Descriptive complexity of Banach spaces

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- a sBS $\equiv$ a separable Banach space
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Why?

- let P be a property of sBSs invariant under isomorphism / isometry
- then we may talk about the set of codes of those sBSs having property P
- we may investigate the complexity of this set (Borel, analytic, ... / open, G_δ, ...)

Example

Theorem (Szlenk, 1968; Bossard, 1993)

There is no reflexive sBS containing an isomorphic copy of every reflexive sBS.

Proof: In a certain coding of sBSs it holds:

- the set of codes of all subspaces of a fixed sBS is analytic
- the set of codes of all reflexive sBSs is not analytic (Bossard, 1993)

□

The ‘classical’ coding of sBSs by a standard Borel space

$C([0,1])$ contains an isometric copy of every sBS

$\mathcal{F}(C([0,1])) \equiv$ the set of all closed subsets of $C([0,1])$

$\mathcal{F}(C([0,1]))$ is equipped with the Effros-Borel structure, that is, with the σ-algebra generated by the sets

$$\{F \in \mathcal{F}(C([0,1])) : F \cap U \neq \emptyset\}, \quad U \subseteq C([0,1]) \text{ open}$$

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Fact

$\mathcal{F}(C([0,1]))$ is a standard Borel space.

Fact

$SB := \{F \in \mathcal{F}(C([0,1])) : F \text{ is linear}\}$ is a Borel subset of $\mathcal{F}(C([0,1]))$. In particular, it is a standard Borel space.

Note that there is no canonical Polish topology on SB compatible with the Effros-Borel structure.

Coding of sBSs by a Polish space (Godefroy, Saint-Raymond, 2018)

A Polish topology on $\mathcal{F}(C([0,1]))$ is called admissible if it satisfies certain natural axioms.

Properties of admissible topologies:

- The set $SB := \{F \in \mathcal{F}(C([0,1])) : F \text{ is linear}\}$ is a G_δ-set. In particular, it is a Polish space.
- The Borel σ-algebra generated by an admissible topology is the Effros-Borel structure.
- For two admissible topologies, the identity function is of Baire class 1.

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Examples of admissible topologies on $\mathcal{F}(C([0,1]))$

- the Vietoris topology 'inherited' from $F(\hat{C}([0,1]))$, where a compatible totally bounded metric on $C([0,1])$ is fixed and $\hat{C}([0,1])$ is the corresponding metric completion
- the Wijsman topology generated by the maps $F \in \mathcal{F}(C([0,1])) \mapsto d(F,f)$, $f \in C([0,1])$, where d is a compatible metric on $\mathcal{C}([0,1])$

Coding of sBSs by the Polish space of (pseudo)norms

Let V be the vector space over $\mathbb{Q}$ of all finitely supported sequences of rational numbers.

Definition

Let $\mathcal{P} \subseteq \mathbb{R}^V$ be the set of all pseudonorms on V.
Then $\mu \in \mathcal{P}$ is a code for the sBS X_μ obtained as follows:

1. extend μ to a pseudonorm on c_{00} (over $\mathbb{R}$)
2. take the quotient $(c_{00}, \mu)/\{x \in c_{00} : \mu(x) = 0\}$
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Fact

- $\mathcal{P} \subseteq \mathbb{R}^V$ *is a closed set*
- $\mathcal{P}_\infty := \{\mu \in \mathcal{P} : X_\mu$ *is infinite-dimensional*$\}$ *is a* G_δ*-set*
- $\mathcal{B} := \{\mu \in \mathcal{P} : \mu$ *is a norm on* $c_{00}\}$ *is a* G_δ*-set*

Fact

For every infinite-dimensional sBS X there is $\mu \in \mathcal{B}$ such that $X \equiv X_\mu$.

Proof: Let $f_1, f_2, \dots$ be linearly independent vectors in X such that $\overline{\operatorname{span}}(f_1, f_2, \dots) = X$.
Define $\mu \colon V \to \mathbb{R}$ by

$$\mu\left(\sum_{i\in F} \alpha_i e_i\right) := \left\|\sum_{i\in F} \alpha_i f_i\right\|_X .$$

□

Three ways of coding infinite-dimensional sBSs by a Polish space: (a) SB_∞ with an admissible topology, (b) $\mathcal{P}_\infty$, (c) $\mathcal{B}$.

Theorem (Informal statement)

For a given class of Banach spaces, the Borel complexity of the set of the corresponding codes depends only a little on the choice of the coding.

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For a given class of Banach spaces, the Borel complexity of the set of the corresponding codes depends only a little on the choice of the coding.

