. ROY, AND A. SCHLÖMERKEMPER

Abstract. In this article we show local-in-time existence of a weak solution to a system of partial differential equations describing the evolution of a compressible isentropic fluid which contains several rigid bodies. The fluid-structure interaction is incorporated by the Navier-slip boundary condition at the interface of the fluid and the rigid bodies. At the boundary of the fluid's container we assume Dirichlet boundary conditions. This work follows an earlier article of the same authors regarding the evolution of a compressible fluid that contains one rigid body and assumes Navier-slip boundary conditions at the interface as well as at the boundary of the container. The novelties comprise a new bound on the maximal time for which we can prove existence of weak solutions, different test functions and a different extension of the fluid velocity from the fluid domain to the whole container.

1. Introduction

We consider a fluid in a bounded smooth domain $\Omega \subset \mathbb{R}^3$, which contains several rigid bodies $\mathcal{S}_i(t) \subset \Omega$, which are also assumed to be regular, bounded domains. Their motion is modeled with the help of linear and angular momentum and the respective balance equations. The fluid is modeled by the compressible Navier-Stokes equation and the fluid domain is denoted by $\mathcal{F}(t) = \Omega \setminus \bigcup_{i=1}^{M} \overline{\mathcal{S}_i(t)}$, where $t \in [0, T)$ for some $T > 0$. The fluid occupies, at $t = 0$, the domain $\mathcal{F}_0 = \Omega \setminus \bigcup_{i=1}^{M} \overline{\mathcal{S}_{0i}}$, where the initial position of the i-th rigid body is given by $\mathcal{S}_{0i}$, $i = 1, 2, \cdots M$.

The mathematical analysis of corresponding fluid-structure problems in the *incompressible* setting has been developed in several articles in the previous decades, see e.g. the introductory article [6] and [16, Section 1.2] for a brief account on related literature. In particular, it was observed that the choice of the boundary conditions is crucial in view of collisions [7, 8, 9, 17].

On the contrary, the theory of fluid-structure interaction in the *compressible* setting is less developed. Regarding the evolution of a system of a rigid body in a compressible fluid with Dirichlet boundary conditions, existence of strong solutions was studied in [1, 10, 21, 12]; the existence of a weak solution up to a collision is proved in [3]. In [4] this result was generalized to allow also for collisions. Recently, weak-strong uniqueness regarding a system of a compressible fluid with a rigid body was investigated [14].

The Dirichlet no-slip boundary condition fits well to various experimental observations of velocity profiles for compressible and incompressible fluids. Still, mathematical analysis yielded the unrealistic result that rigid objects which are immersed in a linearly viscous fluid cannot collide [11, 13]. Hence, the Navier-slip boundary condition came into focus. At the interface between the fluid and the rigid bodies, the normal components of the respective velocity fields are supposed to be identical. An argument in favor of the Navier-slip boundary condition is that interface roughness influences the slip behavior of a viscous fluid, cf. e.g. [19, 20]. However, the discontinuity in the tangential component of the velocity field causes major difficulties.

In this article we assume a mixed type of boundary conditions. We consider Navier-slip boundary conditions at the fluid-structure interface as we did in [16], where we proved local-in-time existence of weak solutions to the problem of compressible fluid with one rigid body in the case of Navier-slip boundary conditions at the interface and at the boundary of the container. At the boundary of the container $\partial\Omega$, we assume Dirichlet boundary conditions. To overcome the mathematical difficulties related to the discontinuity of the velocity field, we rely on the techniques developed in our earlier article [16], see below.

Before presenting the system of partial differential equations under investigation, we fix some more notation: $\rho_{\mathcal{F}}$ and $u_{\mathcal{F}}$ represent respectively the mass density and the velocity of the fluid; the pressure of the fluid is denoted by $p_{\mathcal{F}}$. Further, $v \otimes w = (v_i w_j)_{1 \leqslant i,j \leqslant 3}$ for any $v, w \in \mathbb{R}^3$.

Date: May 24, 2022.

The fluid flow is assumed to be in the barotropic regime, and more specifically in the isentropic case. The constitutive law between $p_{\mathcal{F}}$ and $\rho_{\mathcal{F}}$ is then given by

$$p_{\mathcal{F}} = a_{\mathcal{F}} \rho_{\mathcal{F}}^{\gamma}, \tag{1.1}$$

with $a_{\mathcal{F}} > 0$ and the adiabatic constant $\gamma > \frac{3}{2}$. Currently, these are the canonical assumptions which ensure existence of a weak solution for the compressible Navier-Stokes equation, cf. e.g. [5]. The stress tensor satisfies

$$\mathbb{T}(u_{\mathcal{F}}) = 2\mu_{\mathcal{F}}\mathbb{D}(u_{\mathcal{F}}) + \lambda_{\mathcal{F}} \operatorname{div} u_{\mathcal{F}}\mathbb{I},$$

where $\mathbb{D}(u_{\mathcal{F}}) = \frac{1}{2}\left(\nabla u_{\mathcal{F}} + \nabla u_{\mathcal{F}}^{\top}\right)$ is the symmetric part of the gradient of the velocity $u_{\mathcal{F}}$; here, $\nabla u_{\mathcal{F}}^{\top}$ denotes the transpose of the matrix $\nabla u_{\mathcal{F}}$. The viscosity coefficients $\mu_{\mathcal{F}}, \lambda_{\mathcal{F}}$ satisfy $\mu_{\mathcal{F}} > 0$ and $3\lambda_{\mathcal{F}} + 2\mu_{\mathcal{F}} \geqslant 0$.

The evolutionary model of the system under consideration consists of the continuity for the mass density of the fluid, the compressible Navier-Stokes equation as well as the equations for the linear and angular momenta of the rigid bodies. The latter involve the Eulerian velocity

$$u_{\mathcal{S}_i}(t,x) = h_i'(t) + \omega_i(t) \times (x - h_i(t)), \qquad t \in (0,T),\ x \in \mathcal{S}_i(t), \tag{1.2}$$

with $h_i(t)$ being the centre of mass and $h_i'(t)$, $\omega_i(t)$ denoting the linear and angular velocities of the rigid bodies, $i = 1, \dots, M$. Hence the domain of the ith rigid body at time t is given by

$$\mathcal{S}_i(t) = \left\{ h_i(t) + \mathbb{O}_i(t)x \mid x \in \mathcal{S}_{0i} \right\},$$

where $\mathbb{O}_i(t) \in SO(3)$ is associated to the rotation of the rigid body:

$$\mathbb{O}_i'(t)\mathbb{O}_i(t)x = \omega_i(t) \times x \quad \forall\, x \in \mathbb{R}^3, \quad \mathbb{O}_i(0) = \mathbb{I}.$$

The initial velocity of the rigid body is given by

$$u_{\mathcal{S}_i}(0,x) = u_{\mathcal{S}_{0i}} := \ell_{0i} + \omega_{0i} \times x, \quad x \in \mathcal{S}_{0i}. \tag{1.3}$$

The mass density $\rho_{\mathcal{S}_i}$ of the rigid bodies is governed by the transport equation

$$\frac{\partial \rho_{\mathcal{S}_i}}{\partial t} + u_{\mathcal{S}_i} \cdot \nabla \rho_{\mathcal{S}_i} = 0, \quad t \in (0,T),\ x \in \mathcal{S}_i(t) \quad \text{with } \rho_{\mathcal{S}_i}(0,x) = \rho_{\mathcal{S}_{0i}}(x), \quad \forall\, x \in \mathcal{S}_{0i}. \tag{1.4}$$

Sometimes it is useful to express the mass m_i, the centre of mass h_i and the moment of inertia $J_i(t)$ with the help of the following formulae:

$$m_i = \int\limits_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i}\ dx, \tag{1.5}$$

$$h_i(t) = \frac{1}{m} \int\limits_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i}\ x\ dx, \tag{1.6}$$

$$J_i(t) = \int\limits_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i} \left[|x - h_i(t)|^2 \mathbb{I} - (x - h_i(t)) \otimes (x - h_i(t)) \right] dx. \tag{1.7}$$

We are now in the position to present the system under investigation, a system of four coupled differential equations:

$$\frac{\partial \rho_{\mathcal{F}}}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}}) = 0, \quad t \in (0,T),\ x \in \mathcal{F}(t), \tag{1.8}$$

$$\frac{\partial (\rho_{\mathcal{F}} u_{\mathcal{F}})}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}} \otimes u_{\mathcal{F}}) - \operatorname{div} \mathbb{T}(u_{\mathcal{F}}) + \nabla p_{\mathcal{F}} = \rho_{\mathcal{F}} g_{\mathcal{F}}, \quad t \in (0,T),\ x \in \mathcal{F}(t), \tag{1.9}$$

$$m_i h_i''(t) = -\int\limits_{\partial \mathcal{S}_i(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I})\nu_i\, d\Gamma + \int\limits_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i} g_{\mathcal{S}_i}\, dx, \quad \text{in } (0,T),\ i = 1, ..., M, \tag{1.10}$$

$$(J_i \omega_i)'(t) = -\int\limits_{\partial \mathcal{S}_i(t)} (x - h_i(t)) \times (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I})\nu_i\, d\Gamma + \int\limits_{\mathcal{S}_i(t)} (x - h_i(t)) \times \rho_{\mathcal{S}_i} g_{\mathcal{S}_i}\, dx, \quad \text{in } (0,T),\ i = 1, ..., M, \tag{1.11}$$

where $g_{\mathcal{F}}$, $g_{\mathcal{S}_i}$ are the specific body forces and ν_i is the unit normal to $\partial \mathcal{S}_i(t)$ which is directed to the interior of the rigid body.

The fluid-structure interaction is governed by the Navier-slip boundary conditions at the interface of the fluid and the rigid bodies. We assume Dirichet boundary conditions on $\partial\Omega$. Then the boundary conditions read

$$u_{\mathcal{F}} \cdot \nu_i = u_{\mathcal{S}_i} \cdot \nu_i, \quad t \in (0,T),\ x \in \partial\mathcal{S}_i(t),\ i = 1, ..., M, \tag{1.12}$$

$$(\mathbb{T}(u_{\mathcal{F}})\nu_i) \times \nu_i = -\alpha(u_{\mathcal{F}} - u_{\mathcal{S}_i}) \times \nu_i, \quad t \in (0,T),\ x \in \partial\mathcal{S}_i(t),\ i = 1, ..., M, \tag{1.13}$$

$$u_{\mathcal{F}} = 0, \quad t \in (0,T),\ x \in \partial\Omega, \tag{1.14}$$

where $\alpha > 0$ is a coefficient of friction. Finally, the initial conditions read

$$\rho_{\mathcal{F}}(0,x) = \rho_{\mathcal{F}_0}(x), \quad (\rho_{\mathcal{F}} u_{\mathcal{F}})(0,x) = q_{\mathcal{F}_0}(x), \quad x \in \mathcal{F}_0, \tag{1.15}$$

$$h_i(0) = 0, \quad h_i'(0) = \ell_{0i}, \quad \omega_i(0) = \omega_{0i}, \quad i = 1, ..., M. \tag{1.16}$$

The main result of this article yields local-in-time existence of finite energy weak solutions to the above system. In the remainder of this section we present the definition of a weak solution to the system and the main result. The proof is presented in the remaining sections and the appendix; it is based on several approximations, see the end of this section for more information on the strategy of the proof, its novelties as well as common lines with [16].

1.1. **Weak formulation and main result.** As is standard, we obtain the weak formulation by multiplying with appropriate test functions and integrating by parts, which involves the boundary conditions. Since we assume Navier-slip boundary conditions at the interfaces between the fluid and the rigid bodies, the test functions show discontinuities across the fluid-solid interface. The set of rigid velocity fields is defined as

$$\mathcal{R}(\Omega) = \left\{\zeta : \Omega \to \mathbb{R}^3 \mid \text{There exist } V, r, a \in \mathbb{R}^3 \text{ such that } \zeta(x) = V + r \times (x-a) \text{ for any } x \in \Omega\right\}. \tag{1.17}$$

Let $\mathcal{D}$ denote the set of all infinitely differentiable functions that have compact support. We then introduce, for any $T > 0$, the test function space V_T as follows:

$$V_T = \left\{ \begin{aligned} &\phi \in C([0,T]; L^2(\Omega)) \text{ such that there exist } \phi_{\mathcal{F}} \in \mathcal{D}([0,T); \mathcal{D}(\Omega)),\ \phi_{\mathcal{S}_i} \in \mathcal{D}([0,T); \mathcal{R}(\Omega)) \\ &\text{satisfying } \phi(t,\cdot) = \phi_{\mathcal{F}}(t,\cdot) \text{ on } \mathcal{F}(t), \quad \phi(t,\cdot) = \phi_{\mathcal{S}_i}(t,\cdot) \text{ on } \mathcal{S}_i(t) \text{ with} \\ &\phi_{\mathcal{F}}(t,\cdot)\cdot\nu_i = \phi_{\mathcal{S}_i}(t,\cdot)\cdot\nu_i \text{ on } \partial\mathcal{S}_i(t) \text{ for all } t \in [0,T] \end{aligned} \right\}, \tag{1.18}$$

which allows us to present the definition of finite energy weak solutions to the system under consideration.

In the definition of a weak solution below we work with an extension of the fluid velocity $u_{\mathcal{F}}$ from the fluid domain $\mathcal{F}(t)$ to Ω as is defined in (4.1). Note that this is different to the corresponding extension in our earlier article due to the Dirichlet boundary conditions at $\partial\Omega$. The velocity fields $u_{\mathcal{S}_i} \in \mathcal{R}(\Omega)$ of the rigid bodies denote rigid extensions from $\mathcal{S}_i(t)$ to Ω as in (1.2). The extended solid density $\rho_{\mathcal{S}_i}$ is an extension from $\mathcal{S}_i(t)$ to Ω by zero. Moreover, the extended fluid density $\rho_{\mathcal{F}}$ in (1.20) is obtained by extending the density from $\mathcal{F}(t)$ to Ω by zero.

The initial fluid density $\rho_{\mathcal{F}_0}$ on Ω is obtained by extending $\rho_{\mathcal{F}_0}$ as in (1.15) from $\mathcal{F}_0$ to Ω by zero. Correspondingly, the extended initial solid density $\rho_{\mathcal{S}_{0i}}$ in (1.24) is an extension of (1.4) from $\mathcal{S}_0$ to Ω by zero, as is the extended initial momentum $q_{\mathcal{F}_0}$ in (1.15). The extended initial rigid velocity field $u_{\mathcal{S}_{0i}} \in \mathcal{R}$, however, is a rigid extension from $\mathcal{S}_{0i}$ to Ω as in (1.3).

Definition 1.1. *Let $T > 0$, and let Ω and $\mathcal{S}_{0i} \Subset \Omega$, $i = 1, ..., M$ be regular bounded domains of $\mathbb{R}^3$. A triplet $(\mathcal{S}, \rho, u)$ with $\mathcal{S} = \cup_{i=1}^M \mathcal{S}_i$ is a bounded energy weak solution to system (1.8)–(1.16) if the following holds:*

- *$\mathcal{S}_i(t) \Subset \Omega$ is a bounded domain of $\mathbb{R}^3$ for all $t \in [0,T)$ such that*

$$\chi_{\mathcal{S}_i}(t,x) := \mathbb{1}_{\mathcal{S}_i(t)}(x) \in L^\infty((0,T)\times\Omega). \tag{1.19}$$

- *u belongs to the following space*

$$U_T = \left\{ \begin{aligned} &u \in L^2(0,T; L^2(\Omega)) \text{ such that there exist } u_{\mathcal{F}} \in L^2(0,T; H_0^1(\Omega)),\ u_{\mathcal{S}_i} \in L^2(0,T; \mathcal{R}(\Omega)) \\ &\text{satisfying } u(t,\cdot) = u_{\mathcal{F}}(t,\cdot) \text{ on } \mathcal{F}(t), \quad u(t,\cdot) = u_{\mathcal{S}_i}(t,\cdot) \text{ on } \mathcal{S}_i(t) \text{ with} \\ &u_{\mathcal{F}}(t,\cdot)\cdot\nu_i = u_{\mathcal{S}_i}(t,\cdot)\cdot\nu_i \text{ on } \partial\mathcal{S}_i(t) \text{ for a.e } t \in [0,T] \quad \text{and any } i = 1,\dots,M \end{aligned} \right\}.$$

- $\rho \geqslant 0$, $\rho \in L^\infty(0,T;L^\gamma(\Omega))$ *with* $\gamma > 3/2$, $\rho|u|^2 \in L^\infty(0,T;L^1(\Omega))$, *where*

$$\rho = \sum_{i=1}^{M} [(1-\mathbb{1}_{\mathcal{S}_i})\rho_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}_i}\rho_{\mathcal{S}_i}], \quad u = \sum_{i=1}^{M} [(1-\mathbb{1}_{\mathcal{S}_i})u_{\mathcal{F}} + \mathbb{1}_{\mathcal{S}_i}u_{\mathcal{S}_i}].$$

- *The continuity equation is satisfied in the weak sense, i.e.*

$$\frac{\partial \rho_{\mathcal{F}}}{\partial t} + \operatorname{div}(\rho_{\mathcal{F}} u_{\mathcal{F}}) = 0 \text{ in } \mathcal{D}'([0,T)\times\Omega), \quad \rho_{\mathcal{F}}(0,x) = \rho_{\mathcal{F}_0}(x),\ x\in\Omega. \tag{1.20}$$

Also, a renormalized continuity equation holds in a weak sense, i.e.

$$\partial_t b(\rho_{\mathcal{F}}) + \operatorname{div}(b(\rho_{\mathcal{F}})u_{\mathcal{F}}) + (b'(\rho_{\mathcal{F}}) - b(\rho_{\mathcal{F}}))\operatorname{div} u_{\mathcal{F}} = 0 \text{ in } \mathcal{D}'([0,T)\times\Omega), \tag{1.21}$$

for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ *satisfying*

$$|b'(z)| \leqslant cz^{-\kappa_0},\ z\in(0,1],\ \kappa_0 < 1, \qquad |b'(z)| \leqslant cz^{\kappa_1},\ z \geqslant 1,\ -1 < \kappa_1 < \infty. \tag{1.22}$$

- *The transport of* $\mathcal{S}_i$ *by the rigid vector field* $u_{\mathcal{S}_i}$ *holds (in the weak sense)*

$$\frac{\partial \chi_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(u_{\mathcal{S}_i}\chi_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\Omega, \quad \chi_{\mathcal{S}_i}(0,x) = \mathbb{1}_{\mathcal{S}_{0i}}(x),\ x\in\Omega. \tag{1.23}$$

- *The densities* $\rho_{\mathcal{S}_i}$ *of the rigid bodies* $\mathcal{S}_i$ *satisfy (in the weak sense)*

$$\frac{\partial \rho_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(u_{\mathcal{S}_i}\rho_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\Omega, \quad \rho_{\mathcal{S}_i}(0,x) = \rho_{\mathcal{S}_{0i}}(x),\ x\in\Omega. \tag{1.24}$$

