



INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

**Inviscid limit for the compressible
Navier-Stokes equations with density
dependent viscosity**

Luca Bisconti

Matteo Caggio

Preprint No. 40-2022

PRAHA 2022

Inviscid limit for the compressible Navier-Stokes equations with density dependent viscosity

Luca Bisconti¹ Matteo Caggio^{2*}

1. Università degli Studi di Firenze Dipartimento di Matematica e Informatica “U. Dini”
Viale Morgagni 67/a, I-50134 Firenze, Italia
 2. Institute of Mathematics of the Academy of Sciences of the Czech Republic,
Žitná 25, 11567, Praha 1, Czech Republic
- * corresponding author: matteocaggio@gmail.com
luca.bisconti@unifi.it

Abstract

We consider the compressible Navier-Stokes system describing the motion of a barotropic fluid with density dependent viscosity confined in a three-dimensional bounded domain Ω . We show the convergence of the weak solution to the compressible Navier-Stokes system to the strong solution to the compressible Euler system when the viscosity and the damping coefficients tend to zero.

Key words: compressible Navier-Stokes equations, density dependent viscosity, inviscid limit, boundary layer.

2010 Mathematics Subject Classifications: 35Q30, 35Q35, 76N10.

1 Introduction and main results

In the three-dimensional smooth bounded domain $\Omega \subset \mathbb{R}^3$ we consider the compressible Navier-Stokes system describing the motion of a barotropic fluid with density dependent viscosity,

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \quad (1.1)$$

$$\begin{aligned} \partial_t (\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x p(\varrho_\varepsilon) \\ - 2 \operatorname{div}_x (\mu(\varrho_\varepsilon) \mathbb{D}(\mathbf{u}_\varepsilon)) - \nabla_x (\lambda(\varrho_\varepsilon) \operatorname{div}_x \mathbf{u}_\varepsilon) + r_1 |\mathbf{u}_\varepsilon| \mathbf{u}_\varepsilon = 0, \end{aligned} \quad (1.2)$$

supplemented with the initial conditions

$$\varrho_\varepsilon(0, \cdot) = \varrho_{0,\varepsilon}, \quad \varrho_\varepsilon \mathbf{u}_\varepsilon(0, \cdot) = \varrho_{0,\varepsilon} \mathbf{u}_{0,\varepsilon} \quad (1.3)$$

and the boundary conditions

$$\varrho_\varepsilon \mathbf{u}_\varepsilon|_{\partial\Omega} = 0, \quad [\mu(\varrho_\varepsilon) \nabla \log \varrho_\varepsilon] \times \mathbf{n}|_{\partial\Omega} = 0. \quad (1.4)$$

where \mathbf{n} is the unit vector normal to the boundary.

Here, $\varrho_\varepsilon = \varrho_\varepsilon(x, t)$, $\mathbf{u}_\varepsilon = \mathbf{u}_\varepsilon(x, t)$ and $p = p(\varrho_\varepsilon(x, t))$ represent the mass density, the velocity vector and the pressure of the fluid respectively. This last is given by a power law type

$$p(\varrho) = a\varrho^\gamma, \quad a > 0, \quad \gamma > 1. \quad (1.5)$$

The term $r_1|\mathbf{u}_\varepsilon|\mathbf{u}_\varepsilon$ represents a damping term, $\mathbb{D}(\mathbf{u}_\varepsilon) = (\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon)/2$ and the viscosity coefficients $\mu(\varrho_\varepsilon)$ and $\lambda(\varrho_\varepsilon)$ satisfy the following algebraic relation

$$\lambda(\varrho_\varepsilon) = 2\varrho_\varepsilon\mu'(\varrho_\varepsilon) - 2\mu(\varrho_\varepsilon). \quad (1.6)$$

In the following we consider the case $\mu(\varrho_\varepsilon) = \varepsilon\varrho_\varepsilon$ and $\lambda(\varrho_\varepsilon) = 0$, with $\varepsilon > 0$ constant viscosity coefficient. Consequently, the system (1.1) and (1.2) reads as follows

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \quad (1.7)$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x p(\varrho_\varepsilon) - 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon)) + r_1|\mathbf{u}_\varepsilon|\mathbf{u}_\varepsilon = 0. \quad (1.8)$$

Formally, letting $(\varepsilon, r_1) \rightarrow 0$, one would expect to obtain the compressible Euler equations

$$\partial_t \varrho^E + \operatorname{div}_x(\varrho^E \mathbf{u}^E) = 0, \quad (1.9)$$

$$\partial_t(\varrho^E \mathbf{u}^E) + \operatorname{div}_x(\varrho^E \mathbf{u}^E \otimes \mathbf{u}^E) + \nabla_x p(\varrho^E) = 0, \quad (1.10)$$

for which we prescribe the initial conditions

$$\varrho^E(0, \cdot) = \varrho_0^E, \quad \varrho^E \mathbf{u}^E(0, \cdot) = \varrho_0^E \mathbf{u}_0^E, \quad (1.11)$$

and the boundary condition

$$\mathbf{u}^E \cdot \mathbf{n}|_{\partial\Omega_\delta} = 0. \quad (1.12)$$

In the present analysis we aim to prove the convergence of the weak solution to the compressible Navier-Stokes system (1.7), (1.8) to the strong solution to the compressible Euler system (1.9), (1.10) in the limit of $(\varepsilon, r_1) \rightarrow 0$.

