On Linear Inhomogeneous Boundary-Value Problems for Differential Systems in Sobolev Spaces

Object

Some problems of the modern mathematical physics lead to the study of the most general or generic classes of Fredholm linear boundary-value problems in Sobolev spaces. It includes all known types of classical boundary conditions and numerous nonclassical problems.

We apply

Complex Sobolev space W_p^{n+r} is

$$\{y \in C^{n+r-1}[a,b] \colon y^{(n+r-1)} \in AC[a,b],\ y^{(n+r)} \in L_p[a,b]\},$$

which is Banach one relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \|y^{(k)}\|_p + \|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is norm in the space $L_p([a,b];\mathbb{C})$.

Statement of the problem

Linear boundary-value problem on a compact interval $[a,b] \subset \mathbb{R}$

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \qquad (1)$$

$$By = c, \quad t \in (a,b),$$

where $\{m, n, r, l\} \subset \mathbb{N}$, $1 \le p \le \infty$, $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, $f(\cdot) \in (W_p^n)^m$, $c \in \mathbb{C}^l$, $y(\cdot) \in (W_p^{n+r})^m$, and continuous operator $B\colon (W_p^{n+r})^m \to \mathbb{C}^l$.

Linear operator equation $(L,B)y = (f,c)$, where

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l.$$

$Y_k(\cdot) \in (W_p^{n+r})^{m\times m}$ is an unknown matrix-valued function to the family of matrix Cauchy problems

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \quad \{k,j\} \subset \{1,\ldots,r\}.$$

Characteristic matrix

A numerical matrix $M(L,B) := ([BY_0], \ldots, [BY_{r-1}]) \in \mathbb{C}^{mr\times l}$ is characteristic to problem. It consists of r block columns $[BY_k(\cdot)] \in \mathbb{C}^{m\times l}$. In matrix $[BY_k]$, j-th column is the result of action of B on the j-th column of $Y_k(\cdot)$.

Results

1. (L,B) is a bounded Fredholm operator with index $mr - l$.
2. Each of the problems relates to a certain characteristic matrix, where
$$\dim\ker(L,B) = \dim\ker\big(M(L,B)\big),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\big(M(L,B)\big).$$
3. (L,B) is invertible if and only if $l = mr$ and $\det M(L,B) \neq 0$.

Application

The sequence of inhomogeneous boundary-value problems

$$L(k)y(t,k) := y^{(r)}(t,k) + \sum_{j=1}^{r} A_{r-j}(t,k) y^{(r-j)}(t,k) = f(t,k),$$

$$B(k)y(\cdot,k) = c(k), \quad t \in (a,b), \quad k \in \mathbb{N}.$$

If $\big(L(k),B(k)\big) \xrightarrow{s} (L,B)$ for $k\to\infty$ then $M\big(L(k),B(k)\big) \to M(L,B)$.

$$\Rightarrow$$

$$\dim\ker\big(L(k),B(k)\big) \le \dim\ker(L,B),$$
$$\dim\operatorname{coker}\big(L(k),B(k)\big) \le \dim\operatorname{coker}(L,B).$$

Parameterized boundary-value problem

Fix a number $\varepsilon_0 > 0$, $\varepsilon \in [0,\varepsilon_0)$.

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon) y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon),$$

$$B(\varepsilon)y(\cdot,\varepsilon) = c(\varepsilon), \quad t \in (a,b),$$

where continuous operator $B(\varepsilon)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$.
This problem is a Fredholm one with index zero.

Continuous dependence

A solution to the problem depends continuously on ε at $\varepsilon = 0$ if:

- * there exists $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0,\varepsilon_1)$, arbitrary $f(\cdot;\varepsilon) \in (W_p^n)^m$ and $c(\varepsilon) \in \mathbb{C}^{rm}$ this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;
- ** $f(\cdot;\varepsilon) \to f(\cdot;0)$ in $(W_p^n)^m$ and $c(\varepsilon) \to c(0)$ in $\mathbb{C}^{rm}$ implies $y(\cdot;\varepsilon) \to y(\cdot;0)$ in $(W_p^{n+r})^m$ as $\varepsilon \to 0+$.

Boundary conditions as $\varepsilon \to 0+$:

- 0 homogeneous problem has only the trivial solution;
- I $A_{r-j}(\cdot;\varepsilon) \to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$;
- II $B(\varepsilon)y \to B(0)y$ in $\mathbb{C}^{rm}$ for any $y \in (W_p^{n+r})^m$.

Criterion of the continuous dependence

A solution to the problem depends continuously on ε at $\varepsilon = 0$ if and only if it satisfies the conditions 0, I, II.

The degree of convergence

The error $\|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p}$ and discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \|L(\varepsilon)y(\cdot;0) - f(\cdot;\varepsilon)\|_{n,p} + \|B(\varepsilon)y(\cdot;0) - c(\varepsilon)\|_{\mathbb{C}^{rm}},$$

where $y(\cdot;0)$ is an approximate solutions.

If the problem satisfies conditions 0, I, II, then there exist $\varepsilon_2 < \varepsilon_1$ and γ_1, γ_2 such that, for any $\varepsilon \in (0,\varepsilon_2)$, the estimate holds

$$\gamma_1 \widetilde{d}_{n,p}(\varepsilon) \le \|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p} \le \gamma_2 \widetilde{d}_{n,p}(\varepsilon),$$

where ε_2, γ_1, and γ_2 do not depend of $y(\cdot;\varepsilon)$, and $y(\cdot;0)$.

Multipoint problem

Associate with parameterized system next Fredholm condition

$$B(\varepsilon)y(\cdot,\varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon),\varepsilon) = q(\varepsilon),$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m\times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a,b]$ are arbitrarily given.

Sufficient constructive conditions are established under which the solutions to problems depend continuously on ε at $\varepsilon = 0$ in W_p^{n+r}.

Approximation

Let condition 0 holds. For the problem (1), there is a sequence of multipoint problems such that they are well-posedness

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c$$

for sufficiently large k and the asymptotic property $y_k \to y$ is fulfilled in $(W_p^{n+r})^m$ for $k \to \infty$. The sequence can be chosen independently of f and c, and constructed explicitly.

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