- *Balance of linear momentum holds in a weak sense, i.e. for all* $\phi \in V_T$ *the following relation holds:*

$$\begin{aligned}
-\int_0^T\int_{\mathcal{F}(t)} \rho_{\mathcal{F}} u_{\mathcal{F}}\cdot\frac{\partial}{\partial t}\phi_{\mathcal{F}} - \sum_{i=1}^{M}\int_0^T\int_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i}u_{\mathcal{S}_i}\cdot\frac{\partial}{\partial t}\phi_{\mathcal{S}_i} - \int_0^T\int_{\mathcal{F}(t)} (\rho_{\mathcal{F}}u_{\mathcal{F}}\otimes u_{\mathcal{F}}) : \nabla\phi_{\mathcal{F}} + \int_0^T\int_{\mathcal{F}(t)} (\mathbb{T}(u_{\mathcal{F}}) - p_{\mathcal{F}}\mathbb{I}) : \mathbb{D}(\phi_{\mathcal{F}}) \\
+ \alpha\sum_{i=1}^{M}\int_0^T\int_{\partial\mathcal{S}_i(t)} [(u_{\mathcal{F}} - u_{\mathcal{S}_i})\times\nu]\cdot[(\phi_{\mathcal{F}} - \phi_{\mathcal{S}_i})\times\nu] \\
= \int_0^T\int_{\mathcal{F}(t)} \rho_{\mathcal{F}}g_{\mathcal{F}}\cdot\phi_{\mathcal{F}} + \sum_{i=1}^{M}\int_0^T\int_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i}g_{\mathcal{S}_i}\cdot\phi_{\mathcal{S}_i} + \int_{\mathcal{F}_0}(\rho_{\mathcal{F}}u_{\mathcal{F}}\cdot\phi_{\mathcal{F}})(0) + \int_{\mathcal{S}_{0i}}(\rho_{\mathcal{S}_i}u_{\mathcal{S}_i}\cdot\phi_{\mathcal{S}_i})(0).
\end{aligned} \tag{1.25}$$

- *The following energy inequality holds for almost every* $t\in(0,T)$*:*

$$\begin{aligned}
E(t) + \int_0^t\int_{\mathcal{F}(\tau)} \left(2\mu_{\mathcal{F}}|\mathbb{D}(u_{\mathcal{F}})|^2 + \lambda_{\mathcal{F}}|\operatorname{div} u_{\mathcal{F}}|^2\right) + \alpha\sum_{i=1}^{M}\int_0^t\int_{\partial\mathcal{S}_i(\tau)} |(u_{\mathcal{F}} - u_{\mathcal{S}_i})\times\nu_i|^2 \\
\leqslant \int_0^t\int_{\mathcal{F}(\tau)} \rho_{\mathcal{F}}g_{\mathcal{F}}\cdot u_{\mathcal{F}} + \sum_{i=1}^{M}\int_0^t\int_{\mathcal{S}_i(\tau)} \rho_{\mathcal{S}_i}g_{\mathcal{S}_i}\cdot u_{\mathcal{S}_i} + E_0.
\end{aligned} \tag{1.26}$$

where $E(t)$ *and* E_0 *are given by*

$$E(t) = \int_{\mathcal{F}(t)} \frac{1}{2}\rho_{\mathcal{F}}|u_{\mathcal{F}}(t,\cdot)|^2 + \sum_{i=1}^{M}\int_{\mathcal{S}_i(t)} \frac{1}{2}\rho_{\mathcal{S}_i}|u_{\mathcal{S}_i}(t,\cdot)|^2 + \int_{\mathcal{F}(t)} \frac{a_{\mathcal{F}}}{\gamma-1}\rho_{\mathcal{F}}^{\gamma}$$

$$E_0 = \int_{\mathcal{F}_0} \frac{1}{2}\frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}} + \sum_{i=1}^{M}\int_{\mathcal{S}_{0i}} \frac{1}{2}\rho_{\mathcal{S}_{0i}}|u_{\mathcal{S}_{0i}}|^2 + \int_{\mathcal{F}_0} \frac{a_{\mathcal{F}}}{\gamma-1}\rho_{\mathcal{F}_0}^{\gamma}.$$

We recall that equation (1.20) is different from the continuity equation in [4] since we assume Navier-slip boundary conditions at the interface of the fluid and the rigid bodies. Note that the better regularity $u_{\mathcal{F}} \in L^2(0,T;H_0^1(\Omega))$ of the extended fluid velocity is needed for the continuity equation to hold true in Ω.

Our main assertion reads as follows:

Theorem 1.2. *Let Ω and $\mathcal{S}_{0i} \Subset \Omega$ be regular bounded domains of $\mathbb{R}^3$, $i = 1,...,M$. Assume that for some $\sigma > 0$,*

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t),\mathcal{S}_{0j}(t)) > 2\sigma, \quad i \neq j, \quad i,j = 1,...,M.$$

Let $g_{\mathcal{F}}$, $g_{\mathcal{S}_i} \in L^\infty((0,T)\times\Omega)$ and the pressure $p_{\mathcal{F}}$ be determined by (1.1) with $\gamma > 3/2$. Assume that the initial data satisfy

$$\rho_{\mathcal{F}_0} \in L^\gamma(\Omega), \quad \rho_{\mathcal{F}_0} \geqslant 0 \text{ a.e. in } \Omega, \quad \rho_{\mathcal{S}_{0i}} \in L^\infty(\Omega), \quad \rho_{\mathcal{S}_{0i}} > 0 \text{ a.e. in } \mathcal{S}_{0i}, \tag{1.27}$$

$$q_{\mathcal{F}_0} \in L^{\frac{2\gamma}{\gamma+1}}(\Omega), \quad q_{\mathcal{F}_0}\mathbb{1}_{\{\rho_{\mathcal{F}_0}=0\}} = 0 \text{ a.e. in } \Omega, \quad \frac{|q_{\mathcal{F}_0}|^2}{\rho_{\mathcal{F}_0}}\mathbb{1}_{\{\rho_{\mathcal{F}_0}>0\}} \in L^1(\Omega), \tag{1.28}$$

$$u_{\mathcal{S}_{0i}} = \ell_{0i} + \omega_{0i} \times x \quad \forall\, x \in \Omega \text{ with } \ell_{0i},\ \omega_{0i} \in \mathbb{R}^3. \tag{1.29}$$

Then there exists $T > 0$ (depending only on $\rho_{\mathcal{F}_0}$, $\rho_{\mathcal{S}_{0i}}$, $q_{\mathcal{F}_0}$, $u_{\mathcal{S}_{0i}}$, $g_{\mathcal{F}}$, $g_{\mathcal{S}_i}$, $\operatorname{dist}(\mathcal{S}_{0i},\partial\Omega)$, $\operatorname{dist}(\mathcal{S}_{0i},\mathcal{S}_{0j})$, $i \neq j$) such that a bounded energy weak solution to (1.8)–(1.16) exists on $[0,T)$. Moreover,

$$\mathcal{S}_i(t) \Subset \Omega, \quad \min(\operatorname{dist}(\mathcal{S}_i(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_i(t),\mathcal{S}_j(t))) \geqslant \frac{3\sigma}{2}, \quad \forall\, t \in [0,T], \quad i \neq j, \quad i,j = 1,...,M.$$

Our method as in [16] relies on a combination of (i) the theory of compressible fluids [15, 5] in terms of a renormalized continuity equations, an effective viscous flux and an artificial pressure and (ii) methods developed for penalization of the discontinuity in the velocity field [8, 2], which is needed due to the fluid-structure interaction. In particular, this involves a regularized fluid velocity and several approximation schemes, which require a careful selection of the test functions. The construction is done in such a way that the solution and the test functions do not show a discontinuity at the level of the approximation but recover a discontinuity in the last limit process.

In this article we deal with the setting of several rigid bodies in a compressible fluid; in [16] one rigid body in a compressible fluid was considered. Hence all test functions need to be adapted. Similarly, the bound on the maximal time for which we can prove existence of weak solutions has changed, cf. (3.16). Another change compared to [16] is related to the Dirichlet boundary conditions at $\partial\Omega$ instead of the Navier-slip boundary conditions that was assumed earlier. This results in dealing with spaces like $W_0^{k,p}$ instead of $W^{k,p}$. Due to the Dirichlet boundary condition at $\partial\Omega$, the extension of the fluid velocity field from $\mathcal{F}(t)$ to Ω had to be modified, cf. (4.1). All the other parts of the proof could be adapted from [16] to the current setting, which is outlined in the following sections in more detail.

2. Approximate Solutions

We begin with the introduction of the approximation scheme in the three levels together with stating the existence propositions for every levels of approximation schemes. Further we prove the existence of the appropriate schemes. We start with the δ-level of approximation via an artificial pressure together with the penalization. We will study the following approximate problem:

Let $\delta > 0$. Find a triplet $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ with $\mathcal{S}^\delta = \bigcup_{i=1}^M \mathcal{S}_i^\delta$ such that

- $\mathcal{S}_i^\delta(t) \Subset \Omega$ are bounded, regular domains for all $t \in [0,T]$, $i = 1,...,M$ with

$$\chi_{\mathcal{S}_i}^\delta(t,x) = \mathbb{1}_{\mathcal{S}_i^\delta(t)}(x) \in L^\infty((0,T)\times\Omega) \cap C([0,T];L^p(\Omega)),\ \forall\, 1 \leqslant p < \infty. \tag{2.1}$$

- The velocity field $u^\delta \in L^2(0,T;H_0^1(\Omega))$, and the density function $\rho^\delta \in L^\infty(0,T;L^\beta(\Omega))$, $\rho^\delta \geqslant 0$ satisfy

$$\frac{\partial \rho^\delta}{\partial t} + \operatorname{div}(\rho^\delta u^\delta) = 0 \text{ in } \mathcal{D}'([0,T)\times\Omega). \tag{2.2}$$

- For all $\phi \in H^1(0,T;L^2(\Omega)) \cap L^r(0,T;W_0^{1,r}(\Omega))$, where $r = \max\left\{\beta+1, \frac{\beta+\theta}{\theta}\right\}$, $\beta \geqslant \max\{8,\gamma\}$ and $\theta = \frac{2}{3}\gamma - 1$ with $\phi|_{t=T} = 0$, the following holds:

$$-\int_0^T \int_\Omega \rho^\delta \left(u^\delta \cdot \frac{\partial}{\partial t}\phi + u^\delta \otimes u^\delta : \nabla\phi\right) + \int_0^T \int_\Omega \left(2\mu^\delta \mathbb{D}(u^\delta) : \mathbb{D}(\phi) + \lambda^\delta \operatorname{div} u^\delta \mathbb{I} : \mathbb{D}(\phi) - p^\delta(\rho^\delta)\mathbb{I} : \mathbb{D}(\phi)\right)$$
$$+ \alpha \sum_{i=1}^M \int_0^T \int_{\partial \mathcal{S}_i^\delta(t)} [(u^\delta - P_{\mathcal{S}_i}^\delta u^\delta) \times \nu_i] \cdot [(\phi - P_{\mathcal{S}_i}^\delta \phi) \times \nu]$$
$$+ \frac{1}{\delta}\int_0^T \int_\Omega \sum_{i=1}^M \chi_{\mathcal{S}_i}^\delta (u^\delta - P_{\mathcal{S}_i}^\delta u^\delta) \cdot (\phi - P_{\mathcal{S}_i}^\delta \phi) = \int_0^T \int_\Omega \rho^\delta g^\delta \cdot \phi + \int_\Omega (\rho^\delta u^\delta \cdot \phi)(0), \quad (2.3)$$

where $P_{\mathcal{S}_i}^\delta$ is defined in (2.9) below.
- $\chi_{\mathcal{S}_i}^\delta(t,x)$ satisfies (in the weak sense)

$$\frac{\partial \chi_{\mathcal{S}_i}^\delta}{\partial t} + P_{\mathcal{S}_i}^\delta u^\delta \cdot \nabla \chi_{\mathcal{S}_i}^\delta = 0 \text{ in } (0,T)\times\Omega, \quad \chi_{\mathcal{S}_i}^\delta|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \Omega. \quad (2.4)$$

- $\rho^\delta \chi_{\mathcal{S}_i}^\delta(t,x)$ satisfies (in the weak sense)

$$\frac{\partial}{\partial t}(\rho^\delta \chi_{\mathcal{S}_i}^\delta) + P_{\mathcal{S}_i}^\delta u^\delta \cdot \nabla(\rho^\delta \chi_{\mathcal{S}_i}^\delta) = 0 \text{ in } (0,T)\times\Omega, \quad (\rho^\delta \chi_{\mathcal{S}_i}^\delta)|_{t=0} = \rho_0^\delta \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \Omega. \quad (2.5)$$

- Initial data are given by

$$\rho^\delta(0,x) = \rho_0^\delta(x), \quad \rho^\delta u^\delta(0,x) = q_0^\delta(x), \quad x \in \Omega. \quad (2.6)$$

Above we have used the following quantities:

- The specific body force is defined as

$$g^\delta = \sum_{i=1}^M [(1-\chi_{\mathcal{S}_i}^\delta) g_{\mathcal{F}} + \chi_{\mathcal{S}_i}^\delta g_{\mathcal{S}_i}].$$

- The artificial pressure is given by

$$p^\delta(\rho) = a^\delta \rho^\gamma + \delta\rho^\beta, \quad \text{with} \quad a^\delta = a_{\mathcal{F}} \sum_{i=1}^M (1-\chi_{\mathcal{S}_i}^\delta), \quad (2.7)$$

where $a_{\mathcal{F}} > 0$ and γ and β are exponents (by abuse of notation) and they satisfy $\gamma > 3/2$, $\beta \geqslant \max\{8,\gamma\}$.
- The viscosity coefficients are given by

$$\mu^\delta = \sum_{i=1}^M [(1-\chi_{\mathcal{S}_i}^\delta)\mu_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}_i}^\delta], \quad \lambda^\delta = \sum_{i=1}^M [(1-\chi_{\mathcal{S}_i}^\delta)\lambda_{\mathcal{F}} + \delta^2 \chi_{\mathcal{S}_i}^\delta] \quad \text{so that} \quad \mu^\delta > 0,\ 2\mu^\delta + 3\lambda^\delta \geqslant 0. \quad (2.8)$$

- The orthogonal projection onto rigid fields, $P_{\mathcal{S}_i}^\delta : L^2(\Omega) \to L^2(\mathcal{S}_i^\delta(t)) \cap \mathcal{R}(\mathcal{S}_i^\delta(t))$, is such that, for all $t \in [0,T]$ and $u \in L^2(\Omega)$, it is given by

$$P_{\mathcal{S}_i}^\delta u(t,x) = \frac{1}{m_i^\delta}\int_\Omega \rho^\delta \chi_{\mathcal{S}_i}^\delta u + \left((J_i^\delta)^{-1} \int_\Omega \rho^\delta \chi_{\mathcal{S}_i}^\delta ((y - h_i^\delta(t)) \times u)\ dy\right) \times (x - h_i^\delta(t)), \quad \forall x \in \Omega, \quad (2.9)$$

where h_i^δ, m_i^δ and J_i^δ are defined as

$$h_i^\delta(t) = \frac{1}{m_i^\delta}\int_{\mathbb{R}^3} \rho^\delta \chi_{\mathcal{S}_i}^\delta x\, dx, \quad m_i^\delta = \int_{\mathbb{R}^3} \rho^\delta \chi_{\mathcal{S}_i}^\delta\, dx,$$

$$J_i^\delta(t) = \int_{\mathbb{R}^3} \rho^\delta \chi_{\mathcal{S}_i}^\delta \left[|x - h_i^\delta(t)|^2\mathbb{I} - (x - h_i^\delta(t)) \otimes (x - h_i^\delta(t))\right]\, dx.$$

We get a weak solution of problem (1.8)–(1.16) in the sense of Definition 1.1 as a limit of the solution $(\mathcal{S}^\delta, \rho^\delta, u^\delta)$ of system (2.1)–(2.6) as $\delta \to 0$.

Let us state the existence result of the approximate system:

Proposition 2.1. *Let Ω and $\mathcal{S}_{0i} \Subset \Omega$, $i = 1, ..., M$ be regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$,*

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t), \partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t), \mathcal{S}_{0j}(t)) > 2\sigma, \quad i \neq j, \quad i, j = 1, ..., M.$$

Let $g_{\mathcal{F}}, g_{\mathcal{S}_i} \in L^\infty((0,T) \times \Omega)$ and

$$\delta > 0,\ \gamma > 3/2,\ \beta \geqslant \max\{8, \gamma\}. \tag{2.10}$$

Further, let the pressure p^δ be determined by (2.7) and the viscosity coefficients μ^δ, λ^δ be given by (2.8). Assume that the initial conditions satisfy

$$\rho_0^\delta \in L^\beta(\Omega), \quad \rho_0^\delta \geqslant 0 \ a.e.\ in\ \Omega, \quad \rho_0^\delta \mathbb{1}_{\mathcal{S}_{0i}} \in L^\infty(\Omega), \quad \rho_0^\delta \mathbb{1}_{\mathcal{S}_{0i}} > 0 \ a.e.\ in\ \mathcal{S}_{0i}, \tag{2.11}$$

$$q_0^\delta \in L^{\frac{2\beta}{\beta+1}}(\Omega), \quad q_0^\delta \mathbb{1}_{\{\rho_0^\delta = 0\}} = 0 \ a.e.\ in\ \Omega, \quad \frac{|q_0^\delta|^2}{\rho_0^\delta}\mathbb{1}_{\{\rho_0 > 0\}} \in L^1(\Omega). \tag{2.12}$$

Let the initial energy

$$E^\delta[\rho_0^\delta, q_0^\delta] = \int_\Omega \left(\frac{1}{2}\frac{|q_0^\delta|^2}{\rho_0^\delta}\mathbb{1}_{\{\rho_0^\delta > 0\}} + \frac{a^\delta(0)}{\gamma - 1}(\rho_0^\delta)^\gamma + \frac{\delta}{\beta - 1}(\rho_0^\delta)^\beta\right) := E_0^\delta$$

be uniformly bounded with respect to δ. Then there exists $T > 0$ (depending only on E_0^δ, $g_{\mathcal{F}}$, $g_{\mathcal{S}_i}$, $\operatorname{dist}(\mathcal{S}_0, \partial\Omega)$, $\operatorname{dist}(\mathcal{S}_{0i}, \mathcal{S}_{0j})$, $i \neq j$) such that system (2.1)–(2.6) admits a weak solution $(S^\delta, \rho^\delta, u^\delta)$, which satisfies the following energy inequality for almost every $t \in (0,T)$:

$$E^\delta[\rho^\delta, q^\delta] + \int_0^T\int_\Omega \left(2\mu^\delta|\mathbb{D}(u^\delta)|^2 + \lambda^\delta|\operatorname{div} u^\delta|^2\right) + \alpha\sum_{i=1}^M \int_0^T \int_{\partial\mathcal{S}_i^\delta(t)} |(u^\delta - P_{\mathcal{S}_i}^\delta u^\delta) \times \nu_i|^2$$
$$+ \frac{1}{\delta}\sum_{i=1}^M \int_0^T\int_\Omega \chi_{\mathcal{S}_i}^\delta |u^\delta - P_{\mathcal{S}_i}^\delta u^\delta|^2 \leqslant \int_0^T\int_\Omega \rho^\delta g^\delta \cdot u^\delta + E_0^\delta. \tag{2.13}$$

Moreover,

$$\min(\operatorname{dist}(\mathcal{S}_i^\delta(t), \partial\Omega),\ \operatorname{dist}(\mathcal{S}_i^\delta(t), \mathcal{S}_j^\delta(t))) \geqslant 2\sigma, \quad \forall\ t \in [0,T], \quad i \neq j, \quad i, j = 1, ..., M,$$

and the solution satisfies the following properties:

(1) *For $\theta = \frac{2}{3}\gamma - 1$, $s = \gamma + \theta$,*

$$\|(a^\delta)^{1/s}\rho^\delta\|_{L^s((0,T)\times\Omega)} + \delta^{\frac{1}{\beta+\theta}}\|\rho^\delta\|_{L^{\beta+\theta}((0,T)\times\Omega)} \leqslant c. \tag{2.14}$$

(2) *The couple (ρ^δ, u^δ) satisfies the identity*

$$\partial_t b(\rho^\delta) + \operatorname{div}(b(\rho^\delta)u^\delta) + [b'(\rho^\delta)\rho^\delta - b(\rho^\delta)] \operatorname{div} u^\delta = 0 \tag{2.15}$$

a.e. in $(0,T) \times \Omega$ for any $b \in C([0,\infty)) \cap C^1((0,\infty))$ satisfying (1.22).

