

# The incompressible limit for the Rayleigh-Benard convection problem

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# Navier–Stokes–Fourier system, primitive system

**Mass conservation:**

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

**Momentum balance:**

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x \rho(\varrho, \vartheta)} = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon} \varrho \nabla_x G}$$

**Internal energy balance (heat equation):**

$$\begin{aligned} \partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \\ = \boxed{\varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u}} - \rho(\varrho, \vartheta) \operatorname{div}_x \mathbf{u} \end{aligned}$$

**Newton's law**

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta) \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \operatorname{div}_x \mathbf{u} \mathbb{I}$$

**Fourier's law**

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

## Oberbeck–Boussinesq system, target system

**Incompressibility:**

$$\operatorname{div}_x \mathbf{U} = 0$$

**Momentum balance:**

$$\bar{\rho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \mu(\bar{\vartheta}) \Delta_x \mathbf{U} + r \nabla_x G$$

**Heat equation:**

$$\bar{\rho} c_p(\bar{\rho}, \bar{\vartheta}) \left( \partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \right) - \bar{\rho} \bar{\vartheta} \alpha(\bar{\rho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G = \kappa(\bar{\vartheta}) \Delta_x \Theta$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\rho}, \bar{\vartheta})}{\partial \rho} \nabla_x r + \frac{\partial \rho(\bar{\rho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \Theta = \bar{\rho} \nabla_x G.$$

$$\alpha(\bar{\rho}, \bar{\vartheta}) \equiv \frac{1}{\bar{\rho}} \frac{\partial \rho(\bar{\rho}, \bar{\vartheta})}{\partial \vartheta} \left( \frac{\partial \rho(\bar{\rho}, \bar{\vartheta})}{\partial \rho} \right)^{-1}$$

$$c_p(\bar{\rho}, \bar{\vartheta}) \equiv \frac{\partial e(\bar{\rho}, \bar{\vartheta})}{\partial \vartheta} + \bar{\rho}^{-1} \bar{\vartheta} \alpha(\bar{\rho}, \bar{\vartheta}) \frac{\partial \rho(\bar{\rho}, \bar{\vartheta})}{\partial \vartheta}$$

# Spatial domain

## Periodic strip

$$\Omega = \mathcal{T}^{d-1} \times [0, 1]$$
$$\mathcal{T}^{d-1} = ([-1, 1] |_{\{-1;1\}})^{d-1}$$

## Conservative boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \quad \mathbb{S} \cdot \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0, \quad \nabla_x \vartheta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

## Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \vartheta|_{\partial\Omega} = \vartheta_B$$

## Singular limit in the conservative case

From Navier–Stokes–Fourier to Oberbeck–Boussinesq

$$\begin{aligned}\mathbf{u}_\varepsilon &\rightarrow \mathbf{U} \\ \frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} &\rightarrow r \\ \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} &\rightarrow \Theta\end{aligned}$$

**Convergence for the conservative boundary conditions, EF, A. Novotný 2009**

If the initial data are ill prepared:

$$\mathbf{u}_\varepsilon(0, \cdot), \frac{\varrho_\varepsilon(0, \cdot) - \bar{\varrho}}{\varepsilon}, \frac{\vartheta_\varepsilon(0, \cdot) - \bar{\vartheta}}{\varepsilon} \text{ bounded,}$$

then  $\mathbf{U}$ ,  $r$ , and  $\Theta$  solve the Oberbeck–Boussinesq system with the conservative boundary conditions

$$\mathbf{U} \cdot \mathbf{n}|_{\partial\Omega} = 0, \mathbb{S}(\bar{\vartheta}, \nabla_x \mathbf{U}) \cdot \mathbf{n} \times \mathbf{n}|_{\partial\Omega} = 0, \nabla_x \Theta \cdot \mathbf{n}|_{\partial\Omega} = 0$$

# Singular limit for the Dirichlet boundary conditions

**Scaled boundary conditions for the temperature:**

$$\vartheta_\varepsilon|_{\partial\Omega} = \bar{\vartheta} + \varepsilon\Theta_B$$

