INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

An entropy stable finite volume method for a compressible two phase model

Eduard Feireisl
Mădălina Petcu
Bangwei She

Preprint No. 31-2022

PRAHA 2022

An entropy stable finite volume method for a compressible two phase model*

Eduard Feireisl[1] Mădălina Petcu[2,3,4] Bangwei She[1,5]

[1]Institute of Mathematics of the Academy of Sciences of the Czech Republic
Žitná 25, CZ-115 67 Praha 1, Czech Republic

[2]Laboratoire de Mathématiques et Applications, UMR CNRS 7348 - SP2MI
Université de Poitiers, Boulevard Marie et Pierre Curie - Téléport 2
86962 Chasseneuil, Futuroscope Cedex, France
[3]The Institute of Mathematics of the Romanian Academy, Bucharest, Romania
and
[4]The Institute of Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania

[5]Academy for Multidisciplinary studies, Capital Normal University
West 3rd Ring North Road 105, 100048 Beijing, P. R. China

Abstract

We study a binary mixture of compressible viscous fluids modelled by the Navier-Stokes-Allen-Cahn system with isentropic or ideal gas law. We propose a finite volume method for the approximation of the system based on upwinding and artificial diffusion approaches. We prove the entropy stability of the numerical method and present several numerical experiments to support the theory.

Keywords: compressible Navier-Stokes-Allen-Cahn; finite volume method; entropy stability

1 Introduction

Binary mixture of compressible fluids finds its wide applications in physics. Despite the existence of a rich mathematical theory of such problems [1, 2, 8, 10], the corresponding numerical analysis

*The research of E.F. and B.S. leading to these results has received funding from the Czech Sciences Foundation (GAČR), Grant Agreement 21–02411S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

is far from well understood. In this paper, we are interested in a finite volume approximation of the following model in the time space cylinder $(0,T)\times\Omega$, $\Omega\subset\mathbb{R}^d$, $d=2,3$:

$$\partial_t\varrho+\mathrm{div}_x(\varrho\mathbf{u})=0, \tag{1.1a}$$

$$\partial_t(\varrho\mathbf{u})+\mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u})+\nabla_x p=\mathrm{div}_x\mathbb{S}-\mathrm{div}_x\left(\nabla_x\chi\otimes\nabla_x\chi-\frac{1}{2}|\nabla_x\chi|^2\mathbb{I}\right)+\nabla_x\mathcal{F}(\chi) \tag{1.1b}$$

$$\partial_t\chi+\mathbf{u}\cdot\nabla_x\chi=\Delta_x\chi-\frac{\partial\mathcal{F}(\chi)}{\partial\chi} \tag{1.1c}$$

where $\varrho,\mathbf{u}$ and χ represent the fluid density, velocity, and the order parameter, respectively. Moreover, $\mathcal{F}=\mathcal{F}(\chi)$ is Ginzburg-Landau potential, $\mathbb{S}=\mathbb{S}(\nabla_x\mathbf{u})$ is the viscous stress

$$\mathbb{S}(\mathbb{D}_x\mathbf{u})=2\mu\left[\mathbb{D}_x\mathbf{u}-\frac{1}{d}\mathrm{div}_x\mathbf{u}\mathbb{I}\right]+\eta\mathrm{div}_x\mathbf{u}\mathbb{I},\quad \mathbb{D}_x\mathbf{u}=\frac{1}{2}\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}\right).$$

The fluid pressure p is determined by the state equation. In this paper, we consider two variant of state equations:

- **Isentropic case**. The pressure is a function of density

$$p=p(\varrho)=\varrho^\gamma. \tag{1.2a}$$

- **Non-isothermal case**. The pressure is a function of density and absolute temperature ϑ

$$\begin{aligned}&p=p(\varrho,\vartheta)=\varrho\vartheta \quad \text{with}\\ &c_v\big(\partial_t(\varrho\vartheta)+\mathrm{div}_x(\varrho\vartheta\mathbf{u})\big)-\kappa\Delta\vartheta=-p\mathrm{div}_x\mathbf{u}+\mathbb{S}:\mathbb{D}_x\mathbf{u}+(\Delta_x\chi-\mathcal{F}'(\chi))^2.\end{aligned} \tag{1.2b}$$

For the sake of simplicity, we suppose the no–slip boundary conditions for the velocity, together with the Neumann boundary conditions for the order parameter, i.e.,

$$\begin{gathered}\mathbf{u}|_{\partial\Omega}=0,\quad \nabla_x\chi\cdot\mathbf{n}|_{\partial\Omega}=0 \text{ for the isentropic case,}\\ \text{and, in addition, } \nabla_x\vartheta\cdot\mathbf{n}|_{\partial\Omega}=0 \text{ for the non-isothermal case.}\end{gathered} \tag{1.3}$$

To close the system we impose the initial conditions

$$\begin{gathered}(\varrho,\chi,\mathbf{u})(0)=(\varrho_0,\chi_0,\mathbf{u}_0) \text{ with } \varrho_0>0 \text{ for the isentropic case}\\ \text{and, in addition, } \vartheta(0)=\vartheta_0>0 \text{ for the non-isothermal case.}\end{gathered} \tag{1.4}$$

1.1 Stability of the system

It is easy to check the system (1.1)–(1.4) satisfy the energy balance equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E\,\mathrm{d}x + \int_\Omega \left(\mathbb{S}:\mathbb{D}_x\mathbf{u} + (\Delta_x\chi - \mathcal{F}'(\chi))^2\right)\,\mathrm{d}x = 0 \tag{1.5a}$$

for the isentropic case, and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E\,\mathrm{d}x = 0 \tag{1.5b}$$

for the non-isothermal case, where E is the total energy function of the system

$$E = \begin{cases} \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x\chi|^2 + \mathcal{H}(\varrho) + \mathcal{F}(\chi) & \text{for the isentropic case,} \\ \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{1}{2}|\nabla_x\chi|^2 + c_v\varrho\vartheta + \mathcal{F}(\chi) & \text{for the non-isothermal case,} \end{cases}$$

where $\mathcal{H}(\varrho) = c_v\varrho^\gamma$ and $c_v = 1/(\gamma-1)$.

For the isentropic case, the total energy plays the role of the mathematical entropy. When considering the non-isothermal case, the system satisfies the following entropy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho s\,\mathrm{d}x \geq \int_\Omega \frac{1}{\vartheta}\left[\mathbb{S}:\mathbb{D}_x\mathbf{u} + (\Delta_x\chi - \mathcal{F}'(\chi))^2 + \frac{\kappa|\nabla_x\vartheta|^2}{\vartheta}\right]\,\mathrm{d}x. \tag{1.6}$$

where $s = \log(\vartheta^{c_v}/\varrho)$ is the specific entropy.

The goal of this paper is to propose a finite volume method preserving the discrete variant of the above stabilities (1.5a)–(1.6), namely the entropy/energy stability. The rest of the paper is organized as follows. In Section 2 we introduce a finite volume method for the approximation of the problem. In Section 3 we analyze the energy/entropy stability of the numerical solutions. Further, in Section 4 we validate the theoretical results by numerical experiments. Section 5 is the conclusion.

2 Numerical method

In this section we propose a finite volume method for the approximation of the system (1.1)–(1.4).

Primary grid. Let Ω_h be either a regular and quasi-uniform unstructured triangulation of Ω in the sense of Ciarlet [3] or a uniform structured mesh of Ω with the following notations:

- We denote by K a generic element such that $\Omega = \cup_{K\in\Omega_h} K$, where in the case of unstructured mesh K is either a triangle for $d = 2$ or a tetrahedron for $d = 3$, and in the case of structured mesh K is either a rectangle for $d = 2$ or a cuboid for $d = 3$.
- For any element K we denote by $|K|$ its volume and by h_K its diameter. Further, we define by $h = \max_{K\in\Omega_h} h_K$ the size of the mesh.

- We denote by $\mathcal{E}$ the set of all faces, $\mathcal{E}_B$ the set of all faces on the boundary, $\mathcal{E}_I = \mathcal{E}\backslash\mathcal{E}_B$ the set of all interior faces, and $\mathcal{E}(K)$ the set of all faces of the element $K \in \Omega_h$. Further, we denote by $\sigma = K|L$ the common face of two neighbouring elements K and L.
- For each face $\sigma \in \mathcal{E}$, we denote by $|\sigma|$ its Lebesgue measure, and $\mathbf{n}$ its outer normal vector. If furthermore $\sigma \in \mathcal{E}(K)$ we write it as $\mathbf{n}_K$.
- Let $\mathbb{P}_x = \{x_K | x_K \in K \in \Omega_h\}$ be a set of control points such that for any $\sigma = K|L$ the segment $\overrightarrow{x_K x_L}$ is perpendicular to σ. Then we denote by d_σ the Euclidean distance between x_K and x_L for $\sigma = K|L$.

Hereafter, we call Ω_h the primary grid and introduce a dual grid for convenience of notations.

