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Fuad Kittaneh
Vladimír Müller

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SPECTRAL RADIUS INEQUALITIES FOR MATRICES WITH ENTRIES FROM A BANACH ALGEBRA

FUAD KITTANEH AND VLADIMIR MULLER

Abstract. In this paper, we give spectral radius inequalities for matrices with entries from a Banach algebra. An emphasis will be given to matrices with commuting entries. Our inequalities are natural generalizations of earlier known inequalities for operator matrices.

1. Introduction

Computing the exact values of the spectral radius of a complex matrix is not always possible. The problem becomes more difficult for Hilbert space operator matrices, i.e., matrices whose entries are bounded linear operators on a Hilbert space.

This problem, which has a wide range of applications, stimulates mathematicians to establish estimates of the spectral radii of operator matrices by deriving spectral radius inequalities. For a comprehensive account of this problem, we refer to [6] and references therein.

In this paper, we are interested in spectral radius inequalities for matrices with entries from a Banach algebra. In Section 2, we give estimates of the spectral radius, in Section 3, we give estimates using the numerical radius, and in Section 4, we consider matrices with commuting entries to get better spectral radius estimates. Our new results are natural generalizations of earlier related results for operator matrices given in [1], [2], [5], and [6].

2. Estimates of the spectral radius

Let $\mathcal{A}$ be a unital Banach algebra and $k \in \mathbb{N}$. Let $\mathcal{A}_k = \underbrace{\mathcal{A} \oplus \cdots \oplus \mathcal{A}}_{k}$ be the Banach space with the norm

$$\big\|(x_1 \oplus \cdots \oplus x_k)\big\| = \Big(\sum_{j=1}^{k} \|x_j\|^2\Big)^{1/2}.$$

Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix}$$

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be a $k \times k$-matrix with entries from $\mathcal{A}$. Then A can be considered as an operator acting on $\mathcal{A}_k$ defined by

$$A = \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^k a_{1j} x_j \\ \vdots \\ \sum_{j=1}^k a_{kj} x_j \end{pmatrix}.$$

Let $\|A\|$ be the operator norm of A and $r(A)$ the spectral radius of A in the Banach algebra $B(\mathcal{A}_k)$ of all bounded linear operators on $\mathcal{A}_k$.

Clearly,

$$\max\big\{\|a_{ij}\| : 1 \le i, j \le k\big\} \le \|A\| \le k^2 \cdot \max\big\{\|a_{ij}\| : 1 \le i, j \le k\big\}.$$

Theorem 2.1. *Let $A = (a_{ij})_{i,j=1}^k$ be a $k \times k$ matrix with entries from $\mathcal{A}$. Then*

$$r(A) \le r \begin{pmatrix} \|a_{11}\| & \dots & \|a_{1k}\| \\ \vdots & \ddots & \vdots \\ \|a_{k1}\| & \dots & \|a_{kk}\| \end{pmatrix}.$$

Proof. Let

$$N = \begin{pmatrix} \|a_{11}\| & \dots & \|a_{1k}\| \\ \vdots & \ddots & \vdots \\ \|a_{k1}\| & \dots & \|a_{kk}\| \end{pmatrix}.$$

For $n \in \mathbb{N}$, denote by $(A^n)_{ij}$ and $(N^n)_{ij}$ the (i,j)-entry of the matrix A^n and N^n, respectively. We have

$$r(A) = \lim_{n\to\infty} \|A^n\|^{1/n} \le \liminf_{n\to\infty} \Big(k^2 \cdot \max\{\|(A^n)_{ij}\| : 1 \le i, j \le k\}\Big)^{1/n}$$

$$= \liminf_{n\to\infty} \Big(\max\{\|(A^n)_{ij}\| : 1 \le i, j \le k\}\Big)^{1/n},$$

where

$$\|(A^n)_{ij}\| = \Big\| \sum_{1 \le i_1, \dots, i_{n-1} \le k} a_{i,i_1} a_{i_1,i_2} \cdots a_{i_{n-2},i_{n-1}} a_{i_{n-1},j} \Big\|$$

$$\le \sum_{1 \le i_1, \dots, i_{n-1} \le k} \|a_{i,i_1}\| \cdot \|a_{i_1,i_2}\| \cdots \|a_{i_{n-1},j}\| = (N^n)_{i,j}.$$

So,

$$r(A) \le \liminf_{n\to\infty} \Big(\max\{(N^n)_{i,j} : 1 \le i, j \le k\}\Big)^{1/n} \le \lim_{n\to\infty} \|N^n\|^{1/n} = r(N).$$

∎

As in [5], one can prove the following consequences. The proofs remain unchanged and therefore we omit them.