Remark 1.1. *The boundary condition (1.4)₂ has been introduced tacitly assuming that, in the system (1.1)–(1.2), $\lambda(\varrho_\varepsilon) = 0$. This boundary condition expresses that the density should be constant on each connected component of $\partial\Omega$ and has to be understood in the weak sense (see the Appendix). Bresch et al. [10] introduced this boundary condition in order to preserve the well-known Bresch-Desjardins entropy inequality on smooth enough bounded domains with Dirichlet and Navier boundary conditions. For further details the reader can refer to [10], Section 3.*

The vanishing limit problem dates to the pioneer work of Prandtl [38] that introduced the concept of *boundary layer* and for the Navier-Stokes equations with no-slip boundary conditions derived the so-called *Prandtl equations* describing the boundary layer generated by an incompressible flow near the physical boundary. In the mathematical context, many interesting results concerning the Prandtl equations and the vanishing viscosity limit for the incompressible Navier-Stokes equations have been developed; cf. see [27], [29], [34], [35], [36], [37], [40], [41], [48], [49], [51].

In particular, an approach for proving the convergence from the solutions of the Navier-Stokes equations to the solutions of the Euler equation was introduced by Kato [31] that studied the vanishing viscosity limit of the incompressible viscous flow with no-slip boundary conditions and proved the following conditional result: *if the energy dissipation rate of the viscous flow in a boundary layer of width proportional to the viscosity vanishes, then the solutions of the incompressible Navier–Stokes equations converge to some solutions of the incompressible Euler equations in the energy space.* In other words, the viscous flow can be approximated by the inviscid flow in the energy space under a dissipation condition of energy in a neighborhood of the physical boundary with width proportional to the viscosity, by constructing an artificial or "fake" boundary layer.

Since the Kato works, the result has been improved by several authors. Wang [47] relaxed Kato's dissipation condition of energy to the case only containing the tangential derivatives of the tangential or normal velocity, but requiring a thicker boundary layer. Kelliher [32] extended Kato's result replacing the gradient of velocity of Kato's energy condition by only the vorticity of the flow. Finally, under the assumption of the Oleinik condition of no back-flow in the trace of the Euler flow and of a lower bound for the Navier-Stokes vorticity in a Kato-like boundary layer, Constantin et al. [19] obtained that the inviscid limit from the Navier-Stokes equations to the Euler equations holds in energy space.

In the compressible case, not much is known. Sueur [44] assumed the following (sufficient) conditions in order for the convergence to hold

$$\varepsilon \int_{[0,T] \times \Gamma_\varepsilon} \left(\frac{\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2}{d_\Omega^2(x)} + \frac{\varrho_\varepsilon^2 (\mathbf{u}_\varepsilon \cdot \mathbf{n})^2}{d_\Omega^2(x)} + \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) \right) dxdt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (1.13)$$

Here, $\mathbf{u}_\varepsilon \cdot \mathbf{n}$ is the normal component of \mathbf{u}_ε , $d_\Omega(x)$ the distance of $x \in \Omega$ to the boundary $\partial\Omega$ and $\Gamma_{c\varepsilon} = \{x \in \Omega : d_\Omega(x) \leq c\varepsilon\}$ for a constant $c > 0$ (in the case of (1.13), we have $c = 1$). Besides (1.13), Bardos and Nguyen [5] introduced other criteria (see Theorem 1.8). A similar assumption to that in [44] reads as follows

$$\int_{[0,T] \times \Gamma_\varepsilon} \left(\frac{\varrho_\varepsilon^\gamma}{\gamma - 1} + \varepsilon \frac{\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2}{d_\Omega^2(x)} + \varepsilon \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) \right) dxdt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (1.14)$$

For other results concerning the vanishing viscosity limit in the case of linearized Navier-Stokes equations, one-dimensional case and noncharacteristic boundary layer, the reader can refer to [28], [39] and [52], respectively. The result in [44] has been recently improved by Wang and Zhu [50] in the sense that the authors assumed as sufficient conditions the tangential or the normal component of velocity only (see relations (2.5) and (2.6) in Theorem 2.1) at the cost of increasing the width of the boundary layer.