To show Proposition 2.1, we introduce a problem with another level of approximation: the ε-level approximation is obtained introducing the dissipation in the continuity accompanied by the artificial pressure in the momentum equation. Our aim is to find a triplet $(\mathcal{S}^{\varepsilon},\rho^{\varepsilon},u^{\varepsilon})$ with $\mathcal{S}^{\varepsilon}=\bigcup_{i=1}^{M}\mathcal{S}_i^{\varepsilon}$ such that we can obtain a weak solution $(\mathcal{S}^{\delta},\rho^{\delta},u^{\delta})$ of the system (2.1)–(2.6) as a weak limit of the sequence $(\mathcal{S}^{\varepsilon},\rho^{\varepsilon},u^{\varepsilon})$ as $\varepsilon\to 0$. For $\varepsilon>0$, the triplet is supposed to satisfy:

- $\mathcal{S}_i^{\varepsilon}(t)\Subset\Omega$ are bounded, regular domains for all $t\in[0,T]$, $i=1,..,M$ with
$$\chi^{\varepsilon}_{\mathcal{S}_i}(t,x):=\mathbb{1}_{\mathcal{S}_i^{\varepsilon}(t)}(x)\in L^{\infty}((0,T)\times\Omega)\cap C([0,T];L^p(\Omega)),\ \forall\, 1\leqslant p<\infty. \tag{2.16}$$
- The velocity field $u^{\varepsilon}\in L^2(0,T;H^1_0(\Omega))$ and the density function $\rho^{\varepsilon}\in L^{\infty}(0,T;L^{\beta}(\Omega))\cap L^2(0,T;H^1(\Omega))$, $\rho^{\varepsilon}\geqslant 0$ satisfy
$$\frac{\partial\rho^{\varepsilon}}{\partial t}+\operatorname{div}(\rho^{\varepsilon}u^{\varepsilon})=\varepsilon\Delta\rho^{\varepsilon}\ \text{ in }\ (0,T)\times\Omega,\quad \frac{\partial\rho^{\varepsilon}}{\partial\nu}=0\ \text{on}\ \partial\Omega. \tag{2.17}$$
- For all $\phi\in H^1(0,T;L^2(\Omega))\cap L^{\beta+1}(0,T;W^{1,\beta+1}_0(\Omega))$ with $\phi|_{t=T}=0$, where $\beta\geqslant\max\{8,\gamma\}$, the following holds:

$$-\int_0^T\int_{\Omega}\rho^{\varepsilon}\left(u^{\varepsilon}\cdot\frac{\partial}{\partial t}\phi+u^{\varepsilon}\otimes u^{\varepsilon}:\nabla\phi\right)+\int_0^T\int_{\Omega}\Big(2\mu^{\varepsilon}\mathbb{D}(u^{\varepsilon}):\mathbb{D}(\phi)+\lambda^{\varepsilon}\operatorname{div}u^{\varepsilon}\mathbb{I}:\mathbb{D}(\phi)-p^{\varepsilon}(\rho^{\varepsilon})\mathbb{I}:\mathbb{D}(\phi)\Big)$$
$$+\int_0^T\int_{\Omega}\varepsilon\nabla u^{\varepsilon}\nabla\rho^{\varepsilon}\cdot\phi+\alpha\sum_{i=1}^{M}\int_0^T\int_{\partial\mathcal{S}_i^{\varepsilon}(t)}[(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\times\nu_i]\cdot[(\phi-P^{\varepsilon}_{\mathcal{S}_i}\phi)\times\nu_i]$$
$$+\frac{1}{\delta}\int_0^T\int_{\Omega}\sum_{i=1}^{M}\chi^{\varepsilon}_{\mathcal{S}_i}(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\cdot(\phi-P^{\varepsilon}_{\mathcal{S}_i}\phi)=\int_0^T\int_{\Omega}\rho^{\varepsilon}g^{\varepsilon}\cdot\phi+\int_{\Omega}(\rho^{\varepsilon}u^{\varepsilon}\cdot\phi)(0). \tag{2.18}$$

- $\chi^{\varepsilon}_{\mathcal{S}_i}(t,x)$ satisfies (in the weak sense)
$$\frac{\partial\chi^{\varepsilon}_{\mathcal{S}_i}}{\partial t}+P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon}\cdot\nabla\chi^{\varepsilon}_{\mathcal{S}_i}=0\ \text{in}\ (0,T)\times\Omega,\quad \chi^{\varepsilon}_{\mathcal{S}_i}|_{t=0}=\mathbb{1}_{\mathcal{S}_{0i}}\ \text{in}\ \Omega. \tag{2.19}$$
- $\rho^{\varepsilon}\chi^{\varepsilon}_{\mathcal{S}_i}(t,x)$ satisfies (in the weak sense)
$$\frac{\partial}{\partial t}(\rho^{\varepsilon}\chi^{\varepsilon}_{\mathcal{S}_i})+P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon}\cdot\nabla(\rho^{\varepsilon}\chi^{\varepsilon}_{\mathcal{S}_i})=0\ \text{in}\ (0,T)\times\Omega,\quad (\rho^{\varepsilon}\chi^{\varepsilon}_{\mathcal{S}_i})|_{t=0}=\rho_0^{\varepsilon}\mathbb{1}_{\mathcal{S}_{0i}}\ \text{in}\ \Omega. \tag{2.20}$$
- The initial data are given by
$$\rho^{\varepsilon}(0,x)=\rho_0^{\varepsilon}(x),\quad \rho^{\varepsilon}u^{\varepsilon}(0,x)=q_0^{\varepsilon}(x)\quad\text{in }\Omega,\quad \frac{\partial\rho_0^{\varepsilon}}{\partial\nu}\Big|_{\partial\Omega}=0. \tag{2.21}$$

Above we have used the following quantities:

- The specific body force is defined as
$$g^{\varepsilon}=\sum_{i=1}^{M}[(1-\chi^{\varepsilon}_{\mathcal{S}_i})g_{\mathcal{F}}+\chi^{\varepsilon}_{\mathcal{S}_i}g_{\mathcal{S}_i}]. \tag{2.22}$$
- The artificial pressure is given by
$$p^{\varepsilon}(\rho)=a^{\varepsilon}\rho^{\gamma}+\delta\rho^{\beta},\qquad\text{with}\qquad a^{\varepsilon}=a_{\mathcal{F}}\sum_{i=1}^{M}(1-\chi^{\varepsilon}_{\mathcal{S}_i}), \tag{2.23}$$
where $a_{\mathcal{F}},\delta>0$, and the exponents γ and β satisfy $\gamma>3/2,\ \beta\geqslant\max\{8,\gamma\}$.
- The viscosity coefficients are given by
$$\mu^{\varepsilon}=\sum_{i=1}^{M}[(1-\chi^{\varepsilon}_{\mathcal{S}_i})\mu_{\mathcal{F}}+\delta^2\chi^{\varepsilon}_{\mathcal{S}_i}],\quad \lambda^{\varepsilon}=\sum_{i=1}^{M}[(1-\chi^{\varepsilon}_{\mathcal{S}})\lambda_{\mathcal{F}}+\delta^2\chi^{\varepsilon}_{\mathcal{S}_i}]\quad\text{so that}\quad \mu^{\varepsilon}>0,\ 2\mu^{\varepsilon}+3\lambda^{\varepsilon}\geqslant 0. \tag{2.24}$$

- $P^{\varepsilon}_{\mathcal{S}_i}: L^2(\Omega) \to L^2(\mathcal{S}^{\varepsilon}_i(t)) \cap \mathcal{R}(\mathcal{S}^{\varepsilon}(t))$ is the orthogonal projection onto rigid fields; it is defined as in (2.9) with $\chi^{\delta}_{\mathcal{S}_i}$ is replaced by $\chi^{\varepsilon}_{\mathcal{S}_i}$.

Proposition 2.2. *Let Ω and $\mathcal{S}_{0i} \Subset \Omega$, $i = 1, ..., M$ be regular bounded domains of $\mathbb{R}^3$. Assume that for some $\sigma > 0$,*

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t), \partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t), \mathcal{S}_{0j}(t)) > 2\sigma, \quad i \neq j, \quad i, j = 1, ..., M.$$

Let $g_{\mathcal{F}}, g_{\mathcal{S}_i} \in L^{\infty}((0,T) \times \Omega)$ and β, δ, γ be given as in (2.10). Further, let the pressure p^{ε} be determined by (2.23) and the viscosity coefficients μ^{ε}, λ^{ε} be given by (2.24). The initial conditions satisfy, for some $\underline{\rho}$, $\overline{\rho} > 0$,

$$0 < \underline{\rho} \leqslant \rho^{\varepsilon}_0 \leqslant \overline{\rho} \quad in \quad \Omega, \quad \rho^{\varepsilon}_0 \in W^{1,\infty}(\Omega), \quad q^{\varepsilon}_0 \in L^2(\Omega). \tag{2.25}$$

Let the initial energy

$$E^{\varepsilon}[\rho^{\varepsilon}_0, q^{\varepsilon}_0] = \int_{\Omega} \left(\frac{1}{2} \frac{|q^{\varepsilon}_0|^2}{\rho^{\varepsilon}_0} \mathbb{1}_{\{\rho^{\varepsilon}_0 > 0\}} + \frac{a^{\varepsilon}(0)}{\gamma - 1} (\rho^{\varepsilon}_0)^{\gamma} + \frac{\delta}{\beta - 1} (\rho^{\varepsilon}_0)^{\beta} \right) := E^{\varepsilon}_0$$

be uniformly bounded with respect to δ and ε. Then there exists $T > 0$ (depending only on E^{ε}_0, $g_{\mathcal{F}}$, $g_{\mathcal{S}_i}$, $\operatorname{dist}(\mathcal{S}_{0i}, \partial\Omega)$, $\operatorname{dist}(\mathcal{S}_{0i}, \mathcal{S}_{0j})$, $i \neq j$)) such that system (2.16)–(2.21) admits a weak solution $(S^{\varepsilon}, \rho^{\varepsilon}, u^{\varepsilon})$, which satisfies the following energy inequality for almost every $t \in (0,T)$:

$$\begin{aligned} E^{\varepsilon}[\rho^{\varepsilon}, q^{\varepsilon}] &+ \int_0^T \int_{\Omega} \left(2\mu^{\varepsilon} |\mathbb{D}(u^{\varepsilon})|^2 + \lambda^{\varepsilon} |\operatorname{div} u^{\varepsilon}|^2 \right) + \delta\varepsilon\beta \int_0^T \int_{\Omega} (\rho^{\varepsilon})^{\beta-2} |\nabla\rho^{\varepsilon}|^2 \\ &+ \alpha \sum_{i=1}^{M} \int_0^T \int_{\partial\mathcal{S}^{\varepsilon}_i(t)} |(u^{\varepsilon} - P^{\varepsilon}_{\mathcal{S}_i} u^{\varepsilon}) \times \nu_i|^2 + \frac{1}{\delta} \int_0^T \int_{\Omega} \sum_{i=1}^{M} \chi^{\varepsilon}_{\mathcal{S}_i} |u^{\varepsilon} - P^{\delta}_{\mathcal{S}_i} u^{\varepsilon}|^2 \leqslant \int_0^T \int_{\Omega} \rho^{\varepsilon} g^{\varepsilon} \cdot u^{\varepsilon} + E^{\varepsilon}_0. \end{aligned} \tag{2.26}$$

Moreover,

$$\min(\operatorname{dist}(\mathcal{S}^{\varepsilon}_i(t), \partial\Omega),\ \operatorname{dist}(\mathcal{S}^{\varepsilon}_i(t), \mathcal{S}^{\varepsilon}_j(t))) \geqslant 2\sigma, \quad \forall\, t \in [0,T], \quad i \neq j, \quad i, j = 1, ..., M,$$

and the solution satisfies

$$\begin{gathered} \partial_t \rho^{\varepsilon},\ \Delta\rho^{\varepsilon} \in L^{\frac{5\beta-3}{4\beta}}((0,T) \times \Omega), \\ \sqrt{\varepsilon} \|\nabla\rho^{\varepsilon}\|_{L^2((0,T)\times\Omega)} + \|\rho^{\varepsilon}\|_{L^{\beta+1}((0,T)\times\Omega)} + \|(a^{\varepsilon})^{\frac{1}{\gamma+1}} \rho^{\varepsilon}\|_{L^{\gamma+1}((0,T)\times\Omega)} \leqslant c, \end{gathered} \tag{2.27}$$

where c is a positive constant depending on δ but independent of ε.

To solve the problem (2.16)–(2.21), we introduce another level of approximation - N-level approximation which is obtained via a Faedo-Galerkin approximation scheme.

Let $\{e_k\}_{k \geqslant 1} \subset \mathcal{D}(\Omega)$ is a basis of $L^2(\Omega)$. We denote by X_N the following space:

$$X_N = \operatorname{span}(e_1, \ldots, e_N).$$

X_N is a finite dimensional space with scalar product induced by the scalar product in $L^2(\Omega)$. As X_N is finite dimensional, norms on X_N induced by $W^{k,p}_0$ norms, $k \in \mathbb{N}$, $1 \leqslant p \leqslant \infty$ are equivalent. We also assume that

$$\bigcup_N X_N \text{ is dense in } W^{1,p}_0(\Omega), \text{ for any } 1 \leqslant p < \infty.$$

The aim is to find a triplet $(\mathcal{S}^N, \rho^N, u^N)$ with $\mathcal{S}^N = \cup_{i=1}^M \mathcal{S}^N_i$ satisfying:

- $\mathcal{S}^N_i(t) \Subset \Omega$ are bounded, regular domains for all $t \in [0,T]$, $i = 1, ..., M$ with
$$\chi^N_{\mathcal{S}_i}(t,x) := \mathbb{1}_{\mathcal{S}^N_i(t)}(x) \in L^{\infty}((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \forall\, 1 \leqslant p < \infty. \tag{2.28}$$
- The velocity field $u^N(t,\cdot) = \sum_{k=1}^{N} \alpha_k(t) e_k$ with $(\alpha_1, \alpha_2, \ldots, \alpha_N) \in C([0,T])^N$ and the density function $\rho^N \in L^2(0,T; H^2(\Omega)) \cap H^1(0,T; L^2(\Omega))$, $\rho^N > 0$ satisfies
$$\frac{\partial\rho^N}{\partial t} + \operatorname{div}(\rho^N u^N) = \varepsilon\Delta\rho^N \text{ in } (0,T) \times \Omega, \quad \frac{\partial\rho^N}{\partial\nu} = 0 \text{ on } \partial\Omega. \tag{2.29}$$

- For all $\phi \in \mathcal{D}([0,T);X_N)$, the following holds:

$$-\int_0^T\int_\Omega \rho^N\left(u^N\cdot\frac{\partial}{\partial t}\phi + u^N\otimes u^N:\nabla\phi\right) + \int_0^T\int_\Omega \Big(2\mu^N\mathbb{D}(u^N):\mathbb{D}(\phi) + \lambda^N \operatorname{div} u^N\mathbb{I}:\mathbb{D}(\phi) - p^N(\rho^N)\mathbb{I}:\mathbb{D}(\phi)\Big)$$

$$\int_0^T\int_\Omega \varepsilon\nabla u^N\nabla\rho^N\cdot\phi + \alpha\sum_{i=1}^M\int_0^T\int_{\partial\mathcal{S}_i^N(t)} [(u^N - P^N_{\mathcal{S}_i}u^N)\times\nu_i]\cdot[(\phi - P^N_{\mathcal{S}_i}\phi)\times\nu_i]$$

$$+\frac{1}{\delta}\int_0^T\int_\Omega\sum_{i=1}^M \chi^N_{\mathcal{S}_i}(u^N - P^N_{\mathcal{S}_i}u^N)\cdot(\phi - P^N_{\mathcal{S}_i}\phi) = \int_0^T\int_\Omega \rho^N g^N\cdot\phi + \int_\Omega(\rho^N u^N\cdot\phi)(0). \quad (2.30)$$

- $\chi^N_{\mathcal{S}_i}(t,x)$ satisfies (in the weak sense)

$$\frac{\partial\chi^N_{\mathcal{S}_i}}{\partial t} + P^N_{\mathcal{S}_i}u^N\cdot\nabla\chi^N_{\mathcal{S}_i} = 0 \text{ in } (0,T)\times\Omega, \quad \chi^N_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \Omega. \quad (2.31)$$

- $\rho^N\chi^N_{\mathcal{S}_i}(t,x)$ satisfies (in the weak sense)

$$\frac{\partial}{\partial t}(\rho^N\chi^N_{\mathcal{S}_i}) + P^N_{\mathcal{S}_i}u^N\cdot\nabla(\rho^N\chi^N_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\Omega, \quad (\rho^N\chi^N_{\mathcal{S}_i})|_{t=0} = \rho^N_0\mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \Omega. \quad (2.32)$$

- The initial data are given by

$$\rho^N(0) = \rho^N_0, \quad u^N(0) = u^N_0 \quad \text{in } \Omega, \quad \frac{\partial\rho^N_0}{\partial\nu}\Big|_{\partial\Omega} = 0. \quad (2.33)$$

Above we have used the following quantities:

- The specific body force is defined as

$$g^N = \sum_{i=1}^M[(1-\chi^N_{\mathcal{S}_i})g_{\mathcal{F}} + \chi^N_{\mathcal{S}_i}g_{\mathcal{S}_i}]. \quad (2.34)$$

- The artificial pressure is given by

$$p^N(\rho) = a^N\rho^\gamma + \delta\rho^\beta, \quad \text{with} \quad a^N = \sum_{i=1}^M a_{\mathcal{F}}(1-\chi^N_{\mathcal{S}_i}), \quad (2.35)$$

  where $a_{\mathcal{F}},\delta > 0$ and the exponents γ and β satisfy $\gamma > 3/2,\ \beta \geqslant \max\{8,\gamma\}$.
- The viscosity coefficients are given by

$$\mu^N = \sum_{i=1}^M[(1-\chi^N_{\mathcal{S}_i})\mu_{\mathcal{F}} + \delta^2\chi^N_{\mathcal{S}_i}], \quad \lambda^N = \sum_{i=1}^M[(1-\chi^N_{\mathcal{S}_i})\lambda_{\mathcal{F}} + \delta^2\chi^N_{\mathcal{S}_i}] \quad \text{so that} \quad \mu^N > 0,\ 2\mu^N + 3\lambda^N \geqslant 0. \quad (2.36)$$

- $P^N_{\mathcal{S}_i}: L^2(\Omega)\to L^2(\mathcal{S}^N_i(t))\cap\mathcal{R}(\mathcal{S}^N_i(t))$ is the orthogonal projection onto rigid fields; it is defined as in (2.9) with $\chi^\delta_{\mathcal{S}_i}$ replaced by $\chi^N_{\mathcal{S}_i}$.

