INSTITUTE OF MATHEMATICS

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Preprint No. 34-2022

PRAHA 2022

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Abstract

We consider the Oberbeck–Boussinesq system with non–local boundary conditions arising as a singular limit of the full Navier–Stokes–Fourier system in the regime of low Mach and low Froude number. The existence of strong solutions is shown on a maximal time interval $[0, T_{\max})$. Moreover, $T_{\max} = \infty$ in the two dimensional setting.

Keywords: Oberbeck–Boussinesq system, non–local boundary condition, strong solution

Dedicated to Constantine Dafermos of the occassion of his 80-th birthday

1 Introduction

We consider the Oberbeck–Bousinesq system describing the motion of an incompressible viscous and heat conducting fluid confined to a bounded domain $\Omega \subset R^d$, $d = 2, 3$. The velocity $\mathbf{v} = \mathbf{v}(t, x)$ of the fluid and the temperature $\Theta = \Theta(t, x)$ satisfy the following system of equations:

*The work of A. A. was supported by the ERC-STG Grant n. 759229 HiCoS "Higher Co-dimension Singularities: Minimal Surfaces and the Thin Obstacle Problem".

†The work of E.F. was partially supported by the Czech Sciences Foundation (GAČR), Grant Agreement 21–02411S. The Institute of Mathematics of the Academy of Sciences of the Czech Republic is supported by RVO:67985840.

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = \mu \Delta_x \mathbf{v} - \Theta \nabla_x G, \tag{1.1}$$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{1.2}$$

$$\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{v}) + a \mathrm{div}_x(G\mathbf{v}) = \kappa \Delta_x \Theta; \tag{1.3}$$

supplemented with the no-slip boundary condition for the velocity

$$\mathbf{v}|_{\partial\Omega} = 0; \tag{1.4}$$

and the non–local Dirichlet boundary condition for the temperature

$$\Theta|_{\partial\Omega} = \Theta_B - \lambda \fint_\Omega \Theta \, \mathrm{d}x. \tag{1.5}$$

The problem (1.2)–(1.5) has been identified as a singular limit of the full Navier–Stokes–Fourier system in the low Mach and low Froude number regime, see [2]. In this context, the temperature Θ is interpreted as a deviation from a background temperature and as such need not be non–negative. The symbol $G = G(x)$ stands for the gravitational potential and $\Theta_B = \Theta_B(x)$ is the prescribed boundary temperature. The transport coefficients μ and κ are positive constants, $a \in R$ and $\lambda > 0$. In addition, we assume

$$\fint_\Omega G \, \mathrm{d}x = 0, \ \ \Delta_x G = 0. \tag{1.6}$$

There is a vast amount of literature concerning the Oberbeck–Boussinesq system, see e.g. Constantin and Doering [4], Foias, Manley and Temam [8], Li and Titi [12] or the survey by Zeytounian [16], and the references therein. The main novelty of the present paper is the non–local term in (1.5) arising in the singular limit of the complete Navier–Stokes–Fourier system.

Parabolic problems with non–local boundary conditions have been studied by a number of authors, see e.g. Day [5], Friedman [9], and, more recently, Pao [13] among others. Unfortunately, most of the results concern the case $|\lambda| \leq 1$, while such a restriction cannot be justified in the singular limit process. Accordingly, we focus on the general case $\lambda > 0$. In particular, solutions of (1.3), (1.5) may not comply with the standard maximum principle.

The plan of the paper is as follows. First, following [7, Chapter 5], we construct a weak solution of the problem for any finite energy initial data. These solutions are global in time for both $d = 2$ and $d = 3$. In addition, we show that the temperature Θ of any weak solution is necessarily bounded, see Section 2. As a result, we obtain weak solutions of the Navier–Stokes system (1.2), (1.1) driven by a bounded force. Applying the L^p regularity theory due to Gerhardt [10], Giga and Miyakawa [11], Solonnikov [14], von Wahl [15], together with the abstract weak–strong uniqueness result [1], we conclude that the velocity field $\mathbf{v}$ is in fact regular. This observation together with a standard bootstrap argument yields regularity of the temperature, and, consequently, the existence of strong solutions defined on some maximal time interval $[0, T_{\max})$. In addition, $T_{\max} = \infty$ if $d = 2$, see Section 3. Finally, repeating the bootstrap argument, we conclude the problem admits a classical smooth solutions as soon as the data are smooth satisfying the associated compatibility conditions, see Section 4.