As mentioned above, the proposal of our analysis is to study the vanishing viscosity limit for the compressible Navier-Stokes system with density dependent viscosity. The strategy adopted in order to prove the convergence relies to the issue of weak-strong uniqueness by using relative energy estimates. In particular, we introduce a relative energy functional "measuring" the distance between the weak solution of the compressible Navier-Stokes system and the strong solution of the compressible Euler system. Consequently, we derive a relative energy inequality satisfied by the weak solution of the

Navier-Stokes equations. As the weak formulation, the relative energy inequality involves some test functions that have to satisfy the no-slip conditions, which are not satisfied by the solution of the Euler equations. For this reasons, we will introduce a “correction” based on the Kato’s “fake” boundary layer in order to work with test functions satisfying the no-slip conditions at the boundaries. However, different from the construction of Feireisl et al. [23], [24], our relative energy inequality is derived from an “augmented version” of the compressible Navier-Stokes system (see Bresch et al. [12], [13], [14]; see also [16],[17],[18] for recent applications). The reason for that stands in the fact that a H^1 bound for the velocity is no longer available because of the density dependent viscosity. Consequently, standard application of the Korn’s inequality in the weak-strong uniqueness context is not possible (see, for example, Feireisl et al. [24]). We would like to mention that, as far as the authors are aware, this is the first result in this direction. A recent result of similar type has been proved by Geng et al. [25] where the authors establish the convergence in the vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for three-dimensional compressible isentropic flow in the whole space \mathbb{R}^3 when the viscosity coefficients are given as constant multiples of the density’s power. Moreover, a convergence to dissipative solution of compressible Euler equations has been analyzed in [12] in the three-dimensional torus \mathbb{T}^3 .

Our paper is organized as follows. In this section we introduce the weak solutions to the compressible Navier-Stokes system (1.1)–(1.4) together with the existence result. Subsequently, we discuss the existence of the strong solution to the compressible Euler system (1.9)–(1.12) and the “augmented” version of the compressible Navier-Stokes system (1.1)–(1.4). We conclude the section presenting our main result (see Theorem 1.3 below) together with a preliminary Lemma and the a priori estimates. Section 2 is devoted to the proof of the our result. First, we derive a relative energy inequality satisfied by the weak solutions of the “augmented” version of the compressible Navier-Stokes system (1.1)–(1.4). Second, we introduce the Kato “fake” boundary layer and we discuss its properties. Finally, we perform the inviscid limit.

1.1 Weak solutions to the compressible Navier-Stokes system

We introduce the definition of the weak solution to the compressible Navier-Stokes system.

Definition 1.1. *We say that $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ is a global weak solution of (1.7) and (1.8) with boundary conditions (1.4) if it satisfies the following regularity properties*

$$\begin{aligned} \varrho_\varepsilon &\in L^\infty(0, T; L^\gamma(\Omega)), \quad \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon, \quad \nabla_x \sqrt{\varrho_\varepsilon} \in L^\infty(0, T; L^2(\Omega)), \\ \varrho_\varepsilon^{1/3} \mathbf{u}_\varepsilon &\in L^3((0, T) \times \Omega), \end{aligned} \tag{1.15}$$

as well as (1.4)₁ in $L^2(0, T; L^1(\partial\Omega))$ and (1.4)₂ in $L^2(0, T; L^\infty(\partial\Omega))$. The continuity equation is satisfied in the following sense

$$-\int_0^T \int_\Omega \varrho_\varepsilon \partial_t \varphi dx dt - \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi dx dt = \int_\Omega \varrho_\varepsilon(0, \cdot) \varphi(0, \cdot) dx \tag{1.16}$$

for all $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R})$. The momentum equation is satisfied in the following sense

$$\begin{aligned} & - \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi dx dt - \int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \varphi dx dt + 2\varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon) : \nabla_x \varphi dx dt \\ & - \int_0^T \int_\Omega p(\varrho_\varepsilon) \operatorname{div}_x \varphi dx dt + r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon| \mathbf{u}_\varepsilon \cdot \varphi dx dt = \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon(0, \cdot) \cdot \varphi(0, \cdot) dx \end{aligned} \quad (1.17)$$

for all $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R}^3)$, where, for $(i, j = 1, 2, 3)$ the viscous term reads as follows

$$\begin{aligned} & \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon) : \nabla_x \varphi dx dt \\ & = -\varepsilon \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j \partial_{ii} \varphi_j dx dt - 2\varepsilon \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i \varphi_j dx dt \\ & \quad - \varepsilon \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_i \partial_{ji} \varphi_j dx dt - 2\varepsilon \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_i \partial_j \sqrt{\varrho_\varepsilon} \partial_i \varphi_j dx dt. \end{aligned} \quad (1.18)$$

Moreover, there exists Λ such that $\varrho_\varepsilon \mathbf{u}_\varepsilon = \sqrt{\varrho_\varepsilon} \Lambda$, and $\mathcal{S} \in L^2((0, T) \times \Omega)$ such that $\sqrt{\varrho_\varepsilon} \mathcal{S} = \operatorname{Symm}(\nabla(\varrho_\varepsilon \mathbf{u}_\varepsilon) - 2\nabla \sqrt{\varrho_\varepsilon} \otimes \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon)$ in \mathcal{D}' , satisfying the following energy inequality

$$\begin{aligned} & \sup_{t \in (0, T)} \int_\Omega \frac{1}{2} |\Lambda|^2 + H(\varrho_\varepsilon) dx + 2\varepsilon \int_0^T \int_\Omega |\mathcal{S}|^2 dx dt + r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^3 dx dt \\ & \leq \int_\Omega \frac{1}{2} \varrho_{0, \varepsilon} |\mathbf{u}_{0, \varepsilon}|^2 + H(\varrho_{0, \varepsilon}) dx, \end{aligned} \quad (1.19)$$

