Euler system in fluid dynamics: Good and bad news

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Euler system of gas dynamics

Leonhard Paul Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) = a\varrho^\gamma$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \Omega \subset R^d, \ \ d = 2, 3$$

Initial state (data)

$$\varrho(0,\cdot) = \varrho_0, \ \ (\varrho \mathbf{u})(0,\cdot) = \varrho_0 \mathbf{u}_0$$

Admissibility

Energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma$$

Dissipative (weak) solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho, \mathbf{u})\ \mathrm{d}x \leq 0,\ \int_\Omega E(\varrho, \mathbf{u})(\tau, \cdot)\ \mathrm{d}x \leq \int_\Omega E(\varrho_0, \mathbf{u}_0)\ \mathrm{d}x$$

Admissible (weak) solutions

$$\partial_t E(\varrho, \mathbf{u}) + \mathrm{div}_x \Big(E(\varrho, \mathbf{u})\mathbf{u} + p(\varrho)\mathbf{u} \Big) \leq 0,\ E(\varrho, \mathbf{u})(\tau, \cdot) \nearrow E(\varrho_0, \mathbf{u}_0),\ \tau \to 0$$

Wild data

Initial state

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0$$

The initial data are *wild* if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0,\tau]$, $0 < \tau < T$

Theorem (E. Chiodaroli, EF 2022) The set of wild data is dense in $L^2 \times L^2$

E. Chiodaroli (Pisa)

Related results for the incompressible Euler system by Székelyhidi–Wiedemann, Daneri–Székelyhidy

Related results for the barotropic Euler system by Ming, Vasseur, and You

$$\int_\Omega E(\varrho,\mathbf{u})(\tau)\ \mathrm{d}x \leq \int_\Omega E(\varrho_0,\mathbf{u}_0)\ \mathrm{d}x,\ \tau \geq 0$$

Application of convex integration, I

- **Step 1:** Euler system admits local in time *smooth* solutions
- **Step 2:** Use the smooth solutions as "subsolutions" for convex integration (idea of Ming, Vasseur, You)
- **Step 3:** Construct solutions with prescribed energy profile by "non–constant" version of convex integration

Weak vs. strong continuity

$$\mathbf{U} = [\varrho, \mathbf{m}],\ \mathbf{m} = \varrho \mathbf{u}$$

Weak continuity

$$\mathbf{U} \in C_{\rm weak}([0,T]; L^p(\Omega; R^d)),\ t \mapsto \int_\Omega \mathbf{U} \cdot \boldsymbol{\varphi} \ {\rm d}x \in C[0,T]$$

$$\boldsymbol{\varphi} \in L^{p'}(\Omega; R^d)$$

Strong continuity

$$\tau \in [0,T],\ \| \mathbf{U}(t, \cdot) - \mathbf{U}(\tau, \cdot) \|_{L^p(\Omega; R^d)} \text{ whenever } t \to \tau$$

Strong vs. weak

$$\text{strong } \Rightarrow \text{weak},\ \text{weak} \not\Rightarrow \text{strong}$$

Strong discontinuity

Theorem (A.Abbatiello, EF 2021)

Anna Abbatiello (Roma La Sapienza)

Let $d = 2, 3$. Let $\mathcal{R}$ denote the set of bounded Riemann integrable functions. Let ϱ_0, $\mathbf{m}_0$ be given such that

$$\varrho_0 \in \mathcal{R},\ 0 \leq \underline{\varrho} \leq \varrho_0 \leq \overline{\varrho},$$

$$\mathbf{m}_0 \in \mathcal{R},\ \mathrm{div}_x \mathbf{m}_0 \in \mathcal{R},\ \mathbf{m}_0 \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

Let $\{\tau_i\}_{i=1}^{\infty} \subset (0, T)$ be an arbitrary (countable dense) set of times.
Then the Euler problem admits infinitely many weak solutions ϱ, $\mathbf{m}$ with a strictly decreasing total energy profile such that

$$\varrho \in C_{\rm weak}([0, T]; L^{\gamma}(\Omega)),\ \mathbf{m} \in C_{\rm weak}([0, T]; L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d))$$

but

$$t \mapsto [\varrho(t, \cdot), \mathbf{m}(t, \cdot)] \ \boxed{\text{is not}}\ \text{strongly continuous at any } \tau_i$$