Theorem (Formal statement)

- *There is a continuous map* $\varphi\colon SB_\infty \to \mathcal{P}_\infty$ *such that* $Y \equiv X_{\varphi(Y)}$, $Y \in SB_\infty$.
- *There is a Baire class* 1 *map* $\psi\colon \mathcal{P}_\infty \to \mathcal{B}$ *such that* $X_\mu \equiv X_{\psi(\mu)}$, $\mu \in \mathcal{P}_\infty$.
- *There is a Baire class* 1 *map* $\chi : \mathcal{B} \to SB_\infty$ *such that* $X_\mu \equiv \chi(\mu)$, $\mu \in \mathcal{B}$.

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From now on we use the coding $\mathcal{B}$ only.

Relationship to finite representability

Definition

Let X, Y be Banach spaces. We say that X is finitely representable in Y if for every finite-dimensional subspace $E \subseteq X$ and every $\varepsilon > 0$ there exists an isomorphic embedding $T : E \to Y$ such that $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.

For a sBS X we denote by $\langle X \rangle_{\equiv}^{\mathcal{B}}$ the set $\{\mu \in \mathcal{B} : X_\mu \equiv X\}$.

Fact

Let X be a sBS. Then

$$\overline{\langle X \rangle_{\equiv}^{\mathcal{B}}} = \{\mu \in \mathcal{B} : X_\mu \text{ is finitely representable in } X\}.$$

Complexity of isometry classes and isomorphism classes

For a sBS X we denote

- $\langle X \rangle^{\mathcal{B}}_{\equiv} = \{\mu \in \mathcal{B} : X_\mu \equiv X\}$
- $\langle X \rangle^{\mathcal{B}}_{\simeq} = \{\mu \in \mathcal{B} : X_\mu \simeq X\}$

Fact

Let X be a sBS. Then $\langle X \rangle^{\mathcal{B}}_{\equiv}$ is a Borel set.

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Let X be a sBS. Then $\langle X \rangle^{\mathcal{B}}_{\simeq}$ is an analytic set but not necessarily Borel.

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Fact

Let X be a sBS. Then $\langle X \rangle^{\mathcal{B}}_{\equiv}$ is a Borel set.

Proof:

- the relation of linear isometry is Borel bireducible with an orbit equivalence relation (Melleray, 2007)
- orbit equivalence relations have Borel equivalence classes □

Fact

Let X be a sBS. Then $\langle X \rangle^{\mathcal{B}}_{\simeq}$ is an analytic set but not necessarily Borel.

Proof: These spaces do not have a Borel isomorphism class:
$C(2^\omega)$, $L_p([0,1])$ $(1 < p < \infty, p \neq 2)$, $c_0, \dots$ □

Hilbert space and isometry classes

Theorem

ℓ_2 is the unique, up to isometry, infinite-dimensional sBS X such that $\langle X\rangle^{\mathcal{B}}_{\equiv}$ is a closed set.

Note that $\langle X\rangle^{\mathcal{B}}_{\equiv}$ is never an open set.

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Proof: ℓ_2 is characterized by the parallelogram law

$$2\|x\|^2 + 2\|y\|^2 = \|x+y\|^2 + \|x-y\|^2, \quad x, y \in \ell_2.$$

Thus $\langle \ell_2\rangle^{\mathcal{B}}_{\equiv}$ is a closed set.

Now suppose that $\langle X\rangle^{\mathcal{B}}_{\equiv}$ is a closed set.
By Dvoretzky's theorem, ℓ_2 is finitely representable in X.
So $\overline{\langle X\rangle^{\mathcal{B}}_{\equiv}}$ contains all $\mu \in \mathcal{B}$ for which $X_\mu \equiv \ell_2$.
On the other hand, all elements of $\langle X\rangle^{\mathcal{B}}_{\equiv} = \overline{\langle X\rangle^{\mathcal{B}}_{\equiv}}$ are codes for spaces isometric to X. □

Hilbert space and isomorphism classes

Theorem

ℓ_2 is the unique, up to isomorphism, infinite-dimensional sBS X such that $\langle X\rangle^{\mathcal{B}}_{\simeq}$ is an F_σ set.

Note that $\langle X\rangle^{\mathcal{B}}_{\simeq}$ is never an open/closed set.
We do not know whether $\langle X\rangle^{\mathcal{B}}_{\simeq}$ can be a G_δ set but the only candidate is the Gurariĭ space.

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Proof: By Kwapień's theorem, ℓ_2 is the unique (up to isomorphism) sBS that has type 2
$\left(\exists C>0\ \forall x_1,\dots,x_n\in X:\ \left(\mathbb{E}\,\|\sum_{i=1}^n \pm x_i\|^2\right)^{1/2}\le C\left(\sum_{i=1}^n\|x_i\|^2\right)^{1/2}\right)$ and cotype 2
$\left(\exists C>0\ \forall x_1,\dots,x_n\in X:\ \left(\sum_{i=1}^n\|x_i\|^2\right)^{1/2}\le C\left(\mathbb{E}\,\|\sum_{i=1}^n \pm x_i\|^2\right)^{1/2}\right)$.
'Having type/cotype 2' are F_σ conditions.
So $\langle \ell_2\rangle^{\mathcal{B}}_{\simeq}$ is an F_σ set.