**Initial data:**

$$\mathbf{u}_\varepsilon(0, \cdot) \rightarrow \mathbf{U}_0 \text{ in } L^2(\Omega; R^d), \quad \mathbf{U}_0|_{\partial\Omega} = 0, \quad \operatorname{div}_x \mathbf{U}_0 = 0$$

$$\frac{\varrho_\varepsilon(0, \cdot) - \bar{\varrho}}{\varepsilon} \rightarrow r_0 \text{ in } L^1(\Omega) \text{ and weakly-} (*) \text{ in } L^\infty(\Omega)$$

$$\frac{\vartheta_\varepsilon(0, \cdot) - \bar{\vartheta}}{\varepsilon} \rightarrow \mathfrak{T}_0 \text{ in } L^1(\Omega) \text{ and weakly-} (*) \text{ in } L^\infty(\Omega)$$

**Boussinesq relation**

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r_0 + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T}_0 = \bar{\varrho} \nabla_x G.$$

# Convergence for the Dirichlet boundary conditions

Convergence, P. Bella, EF, F. Oschmann 2022:

$$\lambda(\bar{\varrho}, \bar{\vartheta}) = \frac{\bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta})}{\bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta})} \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \in (0, 1)$$

$$\mathbf{u}_\varepsilon \rightarrow \mathbf{U}$$

$$\frac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \rightarrow r$$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} - \boxed{\lambda(\bar{\varrho}, \bar{\vartheta}) \frac{1}{|\Omega|} \int_\Omega \frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} dx} \rightarrow \Theta,$$

where  $(\mathbf{U}, r, \Theta)$  solves the Oberbeck–Boussinesq system with  
*nonlocal boundary conditions*

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\bar{\varrho}, \bar{\vartheta})}{1 - \lambda(\bar{\varrho}, \bar{\vartheta})} \frac{1}{|\Omega|} \int_\Omega \Theta dx$$

# Weak solutions to the primitive system

**Mass conservation:**

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

**Momentum balance:**

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta) = \operatorname{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon} \varrho \nabla_x G$$

**Gibbs' relation**

$$\vartheta Ds = De + pD \left( \frac{1}{\varrho} \right)$$

**Entropy balance (inequality):**

$$\begin{aligned} \partial_t(\varrho s(\varrho, \vartheta)) + \operatorname{div}_x(\varrho s(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \left( \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) \\ \geq \frac{1}{\vartheta} \left( \varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \end{aligned}$$



## Energy conservation, conservative boundary conditions

**Total energy:**

$$E_\varepsilon(\varrho, \vartheta, \mathbf{u}) = \frac{\varepsilon^2}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

**Energy balance (inequality):**

$$\begin{aligned} \partial_t E_\varepsilon(\varrho, \vartheta, \mathbf{u}) + \operatorname{div}_x(E_\varepsilon(\varrho, \vartheta, \mathbf{u})\mathbf{u}) + \operatorname{div}_x(\mathbf{p}(\varrho, \vartheta)\mathbf{u}) + \operatorname{div}_x \mathbf{q}(\vartheta, \nabla_x \vartheta) \\ = \boxed{(\leq)} \varepsilon^2 \operatorname{div}_x (\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \cdot \mathbf{u}) + \varepsilon \varrho \nabla_x G \cdot \mathbf{u} \end{aligned}$$

**Energy conservation (balance), conservative boundary conditions:**

$$\frac{d}{dt} \int_{\Omega} E_\varepsilon(\varrho, \vartheta, \mathbf{u}) \, dx = (\leq) \varepsilon \int_{\Omega} \varrho \nabla_x G \cdot \mathbf{u} \, dx$$

## Basic properties of weak solutions

- **Global existence.** Under certain physically grounded restrictions imposed on the constitutive relations, the weak solutions exist globally in time for any finite energy initial data
- **Compatibility.** Any sufficiently smooth weak solution is a classical solution of the Navier–Stokes–Fourier system
- **Weak–strong uniqueness.** If  $[\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}]$  is a classical solution and  $[\varrho, \vartheta, \mathbf{u}]$  a weak solution starting from the same initial data, then  $\varrho = \tilde{\varrho}$ ,  $\vartheta = \tilde{\vartheta}$ ,  $\mathbf{u} = \tilde{\mathbf{u}}$ .