2 Weak solutions

Without loss of generality, we may assume $a = 0$. Indeed replacing $\Theta \approx \Theta + aG$ and using hypothesis (1.6) we obtain a new system

$$\partial_t \mathbf{v} + \mathrm{div}_x(\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = \mu \Delta_x \mathbf{v} - \Theta \nabla_x G, \tag{2.1}$$

$$\mathrm{div}_x \mathbf{v} = 0, \tag{2.2}$$

$$\partial_t \Theta + \mathrm{div}_x(\Theta \mathbf{v}) = \kappa \Delta_x \Theta, \tag{2.3}$$

$$\mathbf{v}|_{\partial\Omega} = 0, \tag{2.4}$$

$$\Theta|_{\partial\Omega} = \Theta_B - \lambda \fint_\Omega \Theta \, \mathrm{d}x, \tag{2.5}$$

with $\Theta_B \approx \Theta_B + aG$.

Definition 2.1 (Weak solution). We say that $\mathbf{v}$, Θ is *weak solution* to problem (2.2)–(2.5) with the initial data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \Theta(0, \cdot) = \Theta_0$$

if the following holds:

- Regularity.

$$\begin{aligned} &\mathbf{v} \in L^\infty(0, T; L^2(\Omega; R^d)) \cap L^2(0, T; W_0^{1,2}(\Omega; R^d)), \\ &\Theta \in L^\infty((0, T) \times \Omega) \cap L^2(0, T; W^{1,2}(\Omega)). \end{aligned} \tag{2.6}$$

- Equations of motion.

$$\begin{aligned} &\int_0^T \int_\Omega \Big[\mathbf{v} \cdot \partial_t \boldsymbol{\varphi} + (\mathbf{v} \otimes \mathbf{v}) : \nabla_x \boldsymbol{\varphi} \Big] \ \mathrm{d}x \, \mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \mu \nabla_x \mathbf{v} : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_\Omega \Theta \nabla_x G \cdot \boldsymbol{\varphi} \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{v}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x \end{aligned} \tag{2.7}$$

  for any $\boldsymbol{\varphi} \in C_c^1([0, T) \times \Omega; R^d)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$,

$$\mathrm{div}_x \mathbf{v} = 0 \text{ a.a. in } (0, T) \times \Omega. \tag{2.8}$$

- Heat equation.

$$\int_0^T \int_\Omega \Big[\Theta \partial_t \varphi + \Theta \mathbf{v} \cdot \nabla_x \varphi \Big] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \kappa \nabla_x \Theta \cdot \nabla_x \varphi \ \mathrm{d}x \, \mathrm{d}t - \int_\Omega \Theta_0 \varphi(0, \cdot) \ \mathrm{d}x \tag{2.9}$$

  for any $\varphi \in C^1_c([0,T) \times \Omega)$.

- Boundary conditions.

$$\Theta + \lambda \fint_\Omega \Theta \, \mathrm{d}x - \Theta_B \in L^2(0,T; W^{1,2}_0(\Omega)). \tag{2.10}$$

- Mechanical energy inequality.

$$\frac{1}{2} \int_\Omega |\mathbf{v}|^2(\tau, \cdot) \ \mathrm{d}x + \mu \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}|^2 \ \mathrm{d}x \, \mathrm{d}t \leq \frac{1}{2} \int_\Omega |\mathbf{v}_0|^2 \ \mathrm{d}x - \int_0^\tau \int_\Omega \Theta \nabla_x \mathbf{G} \cdot \mathbf{v} \ \mathrm{d}x \, \mathrm{d}t \tag{2.11}$$

  for any $0 \leq \tau \leq T$.

Remark 2.2. In (2.10), we tacitly assume that Θ_B has been suitably extended inside Ω.