The other implication is based on the solution to the homogeneous subspace problem (Komorowski, Tomczak-Jaegermann, 1995; Gowers, 2002).

The Gurariĭ space

Definition

The Gurariĭ space is the unique, up to isometry, sBS $\mathbb{G}$ such that for every $\varepsilon > 0$, every finite-dimensional Banach spaces $A \subseteq B$ and every isometric embedding $e \colon A \to \mathbb{G}$ there is an extension $f \colon B \to \mathbb{G}$ of e such that f is an ε-isometric embedding.

Theorem

The isometry class $\langle \mathbb{G} \rangle^{\mathcal{B}}_{\equiv}$ is a dense G_δ set.

We do not know the descriptive complexity of the isomorphism class $\langle \mathbb{G} \rangle^{\mathcal{B}}_{\simeq}$.

The isometry class of L_p

Theorem

For every $1 \leq p < \infty$, $p \neq 2$, the isometry class $\langle L_p([0,1]) \rangle_{\equiv}^{\mathcal{B}}$ is a G_δ-complete set.

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Theorem

For every $1 \leq p < \infty$, $p \neq 2$, the isometry class $\langle L_p([0,1]) \rangle_{\equiv}^{\mathcal{B}}$ is a G_δ-complete set.

Proof:

- The class of $\mathscr{L}_{p,1+}$ sBSs is a G_δ set $\big($A sBS X is called an $\mathscr{L}_{p,1+}$-space if for every finite-dimensional $E \subseteq X$ and $\varepsilon > 0$ there is a finite-dimensional $E \subseteq F \subseteq X$ and a linear isomorphism $T : F \to \ell_p^n$ with $\|T\| \|T^{-1}\| \leq 1 + \varepsilon$.$\big)$
- An $\mathscr{L}_{p,1+}$ sBS X is isometric to $L_p([0,1])$ if and only if

$$\forall x \in S_X \; \forall \varepsilon > 0 \; \forall \delta > 0 \; \exists x_1, x_2 \in X : \quad (x_1, x_2) \stackrel{1+\varepsilon}{\sim} \ell_p^2 \text{ and } \|2^{1/p} x - x_1 - x_2\| < \delta.$$

□

The isometry class of ℓ_p

Theorem

For every $1 \leq p < \infty$, $p \neq 2$, the isometry class $\langle \ell_p \rangle^{\mathcal{B}}_{\equiv}$ is an $F_{\sigma\delta}$-complete set.

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Proof:

- The class of $\mathscr{L}_{p,1+}$ sBSs is a G_δ set.
- An $\mathscr{L}_{p,1+}$ sBS X is isometric to ℓ_p if and only if

$$\forall x \in S_X \ \forall \delta \in (0,1) \ \exists \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall x_1, \dots, x_N \in X :$$

$$(N^{1/p} x_i)_{i=1}^N \overset{1+\varepsilon}{\sim} \ell_p^N \ \Rightarrow \ \|x - \sum_{i=1}^N x_i\| > \delta.$$

□

The isometry class of c_0

Theorem

The isometry class $\langle c_0 \rangle^{\mathcal{B}}_{\equiv}$ is an $F_{\sigma\delta}$-complete set.

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The isometry class $\langle c_0 \rangle^{\mathcal{B}}_{\equiv}$ is an $F_{\sigma\delta}$-complete set.

Proof:

- The class of $\mathscr{L}_{\infty,1+}$ sBSs is a G_δ set (A sBS X is called an $\mathscr{L}_{\infty,1+}$-space if for every finite-dimensional $E \subseteq X$ and $\varepsilon > 0$ there is a finite-dimensional $E \subseteq F \subseteq X$ and a linear isomorphism $T : F \to \ell_\infty^n$ with $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$.)
- Let $0 < \varepsilon < 1$. An $\mathscr{L}_{\infty,1+}$ sBS X is isometric to c_0 if and only if

$$(B_{X^*})'_{2\varepsilon} = (1 - \varepsilon) B_{X^*},$$

where $F'_\varepsilon = \{x^* \in F : U \ni x^* \text{ is } w^*\text{-open} \Rightarrow \operatorname{diam}(U \cap F) \geq \varepsilon\}$ is the Szlenk derivative.

- Let $\varepsilon > 0$. The mapping

$$F \mapsto F'_\varepsilon, \quad F \subset X^* \text{ is } w^*\text{-compact},$$

is of Baire class 2.

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