Application of convex integration, II

- **Step 1:** Extending "variable coefficients" convex integration to the class of Riemann coefficients
- **Step 2:** Constructing solutions with prescribed Riemann energy profile

Consistent approximation

Approximate equation of continuity

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \mathbf{m}_n \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = e_{1,n}[\varphi]$$

Approximate momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m}_n \cdot \partial_t \boldsymbol{\varphi} + \frac{\mathbf{m}_n \otimes \mathbf{m}_n}{\varrho_n} : \nabla_x \varphi + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x\mathrm{d}t = e_{2,n}[\boldsymbol{\varphi}]$$

Stability - approximate energy inequality

$$\int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_n|^2}{\varrho_n} + P(\varrho_n) \right] \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \mathrm{d}x + e_{3,n}$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ e_{2,n}[\boldsymbol{\varphi}] \to 0, \ e_{3,n} \to 0 \text{ as } n \to \infty$$

Dissipative solutions – limits of consistent approximations

Dominic Breit (Edinburgh)

Martina Hofmanová (Bielefeld)

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \left(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right) + \nabla_x p(\varrho) = -\mathrm{div}_x \mathfrak{R}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} E(t) \leq 0, \ E(t) \leq E_0, \ E_0 = \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}_0|^2}{\varrho_0} + P(\varrho_0) \right] \ \mathrm{d}x$$

$$E \equiv \int_\Omega \left[\frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right] \ \mathrm{d}x + c(\gamma) \int_{\overline{\Omega}} \mathrm{d} \ \mathrm{trace}[\mathfrak{R}]$$

Reynolds stress

$$\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(\overline{\Omega}; R^{d \times d}_{\mathrm{sym}}))$$

Basic properties of dissipative solutions

Well posedness, weak strong uniqueness

- **Existence.** Dissipative solutions exist globally in time for any finite energy initial data
- **Limits of consistent approximations** Limits of consistent approximations are dissipative solutions, in particular limits of consistent numerical schemes.
- **Compatibility.** Any C^1 dissipative solution $[\varrho, \mathbf{m}]$, $\varrho > 0$ is a classical solution of the Euler system
- **Weak–strong uniqueness.** If $[\widetilde{\varrho}, \widetilde{\mathbf{m}}]$ is a classical solution and $[\varrho, \mathbf{m}]$ a dissipative solution starting from the same initial data, then $\mathfrak{R} = 0$ and $\varrho = \widetilde{\varrho}$, $\mathbf{m} = \widetilde{\mathbf{m}}$.

Semiflow selection

Set of data

$$\mathcal{D} = \left\{ \varrho, \mathbf{m}, E \ \Big| \ \int_\Omega \frac{1}{2} \frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \ \mathrm{d}x \leq E \right\}$$

Set of trajectories

$$\mathcal{T} = \left\{ \varrho(t, \cdot), \mathbf{m}(t, \cdot), E(t-, \cdot) \Big| t \in (0, \infty) \right\}$$

Solution set

$$\mathcal{U}[\varrho_0, \mathbf{m}_0, E_0] = \Big\{ [\varrho, \mathbf{m}, E] \ \Big| [\varrho, \mathbf{m}, E] \text{ dissipative solution}$$

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{m}(0, \cdot) = \mathbf{m}_0, \ E(0+) \leq E_0 \Big\}$$

Semiflow selection – semigroup

$$U[\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{U}[\varrho_0, \mathbf{m}_0, E_0], \ [\varrho_0, \mathbf{m}_0, E_0] \in \mathcal{D}$$

$$U(t_1 + t_2)[\varrho_0, \mathbf{m}_0, E_0] = U(t_1) \circ \Big[U(t_2)[\varrho_0, \mathbf{m}_0, E_0] \Big], \ t_1, t_2 > 0$$

Andrej Markov (1856–1933)

N. V. Krylov

Adaptation of Krylov's method by Cardona and Kapitanski

Jorge Cardona (TU Darmstadt)