2.1 Existence of weak solution

Our first result asserts global–in–time existence of weak solutions.

Theorem 2.3 (Weak solutions). *Suppose that* $\Omega \subset R^d$, $d = 2,3$ *is a bounded Lipschitz domain, and*

$$G \in W^{1,\infty}(\Omega), \ \Theta_B \in C(\overline{\Omega}) \cap W^{1,2}(\Omega).$$

Let the initial data satisfy

$$\mathbf{v}_0 \in L^2(\Omega; R^d), \ \mathrm{div}_x \mathbf{v}_0 = 0, \ \mathbf{v}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0,$$
$$\Theta_0 \in C(\overline{\Omega}), \ \Theta_0 + \lambda \fint_\Omega \Theta_0 \, \mathrm{d}x = \Theta_B \ \textit{on} \ \partial\Omega. \tag{2.12}$$

Then for any $T > 0$*, the problem* (2.2)–(2.5) *admits a weak solution* $\mathbf{v}$*,* Θ *in the sense of Definition 2.1.*

The rest of this section is devoted to the proof of Theorem 2.3.

2.2 Faedo–Galerkin approximation

First, suppose that $\partial\Omega$ is smooth. Similarly to [7, Chapter 5], approximate solutions to problem (2.2)–(2.5) are constructed by means of the Faedo–Galerkin approximation of the momentum equation (2.1) and exact solutions to the heat equation (2.3), (2.5).

Let $\{\mathbf{w}_n\}_{n=1}^{\infty}$ be the orthogonal basis of the Hilbert space of solenoidal functions $L^2_\sigma(\Omega; R^d)$ formed by the eigenfunctions of the Stokes operator, with the associated orthogonal projections Π_N,

$$\Pi_N : L^2_\sigma \to X_N \equiv \mathrm{span}\{\mathbf{w}_1, \dots, \mathbf{w}_N\},\ N = 1, 2, \dots .$$

The approximate velocity fields $\mathbf{v}_N \in C^1([0,T]; X_N)$ are obtained as solutions of the system of ordinary differential equations:

$$\partial_t \mathbf{v}_N + \Pi_N[\mathrm{div}_x(\mathbf{v}_N \otimes \mathbf{v}_N)] = \mu \Delta_x \mathbf{v}_N - \Pi_N[\Theta \nabla_x G],\ \mathbf{v}_N(0,\cdot) = \Pi_N \mathbf{v}_0, N = 1, 2, \dots \tag{2.13}$$

The approximate temperature $\Theta = \Theta_N$ is *exact* solution to the problem

$$\partial_t \Theta_N + \mathrm{div}_x(\mathbf{v}_N \Theta_N) = \kappa \Delta_x \Theta_N,\ \Theta_N|_{\partial\Omega} = \Theta_B - \lambda \fint_\Omega \Theta_N \,\mathrm{d}x,\ \Theta_N(0,\cdot) = \Theta_0. \tag{2.14}$$

Lemma 2.4 (Uniqueness for the heat equation). *Given a velocity*

$$\mathbf{v} \in L^\infty((0,T) \times \Omega; R^d),$$

there is at most one weak solution Θ *of the problem*

$$\partial_t \Theta + \mathrm{div}_x(\mathbf{v}\Theta) = \kappa \Delta_x \Theta,\ \Theta|_{\partial\Omega} = \Theta_B - \lambda \fint_\Omega \Theta \,\mathrm{d}x,\ \Theta(0,\cdot) = \Theta_0, \tag{2.15}$$

in the sense specified in (2.9), (2.10) *in Definition 2.1 in the class*

$$\Theta \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega))$$

Proof. Let Θ_1, Θ_2 be two solutions with the same initial value Θ_0. Accordingly,

$$\hat{\Theta} = \Theta_1 - \Theta_2$$

solves the problem

$$\partial_t \hat{\Theta} + \mathrm{div}_x(\hat{\Theta}\mathbf{v}) = \kappa \Delta_x \hat{\Theta},\ \hat{\Theta}|_\Omega = -\lambda \fint_\Omega \hat{\Theta} \,\mathrm{d}x,\ \hat{\Theta}(0,\cdot) = 0.$$