A weak solution $(\mathcal{S}^\varepsilon,\rho^\varepsilon,u^\varepsilon)$ to the system (2.16)–(2.21) is obtained through the limit of $(\mathcal{S}^N,\rho^N,u^N)$ as $N\to\infty$. The existence result of the approximate solution of the Faedo-Galerkin scheme reads:

Proposition 2.3. *Let Ω and $\mathcal{S}_0 \Subset \Omega$ be regular bounded domains of $\mathbb{R}^3$, $i = 1,...,M$. Assume that for some $\sigma > 0$,*

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t),\mathcal{S}_{0j}(t)) > 2\sigma, \quad i\neq j, \quad i,j = 1,...,M.$$

Let $g_{\mathcal{F}}, g_{\mathcal{S}_i} \in L^\infty((0,T)\times\Omega)$ and β, δ, γ be given by (2.10). Further, let the pressure p^N be determined by (2.35) and the viscosity coefficients μ^N, λ^N be given by (2.36). The initial conditions are assumed to satisfy

$$0 < \underline{\rho} \leqslant \rho^N_0 \leqslant \overline{\rho} \ \ in \ \ \Omega, \quad \rho^N_0 \in W^{1,\infty}(\Omega), \quad u^N_0 \in X_N. \quad (2.37)$$

Let the initial energy

$$E^N(\rho_0^N, q_0^N) = \int_\Omega \left(\frac{1}{\rho_0^N} |q_0^N|^2 \mathbb{1}_{\{\rho_0>0\}} + \frac{a^N(0)}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta \right) := E_0^N$$

be uniformly bounded with respect to N, ε, δ. *Then there exists* $T > 0$ *(depending only on* E_0^N, $g_{\mathcal{F}}$, $g_{\mathcal{S}_i}$, $\overline{\rho}$, $\underline{\rho}$, $\operatorname{dist}(\mathcal{S}_{0i}, \partial\Omega)$, $\operatorname{dist}(\mathcal{S}_{0i}, \mathcal{S}_{0j})$, $i \neq j$*) such that the problem* (2.28)–(2.33) *admits a solution* $(\mathcal{S}^N, \rho^N, u^N)$ *and it satisfies the energy inequality:*

$$E^N[\rho^N, q^N] + \int_0^T \int_\Omega \left(2\mu^N |\mathbb{D}(u^N)|^2 + \lambda^N |\operatorname{div} u^N|^2 \right) + \delta\varepsilon\beta \int_0^T \int_\Omega (\rho^N)^{\beta-2} |\nabla \rho^N|^2$$
$$+ \alpha \sum_{i=1}^M \int_0^T \int_{\partial\mathcal{S}_i^N(t)} |(u^N - P^N_{\mathcal{S}_i} u^N) \times \nu_i|^2 + \sum_{i=1}^M \frac{1}{\delta} \int_0^T \int_\Omega \chi^N_{\mathcal{S}_i} |u^N - P^N_{\mathcal{S}_i} u^N|^2 \leqslant \int_0^T \int_\Omega \rho^N g^N \cdot u^N + E_0^N.$$

Moreover,

$$\min(\operatorname{dist}(\mathcal{S}_i^N(t), \partial\Omega),\ \operatorname{dist}(\mathcal{S}_i^N(t), \mathcal{S}_j^N(t))) \geqslant 2\sigma, \quad \forall\, t \in [0,T], \quad i \neq j, \quad i,j = 1, ..., M.$$

3. Existence proofs of Approximate solutions

In this section, we present the proofs or the sketch of the proof of the existence results of the three approximation levels. Let us begin with the N-level approximation in Section 3.1 and the limit as $N \to \infty$ in Section 3.2, which gives existence at the ε-level.

3.1. **Solution of the Faedo-Galerkin approximation.** In this subsection, we construct a solution $(\mathcal{S}^N, \rho^N, u^N)$ to the problem (2.28)–(2.33).

Proof of Proposition 2.3. The idea is to apply the Galerkin approximation as a fixed point problem and then used the Schauder's fixed point theorem to it. We set

$$B_{R,T} = \{u \in C([0,T]; X_N),\ \|u\|_{L^\infty(0,T;L^2(\Omega))} \leqslant R\},$$

for R and T positive which will be fixed in Step 3.

Step 1: Continuity equation and transport of the body.

For a given $u \in B_{R,T}$, we solve ρ as the solution to

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = \varepsilon \Delta \rho \text{ in } (0,T) \times \Omega, \quad \frac{\partial \rho}{\partial \nu} = 0 \text{ on } \partial\Omega, \quad \rho(0) = \rho_0^N, \quad 0 < \underline{\rho} \leqslant \rho_0^N \leqslant \overline{\rho}, \tag{3.1}$$

and let $\chi_{\mathcal{S}}$ be a solution of the transport equation

$$\frac{\partial \chi_{\mathcal{S}_i}}{\partial t} + P_{\mathcal{S}_i} u \cdot \nabla \chi_{\mathcal{S}} = 0, \quad \chi_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}}, \tag{3.2}$$

and

$$\frac{\partial}{\partial t}(\rho \chi_{\mathcal{S}_i}) + P_{\mathcal{S}_i} u \cdot \nabla(\rho \chi_{\mathcal{S}_i}) = 0, \quad (\rho\chi_{\mathcal{S}_i})|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_0 i}, \tag{3.3}$$

where $P_{\mathcal{S}_i} u \in \mathcal{R}(\Omega)$ and it is given by (5.2).

Since $\rho_0^N \in W^{1,\infty}(\Omega)$, $u \in B_{R,T}$ in (3.1), we can use [18, Proposition 7.39, Page 345] to conclude that $\rho > 0$ and

$$\rho \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega)).$$

Moreover, using the theory of the transport equation, see Proposition 5.1, we get

$$\chi_{\mathcal{S}_i} \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \forall\, 1 \leqslant p < \infty,$$
$$\rho\chi_{\mathcal{S}_i} \in L^\infty((0,T) \times \Omega) \cap C([0,T]; L^p(\Omega)), \forall\, 1 \leqslant p < \infty.$$

Further, we define

$$\mu=\sum_{i=1}^{M}[(1-\chi_{\mathcal{S}_i})\mu_{\mathcal{F}}+\delta^2\chi_{\mathcal{S}_i}],\quad \lambda=\sum_{i=1}^{M}[(1-\chi_{\mathcal{S}_i})\lambda_{\mathcal{F}}+\delta^2\chi_{\mathcal{S}}]\text{ so that }\mu>0,\ 2\mu+3\lambda\geqslant 0,$$
$$g=\sum_{i=1}^{M}[(1-\chi_{\mathcal{S}_i})g_{\mathcal{F}}+\chi_{\mathcal{S}_i}g_{\mathcal{S}_i}],\quad p(\rho)=a\rho^{\gamma}+\delta\rho^{\beta}\quad\text{with}\quad a=\sum_{i=1}^{M}a_{\mathcal{F}}(1-\chi_{\mathcal{S}_i}).$$

Step 2: Momentum equation.

For a given $u\in B_{R,T}$, let us consider the following equation satisfied by $\widetilde{u}:[0,T]\mapsto X_N$:

$$-\int_0^T\int_\Omega \rho\Big(\frac{\partial\widetilde{u}}{\partial t}\cdot e_j+(u\cdot\nabla e_j)\cdot\widetilde{u}\Big)+\int_0^T\int_\Omega\Big(2\mu\mathbb{D}(\widetilde{u}):\mathbb{D}(e_j)+\lambda\operatorname{div}\widetilde{u}\mathbb{I}:\mathbb{D}(e_j)-p(\rho)\mathbb{I}:\mathbb{D}(e_j)\Big)$$
$$+\int_0^T\int_\Omega\varepsilon\nabla\widetilde{u}\nabla\rho\cdot e_j+\alpha\sum_{i=1}^{M}\int_0^T\int_{\partial\mathcal{S}_i^N(t)}[(\widetilde{u}-P_{\mathcal{S}_i}\widetilde{u})\times\nu_i]\cdot[(e_j-P_{\mathcal{S}_i}e_j)\times\nu_i]$$
$$+\frac{1}{\delta}\sum_{i=1}^{M}\int_0^T\int_\Omega\chi_{\mathcal{S}_i}(\widetilde{u}-P_{\mathcal{S}_i}\widetilde{u})\cdot(e_j-P_{\mathcal{S}_i}e_j)=\int_0^T\int_\Omega\rho g\cdot e_j,\tag{3.4}$$

where ρ, $\chi_{\mathcal{S}_i}$ are defined as in Step 1. We can write

$$\widetilde{u}(t,\cdot)=\sum_{i=1}^{N}g_i(t)e_i,\quad \widetilde{u}(0)=u_0^N=\sum_{i=1}^{N}\left(\int_\Omega u_0\cdot e_i\right)e_i.$$

The coefficients $\{g_i\}$ of $\widetilde{u}$ satisfy the ordinary differential equation,

$$\sum_{i=1}^{N}a_{i,j}g_i'(t)+\sum_{i=1}^{N}b_{i,j}g_i(t)=f_j(t),\quad g_i(0)=\int_\Omega u_0^N\cdot e_i,\tag{3.5}$$

where $a_{i,j}$, $b_{i,j}$ and f_j are given by

$$a_{i,j}=\int_0^T\int_\Omega\rho e_ie_j,$$
$$b_{i,j}=\int_0^T\int_\Omega\rho(u\cdot\nabla e_j)\cdot e_i+\int_0^T\int_\Omega\Big(2\mu\mathbb{D}(e_i):\mathbb{D}(e_j)+\lambda\operatorname{div}e_i\mathbb{I}:\mathbb{D}(e_j)\Big)+\int_0^T\int_\Omega\varepsilon\nabla e_j\nabla\rho\cdot e_i$$
$$+\alpha\sum_{k=1}^{M}\int_0^T\int_{\partial\mathcal{S}_k(t)}[(e_i-P_{\mathcal{S}_k}e_i)\times\nu^k]\cdot[(e_j-P_{\mathcal{S}_k}e_j)\times\nu^k]+\frac{1}{\delta}\int_0^T\int_\Omega\chi_{\mathcal{S}_k}(e_i-P_{\mathcal{S}_k}e_i)\cdot(e_j-P_{\mathcal{S}_k}e_j),$$
$$f_j=\int_0^T\int_\Omega\rho g\cdot e_j+\int_0^T\int_\Omega p(\rho)\mathbb{I}:\mathbb{D}(e_j).$$

Let us stress that the positive lower bound of ρ in [18, Proposition 7.39, Page 345] guarantees the invertibility of the matrix $(a_{i,j}(t))_{1\leqslant i,j\leqslant N}$. Using the regularity of ρ ([18, Proposition 7.39, Page 345]), of $\chi_{\mathcal{S}}$ and of the propagator associated to $P_{\mathcal{S}}u$ (Proposition 5.1) we can conclude the continuity of $(a_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $(b_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $(f_i(t))_{1\leqslant i\leqslant N}$.

The existence and uniqueness theorem for ordinary differential equations gives that system (3.5) has a unique solution defined on $[0,T]$ and therefore equation (3.4) has a unique solution

$$\widetilde{u} \in C([0,T]; X_N).$$

Step 3: The map $\mathcal{N}$.
Let us introduce a map

$$\mathcal{N} : B_{R,T} \to C([0,T], X_N)$$
$$u \mapsto \widetilde{u},$$

where $\widetilde{u}$ satisfies (3.4). From the existence of $\widetilde{u} \in C([0,T]; X_N)$ in the problem (3.4), we have that $\mathcal{N}$ is well-defined from $B_{R,T}$ to $C([0,T]; X_N)$. Now we establish the fact that $\mathcal{N}$ maps $B_{R,T}$ to itself for suitable R and T.

We fix

$$0 < \sigma < \min(\frac{1}{2}\operatorname{dist}(\mathcal{S}_{0i}, \partial\Omega), \frac{1}{2}\operatorname{dist}(\mathcal{S}_{0i}, \mathcal{S}_{0j})), i \neq j.$$

Given $u \in B_{R,T}$, we want to estimate $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))}$. Using a simple integration by parts we obtain :

$$\int_0^t \int_\Omega \rho \widetilde{u}' \cdot \widetilde{u} = -\frac{1}{2}\int_0^t \int_\Omega \frac{\partial \rho}{\partial t}|\widetilde{u}|^2 + \frac{1}{2}(\rho|\widetilde{u}|^2)(t) - \frac{1}{2}\rho_0|u_0|^2, \tag{3.6}$$

$$\int_0^T \int_\Omega \rho(u\cdot\nabla)\widetilde{u}\cdot\widetilde{u} = -\frac{1}{2}\int_0^T \int_\Omega \operatorname{div}(\rho u)|\widetilde{u}|^2, \tag{3.7}$$

$$\begin{aligned}\int_\Omega \nabla(\rho^\gamma)\cdot\widetilde{u} &= \frac{\gamma}{\gamma-1}\int_\Omega \nabla(\rho^{\gamma-1})\cdot\rho\widetilde{u} = -\frac{\gamma}{\gamma-1}\int_\Omega \rho^{\gamma-1}\operatorname{div}(\rho\widetilde{u}) = \frac{1}{\gamma-1}\frac{d}{dt}\int_\Omega \rho^\gamma - \frac{\varepsilon\gamma}{\gamma-1}\int_\Omega \rho^{\gamma-1}\Delta\rho \\ &= \frac{1}{\gamma-1}\frac{d}{dt}\int_\Omega \rho^\gamma + \varepsilon\gamma\int_\Omega \rho^{\gamma-2}|\nabla\rho|^2 \geqslant \frac{1}{\gamma-1}\frac{d}{dt}\int_\Omega \rho^\gamma.\end{aligned} \tag{3.8}$$

Similarly,

$$\int_\Omega \nabla(\rho^\beta)\cdot\widetilde{u} = \frac{1}{\beta-1}\frac{d}{dt}\int_\Omega \rho^\beta + \varepsilon\beta\int_\Omega \rho^{\beta-2}|\nabla\rho|^2. \tag{3.9}$$

We multiply equation (3.4) by g_j, add these equations for $j = 1, 2, ..., N$, use the relations (3.6)–(3.9) and the continuity equation (3.1) to obtain the following energy estimate:

$$\begin{aligned}&\int_\Omega \left(\frac{1}{2}\rho|\widetilde{u}|^2 + \frac{a}{\gamma-1}\rho^\gamma + \frac{\delta}{\beta-1}\rho^\beta\right) + \int_0^T\int_\Omega \left(2\mu|\mathbb{D}(\widetilde{u})|^2 + \lambda|\operatorname{div}\widetilde{u}|^2\right) + \delta\varepsilon\beta\int_0^T\int_\Omega \rho^{\beta-2}|\nabla\rho|^2 \\ &+\alpha\sum_{k=1}^M \int_0^T \int_{\partial\mathcal{S}_k(t)} |(\widetilde{u} - P_{\mathcal{S}_k}\widetilde{u})\times\nu|^2 + \frac{1}{\delta}\int_0^T\int_\Omega \chi_{\mathcal{S}_k}|\widetilde{u} - P_{\mathcal{S}_k}\widetilde{u}|^2 \leqslant \int_0^T\int_\Omega \rho g\cdot\widetilde{u} + \int_\Omega \left(\frac{1}{2}\frac{\rho_0^N}{|q_0^N|^2}\mathbb{1}_{\{\rho_0>0\}} + \frac{a}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta\right) \\ &\leqslant \sqrt{\overline{\rho}}T\left(\frac{1}{2\widetilde{\varepsilon}}\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{\widetilde{\varepsilon}}{2}\|\sqrt{\rho}\widetilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))}\right) + \int_\Omega \left(\frac{1}{2}\frac{\rho_0^N}{|q_0^N|^2}\mathbb{1}_{\{\rho_0>0\}} + \frac{a}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta\right).\end{aligned} \tag{3.10}$$

An appropriate choice of $\widetilde{\varepsilon}$ in (3.10) gives us

$$\|\widetilde{u}\|^2_{L^\infty(0,T;L^2(\Omega))} \leqslant \frac{4\overline{\rho}}{\underline{\rho}}T^2\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}}E_0^N,$$

where $\overline{\rho}$ and $\underline{\rho}$ are the upper and lower bounds of ρ. In order to get $\|\widetilde{u}\|_{L^\infty(0,T;L^2(\Omega))} \leqslant R$, we need

$$R^2 \geqslant \frac{4\overline{\rho}}{\underline{\rho}}T^2\|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}}E_0^N. \tag{3.11}$$

Moreover, we have to verify that for T small enough and for any $u \in B_{R,T}$,

$$\inf_{u\in B_{R,T}} (\operatorname{dist}(\mathcal{S}_i(t), \partial\Omega), \operatorname{dist}(\mathcal{S}_i(t), \mathcal{S}_j(t)) \geqslant 2\sigma > 0 \tag{3.12}$$

holds. We can write $\mathcal{S}_i(t) = \eta^i_{t,0}(\mathcal{S}_{0i})$ with the isometric propagator $\eta^i_{t,s}$ (see equation (5.3) for the precise definition) associated to the rigid field $P_{\mathcal{S}_i} u = h'_i(t) + \omega_i(t) \times (y - h_i(t))$. From [8, Proposition 4.6, Step 2], we conclude: proving (3.12) is equivalent to establishing the following bound:

$$\sup_{t\in[0,T]} |\partial_t \eta^i_{t,0}(t,y)| < \frac{\min(\operatorname{dist}(\mathcal{S}_0, \partial\Omega), \operatorname{dist}(\mathcal{S}_i(t), \mathcal{S}_j(t)) - 2\sigma}{T}, \quad t \in [0,T],\ y \in \mathcal{S}_{0i}. \tag{3.13}$$