Dual grid. We denote $\mathcal{D}_h = \bigcup_{\sigma\in\mathcal{E}} D_\sigma$ as the dual grid, where D_σ is the dual cell associated to the face σ. On the one hand, for any exterior face $\sigma \in \mathcal{E}(K)\cap\mathcal{E}_B$ we define its dual cell as $D_\sigma = D_{\sigma,K}$, where $D_{\sigma,K}$ is the domain obtained by connecting the control point x_K with the $(d-1)$ vertices of σ. On the other hand, for any interior face $\sigma = K_1|K_2 \in \mathcal{E}_I$ we define $D_\sigma = D_{\sigma,K_1} \cup D_{\sigma,K_2}$.

Remark 2.1. For uniform structured mesh $\mathbb{P}_x$ can be chosen as the barycenters. Concerning unstructured, we refer to VanderZee et al. [11] for the discussion of well-centered mesh, where the control points are chosen as the circumcenters.

For a piecewise (elementwise) continuous function v we define

$$v^{\text{in}}(x) = \lim_{\delta\to 0+} v(x - \delta\mathbf{n}) \; \forall\, x \in \sigma \in \mathcal{E} \quad \text{and} \quad v^{\text{out}}(x) = \lim_{\delta\to 0+} v(x + \delta\mathbf{n}) \; \forall\, x \in \sigma \in \mathcal{E}_I.$$

Moreover, for $x \in \sigma \in \mathcal{E}_B$, we specify $v^{\text{out}}(x)$ according to the boundary conditions:

$$v^{\text{out}} = \begin{cases} v^{\text{in}} & \text{for no flux condition, s.t. } [\![v]\!] = 0, \\ -v^{\text{in}} & \text{for zero-Dirichlet condition, s.t. } \{\!\{v\}\!\} = 0. \end{cases}$$

where

$$\{\!\{v\}\!\}(x) = \frac{v^{\text{in}}(x) + v^{\text{out}}(x)}{2}, \quad [\![v]\!](x) = v^{\text{out}}(x) - v^{\text{in}}(x).$$

Function spaces. We define Q_h and W_h respectively as the space of piecewise constant functions on the primary grid Ω_h and the dual grid $\mathcal{D}_h$. Moreover, we mean by $\mathbf{v} \in \mathbf{Q}_h$ (resp. $\mathbf{v} \in \mathbf{W}_h$) that $\mathbf{v} \in Q_h(\Omega; R^d)$ (resp. $\mathbf{v} \in W_h(\Omega; R^d)$), i.e., $v_i \in Q_h$ (resp. $v_i \in W_h$) for all $i = 1, \ldots, d$. The interpolation operator associated to Q_h reads

$$\Pi_Q \phi = \sum_{K\in\Omega_h} \frac{1_K(x)}{|K|} \int_K \phi \,\mathrm{d}x, \quad 1_K(x) = \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{otherwise.} \end{cases}$$

Diffusive upwind flux. Given the velocity field $\mathbf{v} \in \mathbf{Q}_h$, the upwind flux for any function $r \in Q_h$ is specified at each face $\sigma \in \mathcal{E}$ by

$$\mathrm{Up}[r, \mathbf{v}]|_\sigma = r^{\mathrm{up}} \mathbf{v}_\sigma \cdot \mathbf{n} = r^{\mathrm{in}} [\mathbf{v}_\sigma \cdot \mathbf{n}]^+ + r^{\mathrm{out}} [\mathbf{v}_\sigma \cdot \mathbf{n}]^- = \{\!\{r\}\!\} \;\; \mathbf{v}_\sigma \cdot \mathbf{n} - \frac{1}{2} |\mathbf{v}_\sigma \cdot \mathbf{n}| \, [\![r]\!] \,,$$

where

$$\mathbf{v}_\sigma = \{\!\{\mathbf{v}\}\!\} \,|_\sigma, \quad [f]^\pm = \frac{f \pm |f|}{2} \quad \text{and} \quad r^{\mathrm{up}} = \begin{cases} r^{\mathrm{in}} & \text{if } \mathbf{v}_\sigma \cdot \mathbf{n} \geq 0, \\ r^{\mathrm{out}} & \text{if } \mathbf{v}_\sigma \cdot \mathbf{n} < 0. \end{cases}$$

Furthermore, we consider a diffusive numerical flux function of the following form

$$F_\varepsilon^{\mathrm{up}}(r, \mathbf{v}) = \mathrm{Up}[r, \mathbf{v}] - h^\varepsilon \, [\![r]\!] \,, \; \varepsilon > -1. \tag{2.1}$$

Discrete operators. For piecewise constant functions we define the discrete gradient, divergence and Laplace operators elementwisely on the primary grid in the following way

$$\nabla_h r_h(x) = \sum_{K \in \Omega_h} \left(\nabla_h r_h\right)_K 1_K(x), \qquad \left(\nabla_h r_h\right)_K = \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \{\!\{r_h\}\!\} \, \mathbf{n},$$

$$\mathrm{div}_h \mathbf{v}_h(x) = \sum_{K \in \Omega_h} \left(\mathrm{div}_h \mathbf{v}_h\right)_K 1_K(x), \qquad \left(\mathrm{div}_h \mathbf{v}_h\right)_K = \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \{\!\{\mathbf{v}_h\}\!\} \cdot \mathbf{n},$$

$$\Delta_h r_h(x) = \sum_{K \in \Omega_h} \left(\Delta_h r_h\right)_K 1_K(x), \qquad \left(\Delta_h r_h\right)_K = \sum_{\sigma \in \mathcal{E}(K)} \frac{|\sigma|}{|K|} \frac{[\![r_h]\!]}{d_\sigma},$$

for any $r_h \in Q_h$ and $\mathbf{v}_h \in \mathbf{Q}_h$. Further, we define discrete difference operator that involves the dual grid. For any $r_h \in Q_h$ and $\mathbf{q} \in \mathbf{W}_h$, we define

$$\nabla_{\mathcal{E}} r_h(x) = \sum_{\sigma \in \mathcal{E}} 1_{D_\sigma} \left(\nabla_{\mathcal{E}} r_h\right)_{D_\sigma}, \qquad \left(\nabla_{\mathcal{E}} r_h\right)_{D_\sigma} := \sqrt{d} \frac{[\![r]\!]}{d_\sigma} \mathbf{n}, \quad \text{for all } \; \sigma \in \mathcal{E}_I.$$

It is easy to check for any $r_h, \varphi_h \in Q_h$ that

$$\int_\Omega \Delta_h r_h \varphi_h \,\mathrm{d}x = -\int_{\mathcal{E}_I} \frac{1}{d_\sigma} [\![r_h]\!] \, [\![\varphi_h]\!] \,\mathrm{d}S_x = -\int_\Omega \nabla_{\mathcal{E}} r_h \cdot \nabla_{\mathcal{E}} \varphi_h \,\mathrm{d}x \quad \text{ if } [\![r_h]\!]_\sigma = 0 \;\forall\, \sigma \in \mathcal{E}_B. \tag{2.2}$$

Time discretization. For a given time step $\Delta t > 0$, we denote the approximation of a generic discrete function v_h at time $t^k = k\Delta t$ by v_h^k for $k = 1, \ldots, N_T (= T/\Delta t)$. and define

$$D_t v_h^k = \frac{v_h^k - v_h^{k-1}}{\Delta t}.$$

2.1 A finite volume method

Now we introduce a finite volume (FV) method for the approximation of phase field model (1.1)–(1.4). First, we consider the isentropic case (1.2a).

Scheme-A: An FV method for the isentropic model.

Let $(\varrho_h^0, \mathbf{u}_h^0, \chi_h^0) = (\Pi_Q \varrho_0, \Pi_Q \chi_0, \Pi_Q \mathbf{u}_0)$. Given $(\varrho_h^{k-1}, \chi_h^{k-1}, \mathbf{u}_h^{k-1}) \in Q_h \times Q_h \times \mathbf{Q}_h$ for any $k = 1, \cdots, N_T$, we say that the triple $(\varrho_h^k, \chi_h^k, \mathbf{u}_h^k) \in Q_h \times Q_h \times \mathbf{Q}_h$ is an FV approximation of the Navier–Stokes–Allen–Cahn system (1.1), (1.2a), (1.3)–(1.4) at time t^k if the following system of algebraic equations holds:

$$\int_\Omega D_t \varrho_h^k \varphi_h \,\mathrm{d}x - \int_{\mathcal{E}} F_\varepsilon^{\mathrm{up}}(\varrho_h^k, \mathbf{u}_h^k) \llbracket \varphi_h \rrbracket \,\mathrm{d}S_x = 0, \quad \text{for all } \varphi_h \in Q_h; \tag{2.3a}$$