Thus, introducing a new quantity

$$V = \hat{\Theta} + \lambda \fint_\Omega \hat{\Theta} \,\mathrm{d}x,$$

we obtain

$$\partial_t \left(V - \frac{\lambda}{1+\lambda} \fint_\Omega V \, {\rm d}x \right) + {\rm div}_x (\mathbf{v} V) = \kappa \Delta_x V, \ V|_{\partial \Omega} = 0, \ V(0, \cdot) = 0, \tag{2.16}$$

where the differential equation is satisfied in the sense of distributions. As

$$V \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),$$

we may justify multiplication of equation (2.16) on V and subsequent integration by parts yielding

$$\frac{1}{2} \left[\fint_\Omega |V|^2 (\tau, \cdot) \, {\rm d}x - \frac{\lambda}{1 + \lambda} \left(\fint_\Omega V \, {\rm d}x \right)^2 \right] + \kappa \int_0^\tau \fint_\Omega |\nabla_x V|^2 \, {\rm d}x = 0 \text{ for any } 0 \leq \tau \leq T.$$

Consequently $V = 0 \ \Rightarrow \ \hat{\Theta} = 0$. □

In view of Lemma 2.4, we may consider a well defined mapping

$$\mathbf{v}_N \mapsto \Theta(\mathbf{v}_N) = \Theta_N,$$

where Θ_N is the unique solution of (2.14) as soon as we show such a solution exists.

The proof of solvability of (2.14) for a given $\mathbf{v}_N$ follows the same line of arguments as the proof of Lemma 2.4. First, we introduce

$$V = \Theta + \lambda \fint_\Omega \Theta \, {\rm d}x$$

transforming the problem to

$$\partial_t \left(V - \frac{\lambda}{1+\lambda} \fint_\Omega V \, {\rm d}x \right) + {\rm div}_x (\mathbf{v}_N V) = \kappa \Delta_x V, \ V|_{\partial \Omega} = \Theta_B, \ V(0, \cdot) = \Theta_0 + \lambda \fint_\Omega \Theta_0 \, {\rm d}x. \tag{2.17}$$

Solutions of (2.17) may be written in the form

$$V = Z + \Theta_B,$$

where

$$\partial_t \left(Z - \frac{\lambda}{1+\lambda} \fint_\Omega Z \, {\rm d}x \right) + {\rm div}_x (\mathbf{v}_N (Z + \Theta_B)) = \kappa \Delta_x Z, \ Z|_{\partial \Omega} = 0, \ Z(0, \cdot) = \Theta_0 + \lambda \fint_\Omega \Theta_0 \, {\rm d}x - \Theta_B. \tag{2.18}$$

Here, for the sake of simplicity, we have considered the harmonic extension of Θ_B inside Ω. Seeing that the mapping

$$\mathcal{L}[V] = V - \frac{\lambda}{1 + \lambda} V, \ \lambda > 0 \text{ is a self–adjoint isomphism on the Hilbert space } L^2(\Omega),$$

we may solve (2.18) by a Faedo–Galerkin method based on the system of eigenfuctions of the Dirichlet Laplacean on the domain Ω. Note that this approximation is compatible with the associated "energy" estimates based on multiplication of (2.18) on Z:

$$\frac{1}{2}\left[\fint_\Omega |Z|^2(\tau,\cdot)\,{\rm d}x - \frac{\lambda}{1+\lambda}\left(\fint_\Omega Z\,{\rm d}x \right)^2 \right] + \kappa \int_0^\tau \fint_\Omega |\nabla_x Z|^2\,{\rm d}x$$

$$\leq \frac{1}{2}\left[\fint_\Omega |Z_0|^2\,{\rm d}x - \frac{\lambda}{1+\lambda}\left(\fint_\Omega Z_0\,{\rm d}x \right)^2 \right] + \int_0^\tau \int_\Omega \Theta_B \mathbf{v}_N \cdot \nabla_x Z \ {\rm d}x \tag{2.19}$$

for any $0 \leq \tau \leq T$, where

$$Z_0 = \Theta_0 + \lambda \fint_\Omega \Theta_0\,{\rm d}x - \Theta_B.$$

Inequality (2.19) yields the desired uniform bounds so that the Faedo–Galerkin approximation converges to a unique solution in the class

$$Z \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega)) \ \Rightarrow \ \Theta_N \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;W^{1,2}(\Omega)).$$