We have

$$|\partial_t \eta^i_{t,0}(t,y)| = |P_{\mathcal{S}_i} u(t, \eta^i_{t,0}(t,y))| \leqslant |h'_i(t)| + |\omega_i(t)||y - h_i(t)|.$$

Furthermore, if $\overline{\rho}$ is the upper bound of ρ, then for $u \in B_{R,T}$

$$\sum_{i=1}^{M} |h'_i(t)|^2 + J_i(t)\omega_i(t)\cdot\omega_i(t) = \sum_{i=1}^{M} \int_{\mathcal{S}_i(t)} \rho_{\mathcal{S}_i} |P_{\mathcal{S}_i} u(t,\cdot)|^2 \leqslant \int_{\Omega} \rho |u(t,\cdot)|^2 \leqslant \overline{\rho} R^2 \tag{3.14}$$

for any R and $t \in (0,T)$. As $J_i(t)$ is congruent to $J_i(0)$, they have the same eigenvalues and we have

$$\lambda_{0i} |\omega_i(t)|^2 \leqslant J_i(t)\omega_i(t)\cdot\omega_i(t),$$

where λ_{0i} is the smallest eigenvalue of $J_i(0)$. Observe that for $t \in [0,T]$, $y \in \mathcal{S}_{0i}$,

$$\begin{aligned} |h'_i(t)| + |\omega_i(t)||y - h_i(t)| &\leqslant \sqrt{2}(|h'_i(t)|^2 + |\omega_i(t)|^2 |y - h_i(t)|^2)^{1/2} \leqslant \sqrt{2} \max\{1, |y - h_i(t)|\} (|h'_i(t)|^2 + |\omega_i(t)|^2)^{1/2} \\ &\leqslant C_0 \left(|h'_i(t)|^2 + J_i(t)\omega_i(t)\cdot\omega_i(t)\right)^{1/2}, \end{aligned} \tag{3.15}$$

where $C_0 = \sqrt{2}\frac{\max\{1,|y-h_i(t)|\}}{\min\{1,\lambda_{0i}\}^{1/2}}$. Thus, with the help of (3.14)–(3.15) and the relation of R in (3.11), we can conclude that any

$$T < \frac{\min(\operatorname{dist}(\mathcal{S}_{0i}, \partial\Omega), \operatorname{dist}(\mathcal{S}_{0i}, \mathcal{S}_{0j})) - 2\sigma}{C_0 |\overline{\rho}|^{1/2} [\frac{4\overline{\rho}}{\underline{\rho}} T^2 \|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \frac{4}{\underline{\rho}} E^N_0]^{1/2}} \tag{3.16}$$

satisfies the relation (3.12). Thus, we choose T satisfying (3.16) and fix it. Then we choose R as in (3.11) to conclude that $\mathcal{N}$ maps $B_{R,T}$ to itself.

Step 4: Continuity of $\mathcal{N}$.

We will show that if a sequence $\{u^k\} \subset B_{R,T}$ is such that $u^k \to u$ in $B_{R,T}$, then $\mathcal{N}(u^k) \to \mathcal{N}(u)$ in $B_{R,T}$.

As $\operatorname{span}(e_1, e_2, ..., e_N)$ is a finite dimensional subspace of $\mathcal{D}(\overline{\Omega})$, we have $u^k \to u$ in $C([0,T]; \mathcal{D}(\overline{\Omega}))$. Given $\{u^k\} \subset B_{R,T}$, we have that $\rho^k \in L^2(0,T; H^2(\Omega)) \cap C([0,T]; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ is the solution to (3.1), $\chi^k_{\mathcal{S}_i}$ is bounded in $L^\infty((0,T)\times\mathbb{R}^3)$ satisfying (3.2) and $\{\rho^k \chi^k_{\mathcal{S}_i}\}$ is a bounded sequence in $L^\infty((0,T)\times\mathbb{R}^3)$ satisfying (3.3). We apply Proposition 5.2 to obtain

$$\begin{aligned} &\chi^k_{\mathcal{S}_i} \to \chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T]; L^p_{loc}(\mathbb{R}^3)),\ \forall\, 1 \leqslant p < \infty, \\ &P^k_{\mathcal{S}_i} u^k \to P_{\mathcal{S}_i} u \text{ strongly in } C([0,T]; C^\infty_{loc}(\mathbb{R}^3)), \\ &\eta^{k,i}_{t,s} \to \eta^i_{t,s} \text{ strongly in } C^1([0,T]^2; C^\infty_{loc}(\mathbb{R}^3)). \end{aligned}$$

We use the continuity argument as in Step 2 to conclude

$$a^k_{i,j} \to a_{i,j}, \quad b^k_{i,j} \to b_{i,j}, \quad f^k_j \to f_j \text{ strongly in } C([0,T]),$$

and so we obtain

$$\mathcal{N}(u^k) = \widetilde{u}^k \to \widetilde{u} = \mathcal{N}(u) \text{ strongly in } C([0,T]; X_N).$$

Step 5: Compactness of $\mathcal{N}$.

If $\widetilde{u}(t) = \sum_{i=1}^{N} g_i(t)e_i$, we can view (3.5) as

$$A(t)G'(t) + B(t)G(t) = F(t),$$

where $A(t) = (a_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $B(t) = (b_{i,j}(t))_{1\leqslant i,j\leqslant N}$, $F(t) = (f_i(t))_{1\leqslant i\leqslant N}$, $G(t) = (g_i(t))_{1\leqslant i\leqslant N}$. We deduce

$$|g_i'(t)| \leqslant R|A^{-1}(t)||B(t)| + |A^{-1}(t)||F(t)|.$$

Thus, we obtain

$$\sup_{t\in[0,T]} \Big(|g_i(t)| + |g_i'(t)|\Big) \leqslant C.$$

It also implies

$$\sup_{u\in B_{R,T}} \|\mathcal{N}(u)\|_{C^1([0,T];X_N)} \leqslant C.$$

The $C^1([0,T];X_N)$-boundedness of $\mathcal{N}(u)$ allows us to apply the Arzela-Ascoli theorem to obtain compactness of $\mathcal{N}$ in $B_{R,T}$.

Now we are in a position to apply Schauder's fixed point theorem to $\mathcal{N}$ to conclude the existence of a fixed point $u^N \in B_{R,T}$. Then we define ρ^N satisfying the continuity equation (2.29) on $(0,T)\times\Omega$, and $\chi^N_{\mathcal{S}_i} = \mathbb{1}_{\mathcal{S}_i^N}$ is the corresponding solution to the transport equation (2.31) on $(0,T)\times\mathbb{R}^3$. It only remains to justify the momentum equation (2.30). We multiply equation (3.4) by $\psi\in\mathcal{D}([0,T))$ to obtain:

$$-\int_0^T\int_\Omega \frac{\partial u^N}{\partial t}\cdot\psi(t)e_j + (u^N\cdot\nabla(\psi(t)e_j))\cdot u^N\Big) + \int_0^T\int_\Omega \varepsilon\nabla(\psi(t)e_j)\nabla\rho^N\cdot u^N$$
$$+\int_0^T\int_\Omega \Big(2\mu^N\mathbb{D}(u^N):\mathbb{D}(\psi(t)e_j) + \lambda^N \operatorname{div} u^N\mathbb{I}:\mathbb{D}(\psi(t)e_j) - p^N(\rho^N)\mathbb{I}:\mathbb{D}(\psi(t)e_j)\Big)$$
$$+\alpha\sum_{i=1}^M\int_0^T\int_{\partial\mathcal{S}_i^N(t)} [(u^N - P^N_{\mathcal{S}_i}u^N)\times\nu_i]\cdot[(\psi(t)e_j - P^N_{\mathcal{S}_i}\psi(t)e_j)\times\nu_i] + \frac{1}{\delta}\int_0^T\int_\Omega\sum_{i=1}^M \chi_{\mathcal{S}_i}(u^N - P^N_{\mathcal{S}_\rangle}u^N)\cdot(\psi(t)e_j - P^N_{\mathcal{S}_i}\psi(t)e_j)$$
$$= \int_0^T\int_\Omega \rho^N g^N\cdot\psi(t)e_j, \quad (3.17)$$

Using the integration by parts we have the following identities:

$$\int_0^T \rho^N\frac{\partial u^N}{\partial t}\cdot\psi(t)e_j = -\int_0^T\frac{\partial\rho^N}{\partial t}u^N\cdot\psi(t)e_j - \int_0^T\rho^N u^N\cdot\psi'(t)e_j - (\rho^N u^N\cdot\psi e_j)(0), \quad (3.18)$$

$$\int_\Omega \rho^N(u^N\cdot\nabla(\psi(t)e_j))\cdot u^N = -\int_\Omega \operatorname{div}(\rho^N u^N)(\psi(t)e_j\cdot u^N) - \int_\Omega\rho^N(u^N\cdot\nabla)u^N\cdot\psi(t)e_j. \quad (3.19)$$

Thus we can use the relations (3.18)–(3.19) and continuity equation (2.29) in the identity (3.17) to obtain equation (2.30) for all $\phi\in\mathcal{D}([0,T);X_N)$. □

3.2. **Convergence of the Faedo-Galerkin scheme and the limiting system.** In Proposition 2.3, we have already constructed a solution $(\mathcal{S}^N,\rho^N,u^N)$ to the problem (2.28)–(2.33). In this section, we establish Proposition 2.2 by passing to the limit in (2.28)–(2.33) as $N\to\infty$ to recover the solution of (2.16)–(2.21), i.e. of the ε-level approximation.

Proof of Proposition 2.2. If we replace ϕ by u^N in (2.30), then as in (3.10), we derive

$$E^N[\rho^N,q^N] + \int_0^T\int_\Omega \left(2\mu^N|\mathbb{D}(u^N)|^2 + \lambda^N|\operatorname{div} u^N|^2\right) + \delta\varepsilon\beta\int_0^T\int_\Omega (\rho^N)^{\beta-2}|\nabla\rho^N|^2$$
$$+\sum_{i=1}^M \alpha\int_0^T\int_{\partial\mathcal{S}_i^N(t)} |(u^N - P^N_{\mathcal{S}_i}u^N)\times\nu|^2 + \frac{1}{\delta}\int_0^T\int_\Omega\sum_{i=1}^M \chi^N_{\mathcal{S}_i}|u^N - P^N_{\mathcal{S}_i}u^N|^2 \leqslant \int_0^T\int_\Omega \rho^N g^N\cdot u^N + E^N_0, \quad (3.20)$$

where

$$E^N[\rho^N,q^N] = \int_\Omega \left(\frac{1}{2}\rho^N|u^N|^2 + \frac{a^N}{\gamma-1}(\rho^N)^\gamma + \frac{\delta}{\beta-1}(\rho^N)^\beta\right).$$

Following the idea of the footnote in [18, Page 368], the initial data (ρ_0^N, u_0^N) is constructed in such a way that

$$\rho_0^N \to \rho_0^\varepsilon \text{ in } W^{1,\infty}(\Omega), \quad \rho_0^N u_0^N \to q_0^\varepsilon \text{ in } L^2(\Omega)$$

and

$$\int_\Omega \left(\frac{1}{2}\rho_0^N|u_0^N|^2\mathbb{1}_{\{\rho_0^N>0\}} + \frac{a^N}{\gamma-1}(\rho_0^N)^\gamma + \frac{\delta}{\beta-1}(\rho_0^N)^\beta\right) \to \int_\Omega \left(\frac{1}{2}\frac{|q_0^\varepsilon|^2}{\rho_0^\varepsilon}\mathbb{1}_{\{\rho_0^\varepsilon>0\}} + \frac{a^\varepsilon}{\gamma-1}(\rho_0^\varepsilon)^\gamma + \frac{\delta}{\beta-1}(\rho_0^\varepsilon)^\beta\right) \text{ as } N\to\infty. \quad (3.21)$$

Precisely, we approximate q_0^ε by a sequence q_0^N satisfying (2.37) and such that (3.21) is valid. It is sufficient to take $u_0^N = P_N(\frac{q_0^\varepsilon}{\rho_0^\varepsilon})$, where by P_N we denote the orthogonal projection of $L^2(\Omega)$ onto X_N. Proposition 2.3 is valid with these new initial data.

The construction of ρ^N imply that $\rho^N > 0$. Thus the energy estimate (3.20) yields that up to a subsequence

(1) $u^N \to u^\varepsilon$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$,
(2) $\rho^N \to \rho^\varepsilon$ weakly-$*$ in $L^\infty(0,T;L^\beta(\Omega))$,
(3) $\nabla\rho^N \to \nabla\rho^\varepsilon$ weakly in $L^2((0,T)\times\Omega)$.

We follow the similar analysis as for the fluid case explained in [18, Section 7.8.1, Page 362] to conclude that

- $\rho^N \to \rho^\varepsilon$ in $C([0,T];L^\beta_{weak}(\Omega))$ and $\rho^N \to \rho^\varepsilon$ strongly in $L^p((0,T)\times\Omega)$, $\forall\, 1\leqslant p<\frac{4}{3}\beta$,
- $\rho^N u^N \to \rho^\varepsilon u^\varepsilon$ weakly in $L^2(0,T;L^{\frac{6\beta}{\beta+6}})$ and weakly-$*$ in $L^\infty(0,T;L^{\frac{2\beta}{\beta+1}})$.

We also know that $\chi^N_{\mathcal{S}_i}$ is a bounded sequence in $L^\infty((0,T)\times\mathbb{R}^3)$ satisfying (2.31) and $\{\rho^N\chi^N_{\mathcal{S}_i}\}$ is a bounded sequence in $L^\infty((0,T)\times\mathbb{R}^3)$ satisfying (2.32). We use Proposition 5.3 to conclude

$$\chi^N_{\mathcal{S}_i} \to \chi^\varepsilon_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3)),\ \forall\, 1\leqslant p<\infty, \quad (3.22)$$

with $\chi^\varepsilon_{\mathcal{S}_i}$ satisfying (2.19) along with (2.16). Thus, we have recovered the transport equation for the body (2.19). From (3.22) and the definitions of g^N and g^ε in (2.34) and (2.22), it follows that

$$g^N \to g^\varepsilon \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ \forall\, 1\leqslant p<\infty. \quad (3.23)$$

These convergence results give us the possibility to pass to the limit $N\to\infty$ in (2.29) to achieve (2.17). Now let us concentrate on the limit of the momentum equation (2.30). The four most difficult terms are:

$$A_i^N(t,e_k) = \int_{\partial\mathcal{S}_i^N(t)} [(u^N - P^N_{\mathcal{S}_i}u^N)\times\nu_i]\cdot[(e_k - P^N_{\mathcal{S}_i}e_k)\times\nu_i], \quad B^N(t,e_k) = \int_\Omega \rho^N u^N\otimes u^N : \nabla e_k,$$
$$C^N(t,e_k) = \int_\Omega \varepsilon\nabla u^N\nabla\rho^N\cdot e_k, \quad D^N(t,e_k) = \int_\Omega (\rho^N)^\beta\mathbb{I}:\mathbb{D}(e_k), \quad 1\leqslant k\leqslant N.$$

To analyze the term $A_i^N(t,e_k)$, we do a change of variables to rewrite it in a fixed domain and use the convergence results from Proposition 5.2 for the projection and the isometric propagator:

$$P^N_{\mathcal{S}_i}u^N \to P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon} \text{ weakly-} * \text{ in } L^{\infty}(0,T;C^{\infty}_{loc}(\mathbb{R}^3)),$$
$$\eta^N_{t,s} \to \eta^{\varepsilon}_{t,s} \text{ weakly-} * \text{ in } W^{1,\infty}((0,T)^2;C^{\infty}_{loc}(\mathbb{R}^3)).$$

We follow a similar analysis as in [8, Page 2047–2048] to conclude that A_i^N converges weakly in $L^1(0,T)$ to

$$A_i(t,e_k) = \int_{\partial\mathcal{S}_i^{\varepsilon}(t)} [(u^{\varepsilon} - P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\times\nu]\cdot[(e_k - P^{\varepsilon}_{\mathcal{S}_i}e_k)\times\nu_i].$$

We proceed as explained in the fluid case [18, Section 7.8.2, Page 363–365] to analyze the limiting process for the other terms $B^N(t,e_k)$, $C^N(t,e_k)$, $D^N(t,e_k)$. The limit of $B^N(t,e_k)$ follows from the fact [18, Equation (7.8.22), Page 364] that

$$\rho^N u^N\otimes u^N \to \rho^{\varepsilon}u^{\varepsilon}\otimes u^{\varepsilon} \text{ weakly in } L^2(0,T;L^{\frac{6\beta}{4\beta+3}}(\Omega)). \tag{3.24}$$

To obtain the limit of $C^N(t,e_k)$, we apply [18, Equation (7.8.26), Page 365]:

$$\varepsilon\nabla u^N\nabla\rho^N \to \varepsilon\nabla u^{\varepsilon}\nabla\rho^{\varepsilon} \text{ weakly in } L^2(0,T;L^{\frac{5\beta-3}{4\beta}}(\Omega)),$$

and the limit of $D^N(t,e_k)$ is obtained by using [18, Equation (7.8.8), Page 362]:

$$\rho^N \to \rho^{\varepsilon} \text{ strongly in } L^p(0,T;\Omega), \quad 1\leqslant p<\frac{4}{3}\beta. \tag{3.25}$$

Thus, using the above convergence results for B^N, C^N, D^N and the fact that

$$\bigcup_N X_N \text{ is dense in } W^{1,p}_0(\Omega),$$

we can conclude the following weak convergences in $L^1(0,T)$:

$$B^N(t,\phi^N) \to B(t,\phi^{\varepsilon}) = \int_{\Omega} \rho^{\varepsilon}u^{\varepsilon}\otimes u^{\varepsilon} : \nabla\phi^{\varepsilon},$$

$$C^N(t,\phi^N) \to C(t,\phi^{\varepsilon}) = \int_{\Omega} \varepsilon\nabla u^{\varepsilon}\nabla\rho^{\varepsilon}\cdot\phi^{\varepsilon},$$

$$D^N(t,\phi^N) \to D(t,\phi^{\varepsilon}) = \int_{\Omega} (\rho^{\varepsilon})^{\beta}\mathbb{I} : \mathbb{D}(\phi^{\varepsilon}).$$

Thus we have achieved (2.18) as a limit of equation (2.30) as $N\to\infty$. Hence, we have established the existence of a solution $(\mathcal{S}^{\varepsilon},\rho^{\varepsilon},u^{\varepsilon})$ to system (2.16)–(2.21). Now we establish energy inequality (2.26) and estimates independent of ε:

- The weak convergence of $\rho^N|u^N|^2$ in (3.24) and strong convergence of ρ^N in (3.25) ensure that, up to a subsequence,

$$\int_{\Omega} \Big(\frac{1}{2}\rho^N|u^N|^2 + \frac{a^N}{\gamma-1}(\rho^N)^{\gamma} + \frac{\delta}{\beta-1}(\rho^N)^{\beta}\Big) \to \int_{\Omega} \Big(\frac{1}{2}\rho^{\varepsilon}|u^{\varepsilon}|^2 + \frac{a^{\varepsilon}}{\gamma-1}(\rho^{\varepsilon})^{\gamma} + \frac{\delta}{\beta-1}(\rho^{\varepsilon})^{\beta}\Big) \text{ in } \mathcal{D}'((0,T)). \tag{3.26}$$

- Due to the weak lower semicontinuity of convex functionals, the weak convergence of u^N in $L^2(0,T;H^1(\Omega))$, the strong convergence of $\chi^N_{\mathcal{S}}$ in $C([0,T];L^p(\Omega))$ and the strong convergence of $P^N_{\mathcal{S}}$ in $C([0,T];C^{\infty}_{loc}(\mathbb{R}^3))$, we obtain

$$\int_0^T \psi\int_{\Omega} \Big(2\mu^{\varepsilon}|\mathbb{D}(u^{\varepsilon})|^2 + \lambda^{\varepsilon}|\operatorname{div}u^{\varepsilon}|^2\Big) \leqslant \liminf_{N\to\infty}\int_0^T \psi\int_{\Omega}\Big(2\mu^N|\mathbb{D}(u^N)|^2 + \lambda^N|\operatorname{div}u^N|^2\Big), \tag{3.27}$$

$$\int_0^T \psi \int_\Omega \chi^\varepsilon_{\mathcal{S}_i}|u^\varepsilon - P^\delta_{\mathcal{S}_i}u^\varepsilon|^2 \leqslant \liminf_{N\to\infty} \int_0^T \psi \int_\Omega \chi^N_{\mathcal{S}_i}|u^N - P_{\mathcal{S}_i}u^N|^2, \tag{3.28}$$

where ψ is a smooth non-negative function on $(0,T)$.