$$\begin{aligned}\int_\Omega D_t(\varrho_h^k \mathbf{u}_h^k) \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x - \int_{\mathcal{E}} F_\varepsilon^{\mathrm{up}}(\varrho_h^k \mathbf{u}_h^k, \mathbf{u}_h^k) \cdot \llbracket \boldsymbol{\varphi}_h \rrbracket \,\mathrm{d}S_x + \int_\Omega \left(2\mu \mathbb{D}_h \mathbf{u}_h^k : \nabla_h \boldsymbol{\varphi}_h + \lambda \mathrm{div}_h \mathbf{u}_h^k \mathrm{div}_h \boldsymbol{\varphi}_h\right) \mathrm{d}x \\ = \int_\Omega p_h^k \mathrm{div}_h \boldsymbol{\varphi}_h \,\mathrm{d}x + \int_\Omega (f_h^k - \Delta_h \chi_h^k) \nabla_h \chi_h^k \cdot \boldsymbol{\varphi}_h \,\mathrm{d}x, \text{ for all } \boldsymbol{\varphi}_h \in \mathbf{Q}_h,\end{aligned} \tag{2.3b}$$

$$\int_\Omega (D_t \chi_h^k + \mathbf{u}_h^k \cdot \nabla_h \chi_h^k) \psi_h \,\mathrm{d}x = \int_\Omega \left(\Delta_h \chi_h^k - f_h^k\right) \psi_h \,\mathrm{d}x, \quad \text{for all } \psi_h \in Q_h; \tag{2.3c}$$

where $p_h^k = (\varrho_h^k)^\gamma$, $\mathbb{D}_h \mathbf{u}_h = (\nabla_h \mathbf{u}_h + \nabla_h^t \mathbf{u}_h)/2$, $\lambda = \eta - \frac{2}{d}\mu$, and the approximation of $\mathcal{F}'$ follows the so-called convex-concave splitting technique

$$f_h^k = \mathcal{F}_a'(\chi_h^k) + \mathcal{F}_b'(\chi_h^{k-1}) \tag{2.4}$$

where $\mathcal{F}_a$ and $\mathcal{F}_b$ are the convex and concave part of $\mathcal{F}$, respectively.

Further, we propose an FV method for the non-isothermal case (1.2b).

Scheme-B: An FV method for the non-isothermal model.

Let $(\varrho_h^0, \vartheta_h^0, \mathbf{u}_h^0, \chi_h^0) = (\Pi_Q \varrho_0, \Pi_Q \vartheta_0, \Pi_Q \chi_0, \Pi_Q \mathbf{u}_0)$. Given $(\varrho_h^{k-1}, \vartheta_h^{k-1}, \chi_h^{k-1}, \mathbf{u}_h^{k-1}) \in Q_h \times Q_h \times Q_h \times \mathbf{Q}_h$ for any $k = 1, \cdots, N_T$, we say that the quadruple $(\varrho_h^k, \vartheta_h^k, \chi_h^k, \mathbf{u}_h^k) \in Q_h \times Q_h \times Q_h \times \mathbf{Q}_h$ is an FV approximation of the Navier–Stokes–Allen–Cahn system (1.1), (1.2b), (1.3)–(1.4) at time t^k if it satisfies (2.3) and

$$c_v \int_{\Omega_h} D_t(\varrho_h^k \vartheta_h^k) \phi_h \,\mathrm{d}x - c_v \int_{\mathcal{E}_I} F_\varepsilon^{\mathrm{up}}(\varrho_h^k \vartheta_h^k, \mathbf{u}_h^k) [\![\phi_h]\!] \,\mathrm{d}S_x + \int_{\mathcal{E}_I} \frac{\kappa}{d_\sigma} [\![\vartheta_h^k]\!] [\![\phi_h]\!] \,\mathrm{d}S_x$$
$$= \int_{\Omega_h} \big(\mathbb{S}(\nabla_h \mathbf{u}_h^k) : \nabla_h \mathbf{u}_h^k - p_h^k \mathrm{div}_h \mathbf{u}_h^k + |\Delta_h \chi_h^k - f_h^k|^2\big) \phi_h \,\mathrm{d}x \text{ for all } \phi_h \in Q_h. \quad (2.5)$$

Here, f_h is the same as in **scheme-A**, and the discrete pressure is defined as $p_h = \varrho_h \vartheta_h$ for $\vartheta_h \geq 0$ with an extension to the non-physical regime for $\vartheta_h < 0$ that $p_h(\varrho_h, \vartheta_h) = 0$.

3 Stability

In this section we show the stability of the FV schemes. Before that we recall some important properties satisfied by our numerical approximation. First, we recall from [7] that the density scheme (2.3a) satisfies the positivity of density and conservation of mass.

Lemma 3.1 (Positivity of density and conservation of mass)**.** *Let $(\varrho_h, \mathbf{u}_h)$ satisfy (2.3a) with $\varrho_0 > 0$. Then for any $t \in (0, T)$ there hold*

$$\varrho_h(t, x) > 0 \text{ for all } x \in \Omega, \quad \text{and} \quad \int_\Omega \varrho_h(t) \,\mathrm{d}x = \int_\Omega \varrho_0 \,\mathrm{d}x.$$

Next, we test the density scheme (2.3a) by $\varphi_h = \mathcal{H}'(\varrho_h^k)$, and find the following lemma, see [7, Lemma 11.2].

Lemma 3.2 (Discrete renormalized continuity equation)**.** *Let $(\varrho_h^k, \mathbf{u}_h^k) \in Q_h \times \mathbf{Q}_h$ satisfy the discrete continuity equation (2.3a) for any $k \in \{1, \ldots, N_T\}$. Then, there exist $\varrho_\xi^k \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$ and $\varrho_\zeta^k \in \mathrm{co}\{\varrho_K^k, \varrho_L^k\}$ for any $\sigma = K|L \in \mathcal{E}_I$ such that*

$$\begin{aligned}&\int_\Omega D_t \mathcal{H}(\varrho_h^k) \,\mathrm{d}x + \int_\Omega (\varrho_h^k)^\gamma \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{d}x \\ &= -\frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\varrho_\xi^k) |D_t \varrho_h^k|^2 \,\mathrm{d}x - \int_{\mathcal{E}_I} \mathcal{H}''(\varrho_\zeta^k) [\![\varrho_h^k]\!]^2 \left(h^\varepsilon + \frac{1}{2}|\mathbf{u}_\sigma^k \cdot \mathbf{n}|\right) \mathrm{d}S_x \leq 0.\end{aligned} \quad (3.1)$$

Further, we report the positivity of the temperature, see [6, Lemma 3.5].

Lemma 3.3 (Positivity of temperature)**.** *Let* $(\varrho_h, \vartheta_h, \chi_h, \mathbf{u}_h)$ *satisfy* (2.5) *with* $\varrho_0 > 0, \vartheta_0 > 0$. *Then for any* $t \in (0,T)$ *there hold*

$$\vartheta_h(t,x) > 0 \text{ for all } x \in \Omega.$$

Theorem 3.4 (Existence of numerical solution.)**.** *For every* $k = 1, \dots, N_T$*, there exist a solution* $(\varrho_h^k, \chi_h^k, \mathbf{u}_h^k) \in Q_h \times Q_h \times \mathbf{Q}_h$ *to the FV method* **scheme-A***, and a solution* $(\varrho_h^k, \vartheta_h^k, \chi_h^k, \mathbf{u}_h^k) \in Q_h \times Q_h \times Q_h \times \mathbf{Q}_h$ *to the FV method* **scheme-B**

The proof can be done analogously as [7, Lemma 11.3].

3.1 Stability of Scheme-A

Here we derive the energy stability of **Scheme-A** for compressible Navier–Stokes–Allen–Cahn system with isentropic state equation (1.2a).

Theorem 3.5 (Discrete energy balance)**.** *Let* $(\varrho_h, \chi_h, \mathbf{u}_h)$ *be a solution of the FV method* (2.3)*. Then we have the following energy estimate*

$$\begin{aligned} &D_t \int_\Omega \left(\frac{1}{2} \varrho_h^k \left|\mathbf{u}_h\right|^2 + \mathcal{H}(\varrho_h^k) + F(\chi_h^k) + \frac{1}{2} |\nabla_{\mathcal{E}} \chi_h|^2 \right) \mathrm{d}x \\ &+ 2\mu \|\mathbb{D}_h \mathbf{u}_h^k\|_{L^2}^2 + \lambda \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2}^2 + \|\Delta_h \chi_h^k - f_h^k\|_{L^2}^2 = -D_A^k, \end{aligned} \tag{3.2}$$

where $D_A^k \geq 0$ *is the numerical dissipation*

$$\begin{aligned} D_A^k =& \frac{\Delta t}{2} \int_{\Omega_h} \left(\varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 + \left(\mathcal{F}_a'(\chi_\xi^k) - \mathcal{F}_b'(\chi_\zeta^k)\right) \left|D_t \chi_h^k\right|^2 + |\nabla_{\mathcal{E}} D_t \chi_h^k|^2 \right) \mathrm{d}x \\ &+ \frac{\Delta t}{2} \int_\Omega \mathcal{H}''(\varrho_\xi^k) |D_t \varrho_h^k|^2 \, \mathrm{d}x + \int_{\mathcal{E}_I} \mathcal{H}''(\varrho_\zeta^k) \left[\!\left[\varrho_h^k \right]\!\right]^2 \left(h^\varepsilon + \frac{1}{2} |\left\{\!\!\left\{ \mathbf{u}_h^k \right\}\!\!\right\} \cdot \mathbf{n}| \right) \mathrm{d}S_x \\ &+ \int_{\mathcal{E}_I} \left(h^\varepsilon \left\{\!\!\left\{ \varrho_h^k \right\}\!\!\right\} + \frac{1}{2} \varrho_h^{k,\mathrm{up}} \left| \left\{\!\!\left\{ \mathbf{u}_h^k \right\}\!\!\right\} \cdot \mathbf{n} \right| \right) | \left[\!\left[\mathbf{u}_h^k \right]\!\right] |^2 \, \mathrm{d}S_x, \end{aligned}$$

where $\chi_\xi^k, \chi_\zeta^k \in \mathrm{co}\{\chi_h^{k-1}, \chi_h^k\}$.