It follows from the previous discussion that local-in-time existence of the approximate solutions $\mathbf{v}_N$, Θ_N can be established exactly as in [7, Chapter 5]. Moreover, multiplying the approximate momentum equation on $\mathbf{v}_N$ we obtain an approximate version of the energy balance,

$$\frac{1}{2}\int_\Omega |\mathbf{v}_N|^2(\tau,\cdot)\ {\rm d}x + \mu \int_0^\tau \int_\Omega |\nabla_x \mathbf{v}_N|^2\ {\rm d}x\,{\rm d}t \leq \frac{1}{2}\int_\Omega |\mathbf{v}_0|^2\ {\rm d}x - \int_0^\tau \int_\Omega \Theta_N \nabla_x \mathbf{G} \cdot \mathbf{v}_N\ {\rm d}x\,{\rm d}t, \tag{2.20}$$

which, together with (2.19) written in terms of Θ_N,

$$\frac{1}{2}\frac{1}{1+\lambda}\int_\Omega \left| \Theta_N + \lambda \fint_\Omega \Theta_N\,{\rm d}x - \Theta_B \right|^2 (\tau,\cdot)\ {\rm d}x + \kappa \int_0^\tau \int_\Omega |\nabla_x(\Theta - \Theta_B)|^2\ {\rm d}x$$

$$\leq \frac{1}{2}\int_\Omega \left| \Theta_0 + \lambda \fint_\Omega \Theta_0\,{\rm d}x - \Theta_B \right|^2\ {\rm d}x + \int_0^\tau \int_\Omega \Theta_B \mathbf{v}_N \cdot \nabla_x(\Theta - \Theta_B)\ {\rm d}x, \tag{2.21}$$

yield the desired uniform bounds that guarantee global existence of the approximate solutions up to any desired time $T > 0$.

Finally, as the approximate velocities $\mathbf{v}_N$ are smooth, the standard parabolic local estimates yield smoothness of the approximate temperatures Θ_N in the interior of the space–time cylinder $(0,T) \times \Omega$. Moreover, thanks to the compatibility condition (2.12) imposed on the initial data, the approximate temperatures are continuous up to the boundary of $[0,T] \times \overline{\Omega}$. In particular, the standard maximum principle yields

$$\underline{\Theta} \leq \Theta_N(t,x) \leq \overline{\Theta} \text{ for } (t,x) \in [0,T] \times \Omega, \tag{2.22}$$

for some constants $\underline{\Theta}$, $\overline{\Theta}$ depending only on the initial and boundary data and T.

With the uniform bounds (2.20)–(2.22), it is a routine matter to perform the limit $N \to \infty$ in the family of approximate solutions $(\mathbf{v}_N; \Theta_N)_{N=1}^{\infty}$ and to show that the limit yields the desired weak solution the existence of which is claimed in Theorem 2.3. Note that, by means of the standard Aubin–Lions argument,

$$\Theta_N \to \Theta \text{ in } L^2((0,T) \times \Omega) \text{ and weakly in } L^2(0,T; W^{1,2}(\Omega));$$

which, by interpolation, yields the convergence of traces on $\partial\Omega$. We have proved Theorem 2.3.

Remark 2.5. The reader will have noticed that Theorem 2.3 requires the domain Ω to be merely Lipschitz, while the proof via Faedo–Galerkin approximation was based on smooth domains. However, the result can be extended to Lipschitz domains by their approximation by smooth ones. The details are left to the reader.

3 Strong solutions

Our next goal is to improve regularity of the weak solutions on condition that the data are smooth.