- Using the fact that $\nabla\rho^N \to \nabla\rho$ strongly in $L^2((0,T)\times\Omega)$ (by [18, Equation (7.8.25), Page 365]), strong convergence of ρ^N in (3.25) and Fatou's lemma, we have
$$\int_0^T \psi \int_\Omega (\rho^\varepsilon)^{\beta-2}|\nabla\rho^\varepsilon|^2 \leqslant \liminf_{N\to\infty} \int_0^T \psi \int_\Omega (\rho^N)^{\beta-2}|\nabla\rho^N|^2. \tag{3.29}$$
- For passing to the limit in the boundary terms, we follow the idea of [8]. We introduce the extended velocities U^N, $U^N_{\mathcal{S}_i}$ to whole $\mathbb{R}^3$ associated to $\mathcal{E}u^N$, $P^N_{\mathcal{S}_i}u^N$ respectively. They are defined by:
$$\mathcal{E}u^N(t,\eta^N_{t,0}(y)) = J_{\eta^N_{t,0}}\Big|_y U^N(t,y), \quad P^N_{\mathcal{S}_i}u^N(t,\eta^N_{t,0}(y)) = J_{\eta^N_{t,0}}\Big|_y U^N_{\mathcal{S}_i}(t,y)$$
where $\mathcal{E}: H^1(\Omega)\to H^1(\mathbb{R}^3)$ is the extension operator and $J_{\eta^N_{t,0}}$ is the Jacobian matrix of $\eta^N_{t,0}$. According to [8, Lemma A.2], we have the weak convergences of U^N, $U^N_{\mathcal{S}_i}$ to U^ε, $U^\varepsilon_{\mathcal{S}_i}$ in $L^2(0,T;H^1_{loc}(\mathbb{R}^3))$. These facts along with the lower semicontinuity of the L^2-norm yield
$$\begin{aligned}\int_0^T \psi \int_{\partial\mathcal{S}^\varepsilon_i(t)} |(u^\varepsilon - P^\varepsilon_{\mathcal{S}_i}u^\varepsilon)\times\nu|^2 = \int_0^T \psi \int_{\partial\mathcal{S}_{0i}} |(U^\varepsilon - U^\varepsilon_{\mathcal{S}_i})\times\nu|^2 &\leqslant \liminf_{N\to\infty}\int_0^T \psi \int_{\partial\mathcal{S}_{0i}} |(U^N - U^N_{\mathcal{S}_i})\times\nu|^2 \\ &= \liminf_{N\to\infty}\int_0^T \psi \int_{\partial\mathcal{S}^N_i(t)} |(u^N - P_{\mathcal{S}_i}u^N)\times\nu|^2.\end{aligned} \tag{3.30}$$
In the above, the first and the last equality in (3.30) follows from the change of variables formula.
- Regarding the term on the right hand side of (3.20), the weak convergence of u^N in $L^2(0,T;H^1(\Omega))$, the strong convergence of ρ^N in (3.25) and the strong convergence of g^N in (3.23) yield
$$\int_0^T \psi \int_\Omega \rho^N g^N\cdot u^N \to \int_0^T \psi \int_\Omega \rho^\varepsilon g^\varepsilon \cdot u^\varepsilon, \quad \text{as } N\to\infty. \tag{3.31}$$

We can obtain the following differential form of energy inequality by using the above results (3.26)–(3.31):

$$\begin{aligned}&\frac{d}{dt}E^\varepsilon[\rho^\varepsilon,q^\varepsilon] + \int_\Omega \Big(2\mu^\varepsilon|\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon|\operatorname{div} u^\varepsilon|^2\Big) + \delta\varepsilon\beta\int_\Omega (\rho^\varepsilon)^{\beta-2}|\nabla\rho^\varepsilon|^2 \\ &\quad + \alpha\sum_{i=1}^M \int_{\partial\mathcal{S}^\varepsilon_i(t)} |(u^\varepsilon - P^\varepsilon_{\mathcal{S}_i}u^\varepsilon)\times\nu^i|^2 + \frac{1}{\delta}\sum_{i=1}^M \int_\Omega \chi^\varepsilon_{\mathcal{S}_i}|u^\varepsilon - P^\delta_{\mathcal{S}_i}u^\varepsilon|^2 \leqslant \int_\Omega \rho^\varepsilon g^\varepsilon\cdot u^\varepsilon \text{ in } \mathcal{D}'((0,T)).\end{aligned} \tag{3.32}$$

Since, $E^\varepsilon[\rho^\varepsilon,q^\varepsilon]\in L^\infty((0,T))$, we can apply the if and only if relation between differential and integral form of energy inequality as stated in [18, Equation (7.1.27)-(7.1.28), Page 317]. Hence, we have established energy inequality (2.26) holds for almost every $t\in(0,T)$:

$$\begin{aligned}&E^\varepsilon[\rho^\varepsilon,q^\varepsilon] + \int_0^T\int_\Omega \Big(2\mu^\varepsilon|\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon|\operatorname{div} u^\varepsilon|^2\Big) + \delta\varepsilon\beta\int_0^T\int_\Omega (\rho^\varepsilon)^{\beta-2}|\nabla\rho^\varepsilon|^2 \\ &\quad + \alpha\sum_{i=1}^M \int_0^T \int_{\partial\mathcal{S}^\varepsilon_i(t)} |(u^\varepsilon - P^\varepsilon_{\mathcal{S}_i}u^\varepsilon)\times\nu|^2 + \frac{1}{\delta}\sum_{i=1}^M \int_0^T\int_\Omega \chi^\varepsilon_{\mathcal{S}_i}|u^\varepsilon - P^\delta_{\mathcal{S}_i}u^\varepsilon|^2 \leqslant \int_0^T\int_\Omega \rho^\varepsilon g^\varepsilon\cdot u^\varepsilon + E^\varepsilon_0,\end{aligned} \tag{3.33}$$

where

$$E^{\varepsilon}[\rho^{\varepsilon},q^{\varepsilon}]=\int\limits_{\Omega}\left(\frac{1}{2}\frac{|q^{\varepsilon}|^2}{\rho^{\varepsilon}}+\frac{a^{\varepsilon}}{\gamma-1}(\rho^{\varepsilon})^{\gamma}+\frac{\delta}{\beta-1}(\rho^{\varepsilon})^{\beta}\right).$$

We obtain as in [18, Equation (7.8.14), Page 363]:

$$\partial_t\rho^{\varepsilon},\ \Delta\rho^{\varepsilon}\in L^{\frac{5\beta-3}{4\beta}}((0,T)\times\Omega).$$

Regarding the $\sqrt{\varepsilon}\|\nabla\rho^{\varepsilon}\|_{L^2((0,T)\times\Omega)}$ estimate in (2.27), we have to multiply (2.17) by ρ^{ε} and integrate by parts to obtain

$$\frac{1}{2}\int\limits_{\Omega}|\rho^{\varepsilon}(t)|^2+\varepsilon\int\limits_0^T\int\limits_{\Omega}|\nabla\rho^{\varepsilon}(t)|^2=\frac{1}{2}\int\limits_{\Omega}|\rho_0^{\varepsilon}|^2-\frac{1}{2}\int\limits_0^T\int\limits_{\Omega}|\rho^{\varepsilon}|^2\operatorname{div}u^{\varepsilon}\leqslant\frac{1}{2}\int\limits_{\Omega}|\rho_0^{\varepsilon}|^2+\sqrt{T}\|\|\rho^{\varepsilon}\|^2_{L^{\infty}(0,T;L^4(\Omega))}\|\operatorname{div}u^{\varepsilon}\|_{L^2(0,T;L^2(\Omega))}.$$

Now, the pressure estimates $\|\rho^{\varepsilon}\|_{L^{\beta+1}((0,T)\times\Omega)}$ and $\|\rho^{\varepsilon}\|_{L^{\gamma+1}((0,T)\times\Omega)}$ in (2.27) can be derived by means of the test function $\phi(t,x)=\psi(t)\Phi(t,x)$ with $\Phi(t,x)=\mathcal{B}[\rho^{\varepsilon}-\overline{m}]$ in (2.18), where

$$\psi\in\mathcal{D}(0,T),\quad \overline{m}=|\Omega|^{-1}\int\limits_{\Omega}\rho^{\varepsilon},$$

and $\mathcal{B}$ is the Bogovskii operator related to Ω (for details about $\mathcal{B}$, see [18, Section 3.3, Page 165]). After taking this special test function and integrating by parts, we obtain

$$\begin{aligned}\int\limits_0^T\psi\int\limits_{\Omega}\left(a^{\varepsilon}(\rho^{\varepsilon})^{\gamma}+\delta(\rho^{\varepsilon})^{\beta}\right)\rho^{\varepsilon}&=\int\limits_0^T\psi\int\limits_{\Omega}\left(a^{\varepsilon}(\rho^{\varepsilon})^{\gamma}+\delta(\rho^{\varepsilon})^{\beta}\right)\overline{m}+\int\limits_0^T2\psi\int\limits_{\Omega}\mu^{\varepsilon}\mathbb{D}(u^{\varepsilon}):\mathbb{D}(\Phi)+\int\limits_0^T\psi\int\limits_{\Omega}\lambda^{\varepsilon}\rho^{\varepsilon}\operatorname{div}u^{\varepsilon}\\&-\overline{m}\int\limits_0^T\psi\int\limits_{\Omega}\lambda^{\varepsilon}\operatorname{div}u^{\varepsilon}+\int\limits_0^T\psi\int\limits_{\Omega}\varepsilon\nabla u^{\varepsilon}\nabla\rho^{\varepsilon}\cdot\Phi+\alpha\int\limits_0^T\psi\sum_{i=1}^{M}\int\limits_{\partial\mathcal{S}_i^{\varepsilon}(t)}[(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\times\nu^i]\cdot[(\Phi-P^{\varepsilon}_{\mathcal{S}_i}\Phi)\times\nu_i]\\&+\frac{1}{\delta}\int\limits_0^T\psi\int\limits_{\Omega}\sum_{i=1}^{M}\chi^{\varepsilon}_{\mathcal{S}_i}(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\cdot(\Phi-P^{\varepsilon}_{\mathcal{S}_i}\Phi)+\int\limits_0^T\psi\int\limits_{\Omega}\rho^{\varepsilon}g^{\varepsilon}\cdot\Phi.\end{aligned}\tag{3.34}$$

We see that all the terms can be estimated as in [18, Section 7.8.4, Pages 366–368] except the penalization term. Using Hölder's inequality and bounds from energy estimate (2.26), the penalization term can be dealt with in the following way

$$\int\limits_0^T\psi\int\limits_{\Omega}\chi^{\varepsilon}_{\mathcal{S}_i}(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})\cdot(\Phi-P^{\varepsilon}_{\mathcal{S}_i}\Phi)\leqslant|\psi|_{C[0,T]}\left(\int\limits_0^T\int\limits_{\Omega}\chi^{\varepsilon}_{\mathcal{S}_i}|(u^{\varepsilon}-P^{\varepsilon}_{\mathcal{S}_i}u^{\varepsilon})|^2\right)^{1/2}\|\Phi\|_{L^2((0,T)\times\Omega)}\leqslant C|\psi|_{C[0,T]},\tag{3.35}$$

where in the last inequality we have used $\|\Phi\|_{L^2(\Omega)}\leqslant c\|\rho^{\varepsilon}\|_{L^2(\Omega)}$ and the energy inequality (2.26). Thus, we have an improved regularity of the density and we have established the required estimates of (2.27).

The only remaining thing is to check the following fact: there exists T small enough such that if

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t),\mathcal{S}_{0j}(t))>2\sigma,\quad i\neq j,\quad i,j=1,...,M,$$

then

$$\min(\operatorname{dist}(\mathcal{S}^{\varepsilon}_i(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}^{\varepsilon}_i(t),\mathcal{S}^{\varepsilon}_j(t)))\geqslant2\sigma,\quad\forall\ t\in[0,T],\quad i\neq j,\quad i,j=1,...,M,\tag{3.36}$$

It is equivalent to establishing the following bound:

$$\sup_{t\in[0,T]}|\partial_t\eta^{\varepsilon,i}_{t,0}(t,y)|<\frac{\min\ (\operatorname{dist}(\mathcal{S}_0,\partial\Omega),\operatorname{dist}(\mathcal{S}_{0i},\mathcal{S}_{0j}))-2\sigma}{T},\quad y\in\mathcal{S}_0.\tag{3.37}$$

We show as in Step 3 of the proof of Proposition 2.3 that (see (3.12)–(3.15)):

$$|\partial_t \eta^{\varepsilon,i}_{t,0}(t,y)| \leqslant |(h^\varepsilon_i)'(t)| + |\omega^\varepsilon_i(t)||y-h^\varepsilon_i(t)| \leqslant C_0 \left(\int_\Omega \rho^\varepsilon |u^\varepsilon(t)|^2\right)^{1/2}, \tag{3.38}$$

where $C_0 = \sqrt{2}\frac{\max\{1,|y-h(t)|\}}{\min\{1,\lambda_0\}^{1/2}}$. Moreover, the energy estimate (3.33) yields

$$\frac{d}{dt}E^\varepsilon[\rho^\varepsilon,q^\varepsilon] + \int_\Omega \left(2\mu^\varepsilon|\mathbb{D}(u^\varepsilon)|^2 + \lambda^\varepsilon|\operatorname{div} u^\varepsilon|^2\right) \leqslant \int_\Omega \rho^\varepsilon g^\varepsilon\cdot u^\varepsilon \leqslant E^\varepsilon[\rho^\varepsilon,q^\varepsilon] + \frac{1}{2\gamma_1}\left(\frac{\gamma-1}{2\gamma}\right)^{\gamma_1/\gamma} \|g^\varepsilon\|^{2\gamma_1}_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)},$$

with $\gamma_1 = 1-\frac{1}{\gamma}$, which implies

$$E^\varepsilon[\rho^\varepsilon,q^\varepsilon] \leqslant e^T E^\varepsilon_0 + CT\|g^\varepsilon\|^{2\gamma_1}_{L^\infty((0,T)\times\Omega)}. \tag{3.39}$$

Thus, with the help of (3.37) and (3.38)–(3.39), we can conclude that for any T satisfying

$$T < \frac{\min\ (\operatorname{dist}(\mathcal{S}_0,\partial\Omega), \operatorname{dist}(\mathcal{S}_{0i},\mathcal{S}_{0j})) - 2\sigma}{C_0\left[e^T E^\varepsilon_0 + CT\|g^\varepsilon\|^{2\gamma_1}_{L^\infty((0,T)\times\Omega)}\right]^{1/2}}, \tag{3.40}$$

the relation (3.36) holds. This completes the proof of Proposition 2.2. □

Proof of Proposition 2.1. First of all, the initial data $(\rho^\varepsilon_0, q^\varepsilon_0)$ is constructed in such a way that

$$\rho^\varepsilon_0 > 0, \quad \rho^\varepsilon_0 \in W^{1,\infty}(\Omega), \quad \rho^\varepsilon_0 \to \rho^\delta_0 \text{ in } L^\beta(\Omega), \quad q^\varepsilon_0 \to q^\delta_0 \text{ in } L^{\frac{2\beta}{\beta+1}}(\Omega)$$

and

$$\int_\Omega \left(\frac{|q^\varepsilon_0|^2}{\rho^\varepsilon_0}\mathbb{1}_{\{\rho^\varepsilon_0>0\}} + \frac{a}{\gamma-1}(\rho^\varepsilon_0)^\gamma + \frac{\delta}{\beta-1}(\rho^\varepsilon_0)^\beta\right) \to \int_\Omega \left(\frac{|q^\delta_0|^2}{\rho^\delta_0}\mathbb{1}_{\{\rho^\delta_0>0\}} + \frac{a}{\gamma-1}(\rho^\delta_0)^\gamma + \frac{\delta}{\beta-1}(\rho^\delta_0)^\beta\right) \text{ as } \varepsilon\to 0.$$

As in [16, Section 4.3] the proof can be similarly done in the following steps: limits of the transport equations, the continuity equation, the momentum equation, limit of the pressure term, the energy inequality. Also we need to take care of the fact that *the rigid bodies are away from the boundary of the domain and they are not touching themselves.* To do so, we follow the same idea as in the proof of Proposition 2.2 (precisely, the calculations in (3.37)–(3.40)) to conclude that there exists T small enough such that if

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t),\mathcal{S}_{0j}(t)) > 2\sigma, \quad i\neq j, \quad i,j=1,...,M,$$

then

$$\min(\operatorname{dist}(\mathcal{S}^\delta_i(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}^\delta_i(t),\mathcal{S}^\delta_j(t))) \geqslant 2\sigma, \quad \forall\, t\in[0,T], \quad i\neq j, \quad i,j=1,...,M. \tag{3.41}$$

□

4. Proof of Theorem 1.2

Proof of Theorem 1.2. The existence of a weak solution $(\mathcal{S}^\delta,\rho^\delta,u^\delta)$ to system (2.1)–(2.6) has already been established in Proposition 2.1. We study the limiting behaviour of the solution as $\delta\to 0$ and recover a weak solution to system (1.8)–(1.16). We mention the main steps to give an outline of the proof. For details, please see [16, Section 5].