and there exists $\mathcal{A} \in L^2((0, T) \times \Omega)$ such that $\sqrt{\varrho_\varepsilon} \mathcal{A} = \operatorname{Asymm}(\nabla(\varrho_\varepsilon \mathbf{u}_\varepsilon) - 2\nabla \sqrt{\varrho_\varepsilon} \otimes \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon)$ in \mathcal{D}' , such that the following Bresch-Desjardins entropy inequality is satisfied

$$\begin{aligned} & \sup_{t \in (0, T)} \int_\Omega \frac{1}{2} |\Lambda + 2\varepsilon \nabla \sqrt{\varrho_\varepsilon}|^2 + H(\varrho_\varepsilon) dx + 2\varepsilon \int_0^T \int_\Omega |\mathcal{A}|^2 dx dt \\ & \quad + \varepsilon \int_0^T \int_\Omega \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla \varrho|^2 dx dt + r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^3 dx dt \\ & \quad + \varepsilon r_1 \int_0^T \int_\Omega |\mathbf{u}_\varepsilon| \mathbf{u}_\varepsilon \nabla_x \varrho_\varepsilon dx dt \\ & \leq \int_\Omega \frac{1}{2} |\sqrt{\varrho_{0, \varepsilon}} \mathbf{u}_{0, \varepsilon} + 2\varepsilon \nabla \sqrt{\varrho_{0, \varepsilon}}|^2 + H(\varrho_{0, \varepsilon}) dx. \end{aligned} \quad (1.20)$$

Here $H(\varrho_\varepsilon)$ is such that

$$\varrho_\varepsilon H'(\varrho_\varepsilon) - H(\varrho_\varepsilon) = p(\varrho_\varepsilon), \quad H''(\varrho_\varepsilon) = \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon}.$$

Consequently, we have

$$H(\varrho_\varepsilon) = \frac{\varrho_\varepsilon^\gamma}{\gamma - 1}.$$

Remark 1.2. In [10], the authors do not define the weak solution introducing Λ , \mathcal{S} and \mathcal{A} . The reason that motivates the Definition 1.1 is related to the fact that the “degenerate” viscosity prevents the velocity field to be uniquely determined in the vacuum regions, namely regions where $\{\varrho_\varepsilon = 0\}$. Indeed, none of the quantities \mathbf{u}_ε , $\nabla_x \mathbf{u}_\varepsilon$ and $1/\sqrt{\varrho_\varepsilon}$ are defined a.e. in Ω . Consequently, the problem is best analyzed in terms of $\sqrt{\varrho_\varepsilon}$, $\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon$ and $\varrho_\varepsilon \mathbf{u}_\varepsilon = \sqrt{\varrho_\varepsilon} \Lambda$ (see, for example, [2], [3], [4]). However, for consistency with the literature concerning the Navier-Stokes equations with density dependent viscosity, weak solutions are defined in terms of $\sqrt{\varrho_\varepsilon}$ and $\sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon$. For these reasons, the viscous stress tensor in the energy inequality (1.19) is thought as

$$\varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon) = \sqrt{\varrho_\varepsilon} \mathcal{S}.$$

Indeed, it is not clear if weak solutions satisfy the energy inequality in the usual sense, namely

$$\begin{aligned} \sup_{t \in (0, T)} \int_{\Omega} \frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + H(\varrho_\varepsilon) dx + 2\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon |\mathbb{D}(\mathbf{u}_\varepsilon)|^2 dx dt + r_1 \int_0^T \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^3 dx dt \\ \leq \int_{\Omega} \frac{1}{2} \varrho_{0, \varepsilon} |\mathbf{u}_{0, \varepsilon}|^2 + H(\varrho_{0, \varepsilon}) dx. \end{aligned} \quad (1.21)$$

The following existence result has been proved by Bresch et al. [10].

Theorem 1.1. *Let the initial data for the compressible Navier-Stokes be given in such a way*

$$\begin{aligned} \varrho_{0, \varepsilon} \in L^\gamma(\Omega), \quad \varrho_{0, \varepsilon} \geq 0, \quad \nabla_x \sqrt{\varrho_{0, \varepsilon}} \in L^2(\Omega), \\ \varrho_{0, \varepsilon} \mathbf{u}_{0, \varepsilon} \in L^1(\Omega), \quad \varrho_{0, \varepsilon} \mathbf{u}_{0, \varepsilon} = 0 \text{ if } \varrho_{0, \varepsilon} = 0, \quad \frac{|\varrho_{0, \varepsilon} \mathbf{u}_{0, \varepsilon}|^2}{\varrho_{0, \varepsilon}} \in L^1(\Omega). \end{aligned}$$

Then, for fixed $\varepsilon > 0$ and r_1 , there exist at least a global weak solution to the compressible Navier-Stokes (1.7), (1.8) with boundary conditions (1.4) in the sense of Definition 1.1.