Theorem 3.1 (Strong solutions). *In addition to the hypotheses of Theorem 2.3, suppose that $\Omega \subset R^d$, $d = 2, 3$ is a bounded domain of class C^2,*

$$G \in W^{1,\infty}(\Omega), \ \Theta_B \in C^2(\overline{\Omega}), \tag{3.1}$$

and the initial data satisfy

$$\Theta_0 \in W^{2,p}(\Omega), \ \mathbf{v}_0 \in W^{2,p}(\Omega; R^d), \ \mathrm{div}_x \mathbf{v}_0 = 0, \ \textit{for any } 1 \leq p < \infty,$$
$$\textit{together with the compatibility conditions}$$
$$\mathbf{v}_0 = 0, \ \Theta_0 + \lambda \fint_{\Omega} \Theta_0 \, \mathrm{d}x|_{\partial\Omega} = \Theta_B \ \textit{on } \partial\Omega \tag{3.2}$$

Then there exists $T_{\max} > 0$, $T_{\max} = \infty$ if $d = 2$, such that any weak solution $\mathbf{v}$, Θ belongs to the regularity class

$$\mathbf{v} \in L^p(0,T; W^{2,p}(\Omega; R^d)), \ \partial_t \mathbf{v} \in L^p(0,T; L^p(\Omega; R^d)),$$
$$\Theta \in L^p(0,T; W^{2,p}(\Omega)), \ \partial_t \Theta \in L^p(0,T; L^p(\Omega; R^d)) \ \textit{for any } 1 \leq p < \infty \tag{3.3}$$

for any $0 < T < T_{\max}$.

The rest of this section is devoted to the proof of Theorem 3.1.

3.1 Regularity of the velocity field

As $\Theta \in L^\infty((0,T)\times\Omega)$ the velocity field $\mathbf{v}$ solves the Navier–Stokes system (2.2), (2.1) with a bounded driving force $\Theta\nabla_x G$, in particular, $\Theta\nabla_x G \in L^p((0,T)\times\Omega; R^d)$ for any $1 \le p < \infty$. In view of the available L^p- theory for the Stokes system due to Solonnikov [14] and its adaptation to the Navier–Stokes system by Giga and Miyakawa [11] (cf. also Bothe and Pruess [3]), there exists a time $T_{\max} > 0$ such that the Navier–Stokes system with the initial data in the class (3.2) and the right–hand side $\Theta\nabla_x G$ admits a strong solution $\mathbf{v}$,

$$\mathbf{v} \in L^p(0,T; W^{2,p}(\Omega; R^d)),\ \partial_t \mathbf{v} \in L^p(0,T; L^p(\Omega; R^d))$$

for any $1 \le p < \infty$ and any $0 < T < T_{\max}$. Moreover, as shown by Gerhardt [10] and von Wahl [15], $T_{\max} = \infty$ if $d = 2$.

Finally, as any weak solution in the sense of Definition 2.1 satisfies the energy inequality (2.11), we conclude, using the general weak–strong uniqueness principle established in [1], that the velocity component $\mathbf{v}$ necessarily enjoys the regularity claimed in (3.3).

3.2 Regularity of the temperature

In view of the previous discussion, we already know that Θ satisfies equation 2.3 with *regular* drift term, in particular Θ is regular inside Ω. Moreover, given $\mathbf{v}$, the solution Θ is unique in the class of weak solutions, see Lemma 2.4. Consequently, it is enough to establish *a priori* bounds that would guarantee Lipschitz continuity of the average $\fint_\Omega \Theta \, dx$ in time. Indeed the abstract theory of Denk, Hieber, and Pruess [6] would then guarantee the required L^p-regularity of Θ. Note that the existing bounds established so far only imply

$$\Theta \in C_{\rm weak}([0,T]; L^2(\Omega)) \ \Rightarrow \ \fint_\Omega \Theta \, dx \in C[0,T].$$

The desired *a priori* bound follow by differentiating equation (2.3) in time. Setting

$$\partial_t \Theta = \vartheta$$

we obtain

$$\partial_t \vartheta + {\rm div}_x(\mathbf{v}\vartheta) = \kappa \Delta_x \vartheta - {\rm div}_x(\partial_t \mathbf{v}\Theta) \text{ in } (0,T)\times\Omega \tag{3.4}$$

with the boundary condition

$$\vartheta = -\lambda \fint_\Omega \vartheta \, dx \text{ on } \partial\Omega, \tag{3.5}$$

and the initial condition

$$\vartheta(0,\cdot) = \vartheta_0 = \kappa\Delta_x \Theta_0 - {\rm div}_x(\mathbf{v}_0 \Theta_0). \tag{3.6}$$