## Problems with the Dirichlet boundary conditions

Lack of control of the boundary heat flux:

$$\frac{d}{dt} \int_{\Omega} E_{\varepsilon}(\varrho, \vartheta, \mathbf{u}) \, dx + \int_{\partial\Omega} \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} \, d\sigma_x = (\leq) \varepsilon \int_{\Omega} \varrho \nabla_x G \cdot \mathbf{u} \, dx$$

Ballistic energy:

$$E_{\varepsilon}(\varrho, \vartheta, \mathbf{u}) - \tilde{\vartheta} \varrho s(\varrho, \vartheta) \\ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B$$

Ballistic energy flux:

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} - \frac{\tilde{\vartheta}}{\vartheta} \mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \mathbf{n} = 0 \text{ on } \partial\Omega$$

## Weak solutions revisited

### Ballistic energy balance

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \varepsilon^2 \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) - \tilde{\vartheta} \varrho s(\varrho, \vartheta) \, dx \\ & + \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( \varepsilon^2 \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right) \, dx \\ & \leq \int_{\Omega} \left( \varepsilon \varrho \nabla_x G \cdot \mathbf{u} - \varrho s(\varrho, \vartheta) \partial_t \tilde{\vartheta} - \varrho s(\varrho, \vartheta) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \tilde{\vartheta}}{\vartheta} \right) \, dx \end{aligned}$$

for any  $\tilde{\vartheta} > 0$ ,  $\tilde{\vartheta}|_{\partial\Omega} = \vartheta_B$

- **Compatibility [Chaudhuri–EF 2021].** Smooth weak solutions are classical solutions
- **Weak–strong uniqueness [Chaudhuri–EF 2021].** A weak solution coincides with the strong solution as long as the latter exists

# Relative energy as Bregman distance

Thermodynamic stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0 \text{ for all } \varrho, \vartheta > 0$$

Relative energy:

$$E_\varepsilon \left( \varrho, \vartheta, \mathbf{u} \mid \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) = \frac{1}{2} \varrho |\mathbf{u} - \tilde{\mathbf{u}}|^2 \\ + \frac{1}{\varepsilon^2} \left[ \varrho e - \tilde{\vartheta} \left( \varrho s - \tilde{\varrho} s(\tilde{\varrho}, \tilde{\vartheta}) \right) - \left( e(\tilde{\varrho}, \tilde{\vartheta}) - \tilde{\vartheta} s(\tilde{\varrho}, \tilde{\vartheta}) + \frac{p(\tilde{\varrho}, \tilde{\vartheta})}{\tilde{\varrho}} \right) (\varrho - \tilde{\varrho}) - \tilde{\varrho} e(\tilde{\varrho}, \tilde{\vartheta}) \right]$$

Bregmann distance:

$$E_\varepsilon \left( \varrho, S, \mathbf{m} \mid \tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}} \right) = E_\varepsilon(\varrho, S, \mathbf{m}) \\ - \left\langle \partial_{\varrho, S, \mathbf{m}} E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}); (\varrho - \tilde{\varrho}, S - \tilde{S}, \mathbf{m} - \tilde{\mathbf{m}}) \right\rangle - E_\varepsilon(\tilde{\varrho}, \tilde{S}, \tilde{\mathbf{m}}) \\ (\varrho, S, \mathbf{m}) \mapsto E_\varepsilon(\varrho, S, \mathbf{m}) \text{ (strictly) convex}$$