Proof. First, we sum up (2.3a) and (2.3b) with $\varphi_h = -\frac{1}{2}|\mathbf{u}_h^k|^2 \in Q_h$ and $\boldsymbol{\varphi}_h = \mathbf{u}_h^k \in \mathbf{Q}_h$ to get the kinetic energy balance

$$\begin{aligned} D_t \int_{\Omega_h} \frac{1}{2} \varrho_h^k |\mathbf{u}_h^k|^2 \, \mathrm{d}x + \frac{\Delta t}{2} \int_{\Omega_h} \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \, \mathrm{d}x + \int_{\mathcal{E}_I} \left(h^\varepsilon \left\{\!\!\left\{ \varrho_h^k \right\}\!\!\right\} + \frac{1}{2} \varrho_h^{k,\mathrm{up}} \left| \left\{\!\!\left\{ \mathbf{u}_h^k \right\}\!\!\right\} \cdot \mathbf{n} \right| \right) | \left[\!\left[\mathbf{u}_h^k \right]\!\right] |^2 \, \mathrm{d}S_x \\ + 2\mu \|\mathbb{D}_h \mathbf{u}_h^k\|_{L^2}^2 + \lambda \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2}^2 = \int_{\Omega_h} (\varrho_h^k)^\gamma \mathrm{div}_h \mathbf{u}_h^k \, \mathrm{d}x + \int_{\Omega_h} (f_h^k - \Delta_h \chi_h^k) \nabla_h \chi_h^k \cdot \mathbf{u}_h^k \, \mathrm{d}x, \end{aligned} \tag{3.3}$$

see e.g. [5, equation 3.4] for more details.
Next, by choosing $\psi_h = (\Delta_h \chi_h^k - f_h^k) \in Q_h$ in (2.3c) we derive

$$
\begin{aligned}
&\int_{\Omega_h} (\Delta_h \chi_h^k - f_h^k)^2 \,\mathrm{d}x = \int_{\Omega} \left(\Delta_h \chi_h^k - f_h^k\right)(D_t \chi_h^k + \mathbf{u}_h^k \cdot \nabla_h \chi_h^k) \,\mathrm{d}x \\
&= -D_t \int_{\Omega} \frac{1}{2} |\nabla_{\mathcal{E}} \chi_h^k|^2 \,\mathrm{d}x - \frac{\Delta t}{2} \int_{\Omega} |\nabla_{\mathcal{E}} D_t \chi_h^k|^2 \,\mathrm{d}x - D_t \int_{\Omega_h} \mathcal{F}(\chi_h^k) \,\mathrm{d}x \\
&\quad - \frac{\Delta t}{2} \int_{\Omega_h} \left(\mathcal{F}_a''(\chi_\xi^k) - \mathcal{F}_b''(\chi_\zeta^k)\right) |D_t \chi_h^k|^2 \,\mathrm{d}x + \int_{\Omega} \left(\Delta_h \chi_h^k - f_h^k\right) \mathbf{u}_h^k \cdot \nabla_h \chi_h^k \,\mathrm{d}x
\end{aligned}
\tag{3.4}
$$

where $\chi_\xi^k, \chi_\zeta^k \in \mathrm{co}\{\chi_h^{k-1}, \chi_h^k\}$.

Finally, summing up (3.3) and (3.4) together with (3.1) completes the proof. □

3.2 Stability of Scheme-B

Considering the compressible Navier–Stokes–Allen–Cahn system with non-isothermal state equation (1.2b), its stability includes not only the energy stability but also the entropy stability. First, we derive the energy stability of **Scheme-B**.

Theorem 3.6 (Energy stability)**.** *Let $(\varrho_h, \vartheta_h, \chi_h, \mathbf{u}_h)$ be a solution of* **scheme-B**. *Then there holds*

$$
D_t \int_{\Omega_h} \left(\frac{1}{2} \varrho_h^k \left|\mathbf{u}_h\right|^2 + c_v \varrho_h^k \vartheta_h^k + F(\chi_h^k) + \frac{1}{2} |\nabla_{\mathcal{E}} \chi_h^k|^2 \right) \mathrm{d}x = -D_B^k,
\tag{3.5}
$$

where $D_B^k \geq 0$ is the numerical dissipation

$$
\begin{aligned}
D_B^k = &\frac{\Delta t}{2} \int_{\Omega_h} \left(\varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 + \left(\mathcal{F}_a''(\chi_\xi^k) - \mathcal{F}_b''(\chi_\zeta^k)\right) \left|D_t \chi_h^k\right|^2 + |\nabla_{\mathcal{E}} D_t \chi_h^k|^2 \right) \mathrm{d}x \\
&+ \int_{\mathcal{E}_I} \left(h^\varepsilon \left\{\!\!\left\{ \varrho_h^k \right\}\!\!\right\} + \frac{1}{2} \varrho_h^{k,\mathrm{up}} \left| \left\{\!\!\left\{ \mathbf{u}_h^k \right\}\!\!\right\} \cdot \mathbf{n} \right| \right) \left| \left[\!\left[\mathbf{u}_h^k \right]\!\right] \right|^2 \mathrm{d}S_x,
\end{aligned}
$$

where $\chi_\xi^k, \chi_\zeta^k \in \mathrm{co}\{\chi_h^{k-1}, \chi_h^k\}$.

Proof. First, following the proof of Theorem 3.5, we set $\varphi_h = -\frac{1}{2}|\mathbf{u}_h^k|^2 Q_h$ in (2.3a), $\boldsymbol{\varphi}_h = \mathbf{u}_h^k \in \mathbf{Q}_h$ in (2.3b) and $\psi_h = (\Delta_h \chi_h^k - f_h^k) \in Q_h$ in (2.3c) to get

$$
\begin{aligned}
&D_t \int_{\Omega_h} \frac{1}{2} \varrho_h^k |\mathbf{u}_h^k|^2 \,\mathrm{d}x + \frac{\Delta t}{2} \int_{\Omega_h} \varrho_h^{k-1} |D_t \mathbf{u}_h^k|^2 \,\mathrm{d}x + 2\mu \|\mathbb{D}_h \mathbf{u}_h^k\|_{L^2}^2 + \lambda \|\mathrm{div}_h \mathbf{u}_h^k\|_{L^2}^2 + \|\Delta_h \chi_h^k - f_h^k\|_{L^2}^2 \\
&= \int_{\Omega_h} p_h^k \mathrm{div}_h \mathbf{u}_h^k \,\mathrm{d}x - D_t \int_{\Omega} \frac{1}{2} |\nabla_{\mathcal{E}} \chi_h^k|^2 \,\mathrm{d}x - \frac{\Delta t}{2} \int_{\Omega} |\nabla_{\mathcal{E}} D_t \chi_h^k|^2 \,\mathrm{d}x - D_t \int_{\Omega_h} \mathcal{F}(\chi_h^k) \,\mathrm{d}x \\
&\quad - \frac{\Delta t}{2} \int_{\Omega_h} \left(\mathcal{F}_a''(\chi_\xi^k) - \mathcal{F}_b''(\chi_\zeta^k)\right) |D_t \chi_h^k|^2 \,\mathrm{d}x - \int_{\mathcal{E}_I} \left(h^\varepsilon \left\{\!\!\left\{ \varrho_h^k \right\}\!\!\right\} + \frac{1}{2} \varrho_h^{k,\mathrm{up}} \left| \left\{\!\!\left\{ \mathbf{u}_h^k \right\}\!\!\right\} \cdot \mathbf{n} \right| \right) \left| \left[\!\left[\mathbf{u}_h^k \right]\!\right] \right|^2 \mathrm{d}S_x
\end{aligned}
\tag{3.6}
$$

Next, by setting $\phi_h = 1 \in Q_h$ in (2.5) we obtain

$$D_t \int_{\Omega_h} c_v \varrho_h^k \vartheta_h^k \,\mathrm{d}x = \int_{\Omega_h} \left(\mathbb{S}(\nabla_h \mathbf{u}_h^k) : \nabla_h \mathbf{u}_h^k - p_h(\varrho_h^k, \vartheta_h^k) \mathrm{div}_h \mathbf{u}_h^k \right) \,\mathrm{d}x + \|\Delta_h \chi_h^k - f_h^k\|_{L^2}^2 \tag{3.7}$$

Further, summing up (3.6) and (3.7) yields the desired result. □

Next, following [6, Theorem 3.7] we report the entropy inequality for **Scheme-B**.