Remark 1.3. *Theorem 1.1 extends to smooth enough bounded domains existence results about barotropic compressible Navier–Stokes systems with density dependent viscosity coefficients. The authors in [10] proved the existence of global weak solutions for Dirichlet and Navier boundary conditions on the velocity. An additional turbulent drag term in the momentum equation is used to handle the construction of approximate solutions.*

1.2 Strong solution to the compressible Euler system

We recall the local existence of strong solution for the compressible Euler system (see [1], [6], [21], [22], [43])

Theorem 1.2. *Let $(\varrho_0^E, \mathbf{u}_0^E) \in C^{1+\delta}$, $\delta > 0$ be some compatible initial data with $0 < \inf_{\Omega} \varrho_0^E$ and $\sup_{\Omega} \varrho_0^E < \infty$. Then, there exists $T > 0$ and a unique solution*

$$(\varrho^E, \mathbf{u}^E) \in C_w([0, T]; C^{1+\delta}(\Omega)) \cap C^1([0, T] \times \bar{\Omega}) \quad (1.22)$$

of (1.9)–(1.12) such that

$$0 < \inf_{(0,T) \times \Omega} \varrho^E \text{ and } \sup_{(0,T) \times \Omega} \varrho^E < \infty. \quad (1.23)$$

Remark 1.4. As remarked by Sueur [44], "compatible" refers to some conditions satisfied by the initial data on the boundary $\partial\Omega$ which are necessary for the existence of strong solution (see [42], [43] for more details).

1.3 "Augmented" version of the compressible Navier-Stokes system

As observed in [13], the system (1.7)–(1.8) can be reformulated through an "augmented" version. Indeed, defining the velocity

$$\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon + \varepsilon \nabla_x \log \varrho_\varepsilon \quad (1.24)$$

with

$$\mathbf{w}_\varepsilon = \varepsilon \nabla_x \log \varrho_\varepsilon, \quad (1.25)$$

the "augmented" version of the Navier-Stokes system reads as follows

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0, \quad (1.26)$$

$$\begin{aligned} \partial_t(\varrho_\varepsilon \mathbf{v}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x p(\varrho_\varepsilon) + r_1 \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \\ - \varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon)) - \varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon)) + \varepsilon \operatorname{div}_x(\varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon) = 0, \end{aligned} \quad (1.27)$$

$$\partial_t(\varrho_\varepsilon \mathbf{w}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{w}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \varepsilon \operatorname{div}_x(\varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon) = 0 \quad (1.28)$$

supplemented with the following boundary conditions

$$[\varrho_\varepsilon (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon)]|_{\partial\Omega} = 0, \quad [\varrho_\varepsilon \mathbf{w}_\varepsilon] \times \mathbf{n}|_{\partial\Omega} = 0, \quad (1.29)$$

meant in the sense of distributions on $\partial\Omega$ (see the Appendix).

Remark 1.5. We discuss here how to derive the equations (1.27) and (1.28). We start with (1.28). From the continuity equation, we have

$$2\varepsilon \partial_t(\varrho_\varepsilon \nabla_x \log \varrho_\varepsilon) = -2\varepsilon \nabla_x \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon).$$

Now, the following identity holds (see Antonelli and Spirito [2])

$$\begin{aligned} 2\varepsilon \nabla_x \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) &= 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) \\ &\quad - 2\varepsilon \Delta(\varrho_\varepsilon \mathbf{u}_\varepsilon) + 4\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon)). \end{aligned}$$

Thus

$$\begin{aligned} 2\varepsilon \partial_t(\varrho_\varepsilon \nabla_x \log \varrho_\varepsilon) &= -2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) \\ &\quad + 2\varepsilon \Delta(\varrho_\varepsilon \mathbf{u}_\varepsilon) - 4\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon)) \end{aligned}$$

$$\begin{aligned}
&= -2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) \\
&\quad + 2\varepsilon \Delta(\varrho_\varepsilon \mathbf{u}_\varepsilon) - 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon)), \\
&= -2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) \\
&\quad + 2\varepsilon \operatorname{div}_x(\mathbf{u}_\varepsilon \otimes \nabla_x \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon) - 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon(\nabla_x \mathbf{u}_\varepsilon + \nabla_x^\top \mathbf{u}_\varepsilon)) \\
&= -2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon + \varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) \\
&\quad + 2\varepsilon \operatorname{div}_x(\mathbf{u}_\varepsilon \otimes \nabla_x \varrho_\varepsilon) - 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon).
\end{aligned}$$

Now, we can write

$$2\varepsilon \operatorname{div}_x(\mathbf{u}_\varepsilon \otimes \nabla_x \varrho_\varepsilon) = 2\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \nabla_x \log \varrho_\varepsilon),$$

and it cancels with its counterpart. Consequently

$$\varepsilon \partial_t(\varrho_\varepsilon \nabla_x \log \varrho_\varepsilon) + \varepsilon \operatorname{div}_x(\varrho_\varepsilon \nabla_x \log \varrho_\varepsilon \otimes \mathbf{u}_\varepsilon) + \varepsilon \operatorname{div}_x(\varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon) = 0.$$

From the definition of \mathbf{w}_ε we obtain relation (1.28). Equation (1.27) follows from summing up (1.8) with (1.28).