## Relative energy inequality

$$\begin{aligned}
 & \left[ \int_{\Omega} E_{\varepsilon}(\varrho, \vartheta, \mathbf{u} | \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) \, dx \right]_{t=0}^{t=\tau} \\
 & + \int_0^{\tau} \int_{\Omega} \frac{\tilde{\vartheta}}{\vartheta} \left( \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta) |\nabla_x \vartheta|^2}{\vartheta} \right) \, dx dt \\
 \leq & -\frac{1}{\varepsilon^2} \int_0^{\tau} \int_{\Omega} \left( \varrho (s - s(\tilde{\varrho}, \tilde{\vartheta})) \partial_t \tilde{\vartheta} + \varrho (s - s(\tilde{\varrho}, \tilde{\vartheta})) \mathbf{u} \cdot \nabla_x \tilde{\vartheta} - \frac{\kappa(\vartheta) \nabla_x \vartheta}{\vartheta} \cdot \nabla_x \tilde{\vartheta} \right) \, dx dt \\
 & - \int_0^{\tau} \int_{\Omega} \left[ \varrho (\mathbf{u} - \tilde{\mathbf{u}}) \otimes (\mathbf{u} - \tilde{\mathbf{u}}) + \frac{1}{\varepsilon^2} \rho(\varrho, \vartheta) \mathbb{I} - \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) \right] : \nabla_x \tilde{\mathbf{u}} \, dx dt \\
 & + \int_0^{\tau} \int_{\Omega} \varrho \left[ \frac{1}{\varepsilon} \nabla_x G - \partial_t \tilde{\mathbf{u}} - (\tilde{\mathbf{u}} \cdot \nabla_x) \tilde{\mathbf{u}} \right] \cdot (\mathbf{u} - \tilde{\mathbf{u}}) \, dx dt \\
 & + \frac{1}{\varepsilon^2} \int_0^{\tau} \int_{\Omega} \left[ \left( 1 - \frac{\varrho}{\tilde{\varrho}} \right) \partial_t \rho(\tilde{\varrho}, \tilde{\vartheta}) - \frac{\varrho}{\tilde{\varrho}} \mathbf{u} \cdot \nabla_x \rho(\tilde{\varrho}, \tilde{\vartheta}) \right] \, dx dt
 \end{aligned}$$

any trio of continuously differentiable functions  $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$

$$\tilde{\varrho} > 0, \tilde{\vartheta} > 0, \tilde{\vartheta}|_{\partial\Omega} = \tilde{\vartheta}, \tilde{\mathbf{u}}|_{\partial\Omega} = \mathbf{0}.$$

# Constitutive theory, I

## Gibbs' law

$$\vartheta Ds = De + pD\left(\frac{1}{\varrho}\right)$$

## Thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

## Pressure EOS

$$p(\varrho, \vartheta) = p_m(\varrho, \vartheta) + p_{\text{rad}}(\vartheta)$$

$$p_m(\varrho, \vartheta) = \frac{2}{3}\varrho e_m(\varrho, \vartheta), \quad p_{\text{rad}}(\vartheta) = \frac{a}{3}\vartheta^4, \quad a > 0$$

## Internal energy

$$e(\varrho, \vartheta) = e_m(\varrho, \vartheta) + e_{\text{rad}}(\varrho, \vartheta), \quad e_{\text{rad}}(\varrho, \vartheta) = \frac{a}{\varrho}\vartheta^4.$$

## Constitutive theory, II



$$\text{Gibbs' law} \Rightarrow p_m(\varrho, \vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)$$



$$\text{Thermodynamic stability} \Rightarrow \frac{P(Z)}{Z^{\frac{5}{3}}} \searrow p_\infty > 0 \text{ as } Z \rightarrow \infty$$

■ **Entropy.**

$$s(\varrho, \vartheta) = \mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{4a}{3} \frac{\vartheta^3}{\varrho}, \quad \mathcal{S}' < 0$$

■ **Third law of thermodynamics.**

$$\mathcal{S}(Z) \searrow 0 \text{ as } Z \rightarrow \infty$$



## Constitutive theory, III

### Viscosity.

$$0 < \underline{\mu}(1 + \vartheta) \leq \mu(\vartheta), \quad |\mu'(\vartheta)| \leq \bar{\mu}$$

$$0 \leq \eta(\vartheta) \leq \bar{\eta}(1 + \vartheta)$$

### Thermal conductivity.

$$0 < \underline{\kappa}(1 + \vartheta^\beta) \leq \kappa(\vartheta) \leq \bar{\kappa}(1 + \vartheta^\beta), \quad \beta > 6$$

### Global existence [Chaudhuri, EF, 2021].

Under the above constitutive restrictions, the problem admits global-in-time weak solutions for any finite energy initial data and any sufficiently smooth boundary data