Theorem 2.2. *Let $a_1, a_2, b_1, b_2 \in \mathcal{A}$. Then*

$$r(a_1 b_1 + a_2 b_2) \le \frac{1}{2}\Big(\|b_1 a_1\| + \|b_2 a_2\| + \sqrt{\big(\|b_1 a_1\| - \|b_2 a_2\|\big)^2 + 4\|b_1 a_2\| \cdot \|b_2 a_1\|}\Big).$$

Corollary 2.3. *Let $a, b \in \mathcal{A}$. Then:*

(i)

$$r(a+b) \le \frac{1}{2}\left(\|a\| + \|b\| + \sqrt{(\|a\| - \|b\|)^2 + 4\min\{\|ab\|, \|ba\|\}}\right),$$

(ii)

$$r(ab \pm ba) \le \frac{1}{2}\Big(\|ab\| + \|ba\| + \sqrt{\big(\|ab\| - \|ba\|\big)^2 + 4\|a^2\| \cdot \|b^2\|}\Big),$$

(iii)

$$r(ab \pm ba) \le \|ab\| + \sqrt{\min\{\|a\| \cdot \|ab^2\|, \|b\| \cdot \|a^2 b\|\}},$$

(iv)

$$r(ab) \le \frac{1}{4}\left(\|ab\| + \|ba\| + \sqrt{(\|ab\| - \|ba\|)^2 + 4\min\{\|a\| \cdot \|bab\|, \|b\| \cdot \|aba\|\}}\right).$$

Corollary 2.4. *Let $a, b, c, d \in \mathcal{A}$ and $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then*

$$\max\{r(a), r(d)\} \le \frac{1}{2}\big(\|T\| + \|T^2\|^{1/2}\big).$$

and

$$\sqrt{r(bc)} \le \big(\|T\| + \|T^2\|^{1/2}\big).$$

3. Estimates using the numerical radius

Let X be a Banach space and $T \in B(X)$ a bounded linear operator acting on X. Recall that the numerical range of T is defined by

$$W(T) = \big\{\langle Tx, x^*\rangle : x \in X, x^* \in X^*, \|x\| = \|x^*\| = \langle x, x^*\rangle = 1\big\}$$

and the numerical radius by

$$\begin{aligned} w(T) &= \sup\big\{|\lambda| : \lambda \in W(T)\big\} \\ &= \sup\big\{|\langle Tx, x^*\rangle| : x \in X, x^* \in X^*, \|x\| = \|x^*\| = \langle x, x^*\rangle = 1\big\}. \end{aligned}$$

An equivalent definition of the numerical radius is using the algebraic numerical range in the Banach algebra $B(X)$:

$$V(T, B(X)) = \big\{|f(T)| : f \in B(X)^*, \|f\| = f(I_X) = 1\big\},$$

where I_X denotes the identity operator on X. Then $w(T) = \max\big\{|\lambda| : \lambda \in V(T, B(X))\big\}$, see [3], Theorem 9.3.

Similarly, for an element a of a unital Banach algebra $\mathcal{A}$ with unit $1_{\mathcal{A}}$ we have

$$w(a) = \sup\big\{|f(a)| : f \in \mathcal{A}^*, \|f\| = f(1_{\mathcal{A}}) = 1\} = \sup\{|\lambda| : \lambda \in V(a, \mathcal{A})\big\}.$$

Theorem 3.1. *Let $A = (a_{ij})_{i,j=1}^k$ be a $k \times k$ matrix with entries from $\mathcal{A}$. Then*

$$w(A) \le w\begin{pmatrix} w(a_{11}) & \|a_{12}\| & \|a_{13}\| & \cdots & \|a_{1k}\| \\ \|a_{21}\| & w(a_{22}) & \|a_{23}\| & \cdots & \|a_{2k}\| \\ \|a_{31}\| & \|a_{32}\| & w(a_{33}) & \cdots & \|a_{3k}\| \\ \vdots & \vdots & & \ddots & \vdots \\ \|a_{k1}\| & \|a_{k2}\| & \|a_{k3}\| & \cdots & w(a_{kk}) \end{pmatrix}.$$