Let us now introduce the notion of the weak solution to the system (1.26)–(1.28).

Definition 1.2. We say that $(\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon)$ is a global weak solution of (1.26) and (1.28) with boundary conditions (1.29) if it satisfies the following regularity properties

$$\varrho_\varepsilon \in L^\infty(0, T; L^\gamma(\Omega)), \quad \sqrt{\varrho_\varepsilon} \mathbf{v}_\varepsilon, \quad \nabla_x \sqrt{\varrho_\varepsilon} \in L^\infty(0, T; L^2(\Omega)). \quad (1.30)$$

The continuity equation is satisfied in the following sense

$$\begin{aligned}
& - \int_\Omega \varrho_\varepsilon(T, \cdot) \varphi(T, \cdot) dx + \int_\Omega \varrho_\varepsilon(0, \cdot) \varphi(0, \cdot) dx \\
& \quad + \int_0^T \int_\Omega \varrho_\varepsilon \partial_t \varphi dx dt + \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \varphi dx dt = 0,
\end{aligned} \quad (1.31)$$

for all $\varphi \in C_c^\infty([0, T] \times \Omega; \mathbb{R})$. The momentum equations are satisfied in the following sense

$$\begin{aligned}
& - \int_\Omega \varrho_\varepsilon \mathbf{v}_\varepsilon(T, \cdot) \cdot \boldsymbol{\varphi}(T, \cdot) dx + \int_\Omega \varrho_\varepsilon \mathbf{v}_\varepsilon(0, \cdot) \cdot \boldsymbol{\varphi}(0, \cdot) dx + \int_0^T \int_\Omega \varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \boldsymbol{\varphi} dx dt \\
& \quad + \int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \nabla_x \boldsymbol{\varphi} dx dt - \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \boldsymbol{\varphi} dx dt \\
& \quad - \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \boldsymbol{\varphi} dx dt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon : \nabla_x \boldsymbol{\varphi} dx dt \\
& \quad - r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \cdot \boldsymbol{\varphi} + \int_0^T \int_\Omega p(\varrho_\varepsilon) \operatorname{div}_x \boldsymbol{\varphi} dx dt = 0,
\end{aligned} \quad (1.32)$$

and

$$\begin{aligned}
& - \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(T, \cdot) \cdot \boldsymbol{\varphi}(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(0, \cdot) \cdot \boldsymbol{\varphi}(0, \cdot) dx + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \cdot \partial_t \boldsymbol{\varphi} dx dt \\
& + \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \boldsymbol{\varphi} dx dt = 0,
\end{aligned} \tag{1.33}$$

for all $\boldsymbol{\varphi} \in C_c^{\infty}([0, T] \times \Omega; \mathbb{R}^3)$, where, for $i, j = 1, 2, 3$, the viscous terms reads as follows

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt \\
& = -\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_{ii} \varphi_j dx dt - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt \\
& - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_i \partial_{ji} \varphi_j dx dt - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_i \partial_j \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt,
\end{aligned} \tag{1.34}$$

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt \\
& = -\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_{ii} \varphi_j dx dt - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_i \partial_{ji} \varphi_j dx dt + 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_i \partial_j \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt,
\end{aligned} \tag{1.35}$$

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \boldsymbol{\varphi} dx dt \\
& = -\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_{ii} \varphi_j dx dt - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt,
\end{aligned} \tag{1.36}$$

and

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \boldsymbol{\varphi} dx dt \\
& = -\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_i \partial_{ji} \varphi_j dx dt - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_i \partial_j \sqrt{\varrho_{\varepsilon}} \partial_i \varphi_j dx dt \\
& \leq \int_{\Omega} \frac{1}{2} \left(\varrho_{0, \varepsilon} |\mathbf{u}_{0, \varepsilon}|^2 + H(\varrho_{0, \varepsilon}) \right) dx.
\end{aligned} \tag{1.37}$$

Moreover, there exists Λ such that $\varrho_{\varepsilon} \mathbf{u}_{\varepsilon} = \sqrt{\varrho_{\varepsilon}} \Lambda$, and there exists $\mathcal{A} \in L^2((0, T) \times \Omega)$ such that $\sqrt{\varrho_{\varepsilon}} \mathcal{A} = \text{Asymm}(\nabla(\varrho_{\varepsilon} \mathbf{u}_{\varepsilon}) - 2\nabla \sqrt{\varrho_{\varepsilon}} \otimes \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon})$ in \mathcal{D}' , such that the following