## Target problem – smooth solutions

**Incompressibility:**

$$\operatorname{div}_x \mathbf{U} = 0$$

**Momentum balance:**

$$\bar{\varrho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \mu(\bar{\vartheta}) \Delta_x \mathbf{U} + r \nabla_x G$$

**Heat equation:**

$$\bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta}) \left( \partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \right) - \bar{\varrho} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G = \kappa(\bar{\vartheta}) \Delta_x \Theta$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \Theta = \bar{\varrho} \nabla_x G.$$

**Nonlocal boundary conditions**

$$\Theta|_{\partial\Omega} = \Theta_B - \frac{\lambda(\bar{\varrho}, \bar{\vartheta})}{1 - \lambda(\bar{\varrho}, \bar{\vartheta})} \frac{1}{|\Omega|} \int_{\Omega} \Theta \, dx, \quad 0 < \lambda(\bar{\varrho}, \bar{\vartheta}) < 1$$

## Strong solutions of the target problem

**A. Abbatiello, EF , 2022:**

Suppose that

$$G \in W^{1,\infty}(\Omega), \Theta_B \in C^2(\bar{\Omega}),$$

$\Theta_0 \in W^{2,p}(\Omega)$ ,  $\mathbf{U}_0 \in W^{2,p}(\Omega; R^d)$ ,  $\operatorname{div}_x \mathbf{U}_0 = 0$ , for any  $1 \leq p < \infty$ ,  
together with the compatibility conditions

$$\mathbf{U}_0 = 0, \Theta_0 + \frac{\lambda}{1-\lambda} \frac{1}{|\Omega|} \int_{\Omega} \Theta_0 \, dx = \Theta_B \text{ on } \partial\Omega$$

Then there exists  $T_{\max} > 0$ ,  $T_{\max} = \infty$  if  $d = 2$ , such that the target OB system with the initial data

$$\mathbf{U}(0, \cdot) = \mathbf{U}_0, \Theta(0, \cdot) = \Theta_0,$$

admits a (unique) strong solution  $\mathbf{U}$ ,  $\Theta$  in the regularity class

$$\begin{aligned} \mathbf{U} \in L^p(0, T; W^{2,p}(\Omega; R^d)), \partial_t \mathbf{U} \in L^p(0, T; L^p(\Omega; R^d)), \Pi \in L^p(0, T; W^{1,p}(\Omega)), \\ \Theta \in L^p(0, T; W^{2,p}(\Omega)), \partial_t \Theta \in L^p(0, T; L^p(\Omega; R^d)) \end{aligned}$$

for any  $1 \leq p < \infty$  and any  $0 < T < T_{\max}$

# Convergence to the target problem, I

## Well prepared initial data

$$\varrho_\varepsilon(0, \cdot) = \bar{\varrho} + \varepsilon \varrho_{0,\varepsilon}, \quad \bar{\varrho} > 0 \text{ constant}, \quad \int_{\Omega} \varrho_{0,\varepsilon} \, dx = 0$$

$$\vartheta_\varepsilon(0, \cdot) = \bar{\vartheta} + \varepsilon \vartheta_{0,\varepsilon}, \quad \mathbf{u}_\varepsilon(0, \cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\|\varrho_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1, \quad \varrho_{0,\varepsilon} \rightarrow r_0 \text{ in } L^1(\Omega),$$

$$\|\vartheta_{0,\varepsilon}\|_{L^\infty(\Omega)} \lesssim 1, \quad \vartheta_{0,\varepsilon} \rightarrow \mathfrak{T}_0 \text{ in } L^1(\Omega),$$

$$\|\mathbf{u}_{0,\varepsilon}\|_{L^\infty(\Omega; \mathbb{R}^d)} \lesssim 1, \quad \mathbf{u}_{0,\varepsilon} \rightarrow \mathbf{U}_0 \text{ in } L^1(\Omega; \mathbb{R}^d),$$