Successive minimization of cost functionals

$$I_{\lambda,F}[\varrho, \mathbf{m}, E] = \int_0^\infty \exp(-\lambda t) F(\varrho, \mathbf{m}, E)\ \mathrm{d}t, \ \ \lambda > 0$$

where

$$F : X = W^{-\ell,2}(\Omega) \times W^{-\ell,2}(\Omega; R^d) \times R \to R$$

is a bounded and continuous functional

Maximal dissipation principle

$$\tilde{E}(t) \le E(t) \ \Rightarrow E(t) = \tilde{E}(t)$$

$$\text{whenever, } \ \varrho_0 = \widetilde{\varrho}_0, \ \mathbf{m}_0 = \widetilde{\mathbf{m}}_0, \ E_0 = \widetilde{E}_0$$

Lev Kapitanskii (Miami)

Euler system and turbulence

Fluid domain and obstacle

$$Q = R^d \setminus B,\ d = 2,3$$

B compact, convex

Navier–Stokes system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$p(\varrho) \approx a\varrho^\gamma,\ \gamma > 1,\ \mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I},$$

Boundary and far field conditions

$$\mathbf{u}|_{\partial Q} = 0,\ \varrho \to \varrho_\infty,\ \mathbf{u} \to \mathbf{u}_\infty \text{ as } |x| \to \infty$$

High Reynolds number (vanishing viscosity) limit

Vanishing viscosity

$$\varepsilon_n \searrow 0,\ \ \mu_n = \varepsilon_n \mu, \mu > 0,\ \ \lambda_n = \varepsilon_n \lambda, \lambda \geq 0$$

Questions

- Identify the limit of the corresponding solutions $(\varrho_n, \mathbf{u}_n)$ as $n \to \infty$ in the fluid domain Q
- **Yakhot and Orszak [1986]:** *"The effect of the boundary in the turbulence regime can be modeled in a statistically equivalent way by fluid equations driven by stochastic forcing"*

  Clarify the meaning of "statistically equivalent way"

  Is the (compressible) Euler system driven by a general cylindrical white noise force adequate to describe the limit of $(\varrho_n, \mathbf{u}_n)$?

Statistical limit

Trajectory space

$$(\varrho_n, \mathbf{m}_n) \in \mathcal{T} \equiv C_{\rm weak}([0,T]; L^{\gamma}_{\rm loc}(\Omega) \times L^{\frac{2\gamma}{\gamma+1}}_{\rm loc}(\Omega; R^d))$$

$$\mathcal{V}_N = \frac{1}{N} \sum_{n=1}^{N} \delta_{(\varrho_n, \mathbf{m}_n)}, \ \mathbf{m}_n = \varrho_n \mathbf{u}_n$$

Prokhorov theorem $\Rightarrow$ $\mathcal{V}_N \to \mathcal{V}$ narrowly in $\mathfrak{P}[\mathcal{T}]$

$(\varrho, \mathbf{m}) \approx \mathcal{V}$ a random process with paths in $\mathcal{T}$

Hypothetical limit system

Euler system with stochastic forcing

$$\mathrm{d}\widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} \mathrm{d}t = 0$$

$$\mathrm{d}\widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) \mathrm{d}t + \nabla_x p(\widetilde{\varrho}) \mathrm{d}t = \mathbf{F} \mathrm{d}W$$

$W = (W_k)_{k \geq 1}$ cylindrical Wiener process

$\mathbf{F} = (\mathbf{F}_k)_{k \geq 1}$ – diffusion coefficient

$$\mathbb{E} \left[\int_0^T \sum_{k \geq 1} \| \mathbf{F}_k \|^2_{W^{-\ell,2}(Q;R^d)} \mathrm{d}t \right] < \infty$$

we allow $\mathbf{F} = \mathbf{F}(\varrho, \mathbf{m})$

Statistical equivalence

statistical equivalence $\Leftrightarrow$ identity in expectation of some quantities

$$(\varrho, \mathbf{m}) \text{ statistically equivalent to } (\widetilde{\varrho}, \widetilde{\mathbf{m}})$$

$$\Leftrightarrow$$

- **density and momentum**

$$\mathbb{E}\left[\int_D \varrho\right] = \mathbb{E}\left[\int_D \widetilde{\varrho}\right], \ \mathbb{E}\left[\int_D \mathbf{m}\right] = \mathbb{E}\left[\int_D \widetilde{\mathbf{m}}\right]$$