Firstly, it is necessary to construct the approximate initial data $(\rho^\delta_0, q^\delta_0)$ satisfying (2.11)–(2.12) so that, in the limit $\delta\to 0$, we can recover the initial data $\rho_{\mathcal{F}_0}$, $q_{\mathcal{F}_0}$, $\rho_{\mathcal{S}_{0i}}$, $u_{\mathcal{S}_{0i}}$ satisfying the conditions (1.27)–(1.29). Next, to analyze the behaviour of the velocity field in the fluid part, we introduce the following continuous extension operator:

$$\mathcal{E}^\delta_u(t) : \left\{u\in H^1_0(\mathcal{F}^\delta(t)),\ u=0 \text{ on } \partial\Omega\right\} \to H^1_0(\Omega). \tag{4.1}$$

We set

$$u^\delta_{\mathcal{F}}(t,\cdot) = \mathcal{E}^\delta_u(t)\left[u^\delta(t,\cdot)|_{\mathcal{F}^\delta}\right], \tag{4.2}$$

such that

$$\{u^\delta_{\mathcal{F}}\} \text{ is bounded in } L^2(0,T;H^1_0(\Omega)), \quad u^\delta_{\mathcal{F}} = u^\delta \text{ on } \mathcal{F}^\delta, \text{ i.e. } \sum_{i=1}^M (1-\chi^\delta_{\mathcal{S}_i})(u^\delta - u^\delta_{\mathcal{F}}) = 0. \tag{4.3}$$

Thus, the strong convergence of $\chi^\delta_{\mathcal{S}_i}$ and the weak convergence of $u^\delta_{\mathcal{F}} \to u_{\mathcal{F}}$ in $L^2(0,T;H^1_0(\Omega))$ yield

$$\sum_{i=1}^{M}(1-\chi_{\mathcal{S}_i})\,(u-u_{\mathcal{F}})=0. \tag{4.4}$$

We use the strong convergence of density, weak convergence of velocity, convergence of $\rho^\delta\chi^\delta_{\mathcal{S}_i}$ to analyze the limiting behaviour of the continuity equations as $\delta\to 0$ and we obtain the transport equations (1.20), (1.24). The method of an effective viscous flux and boundedness of oscillations of the density sequence help us to establish the renormalized continuity equation (1.21).

It remains to check that if

$$\min(\operatorname{dist}(\mathcal{S}_{0i}(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_{0i}(t),\mathcal{S}_{0j}(t))>2\sigma,\quad i\neq j,\quad i,j=1,...,M,$$

then

$$\min(\operatorname{dist}(\mathcal{S}_i(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}_i(t),\mathcal{S}_j(t)))\geqslant\frac{3\sigma}{2}>0,\quad\forall\,t\in[0,T],\quad i\neq j,\quad i,j=1,...,M. \tag{4.5}$$

Let us introduce the following notation:

$$(\mathcal{U})_\sigma=\left\{x\in\mathbb{R}^3\mid \operatorname{dist}(x,\mathcal{U})<\sigma\right\},$$

for an open set $\mathcal{U}$ and $\sigma>0$. We recall the following result [8, Lemma 5.4]: Let $\sigma>0$. There exists $\delta_0>0$ such that for all $0<\delta\leqslant\delta_0$,

$$\mathcal{S}^\delta_i(t)\subset(\mathcal{S}_i(t))_{\sigma/4}\subset(\mathcal{S}^\delta_i(t))_{\sigma/2},\quad\forall\,t\in[0,T],\,i=1,...,M, \tag{4.6}$$

Note that condition (4.6) and the relation (3.41), i.e.,

$$\min(\operatorname{dist}(\mathcal{S}^\delta_i(t),\partial\Omega),\ \operatorname{dist}(\mathcal{S}^\delta_i(t),\mathcal{S}^\delta_j(t)))\geqslant 2\sigma>0,\quad\forall\,t\in[0,T],\quad i\neq j,\quad i,j=1,...,M,$$

imply estimate (4.5). Thus we can conclude Theorem 1.2.

□

Proposition 4.1. *Let $\phi\in V_T$ and $\vartheta>0$. Then there exists a sequence*

$$\phi^\delta\in H^1(0,T;L^2(\Omega))\cap L^r(0,T;W^{1,r}_0(\Omega)),\ \textit{where } r=\max\left\{\beta+1,\frac{\beta+\theta}{\theta}\right\},\ \beta\geqslant\max\{8,\gamma\}\ \textit{and } \theta=\frac{2}{3}\gamma-1$$

of the form

$$\phi^\delta=\sum_{i=1}^{M}(1-\chi^\delta_{\mathcal{S}_i})\phi_{\mathcal{F}}+\chi^\delta_{\mathcal{S}_i}\phi^\delta_{\mathcal{S}_i} \tag{4.7}$$

that satisfies for all $p\in[1,\infty)$:

(1) $\|\chi^\delta_{\mathcal{S}_i}(\phi^\delta_{\mathcal{S}_i}-\phi_{\mathcal{S}_i})\|_{L^p((0,T)\times\Omega))}=\mathcal{O}(\delta^{\vartheta/p})$,
(2) $\phi^\delta\to\phi$ *strongly in* $L^p((0,T)\times\Omega)$,
(3) $\|\phi^\delta\|_{L^p(0,T;W^{1,p}(\Omega))}=\mathcal{O}(\delta^{-\vartheta(1-1/p)})$,
(4) $\|\chi^\delta_{\mathcal{S}_i}(\partial_t+P^\delta_{\mathcal{S}_i}u^\delta\cdot\nabla)(\phi^\delta-\phi_{\mathcal{S}_i})\|_{L^2(0,T;L^p(\Omega))}=\mathcal{O}(\delta^{\vartheta/p})$,
(5) $(\partial_t+P^\delta_{\mathcal{S}_i}u^\delta\cdot\nabla)\phi^\delta\to(\partial_t+P_{\mathcal{S}_i}u\cdot\nabla)\phi$ *weakly in* $L^2(0,T;L^p(\Omega))$.

Proof. The idea is to write the test functions in Lagrangian coordinates through the isometric propagator $\eta^\delta_{t,s}$ so that we can work on the fixed domain. Let $\Phi_{\mathcal{F}}$, $\Phi_{\mathcal{S}_i}$ and $\Phi^\delta_{\mathcal{S}_i}$ be the transformed quantities in the fixed domain related to $\phi_{\mathcal{F}}$, $\phi_{\mathcal{S}_i}$ and $\phi^\delta_{\mathcal{S}_i}$ respectively:

$$\phi_{\mathcal{S}_i}(t,\eta^{\delta,i}_{t,0}(y))=J_{\eta^{\delta,i}_{t,0}}\Big|_y(\Phi_{\mathcal{S}_i}(t,y)),\quad \phi_{\mathcal{F}}(t,\eta^{\delta,i}_{t,0}(y))=J_{\eta^{\delta,i}_{t,0}}\Big|_y\Phi_{\mathcal{F}}(t,y)\quad\text{and}\quad \phi^\delta_{\mathcal{S}_i}(t,\eta^{\delta,i}_{t,0}(y))=J_{\eta^{\delta,i}_{t,0}}\Big|_y\Phi^\delta_{\mathcal{S}_i}(t,y), \tag{4.8}$$

where $J_{\eta^{\delta,i}_{t,0}}$ is the Jacobian matrix of $\eta^{\delta,i}_{t,0}$. Note that if we define

$$\Phi^{\delta,i}(t,y)=(1-\chi^\delta_{\mathcal{S}_i})\Phi_{\mathcal{F}}+\chi^\delta_{\mathcal{S}_i}\Phi^\delta_{\mathcal{S}_i},$$

then the definition of ϕ^δ in (4.7) gives $\phi^\delta=\sum_{i=1}^{M}\phi^{\delta,i}$, where

$$\phi^{\delta,i}(t,\eta^{\delta,i}_{t,0}(y))=J_{\eta^{\delta,i}_{t,0}}\Big|_y(\Phi^{\delta,i}(t,y)). \tag{4.9}$$

Thus, the construction of the approximation $\phi^\delta_{\mathcal{S}_i}$ satisfying

$$\phi^\delta_{\mathcal{S}_i}(t,x) = \phi_{\mathcal{F}}(t,x) \quad \forall\, t\in(0,T),\ x\in\partial\mathcal{S}^\delta_i(t), \tag{4.10}$$

and

$$\phi^\delta_{\mathcal{S}_i}(t,\cdot) \approx \phi_{\mathcal{S}_i}(t,\cdot) \text{ in } \mathcal{S}^\delta_i(t) \text{ away from a } \delta^\vartheta \text{ neighborhood of } \partial\mathcal{S}^\delta_i(t) \text{ with } \vartheta>0, \tag{4.11}$$

is equivalent to building the approximation $\Phi^\delta_{\mathcal{S}_i}$ so that there is no jump for the function Φ^δ at the interface and the following holds:

$$\Phi^\delta_{\mathcal{S}_i}(t,x) = \Phi_{\mathcal{F}}(t,x) \quad \forall\, t\in(0,T),\ x\in\partial\mathcal{S}_{0i},$$

and

$$\Phi^\delta_{\mathcal{S}_i}(t,\cdot) \approx \Phi_{\mathcal{S}_i}(t,\cdot) \text{ in } \mathcal{S}_{0i} \text{ away from a } \delta^\vartheta \text{ neighborhood of } \partial\mathcal{S}_{0i} \text{ with } \vartheta>0.$$

Explicitly, we set (for details, see [8, Pages 2055-2058]):

$$\Phi^\delta_{\mathcal{S}_i} = \Phi^\delta_{\mathcal{S}_i,1} + \Phi^\delta_{\mathcal{S}_i,2}, \tag{4.12}$$

with

$$\Phi^\delta_{\mathcal{S}_i,1} = \Phi_{\mathcal{S}_i} + \chi(\delta^{-\vartheta}z)\left[(\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i}) - ((\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i})\cdot e_z)e_z\right], \tag{4.13}$$

where $\chi:\mathbb{R}\to[0,1]$ is a smooth truncation function which is equal to 1 in a neighborhood of 0 and z is a coordinate transverse to the boundary $\partial\mathcal{S}_{0i}=\{z=0\}$. Moreover, to make $\Phi^\delta_{\mathcal{S}_i}$ divergence-free in $\mathcal{S}_{0i}$, we need to take $\Phi^\delta_{\mathcal{S}_i,2}$ such that

$$\operatorname{div}\Phi^\delta_{\mathcal{S}_i,2} = -\operatorname{div}\Phi^\delta_{\mathcal{S}_i,1} \quad \text{in } \mathcal{S}_{0i}, \quad \Phi^\delta_{\mathcal{S}_i,2}=0 \quad \text{on } \partial\mathcal{S}_{0i}.$$

Observe that the explicit form (4.13) of $\Phi^\delta_{\mathcal{S}_i,1}$ yields

$$\operatorname{div}\Phi^\delta_{\mathcal{S}_i,2} = -\operatorname{div}\Phi^\delta_{\mathcal{S}_i,1} = -\chi(\delta^{-\vartheta}z)\operatorname{div}\left[(\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i}) - ((\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i})\cdot e_z)e_z\right]. \tag{4.14}$$

Thus, the expressions (4.13)–(4.14) give us: for all $p<\infty$,

$$\|\Phi^\delta_{\mathcal{S}_i,1}-\Phi_{\mathcal{S}_i}\|_{H^1(0,T;L^p(\mathcal{S}_{0i}))} \leqslant C\delta^{\vartheta/p}, \tag{4.15}$$

$$\|\Phi^\delta_{\mathcal{S}_i,1}-\Phi_{\mathcal{S}_i}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_{0i}))} \leqslant C\delta^{-\vartheta(1-1/p)}, \tag{4.16}$$

and

$$\|\Phi^\delta_{\mathcal{S}_i,2}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_{0i}))} \leqslant C\|\chi(\delta^{-\vartheta}z)\operatorname{div}\left[(\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i}) - ((\Phi_{\mathcal{F}}-\Phi_{\mathcal{S}_i})\cdot e_z)e_z\right]\|_{H^1(0,T;L^p(\mathcal{S}_{0i}))} \leqslant C\delta^{\vartheta/p}. \tag{4.17}$$

Using the decomposition (4.12) of $\Phi^\delta_{\mathcal{S}_i}$ and the estimates (4.15)–(4.16), (4.17), we obtain

$$\|\Phi^\delta_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i}\|_{H^1(0,T;L^p(\mathcal{S}_{0i}))} \leqslant C\delta^{\vartheta/p},$$

$$\|\Phi^\delta_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i}\|_{H^1(0,T;W^{1,p}(\mathcal{S}_{0i}))} \leqslant C\delta^{-\vartheta(1-1/p)}.$$

Furthermore, we combine the above estimates with the uniform bound of the propagator $\eta^{\delta,i}_{t,0}$ in $H^1(0,T;C^\infty(\Omega))$ to obtain

$$\left\|J_{\eta^{\delta,i}_{t,0}}|_y(\Phi^\delta_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i})\right\|_{H^1(0,T;L^p(\mathcal{S}_{0i}))} \leqslant C\delta^{\vartheta/p}, \tag{4.18}$$

$$\left\|J_{\eta^{\delta,i}_{t,0}}|_y(\Phi^\delta_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i})\right\|_{H^1(0,T;W^{1,p}(\mathcal{S}_{0i}))} \leqslant C\delta^{-\vartheta(1-1/p)}. \tag{4.19}$$

Observe that due to the change of variables (4.8) and estimate (4.18):

$$\|\chi^\delta_{\mathcal{S}_i}(\phi^\delta_{\mathcal{S}_i}-\phi_{\mathcal{S}_i})\|_{L^p((0,T)\times\Omega))} \leqslant C\|J_{\eta^{\delta,i}_{t,0}}|_y(\Phi^\delta_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i})\|_{L^p((0,T)\times\mathcal{S}_{0i})} \leqslant C\delta^{\vartheta/p}. \tag{4.20}$$

As $\phi\in V_T$, we have $\phi=\sum_{i=1}^M (1-\chi_{\mathcal{S}_i})\phi_{\mathcal{F}} + \chi_{\mathcal{S}_i}\phi_{\mathcal{S}_i}$. We can estimate

$$\|\phi^\delta-\phi\|_{L^p((0,T)\times\Omega))} \leqslant \sum_{i=1}^M \left(\|(\chi^\delta_{\mathcal{S}_i}-\chi_{\mathcal{S}_i})\phi_{\mathcal{F}}\|_{L^p((0,T)\times\Omega))} + \|\chi^\delta_{\mathcal{S}_i}(\phi^\delta_{\mathcal{S}_i}-\phi_{\mathcal{S}_i})\|_{L^p((0,T)\times\Omega))} + \|(\chi^\delta_{\mathcal{S}_i}-\chi_{\mathcal{S}_i})\phi_{\mathcal{S}_i}\|_{L^p((0,T)\times\Omega))}\right).$$

We use the strong convergence of $\chi^\delta_{\mathcal{S}_i}$ and the estimate (4.20) to conclude that

$$\phi^\delta \to \phi \text{ strongly in } L^p((0,T)\times\Omega).$$

We use estimate (4.16) and the relation (4.9) to obtain

$$\|\phi^{\delta}\|_{L^p(0,T;W^{1,p}(\Omega))} \leqslant \delta^{-\vartheta(1-1/p)}.$$

Moreover, the change of variables (4.8) and estimate (4.18) give

$$\begin{aligned}\|\chi^{\delta}_{\mathcal{S}_i}(\partial_t + P^{\delta}_{\mathcal{S}_i}u^{\delta}\cdot\nabla)(\phi^{\delta}-\phi_{\mathcal{S}_i})\|_{L^2(0,T;L^p(\Omega))} &\leqslant C\left\|\frac{d}{dt}\left(J_{\eta^{\delta,i}_{t,0}}\Big|_y(\Phi^{\delta}_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i})\right)\right\|_{L^2(0,T;L^p(\mathcal{S}_{0i}))} \\ &\leqslant C\left\|J_{\eta^{\delta,i}_{t,0}}|_y(\Phi^{\delta}_{\mathcal{S}_i}-\Phi_{\mathcal{S}_i})\right\|_{H^1(0,T;L^p(\mathcal{S}_{0i}))} \leqslant C\delta^{\vartheta/p}.\end{aligned} \tag{4.21}$$

The above estimate (4.21), strong convergence of $\chi^{\delta}_{\mathcal{S}_i}$ to $\chi_{\mathcal{S}_i}$ in $C(0,T;L^p(\Omega))$ and weak convergence of $P^{\delta}_{\mathcal{S}_i}u^{\delta}$ to $P_{\mathcal{S}_i}u$ weakly in $L^2(0,T;C^{\infty}_{loc}(\mathbb{R}^3))$, give us

$$(\partial_t + P^{\delta}_{\mathcal{S}_i}u^{\delta}\cdot\nabla)\phi^{\delta} \to (\partial_t + P_{\mathcal{S}_i}u\cdot\nabla)\phi \text{ weakly in } L^2(0,T;L^p(\Omega)),$$

where

$$\phi^{\delta} = \sum_{i=1}^{M}(1-\chi^{\delta}_{\mathcal{S}_i})\phi_{\mathcal{F}} + \chi^{\delta}_{\mathcal{S}_i}\phi^{\delta}_{\mathcal{S}_i} \quad \text{and} \quad \phi = \sum_{i=1}^{M}(1-\chi_{\mathcal{S}_i})\phi_{\mathcal{F}} + \chi_{\mathcal{S}_i}\phi_{\mathcal{S}_i}.$$

□

5. Appendix

In this section, we state some results regarding the transport equation that we use in our analysis. The proofs follow directly from [16, Section 3]. We will consider the following equation:

$$\frac{\partial\chi_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(P_{\mathcal{S}_i}u\chi_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \chi_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_0} \text{ in } \mathbb{R}^3, \tag{5.1}$$

where $P_{\mathcal{S}_i}u \in \mathcal{R}(\Omega)$. Note that here $\mathcal{R}(\Omega)$ is referring to the set of rigid fields on $\mathbb{R}^3$ in the spirit of (1.17). It is given by

$$P_{\mathcal{S}_i}u(t,x) = \frac{1}{m}\int_{\Omega}\rho\chi_{\mathcal{S}_i}u + \left(J^{-1}\int_{\Omega}\rho\chi_{\mathcal{S}_i}((y-h_i(t))\times u)\,dy\right)\times(x-h_i(t)), \quad \forall\,(t,x)\in(0,T)\times\mathbb{R}^3. \tag{5.2}$$

Proposition 5.1. *Let $u \in C([0,T];\mathcal{D}(\overline{\Omega}))$ and $\rho \in L^2(0,T;H^2(\Omega))\cap C([0,T];H^1(\Omega))$. Then the following holds true:*

(1) *There is a unique solution $\chi_{\mathcal{S}_i} \in L^{\infty}((0,T)\times\mathbb{R}^3)\cap C([0,T];L^p(\mathbb{R}^3))$ $\forall\, 1\leqslant p<\infty$ to (5.1). More precisely,*

$$\chi_{\mathcal{S}_i}(t,x) = \mathbb{1}_{\mathcal{S}_i(t)}(x), \quad \forall\, t\geqslant 0,\ \forall\, x\in\mathbb{R}^3.$$

If the isometric propagator $\eta^i_{t,s}$, associated to $P_{\mathcal{S}_i}u$ is defined by

$$\frac{\partial\eta^i_{t,s}}{\partial t}(y) = P_{\mathcal{S}_i}u(t,\eta^i_{t,s}(y)), \quad \forall\,(t,s,y)\in(0,T)^2\times\mathbb{R}^3, \quad \eta^i_{s,s}(y)=y, \quad \forall\, y\in\mathbb{R}^3, \tag{5.3}$$

then

$$(t,s)\mapsto\eta^i_{t,s}\in C^1([0,T]^2;C^{\infty}_{loc}(\mathbb{R}^3)).$$

Moreover, we also have $\mathcal{S}_i(t) = \eta^i_{t,0}(\mathcal{S}_{0i})$.