$\mathfrak{T}_0 \in W^{2,p}(\Omega)$ ,  $\mathbf{U}_0 \in W^{2,p}(\Omega; \mathbb{R}^d)$ , for any  $1 \leq p < \infty$ ,  $\operatorname{div}_x \mathbf{U}_0 = 0$ ,  
 $\mathbf{U}_0 = 0$ ,  $\mathfrak{T}_0 = \Theta_B$  on  $\partial\Omega$ ,

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r_0 + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T}_0 = \bar{\varrho} \nabla_x G$$

# Energy estimates

**1-st ansatz in the relative energy inequality**

$$\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}(0, \cdot), \vartheta_{\varepsilon}(0, \cdot), \mathbf{u}_{\varepsilon}(0, \cdot) \middle| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) dx$$

**Boundedness of initial values:**

$$\int_{\Omega} E_{\varepsilon} \left( \varrho_{\varepsilon}(0, \cdot), \vartheta_{\varepsilon}(0, \cdot), \mathbf{u}_{\varepsilon}(0, \cdot) \middle| \bar{\varrho}, \bar{\vartheta} + \varepsilon \Theta_B, 0 \right) dx \lesssim 1$$

independently of  $\varepsilon \rightarrow 0$

## Convergence to the target problem, II

Convergence as a consequence of energy estimates

$$\frac{\underline{\varrho}_\varepsilon - \bar{\varrho}}{\varepsilon} \rightarrow \mathfrak{R} \text{ weakly-} (*) \text{ in } L^\infty(0, T; L^p(\Omega)), \quad p > 1$$

$$\frac{\vartheta_\varepsilon - \bar{\vartheta}}{\varepsilon} \rightarrow \mathfrak{T} \text{ weakly in } L^2(0, T; W^{1,2}(\Omega)) \text{ and weakly-} (*) \text{ in } L^\infty(0, T; L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^d))$$

where

$$\operatorname{div}_x \mathbf{u} = 0$$

$$\mathfrak{T}|_{\partial\Omega} = \Theta_B$$

and

$$\frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x \mathfrak{R} + \frac{\partial p(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathfrak{T} = \bar{\varrho} \nabla_x G$$

## Target problem revisited

**New variables:**

$$\mathcal{T}, \quad \mathcal{T} - \lambda(\bar{\varrho}, \bar{\vartheta}) \frac{1}{|\Omega|} \int_{\Omega} \mathcal{T} \, dx = \Theta.$$

**Transformed system:**

$$\operatorname{div}_x \mathbf{U} = 0$$

$$\bar{\varrho} \left( \partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \right) + \nabla_x \Pi = \operatorname{div}_x \mathbb{S}(\bar{\vartheta}, \nabla_x \mathbf{U}) + r \nabla_x G$$

$$\begin{aligned} & \bar{\varrho} c_p(\bar{\varrho}, \bar{\vartheta}) (\partial_t \mathcal{T} + \mathbf{U} \cdot \nabla_x \mathcal{T}) - \bar{\varrho} \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \mathbf{U} \cdot \nabla_x G \\ & = \kappa(\bar{\vartheta}) \Delta_x \mathcal{T} + \bar{\vartheta} \alpha(\bar{\varrho}, \bar{\vartheta}) \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \partial_t \frac{1}{|\Omega|} \int_{\Omega} \mathcal{T} \, dx \end{aligned}$$

**Boussinesq relation:**

$$\frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial \rho(\bar{\varrho}, \bar{\vartheta})}{\partial \vartheta} \nabla_x \mathcal{T} = \bar{\varrho} \nabla_x G, \quad \int_{\Omega} r \, dx = 0$$

**Boundary conditions:**

$$\mathbf{U}|_{\partial\Omega} = 0, \quad \mathcal{T}|_{\partial\Omega} = \Theta_B,$$

## Convergence to the target problem, III

**2-nd ansatz in the relative energy inequality**

$$E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \mid \bar{\varrho} + \varepsilon r, \bar{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right)$$

**Convergence of the initial values:**

$$\int_{\Omega} E_\varepsilon \left( \varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \mid \bar{\varrho} + \varepsilon r, \bar{\vartheta} + \varepsilon \mathcal{T}, \mathbf{U} \right) (0, \cdot) \, dx \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$