Lemma 3.7 (Entropy inequality)**.** *Let* $(\varrho_h, \vartheta_h, \chi_h, \mathbf{u}_h)$ *be a solution of* **Scheme-B**. *Then there holds*

$$\begin{aligned} D_t \int_{\Omega} \varrho_h^k s_h^k \,\mathrm{d}x = &- \int_{\Omega} \kappa \nabla_{\mathcal{E}} \vartheta_h^k \cdot \nabla_{\mathcal{E}} \left(\frac{1}{\vartheta_h^k} \right) \,\mathrm{d}x + \int_{\Omega} \frac{1}{\vartheta_h^k} \left(2\mu |\mathbb{D}_h(\mathbf{u}_h^k)|^2 + \lambda |\mathrm{div}_h \mathbf{u}_h^k|^2 \right) \,\mathrm{d}x \\ &+ \int_{\Omega_h} \frac{1}{\vartheta_h^k} (\Delta_h \chi_h^k - f_h^k)^2 \,\mathrm{d}x + D_S^k \geq 0. \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} D_S^k = &\frac{\Delta t}{2\xi_{\varrho,h}^k} |D_t \varrho_h^k|^2 + \frac{h}{2\eta_{\varrho,h}^k} |\nabla_{\mathcal{E}} \varrho_h^k|^2 |\overline{\mathbf{u}_h^k} \cdot \mathbf{n}| + \frac{c_v \Delta t}{2|\xi_{\vartheta,h}^k|^2} \varrho_h^{k-1} |D_t \vartheta_h^k|^2 - \frac{c_v h}{2|\eta_{\vartheta,h}^k|^2} |\nabla_{\mathcal{E}} \vartheta_h^k|^2 (\varrho_h^k)^{\mathrm{out}} [\overline{\mathbf{u}_h^k} \cdot \mathbf{n}]^- \\ &+ h^{\varepsilon+1} \nabla_{\mathcal{E}} \varrho_h^k \cdot \nabla_{\mathcal{E}} \left(\nabla_{\varrho}(-\varrho_h^k s_h^k) \right) + h^{\varepsilon+1} \nabla_{\mathcal{E}} p_h^k \cdot \nabla_{\mathcal{E}} \left(\nabla_p(-\varrho_h^k s_h^k) \right) \geq 0 \end{aligned}$$

where $\xi_{\varrho_h}^k \in \mathrm{co}\{\varrho_h^{k-1}, \varrho_h^k\}$, $\xi_{\vartheta_h}^k \in \mathrm{co}\{\vartheta_h^{k-1}, \vartheta_h^k\}$, *and* $\eta_{\varrho_h}^k \in \mathrm{co}\{\varrho_h^{k,\mathrm{in}}, \varrho_h^{k,\mathrm{out}}\}$, $\eta_{\vartheta_h}^k \in \mathrm{co}\{\vartheta_h^{k,\mathrm{in}}, \vartheta_h^{k,\mathrm{out}}\}$ *for any* $\sigma \in \mathcal{E}$ *.*

Remark 3.8. The inequality in Lemma 3.7 implies that the total (physical) entropy is non-decreasing. Here after we call the above inequality as entropy stability.

4 Numerical experiments

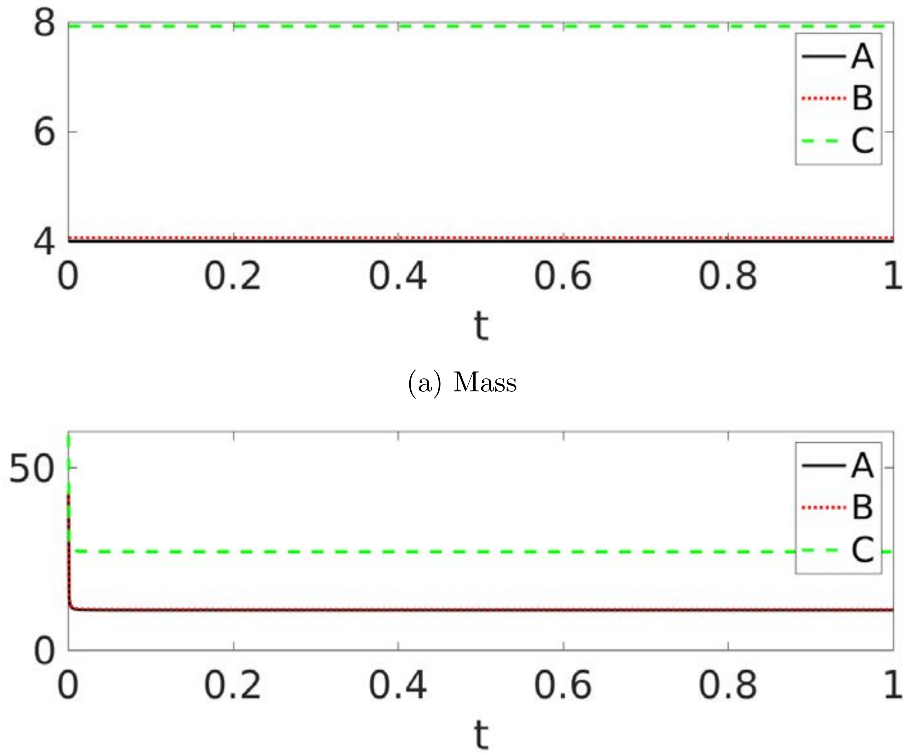
In our experiments, we consider the computational domain to be $\Omega = [-1, 1]^2$, divided into 80^2 uniform squares. To solve the nonlinear schemes, we use the fix-point iteration method and solve an explicit and linear system at each sub-iteration, which requires the so-called CFL condition. To fulfill this condition, we take a small time step size $\Delta t = 1.0e-4$. The potential function $\mathcal{F}(\chi)$ and its discrete derivative f_h denoted in (2.4) are respectively taken as

$$\mathcal{F}(\chi) = \frac{1}{4}(\chi^2 - 1)^2 \quad \text{and} \quad f_h^k = (\chi_h^k)^3 - \chi_h^{k-1}.$$

Experiment 1 – barotropic case. In this experiment, we consider the barotropic case. Initially, one of the fluids occupies two circular area located at $x_l = (-0.12, 0)$ and $x_r = (0.1, 0)$ of the radii 0.08 and 0.1 respectively, while the other fluid stays in the rest of the domain. The initial data read

$$\mathbf{u} = 0, (\varrho_0, \chi_0) = \begin{cases} (\varrho_1, 1) & \text{if } |x - x_l| < 0.08 \text{ or } |x - x_r| < 0.12, \\ (\varrho_2, 0) & \text{otherwise,} \end{cases} \quad (\varrho_1, \varrho_2) = \begin{cases} (1,1) & \text{Case A,} \\ (1,2) & \text{Case B,} \\ (2,1) & \text{Case C.} \end{cases}$$

In 1 we present the time evolution of the total mass and energy, which clearly supports the conservation of mass and stability of energy. Further, we show the time evolution of ϱ and χ in Figure 2 and Figure 3, respectively.