- **kinetic and internal energy**

$$\mathbb{E}\left[\int_D \frac{|\mathbf{m}|^2}{\varrho}\right] = \mathbb{E}\left[\int_D \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\right], \ \mathbb{E}\left[\int_D p(\varrho)\right] = \mathbb{E}\left[\int_D p(\widetilde{\varrho})\right]$$

- **angular energy**

$$\mathbb{E}\left[\int_D \frac{1}{\varrho}(\mathbb{J}_{x_0} \cdot \mathbf{m}) \cdot \mathbf{m}\right] = \mathbb{E}\left[\int_D \frac{1}{\widetilde{\varrho}}(\mathbb{J}_{x_0} \cdot \widetilde{\mathbf{m}}) \cdot \widetilde{\mathbf{m}}\right]$$

$$D \subset (0,T) \times Q, \ x_0 \in R^d, \ \mathbb{J}_{x_0}(x) \equiv |x - x_0|^2 \mathbb{I} - (x - x_0) \otimes (x - x_0)$$

Results (EF, M. Hofmanová 2022)

Hypothesis:

$(\varrho, \mathbf{m})$ statistically equivalent to a solution of the stochastic Euler system $(\widetilde{\varrho}, \widetilde{\mathbf{m}})$

Conclusion:

- **Noise inactive**
  $(\varrho, \mathbf{m})$ is a statistical solution to a deterministic Euler system
- **S-convergence (up to a subsequence) to the limit system**

$$\frac{1}{N}\sum_{n=1}^{N} b(\varrho_n, \mathbf{m}_n) \to \mathbb{E}\left[b(\varrho, \mathbf{m})\right] \text{ strongly in } L^1_{\rm loc}((0,T)\times Q)$$

  for any $b \in C_c(R^{d+1})$, $\varphi \in C_c^\infty((0,T)\times Q)$

- **Conditional statistical convergence**

$$\text{barycenter } (\overline{\varrho}, \overline{\mathbf{m}}) \equiv \mathbb{E}\left[(\varrho, \mathbf{m})\right] \text{ solves the Euler system}$$

$$\Rightarrow$$

$$\frac{1}{N}\#\left\{ n \leq N \middle| \|\varrho_n - \overline{\varrho}\|_{L^\gamma(K)} + \|\mathbf{m}_n - \overline{\mathbf{m}}\|_{L^{\frac{2\gamma}{\gamma+1}}(K;R^d)} > \varepsilon \right\} \to 0$$

as $N \to \infty$ for any $\varepsilon > 0$, and any compact $K \subset [0,T]\times Q$

Conclusion

- Stochastically driven Euler system irrelevant in the description of compressible turbulence (slightly extrapolated statement)

Possible scenarios:

- **Oscillatory limit.** The sequence $(\varrho_n, \mathbf{m}_n)$ generates a Young measure. Its barycenter (weak limit of $(\varrho_n, \mathbf{m}_n)$) is not a weak solution of the Euler system. Statistically, however, the limit is a single object. This scenario is compatible with the hypothesis that the limit is independent of the choice of $\varepsilon_n \searrow 0 \Rightarrow$ computable numerically.
- **Statistical limit.** The limit is a statistical solution of the Euler system. In agreement with Kolmogorov hypothesis concerning turbulent flow advocated in the compressible setting by Chen and Glimm. This scenario is not compatible with the hypothesis that the limit is independent of $\varepsilon_n \searrow 0$ ($\Rightarrow$ numerically problematic) unless the limit is a monoatomic measure in which case the convergence must be strong.

Idea of the proof

- **Step 1:** Reynolds stress $\mathfrak{R} \in L^\infty(0, T; \mathcal{M}^+(Q, R^{d\times d}_{\rm sym})$
- **Step 2:**

$$\mathbb{E}\left[\mathrm{div}_x \mathfrak{R}\right] = \mathbb{E}\left[\mathrm{div}_x\left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}\right)\right] + \mathbb{E}\left[\text{stochastic integral}\right]$$

in $\mathcal{D}'((0, T) \times Q)$

- **Step 3** Use the fact that the stochastic integral is a martingale to show $\mathfrak{R} = 0$