Bresch-Desjardins entropy inequality is satisfied

$$\begin{aligned}
& \sup_{t \in (0, T)} \int_{\Omega} \left(\frac{1}{2} \left(|\Lambda + 2\varepsilon \nabla \sqrt{\varrho_\varepsilon}|^2 + |2\varepsilon \nabla \sqrt{\varrho_\varepsilon}|^2 \right) + H(\varrho_\varepsilon) \right) dx \\
& + \varepsilon \int_0^T \int_{\Omega} |\mathcal{S}|^2 dxdt + \varepsilon \int_0^T \int_{\Omega} |\mathcal{A}|^2 dxdt + \varepsilon \int_0^T \int_{\Omega} \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla \varrho|^2 dxdt \\
& + r_1 \int_{\Omega} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^3 dxdt + \varepsilon r_1 \int_0^T \int_{\Omega} |\mathbf{u}_\varepsilon| \mathbf{u}_\varepsilon \nabla_x \varrho_\varepsilon dxdt \\
& \leq \int_{\Omega} \left(\frac{1}{2} \left(|\sqrt{\varrho_{0, \varepsilon}} \mathbf{u}_{0, \varepsilon} + 2\varepsilon \nabla \sqrt{\varrho_{0, \varepsilon}}|^2 + |2\varepsilon \nabla \sqrt{\varrho_{0, \varepsilon}}|^2 \right) + H(\varrho_{0, \varepsilon}) \right) dx.
\end{aligned} \tag{1.38}$$

As the authors remarked in [14], a global weak solution $(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$ of the compressible Navier-Stokes system is also a solution of the augmented version. Consequently, Theorem 1.1 holds for weak solutions to the system (1.26)–(1.28) in the sense of Definition 1.2.

Remark 1.6. *Relation (1.38) has been originally derived on a three-dimensional torus, \mathbb{T}^3 , in [12]. In the case of a bounded domain, it is possible to obtain the same relation thanks to the boundary conditions (1.4). Indeed, equation (1.27) can be rewritten as follows*

$$\begin{aligned}
& \partial_t(\varrho_\varepsilon \mathbf{v}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \nabla_x p(\varrho_\varepsilon) + r_1 \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \\
& - \varepsilon \operatorname{div}(\varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon)) - \varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbb{A}(\mathbf{u}_\varepsilon)) = 0.
\end{aligned} \tag{1.39}$$

We multiply equation (1.39) by \mathbf{v}_ε . Using the continuity equation and integration by parts, we obtain

$$\begin{aligned}
& \frac{d}{dt} \int_{\Omega} \left(\varrho_\varepsilon \frac{|\mathbf{v}_\varepsilon|^2}{2} + H(\varrho_\varepsilon) \right) dx + \int_{\Omega} \nabla p(\varrho_\varepsilon) \cdot \mathbf{w}_\varepsilon dx \\
& + r_1 \int_{\Omega} \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \mathbf{v}_\varepsilon dx + \int_{\Omega} \varrho_\varepsilon \nabla \mathbf{v}_\varepsilon \cdot (\mathbb{D}(\mathbf{u}_\varepsilon) + \mathbb{A}(\mathbf{u}_\varepsilon)) dx \\
& + \int_{\partial\Omega} \frac{|\mathbf{v}_\varepsilon|^2}{2} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{n} ds + \int_{\partial\Omega} H(\varrho_\varepsilon) \mathbf{u}_\varepsilon \cdot \mathbf{n} ds - \varepsilon \int_{\partial\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{u}_\varepsilon) \mathbf{v}_\varepsilon \mathbf{n} ds \\
& - \varepsilon \int_{\partial\Omega} \varrho_\varepsilon \mathbb{A}(\mathbf{u}_\varepsilon) \mathbf{v}_\varepsilon \mathbf{n} ds = 0.
\end{aligned} \tag{1.40}$$

Thanks to the boundary condition (1.4), we have

$$\int_{\partial\Omega} \frac{|\mathbf{v}_\varepsilon|^2}{2} \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathbf{n} ds = \int_{\partial\Omega} H(\varrho_\varepsilon) \mathbf{u}_\varepsilon \cdot \mathbf{n} ds = 0,$$

while, because $\varrho_\varepsilon \mathbf{v}_\varepsilon = \varrho_\varepsilon \mathbf{w}_\varepsilon$ on $\partial\Omega$, we have

$$- \int_{\partial\Omega} \varrho_\varepsilon \mathbb{A}(\mathbf{u}_\varepsilon) \mathbf{v}_\varepsilon \mathbf{n} ds = -\varepsilon \int_{\partial\Omega} \nabla_x \varrho_\varepsilon \cdot (\operatorname{curl} \mathbf{u}_\varepsilon \times \mathbf{n}) ds = \varepsilon \int_{\partial\Omega} (\nabla \varrho_\varepsilon \times \mathbf{n}) \cdot \operatorname{curl} \mathbf{u}_\varepsilon ds$$

that is equal zero thanks to the boundary condition (1.4)₂. Consequently, we rewrite

relation (1.40) as follows

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\varrho_{\varepsilon} \frac{|\mathbf{v}_{\varepsilon}|^2}{2} + H(\varrho_{\varepsilon}) \right) dx + \int_{\Omega} \nabla p(\varrho_{\varepsilon}) \cdot \mathbf{w}_{\varepsilon} dx + r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \mathbf{v}_{\varepsilon} dx \\ + \varepsilon \int_{\Omega} \varrho_{\varepsilon} \nabla \mathbf{v}_{\varepsilon} \left(\nabla_x \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon} \right) dx - \varepsilon \int_{\partial\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{u}_{\varepsilon}) \mathbf{v}_{\varepsilon} \mathbf{n} ds = 0, \end{aligned} \quad (1.41)$$