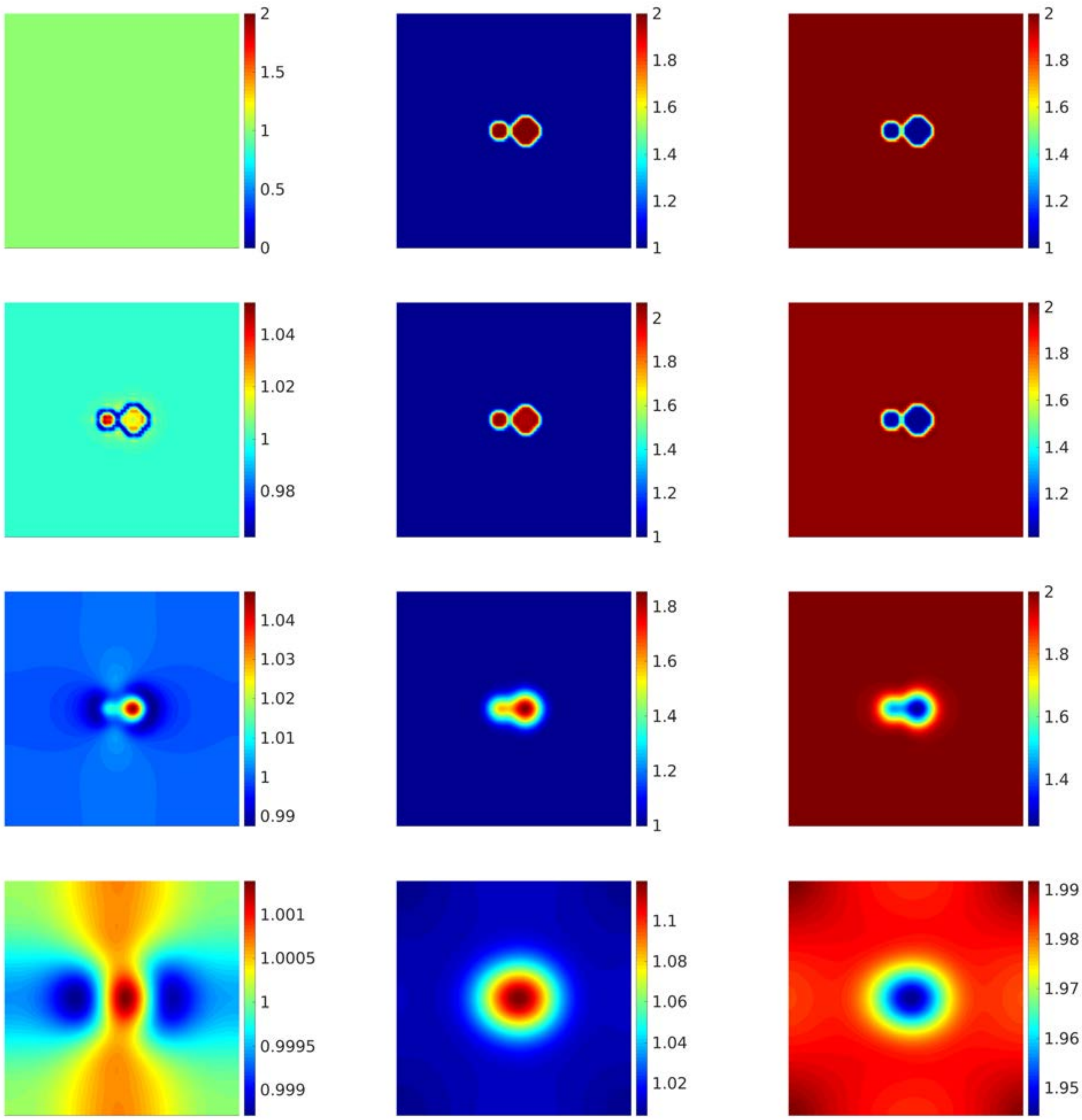


(a) Mass

(b) Energy

Figure 1: Time evolution of the total mass and energy

Experiment 2 – Non-isothermal case. In this experiment, we consider the non-isothermal case. Analogously to experiment 1, one of the fluids occupies two circular area located at $x_l = (-0.12, 0)$ and $x_r = (0.1, 0)$ of the radii 0.08 and 0.1 respectively, while the other fluid stays in

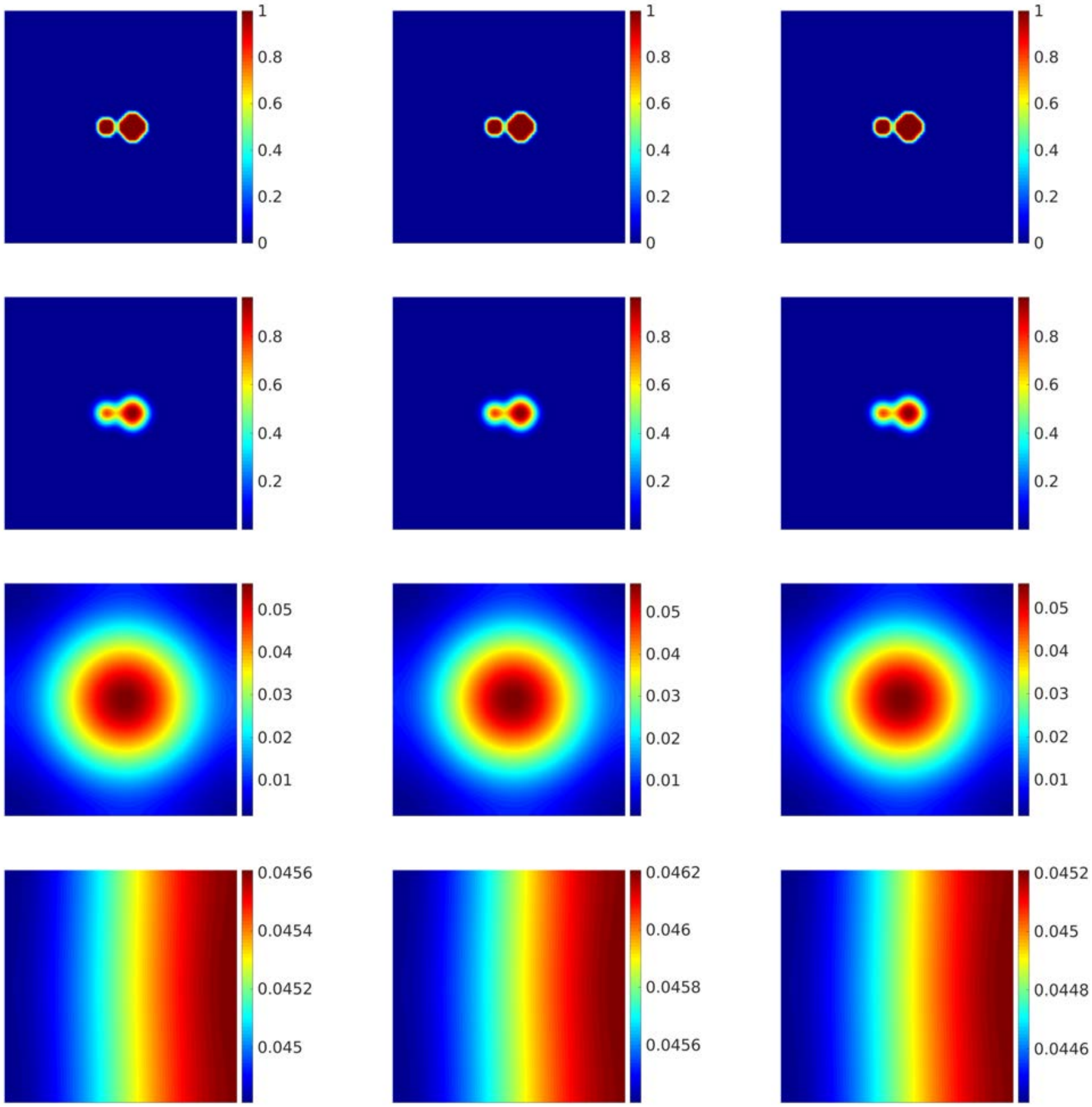


(a) Case A (b) Case B (c) Case C

Figure 2: Experiment 1: time evolution of ϱ, from top to bottom are $t = 0, 0.001, 0.1, 1$

the rest of the domain. We take the initial data for density, velocity, and order parameter, and moreover $\vartheta_0 = 1$.

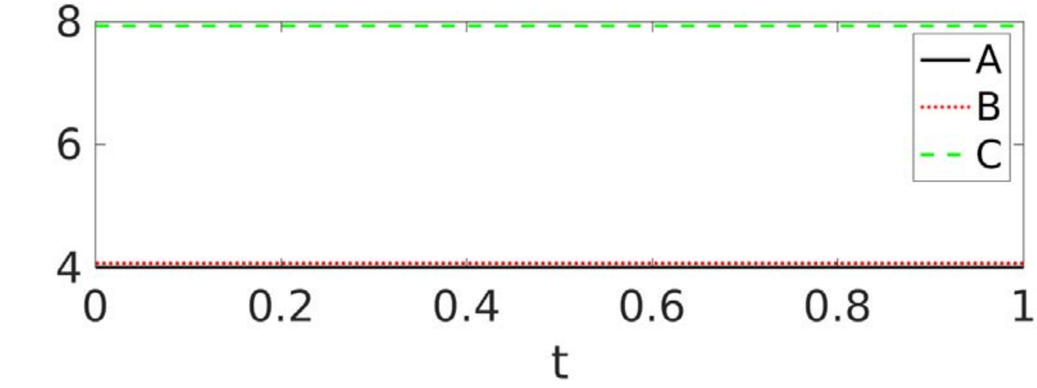
In Figure 4 we present the time evolution of the total mass, energy, and entropy, which clearly