Proof. Note that $(\mathcal{A}_k)^* = \underbrace{\mathcal{A}^* \oplus \cdots \oplus \mathcal{A}^*}_{k}$ with the norm $\|(x_1^* \oplus \cdots \oplus x_k^*)\| = \left(\sum_{j=1}^k \|x_j^*\|^2\right)^{1/2}$. So we have

$$w((a_{ij})) = \sup\Big|\sum_{i=1}^k \sum_{j=1}^k \langle a_{ij}x_j, x_i^*\rangle\Big|,$$

where the supremum is taken over all $x_1, \dots, x_k \in A$ and $x_1^*, \dots, x_k^* \in A^*$ with $\sum_{i=1}^k \|x_i\|^2 = 1 = \sum_{i=1}^k \|x_i^*\|^2 = \sum_{i=1}^k \langle x_i, x_i^*\rangle$. Note that this implies that $\|x_i\| = \|x_i^*\| := r_i$ for all i. So, for $i = 1, \dots, k$, we have $|\langle a_{ii}x_i, x_i^*\rangle| \le r_i^2 w(a_{ii})$, and for $i \ne j$, we have $|\langle a_{ij}x_j, x_i^*\rangle| \le \|a_{ij}\| r_i r_j$. Hence,

$$\Big|\sum_{i=1}^n \sum_{j=1}^n \langle a_{ij}x_j, x_i^*\rangle\Big| \le \left\langle \begin{pmatrix} w(a_{11}) & \|a_{12}\| & \cdots & \|a_{1k}\| \\ \|a_{21}\| & w(a_{22}) & \cdots & \|a_{2k}\| \\ \vdots & \vdots & \ddots & \vdots \\ \|a_{k1}\| & \|a_{k2}\| & \cdots & w(a_{kk}) \end{pmatrix} \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix}, \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_k \end{pmatrix} \right\rangle$$

$$\le w \begin{pmatrix} w(a_{11}) & \|a_{12}\| & \cdots & \|a_{1k}\| \\ \|a_{21}\| & w(a_{22}) & \cdots & \|a_{2k}\| \\ \vdots & \vdots & \ddots & \vdots \\ \|a_{k1}\| & \|a_{k2}\| & \cdots & w(a_{kk}) \end{pmatrix}.$$

∎

Corollary 3.2.

$$w(A) \le 2w\big((w(a_{ij})_{i,j=1}^k\big).$$

Remark 3.3. *The estimate $w\big((a_{ij})_{i,j=1}^k\big) \le w\big((w(a_{ij})_{i,j=1}^k\big)$ is not true. An example is the following 2×2 matrix with blocks of size 2:*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $w(A) = 1$ and

$$w \begin{pmatrix} w(A_{11}) & w(A_{12}) \\ w(A_{21}) & w(A_{22}) \end{pmatrix} = w \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} = 1/2.$$

So the estimate in Corollary 3.2 is sharp.

As in [1], one can obtain improvements of the estimates from the previous section.

Theorem 3.4. *Let $a_1, a_2, b_1, b_2 \in \mathcal{A}$. Then*

$$r(a_1b_1+a_2b_2) \le \frac{1}{2}\Big(w(b_1a_1)+w(b_2a_2)+\sqrt{\big(w(b_1a_1) - w(b_2a_2)\big)^2 + 4\|b_1a_2\| \cdot \|b_2a_1\|}\Big).$$

Corollary 3.5. *Let $a, b \in \mathcal{A}$. Then:*

(i)

$$r(a+b) \le \frac{1}{2}\Big(w(a) + w(b) + \sqrt{\big(w(a) - w(b)\big)^2 + 4\min\big\{\|ab\|, \|ba\|\big\}}\Big),$$

(ii)
$$r(ab \pm ba) \le w(ab) + \sqrt{\min\{\|a\| \cdot \|ab^2\|, \|b\| \cdot \|a^2b\|\}},$$
(iii)
$$r(ab) \le \frac{1}{4}\Big(w(ab) + w(ba) + \sqrt{(w(ab) - w(ba))^2 + 4\min\{\|a\| \cdot \|bab\|, \|b\| \cdot \|aba\|\}}\Big).$$

4. Commuting elements

Better estimates can be obtained for matrices with mutually commuting entries. The basic tool is the following classical result:

Theorem 4.1. *([4], p. 18) Let $(B, \|\cdot\|)$ be a Banach algebra, let $S \subset B$ be a bounded semigroup. Then there exists an equivalent algebra norm $\|\cdot\|'$ on B such that $\|s\|' \le 1$ for all $s \in S$.*

Corollary 4.2. *Let $(B, \|\cdot\|)$ be a Banach algebra, let $b_1, \dots, b_n \in B$ be mutually commuting elements. Then there exists a sequence of algebra norms $\|\cdot\|_m$ equivalent to $\|\cdot\|$ such that*
$$r(b_j) = \lim_{m\to\infty} \|b_j\|_m \qquad (j = 1, \dots, n).$$

Proof. Let $m \in \mathbf{N}$. Let $b_j' = \frac{b_j}{r(b_j)+m^{-1}}$ $(j = 1, \dots, n)$. Then $r(b_j') < 1$, and so $\sup\{\|b_j'^k\| : k = 0, 1, \dots\} < \infty$ for all j. Let S_m be the semigroup generated by $b_1', \dots, b_n'$. Since the elements b_j' are mutually commuting,
$$S_m = \{b_1'^{k_1} \cdots b_n'^{k_n} : k_1, \dots, k_n = 0, 1, \dots\}$$
and S_m is a bounded semigroup,
$$\sup\{\|x\| : x \in S_m\} \le \prod_{j=1}^n \sup\{\|b_j'^k\| : k = 0, 1, \dots\}.$$
So, there exists an equivalent algebra norm $\|\cdot\|_m$ on B such that $\|b_j'\|_m \le 1$. Thus, $r(b_j) \le \|b_j\|_m \le r(b_j) + m^{-1}$, and so
$$\lim_{m\to\infty} \|b_j\|_m = r(b_j)$$
for all $j = 1, \dots, n$. ∎

Theorem 4.3. *Let $\mathcal{A}$ be a unital Banach algebra and $A = ((a_{ij})_{i,j=1}^k)$ a $k\times k$-matrix with mutually commuting entries from $\mathcal{A}$. Then*
$$r(A) \le r\begin{pmatrix} r(a_{11}) & \dots & r(a_{1k}) \\ \vdots & \ddots & \vdots \\ r(a_{k1}) & \dots & r(a_{kk}) \end{pmatrix}.$$

Proof. By the previous corollary, there exists a sequence of norms $\|\cdot\|_m$ on $\mathcal{A}$ equivalent to the original norm $\|\cdot\|$ such that
$$r(a_{ij}) = \lim_{m\to\infty} \|a_{ij}\|_m$$
for all $i, j, 1 \le i, j \le k$. Since the spectral radius of A does not depend on the choice of an equivalent norm, we have
$$r(A) \le r\begin{pmatrix} \|a_{11}\|_m & \dots & \|a_{1k}\|_m \\ \vdots & \ddots & \vdots \\ \|a_{k1}\|_m & \dots & \|a_{kk}\|_m \end{pmatrix}$$

for all m. So, by the continuity of the spectral radius in the algebra of $k \times k$ complex matrices, we have

$$r(A) \leq \lim_{m \to \infty} r \begin{pmatrix} \|a_{11}\|_m & \dots & \|a_{1k}\|_m \\ \vdots & \ddots & \vdots \\ \|a_{k1}\|_m & \dots & \|a_{kk}\|_m \end{pmatrix} = r \begin{pmatrix} r(a_{11}) & \dots & r(a_{1k}) \\ \vdots & \ddots & \vdots \\ r(a_{k1}) & \dots & r(a_{kk}) \end{pmatrix}.$$

∎

As in [6] and [2], one can obtain the following results.

Corollary 4.4. *Let $a, b \in \mathcal{A}$, $ab = ba$. Then*

$$r(a+b) \leq \frac{1}{2} \left(r(a) + r(b) + \sqrt{\big(r(a) - r(b)\big)^2 + 4r(ab)} \right).$$

Corollary 4.5. *Let $a, b \in \mathcal{A}$, $ab = ba$. If $r(a+b) = r(a) + r(b)$, then $r(ab) = r(a) \cdot r(b)$.*

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Department of Mathematics, The University of Jordan, Amman, Jordan
E-mail address: fkitt@ju.edu.jo

Institute of Mathematics, Czech Academy of Sciences, Zitna 25, Prague, Czech Republic
E-mail address: muller@math.cas.cz