(2) *Let $\rho_0\mathbb{1}_{\mathcal{S}_{0i}} \in L^{\infty}(\mathbb{R}^3)$. Then there is a unique solution $\rho\chi_{\mathcal{S}_i} \in L^{\infty}((0,T)\times\mathbb{R}^3)\cap C([0,T];L^p(\mathbb{R}^3))$, $\forall\, 1\leqslant p<\infty$ to the following equation:*

$$\frac{\partial(\rho\chi_{\mathcal{S}_i})}{\partial t} + \operatorname{div}((\rho\chi_{\mathcal{S}_i})P_{\mathcal{S}_i}u) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \rho\chi_{\mathcal{S}_i}|_{t=0} = \rho_0\mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3. \tag{5.4}$$

Proposition 5.2. *Let $\rho_0^N \in W^{1,\infty}(\Omega)$, let $\rho^k \in L^2(0,T;H^2(\Omega))\cap C([0,T];H^1(\Omega))\cap H^1(0,T;L^2(\Omega))$ be the solution to*

$$\frac{\partial\rho^k}{\partial t} + \operatorname{div}(\rho^k u^k) = \Delta\rho^k \text{ in } (0,T)\times\Omega, \quad \frac{\partial\rho^k}{\partial\nu} = 0 \text{ on } \partial\Omega, \quad \rho^k(0,x) = \rho_0^N(x) \text{ in } \Omega, \quad \frac{\partial\rho_0^k}{\partial\nu}\Big|_{\partial\Omega} = 0. \tag{5.5}$$

$$u^k \to u \text{ strongly in } C([0,T];\mathcal{D}(\overline{\Omega})), \quad \chi^k_{\mathcal{S}_i} \text{ is bounded in } L^{\infty}((0,T)\times\mathbb{R}^3) \text{ satisfying}$$

$$\frac{\partial \chi^k_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(P^k_{\mathcal{S}_i} u^k \chi^k_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \chi^k_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.6}$$

and let $\{\rho^k \chi^k_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^\infty((0,T)\times\mathbb{R}^3)$ *satisfying*

$$\frac{\partial}{\partial t}(\rho^k \chi^k_{\mathcal{S}_i}) + \operatorname{div}(P^k_{\mathcal{S}_i} u^k (\rho^k \chi^k_{\mathcal{S}_i})) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \rho^k\chi^k_{\mathcal{S}_i}|_{t=0} = \rho_0^N \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.7}$$

where $P^k_{\mathcal{S}_i} : L^2(\Omega) \to L^2(\mathcal{S}^k_i(t))$ *is the orthogonal projection to rigid fields with* $\mathcal{S}^k_i(t) \Subset \Omega$ *being a bounded, regular domain for all* $t\in[0,T]$. *Then*

$$\chi^k_{\mathcal{S}_i} \to \chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3)),\ \forall\ 1\leqslant p<\infty,$$
$$\rho^k\chi^k_{\mathcal{S}_i} \to \rho\chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3)),\ \forall\ 1\leqslant p<\infty,$$

where $\chi_{\mathcal{S}_i}$ *and* $\rho\chi_{\mathcal{S}_i}$ *satisfy* (5.1) *and* (5.4) *with initial data* $\mathbb{1}_{\mathcal{S}_{0i}}$ *and* $\rho_0^N\mathbb{1}_{\mathcal{S}_{0i}}$, *respectively. Moreover,*

$$P^k_{\mathcal{S}_i}u^k \to P_{\mathcal{S}_i}u \text{ strongly in } C([0,T];C^\infty_{loc}(\mathbb{R}^3)),$$
$$\eta^{k,i}_{t,s} \to \eta^i_{t,s} \text{ strongly in } C^1([0,T]^2;C^\infty_{loc}(\mathbb{R}^3)).$$

Proposition 5.3. *Let us assume that* $\rho_0^N \in W^{1,\infty}(\Omega)$ *with* $\rho_0^N \to \rho_0$ *in* $W^{1,\infty}(\Omega)$, ρ^N *satisfies* (5.5) *and*

$$\rho^N \to \rho \text{ strongly in } L^p((0,T)\times\Omega),\ 1\leqslant p<\frac{4}{3}\beta \text{ with } \beta\geqslant\max\{8,\gamma\},\ \gamma>3/2.$$

Let $\{u^N, \chi^N_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^\infty(0,T;L^2(\Omega))\times L^\infty((0,T)\times\mathbb{R}^3)$ *satisfying* (5.6). *Let* $\{\rho^N\chi^N_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^\infty((0,T)\times\mathbb{R}^3)$ *satisfying* (5.7). *Then, up to a subsequence, we have*

$$u^N \to u \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)),$$
$$\chi^N_{\mathcal{S}_i} \to \chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3)),\ \forall\ 1\leqslant p<\infty,$$
$$\rho^N\chi^N_{\mathcal{S}_i} \to \rho\chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3)),\ \forall\ 1\leqslant p<\infty,$$

where $\chi_{\mathcal{S}_i}$ *and* $\rho\chi_{\mathcal{S}_i}$ *satisfy* (5.1) *and* (5.4), *respectively. Moreover,*

$$P^N_{\mathcal{S}_i}u^N \to P_{\mathcal{S}_i}u \text{ weakly-}* \text{ in } L^\infty(0,T;C^\infty_{loc}(\mathbb{R}^3)),$$
$$\eta^{N,i}_{t,s} \to \eta^i_{t,s} \text{ weakly-}* \text{ in } W^{1,\infty}((0,T)^2;C^\infty_{loc}(\mathbb{R}^3)).$$

Proposition 5.4. *Let* $\rho_0^\varepsilon \in W^{1,\infty}(\Omega)$ *with* $\rho_0^\varepsilon\to\rho_0$ *in* $L^\beta(\Omega)$, ρ^ε *satisfies*

$$\frac{\partial\rho^\varepsilon}{\partial t} + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = \Delta\rho^\varepsilon \text{ in } (0,T)\times\Omega, \quad \frac{\partial\rho^\varepsilon}{\partial\nu} = 0 \text{ on } \partial\Omega, \quad \rho^\varepsilon(0,x) = \rho_0^\varepsilon(x) \text{ in } \Omega, \quad \frac{\partial\rho_0^\varepsilon}{\partial\nu}|_{\partial\Omega} = 0.,$$

and

$$\rho^\varepsilon \to \rho \text{ weakly in } L^{\beta+1}((0,T)\times\Omega), \text{ with } \beta\geqslant\max\{8,\gamma\},\ \gamma>3/2. \tag{5.8}$$

Let $\{u^\varepsilon,\chi^\varepsilon_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^2(0,T;H^1(\Omega))\times L^\infty((0,T)\times\mathbb{R}^3)$ *satisfying*

$$\frac{\partial \chi^\varepsilon_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(P^\varepsilon_{\mathcal{S}_i} u^\varepsilon \chi^\varepsilon_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \chi^\varepsilon_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.9}$$

and let $\{\rho^\varepsilon\chi^\varepsilon_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^\infty((0,T)\times\mathbb{R}^3)$ *satisfying*

$$\frac{\partial}{\partial t}(\rho^\varepsilon \chi^\varepsilon_{\mathcal{S}_i}) + \operatorname{div}(P^\varepsilon_{\mathcal{S}_i} u^\varepsilon (\rho^\varepsilon \chi^\varepsilon_{\mathcal{S}_i})) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \rho^\varepsilon\chi^\varepsilon_{\mathcal{S}_i}|_{t=0} = \rho_0^\varepsilon \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.10}$$

where $P^\varepsilon_{\mathcal{S}_i} : L^2(\Omega)\to L^2(\mathcal{S}^\varepsilon_i(t))$ *is the orthogonal projection onto rigid fields with* $\mathcal{S}^\varepsilon_i(t)\Subset\Omega$ *being a bounded, regular domain for all* $t\in[0,T]$. *Then up to a subsequence, we have*

$$u^\varepsilon \to u \text{ weakly in } L^2(0,T;H^1(\Omega)),$$
$$\chi^\varepsilon_{\mathcal{S}_i} \to \chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ (1\leqslant p<\infty),$$
$$\rho^\varepsilon\chi^\varepsilon_{\mathcal{S}_i} \to \rho\chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^\infty((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ (1\leqslant p<\infty),$$

with $\chi_{\mathcal{S}_i}$ *and* $\rho\chi_{\mathcal{S}_i}$ *satisfying* (5.1) *and* (5.4), *respectively. Moreover,*

$$\begin{aligned} &P^{\varepsilon}_{\mathcal{S}_i} u^{\varepsilon} \to P_{\mathcal{S}_i} u \text{ weakly in } L^2(0,T;C^{\infty}_{loc}(\mathbb{R}^3)),\\ &\eta^{\varepsilon,i}_{t,s} \to \eta^i_{t,s} \text{ weakly in } H^1((0,T)^2;C^{\infty}_{loc}(\mathbb{R}^3)). \end{aligned}$$

Proposition 5.5. *Let* $\rho^{\delta}_0 \in L^{\beta}(\Omega)$ *with* $\rho^{\delta}_0 \to \rho_0$ *in* $L^{\gamma}(\Omega)$, *let* ρ^{δ} *satisfy*

$$\frac{\partial \rho^{\delta}}{\partial t} + \operatorname{div}(\rho^{\delta} u^{\delta}) = 0 \text{ in } (0,T)\times\Omega, \quad \rho^{\delta}(0,x) = \rho^{\delta}_0(x) \text{ in } \Omega,$$

and

$$\rho^{\delta} \to \rho \text{ weakly in } L^{\gamma+\theta}((0,T)\times\Omega), \text{ with } \gamma > 3/2,\ \theta = \frac{2}{3}\gamma - 1. \tag{5.11}$$

Let $\{u^{\delta}, \chi^{\delta}_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^2(0,T;L^2(\Omega)) \times L^{\infty}((0,T)\times\mathbb{R}^3)$ *satisfying*

$$\frac{\partial \chi^{\delta}_{\mathcal{S}_i}}{\partial t} + \operatorname{div}(P^{\delta}_{\mathcal{S}_i} u^{\delta} \chi^{\delta}_{\mathcal{S}_i}) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \chi^{\delta}_{\mathcal{S}_i}|_{t=0} = \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.12}$$

and let $\{\rho^{\delta}\chi^{\delta}_{\mathcal{S}_i}\}$ *be a bounded sequence in* $L^{\infty}((0,T)\times\mathbb{R}^3)$ *satisfying*

$$\frac{\partial}{\partial t}(\rho^{\delta}\chi^{\delta}_{\mathcal{S}_i}) + \operatorname{div}(P^{\delta}_{\mathcal{S}_i} u^{\delta}(\rho^{\delta}\chi^{\delta}_{\mathcal{S}_i})) = 0 \text{ in } (0,T)\times\mathbb{R}^3, \quad \rho^{\delta}\chi^{\delta}_{\mathcal{S}_i}|_{t=0} = \rho^{\delta}_0 \mathbb{1}_{\mathcal{S}_{0i}} \text{ in } \mathbb{R}^3, \tag{5.13}$$

where $P^{\delta}_{\mathcal{S}_i} : L^2(\Omega) \to L^2(\mathcal{S}^{\delta}_i(t))$ *is the orthogonal projection onto rigid fields with* $\mathcal{S}^{\delta}_i(t) \Subset \Omega$ *being a bounded, regular domain for all* $t \in [0,T]$. *Then, up to a subsequence, we have*

$$\begin{aligned} &u^{\delta} \to u \text{ weakly in } L^2(0,T;L^2(\Omega)),\\ &\chi^{\delta}_{\mathcal{S}_i} \to \chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^{\infty}((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ (1 \leqslant p < \infty),\\ &\rho^{\delta}\chi^{\delta}_{\mathcal{S}_i} \to \rho\chi_{\mathcal{S}_i} \text{ weakly-}* \text{ in } L^{\infty}((0,T)\times\mathbb{R}^3) \text{ and strongly in } C([0,T];L^p_{loc}(\mathbb{R}^3))\ (1 \leqslant p < \infty), \end{aligned}$$

with $\chi_{\mathcal{S}_i}$ *and* $\rho\chi_{\mathcal{S}_i}$ *satisfying* (5.1) *and* (5.4), *respectively. Moreover,*

$$\begin{aligned} &P^{\delta}_{\mathcal{S}_i} u^{\delta} \to P_{\mathcal{S}_i} u \text{ weakly in } L^2(0,T;C^{\infty}_{loc}(\mathbb{R}^3)),\\ &\eta^{\delta,i}_{t,s} \to \eta^i_{t,s} \text{ weakly in } H^1((0,T)^2;C^{\infty}_{loc}(\mathbb{R}^3)). \end{aligned}$$

Acknowledgment

Š. N. has been supported by Praemium Academiæ of Š. Nečasová and by the Czech Science Foundation (GAČR) project 22-01591S. M.R. acknowledges the financial support from National Academy of Sciences, India(NASI). A.R. has been supported by the Basque Government through the BERC 2022-2025 program and by the Spanish State Research Agency through BCAM Severo Ochoa excellence accreditation SEV-2017-0718 and through project PID2020-114189RB-I00 funded by Agencia Estatal de Investigación (PID2020-114189RB-I00 / AEI / 10.13039/501100011033).

References

[1] M. Boulakia and S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 26 (2009), pp. 777–813.
[2] N. Chemetov and Š. Nečasová, *The motion of the rigid body in the viscous fluid including collisions. Global solvability result*, Nonlinear Anal. Real World Appl., 34 (2017), pp. 416–445.
[3] B. Desjardins and M. J. Esteban, *On weak solutions for fluid-rigid structure interaction: Compressible and incompressible model*, Commun. Partial Differential Equations, 25 (2000), pp. 1399–1413.
[4] E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid*, Arch. Rational Mech. Anal., 167 (2003), pp. 281–308.
[5] ———, *Dynamics of viscous compressible fluids*, Oxford University Press, Oxford, 2004.
[6] G. P. Galdi, *On the motion of a rigid body in a viscous liquid: a mathematical analysis with applications*, in Handbook of mathematical fluid dynamics, Vol. I, North-Holland, Amsterdam, 2002, pp. 653–791.
[7] D. Gérard-Varet and M. Hillairet, *Regularity issues in the problem of fluid structure interaction*, Archive for Rational Mechanical Analysis, 2010 (195), pp. 375–407.
[8] D. Gérard-Varet and M. Hillairet, *Existence of weak solutions up to collision for viscous fluid-solid systems with slip*, Comm. Pure Appl. Math., 67 (2014), pp. 2022–2075.
[9] D. Gérard-Varet, M. Hillairet, and C. Wang, *The influence of boundary conditions on the contact problem in a 3d navier-stokes flow*, J. Math. Pures Appl., 103 (2015), pp. 1–38.

[10] B. H. Haak, D. Maity, T. Takahashi, and M. Tucsnak, *Mathematical analysis of the motion of a rigid body in a compressible Navier-Stokes-Fourier fluid*, Math. Nachr., 292 (2019), pp. 1972–2017.
[11] T. Hesla, *Collision of smooth bodies in a viscous fluid: A mathematical investigation*, PhD thesis, Minnesota, 2005.
[12] M. Hieber and M. Murata, *The L^p-approach to the fluid-rigid body interaction problem for compressible fluids*, Evol. Equ. Control Theory, 4 (2015), pp. 69–87.
[13] M. Hillairet, *Lack of collision between solid bodies in a 2d incompressible viscous flow*, Comm. Partial Differential Equations, 32 (2007), pp. 1345–1371.
[14] O. Kreml, Š. Nečasová, and T. Piasecki, *Weak-strong uniqueness for the compressible fluid-rigid body interaction*, J. Differential Equations, 268 (2020), pp. 4756–4785.
[15] P.-L. Lions, *Mathematical topics in fluid dynamics, Vol.2, Compressible models*, Oxford Science Publication, Oxford, 1998.
[16] Š. Nečasová, M. Ramaswamy, A. Roy, and A. Schlömerkemper, *Motion of a rigid body in a compressible fluid with Navier-slip boundary condition*, arXiv preprint arXiv:2103.08762, (2021).
[17] J. Neustupa and P. Penel, *A weak solvability of the Navier-Stokes equation with Navier's boundary condition around a ball striking the wall*, in Advances in mathematical fluid mechanics, Springer, Berlin, 2010, pp. 385–407.
[18] A. Novotný and I. Straškraba, *Introduction to the mathematical theory of compressible flow*, vol. 27 of Oxford Lecture Series in Mathematics and its Applications, Oxford University Press, Oxford, 2004.
[19] N. V. Priezjev, A. A. Darhuber, and S. M. Troian, *Slip behavior in liquid films on surfaces of patterned wettability:Comparison between continuum and molecular dynamics simulations*, Phys. Rev. E, 71 (2005).
[20] N. V. Priezjev and S. M. Troian, *Influence of periodic wall roughness on the slip behaviour at liquid/solid interfaces: molecular versus continuum predictions*, J. Fluid Mech., 554 (2006), pp. 25–46.
[21] A. Roy and T. Takahashi, *Stabilization of a rigid body moving in a compressible viscous fluid*, Journal of Evolution Equations, (2020).

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 11567 Praha 1, Czech Republic
Email address: matus@math.cas.cz

NASI Senior Scientist, ICTS-TIFR, Survey No. 151, Sivakote, Bangalore, 560089, India
Email address: mythily.r@icts.res.in

BCAM, Basque Center for Applied Mathematics, Mazarredo 14, E48009 Bilbao, Bizkaia, Spain
Email address: royarnab244@gmail.com

Institute of Mathematics, University of Würzburg, Emil-Fischer-Str. 40, 97074 Würzburg, Germany
Email address: anja.schloemerkemper@mathematik.uni-wuerzburg.de