Similarly to the above, we consider

$$V = \vartheta + \lambda\vartheta,$$

rewriting (3.4) in the form

$$\partial_t \left(V - \frac{\lambda}{1+\lambda} \fint_\Omega V \,\mathrm{d}x \right) + \mathrm{div}_x(\mathbf{v}V) = \kappa \Delta_x V - \mathrm{div}_x(\partial_t \mathbf{v} \Theta),\ V|_{\partial\Omega} = 0. \tag{3.7}$$

Evoking the energy estimate (2.19) we get

$$\frac{1}{2}\frac{1}{1+\lambda}\int_\Omega |V|^2 (\tau,\cdot)\ \mathrm{d}x + \kappa \int_0^\tau \int_\Omega |\nabla_x V|^2\ \mathrm{d}x \leq \frac{1}{2}\int_\Omega |V_0|^2\ \mathrm{d}x + \int_0^\tau \int_\Omega \Theta \partial_t \mathbf{v} \cdot \nabla_x V\ \mathrm{d}x \tag{3.8}$$

for any $0 \leq \tau \leq T$. Seeing that $\Theta \in L^\infty((0,T)\times\Omega)$, $\partial_t \mathbf{v} \in L^2(0,T;L^2(\Omega;R^d))$ we deduce the desired bound

$$V \in L^\infty(0,T;L^2(\Omega)) \ \Rightarrow\ \partial_t \Theta \in L^\infty(0,T;L^2(\Omega)) \ \Rightarrow\ \fint_\Omega \Theta \,\mathrm{d}x \in W^{1,\infty}(0,T).$$

We have proved Theorem 3.1.

4 Classical solutions

Finally, we claim the following result that can be obtained as a consequence of the standard parabolic theory given the regularity of the weak solution established in Theorem 3.1.

Theorem 4.1 (Classical solutions). *In addition to the hypotheses of Theorem 2.3, suppose that $\Omega \subset R^d$, $d = 2,3$ is a bounded domain of class $C^{2+\nu}$, $\nu > 0$,*

$$G \in C^{1+\mu}(\overline{\Omega}),\ \Theta_B \in C^{2+\nu}(\overline{\Omega}).$$

and the initial data satisfy

$$\begin{aligned} \Theta_0 \in C^{2+\nu}(\Omega),\ \mathbf{v}_0 \in C^{2+\nu}(\Omega;R^d),\ \mathrm{div}_x \mathbf{v}_0 = 0, \\ \mathbf{v}_0 = 0,\ \Theta_0 + \lambda \fint_\Omega \Theta_0 \,\mathrm{d}x = \Theta_B \ \text{on } \partial\Omega, \\ -\mu \Delta_x \mathbf{v}_0 + \Theta_0 \nabla_x G = -\nabla_x \Pi_0 \ \text{on } \partial\Omega, \\ \kappa \Delta_x \Theta_0 = -\frac{\lambda\kappa}{|\Omega|}\int_{\partial\Omega} \nabla_x \Theta_0 \cdot \mathbf{n} \,\mathrm{d}\sigma_x \ \text{on } \partial\Omega. \end{aligned} \tag{4.1}$$

Then there exists $T_{\max} > 0$, $T_{\max} = \infty$ if $d = 2$, such that any weak solution $\mathbf{v}$, Θ is a classical solution, specifically,

$$\mathbf{v},\ \nabla_x^2 \mathbf{v},\ \partial_t \mathbf{v} \in C^\beta([0,T]\times\overline{\Omega};R^d),$$

$$\Theta,\ \nabla_x^2\Theta,\ \partial_t\Theta \in C^\beta([0,T]\times\overline{\Omega}) \text{ for some } \beta > 0, \tag{4.2}$$

for any $0 < T < T_{\max}$.

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