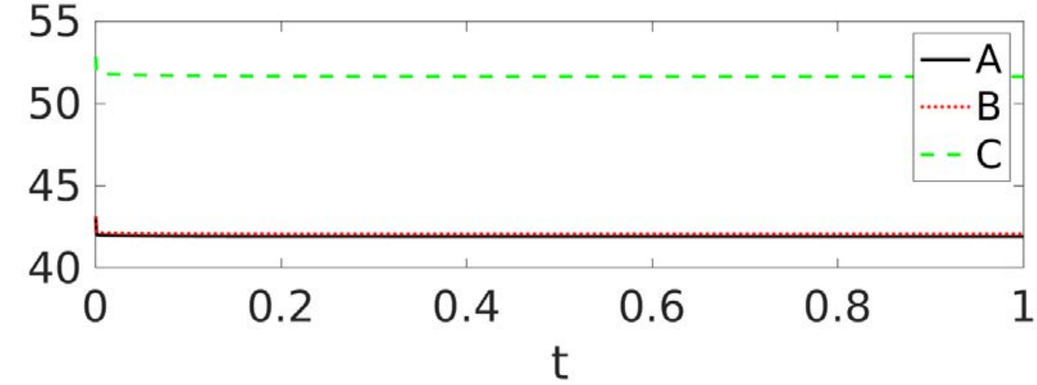
(a) Case A (b) Case B (c) Case C

Figure 3: Experiment 1: time evolution of χ, from top to bottom are $t = 0, 0.001, 0.1, 1$

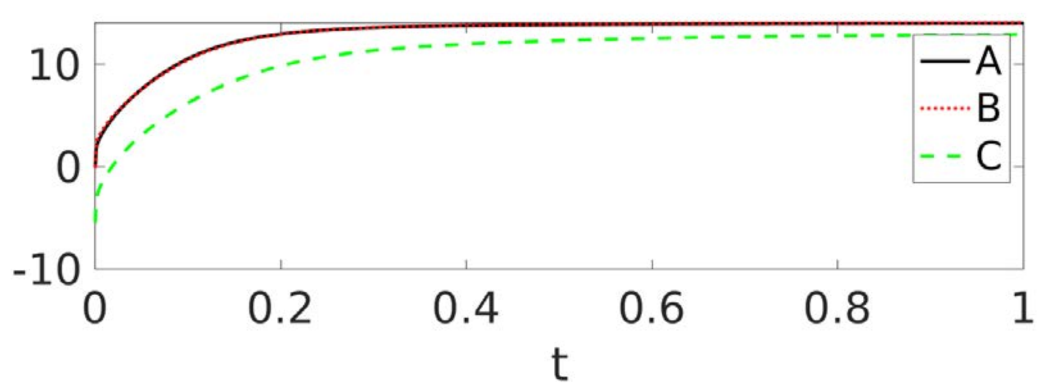
supports the conservation of mass and stability of energy, and entropy. Further, we show the time evolution of ϱ_h and χ_h and ϑ_h in Figure 5, 6, and 7, respectively.



(a) Mass

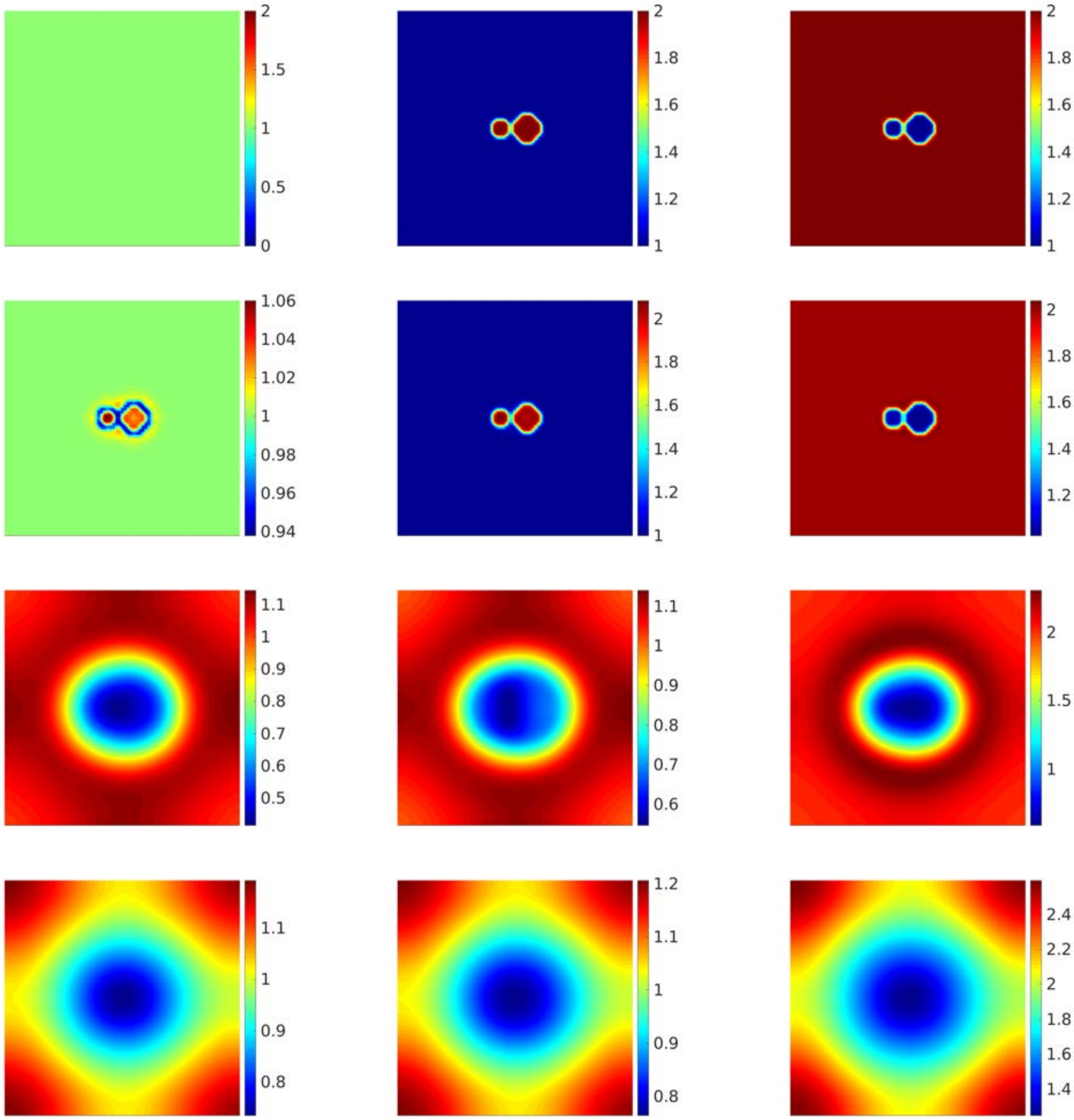


(b) Energy



(c) Entropy

Figure 4: Time evolution of the total mass, energy and entropy

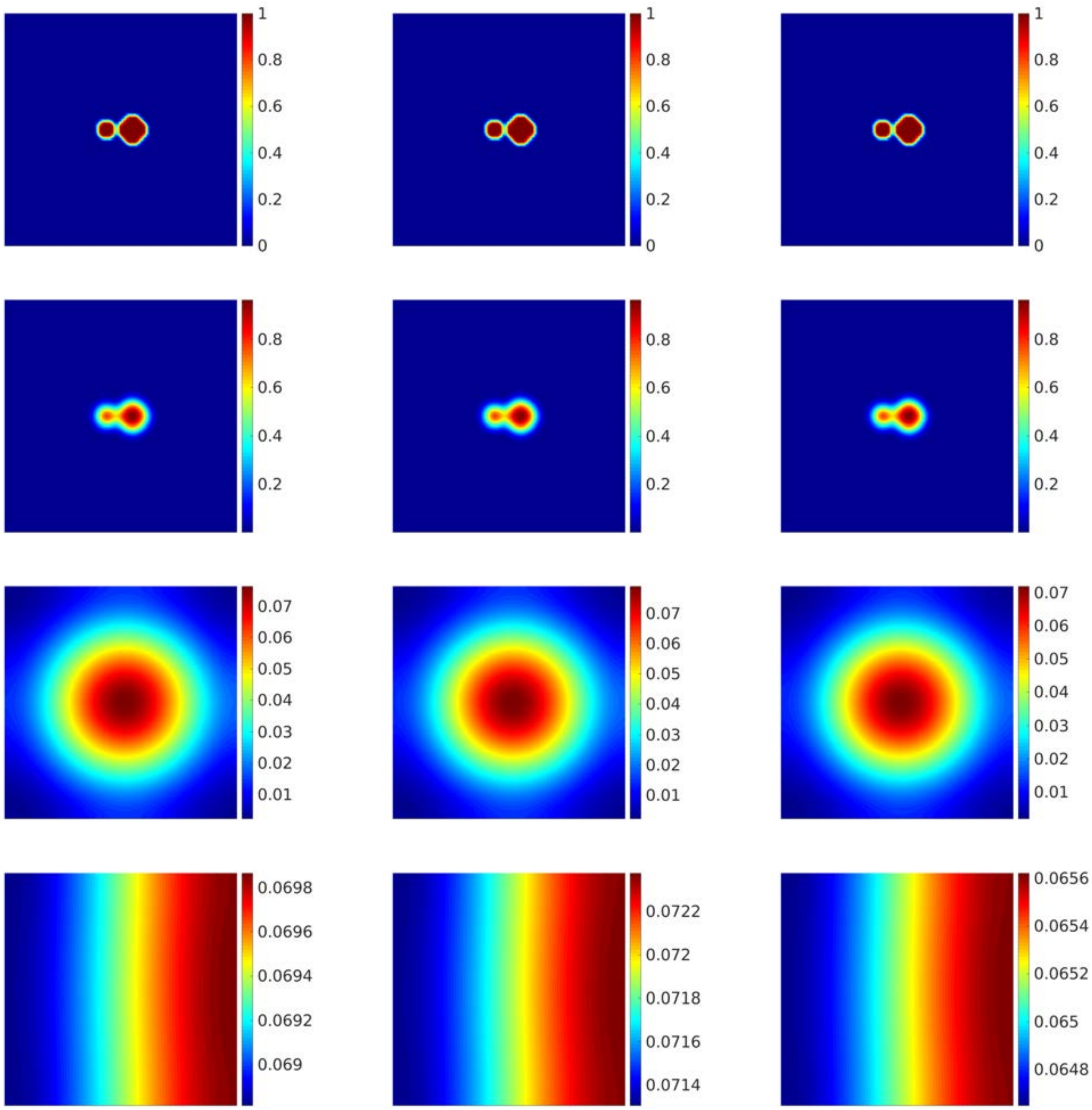


(a) Case A (b) Case B (c) Case C

Figure 5: Experiment 2: time evolution of ϱ, from top to bottom are $t = 0, 0.001, 0.1, 1$

5 Conclusion

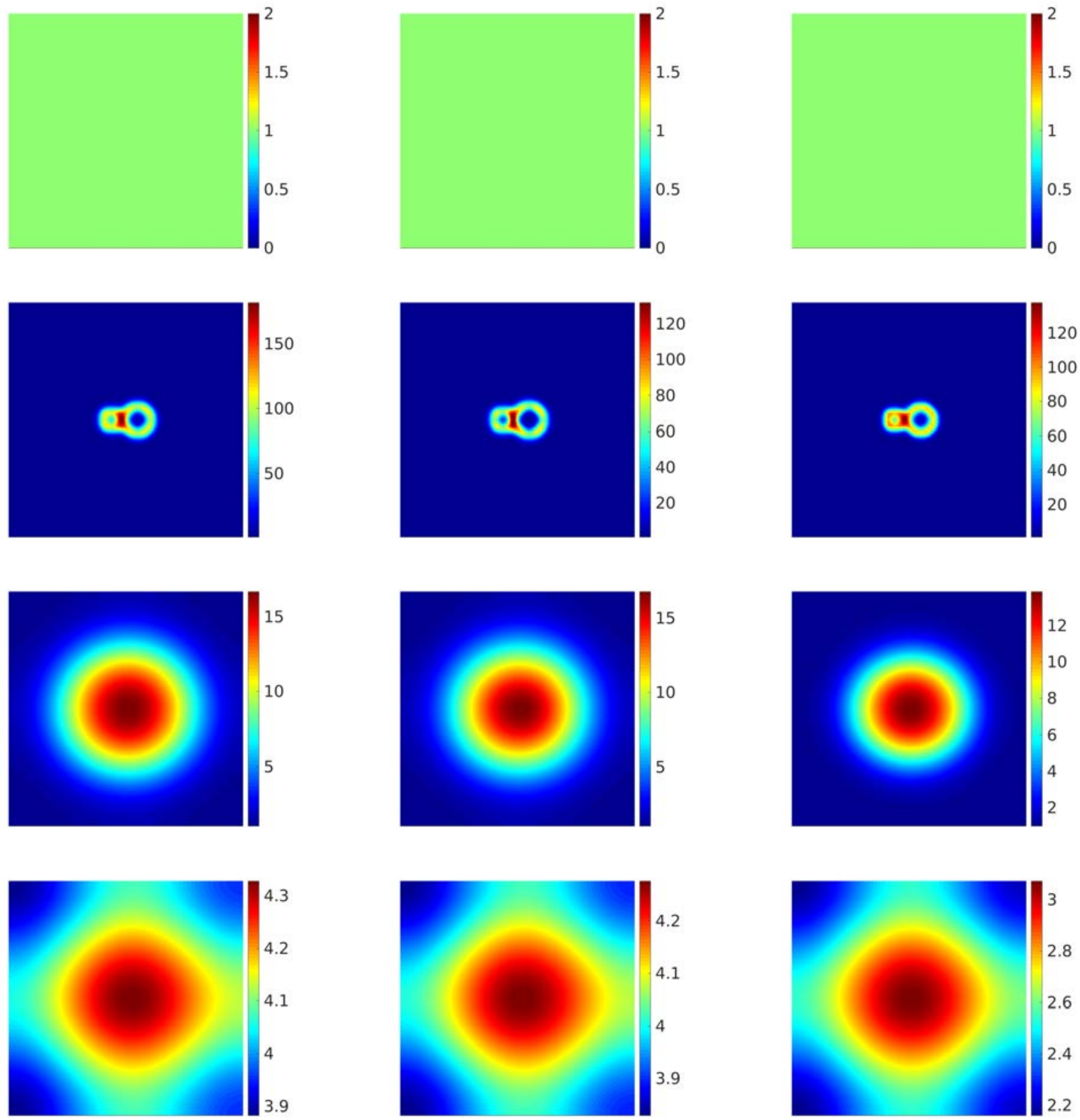
In this paper, we have studied the compressible Navier-Stokes-Allen-Cahn system with both isentropic gas law and ideal gas law. By using central difference, upwinding, and artificial diffusion



(a) Case A (b) Case B (c) Case C

Figure 6: Experiment 2: time evolution of χ, from top to bottom are $t = 0, 0.001, 0.1, 1$

techniques, we have proposed a finite volume method. We have shown that the finite volume method is entropy stable for both isentropic and ideal gas laws. We have also validated the theoretical results by numerical experiments.



(a) Case A (b) Case B (c) Case C

Figure 7: Experiment 2: time evolution of ϑ_h, from top to bottom are $t = 0, 0.001, 0.1, 1$

Remark 5.1. Here we open a few technical discussions.

- We point out that the artificial diffusion term $h^\varepsilon [\![r_h]\!]$ is not necessary for the proof of stability, but plays an important role if one wants to show the consistency of the methods, see [7].

Indeed, when setting $\varepsilon = \infty$, we have $h^\varepsilon = 0$ and the diffusive flux $F_\varepsilon^{\rm up}$ defined in (2.1) becomes the standard upwind flux.

- The current paper can be viewed as the preceding chapter of the convergence analysis of the method, see our recent work on the barotropic Navier–Stokes–Allen-Cahn [9] and Navier–Stokes–Fourier [6].

References

[1] H. Abels and E. Feireisl. On a diffuse interface model for a two-phase flow of compressible viscous fluids. *Indiana University Mathematics Journal* **57**: No.2, 2008.

[2] T. Blesgen. A generalization of the Navier-Stokes equations to two-phase flow. *J. Phys. D Appl. Phys.* **32**:1119–1123, 1999.

[3] P. G. Ciarlet. The Finite Element Method for Elliptic Problems. *Classics in Applied Mathematics, Society for Industrial and Applied Mathematics*, 2002.

[4] W. Dreyer, J. Giesselmann, and C. Kraus. A compressible mixture model with phase transition. *Physica D*, **273-274**:1–13, 2014.

[5] E. Feireisl M. Lukáčová-Medviďová, H. Mizerová, and B. She. Convergence of a finite volume scheme for the compressible Navier–Stokes system. *ESAIM: M2AN* **53** (6): 1957–1979, 2019.

[6] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová and B. She. On the convergence of a finite volume method for the Navier–Stokes–Fourier system. *IMA J. Numer. Anal.* **41**(4): 2388–2422, 2021.

[7] E. Feireisl, M. Lukáčová-Medviďová, H. Mizerová and B. She. *Numerical analysis of compressible fluid flows.* Springer Cham, 2021.

[8] E. Feireisl, M. Petcu and D. Pražák. Relative energy approach to a diffuse interface model of a compressible two-phase flow. *Math. Methods Appl. Sci.*, **42**(5):1465–1479, 2019.

[9] E. Feireisl, M. Petcu and B. She. Numerical analysis of a model of two phase compressible fluid flow. *J. Sci. Comput.*, **89**(1): No.14, 2021.

[10] E. Feireisl, H. Petzeltová, E. Rocca, G. Schimperna. Analysis of a phase-field model for two-phase compressible fluids, *Math. Models Methods Appl. Sci.* **20**: 1129–1160, 2010.

[11] E. VanderZee, A. N. Hirani, D. Guoy and E. A. Ramos. Well-centered triangulation. *SIAM J. Sci. Comput.* **31**(6): 4497–4523, 2010.