where we formally assumed $\nabla \mathbf{w}_{\varepsilon} = \nabla^{\top} \mathbf{w}_{\varepsilon}$. Now, we multiply (1.28) by \mathbf{w}_{ε} . Similarly as before, we obtain

$$\frac{d}{dt} \int_{\Omega} \varrho_{\varepsilon} \frac{|\mathbf{w}_{\varepsilon}|^2}{2} dx + \varepsilon \int_{\Omega} \operatorname{div}_x \left(\varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} \right) \cdot \mathbf{w}_{\varepsilon} dx = 0. \quad (1.42)$$

Integrating by parts the viscous term, we have

$$\begin{aligned} \int_{\Omega} \operatorname{div}_x \left(\varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} \right) \cdot \mathbf{w}_{\varepsilon} dx \\ = - \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} dx + \int_{\partial\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} \mathbf{w}_{\varepsilon} \mathbf{n} ds \\ = - \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \nabla_x \mathbf{w}_{\varepsilon} dx + \int_{\partial\Omega} \varrho_{\varepsilon} (\mathbb{D}(\mathbf{u}_{\varepsilon}) - \mathbb{A}(\mathbf{u}_{\varepsilon})) \mathbf{w}_{\varepsilon} \mathbf{n} ds. \end{aligned}$$

Now, again formally, we consider $\nabla \mathbf{w}_{\varepsilon} = \nabla^{\top} \mathbf{w}_{\varepsilon}$, and we rewrite the relation above as

$$\begin{aligned} - \int_{\Omega} \varrho_{\varepsilon} (\nabla_x^{\top} \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon}) \nabla_x^{\top} \mathbf{w}_{\varepsilon} dx + \int_{\partial\Omega} \varrho_{\varepsilon} (\mathbb{D}(\mathbf{u}_{\varepsilon}) - \mathbb{A}(\mathbf{u}_{\varepsilon})) \mathbf{w}_{\varepsilon} \mathbf{n} ds \\ = - \int_{\Omega} \varrho_{\varepsilon} (\nabla_x^{\top} \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon}) \nabla_x^{\top} \mathbf{w}_{\varepsilon} dx + \int_{\partial\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{u}_{\varepsilon}) \mathbf{w}_{\varepsilon} \mathbf{n} - \int_{\partial\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{u}_{\varepsilon}) \mathbf{w}_{\varepsilon} \mathbf{n} ds, \end{aligned}$$

where the last term is equal zero for the same arguments as above. Consequently, relation (1.42) could be rewritten as follows

$$\frac{d}{dt} \int_{\Omega} \varrho_{\varepsilon} \frac{|\mathbf{w}_{\varepsilon}|^2}{2} dx - \varepsilon \int_{\Omega} \varrho_{\varepsilon} (\nabla_x^{\top} \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon}) \nabla_x^{\top} \mathbf{w}_{\varepsilon} dx + \int_{\partial\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{u}_{\varepsilon}) \mathbf{w}_{\varepsilon} \mathbf{n} ds = 0. \quad (1.43)$$

Summing up (1.41) and (1.43), we obtain

$$\begin{aligned} \frac{d}{dt} \int_{\Omega} \left(\varrho_{\varepsilon} \left(\frac{|\mathbf{v}_{\varepsilon}|^2}{2} + \frac{|\mathbf{w}_{\varepsilon}|^2}{2} \right) + H(\varrho_{\varepsilon}) \right) dx + \int_{\Omega} \nabla p(\varrho_{\varepsilon}) \cdot \mathbf{w}_{\varepsilon} \\ + r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \mathbf{v}_{\varepsilon} dx + \varepsilon \int_{\Omega} \varrho_{\varepsilon} \nabla \mathbf{v}_{\varepsilon} \left(\nabla_x \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon} \right) dx \\ - \varepsilon \int_{\Omega} \varrho_{\varepsilon} (\nabla_x^{\top} \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon}) \nabla_x^{\top} \mathbf{w}_{\varepsilon} dx = 0. \end{aligned} \quad (1.44)$$

Now,

$$\varepsilon \int_{\Omega} \varrho_{\varepsilon} \nabla \mathbf{v}_{\varepsilon} \left(\nabla_x \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon} \right) dx - \varepsilon \int_{\Omega} \varrho_{\varepsilon} (\nabla_x^{\top} \mathbf{v}_{\varepsilon} - \nabla_x^{\top} \mathbf{w}_{\varepsilon}) \nabla_x^{\top} \mathbf{w}_{\varepsilon} dx$$

$$\begin{aligned}
&= \varepsilon \int_{\Omega} \varrho_{\varepsilon} \left(|\mathbb{D}(\mathbf{v}_{\varepsilon}) + \mathbb{A}(\mathbf{v}_{\varepsilon})|^2 + |\nabla_x^{\top} \mathbf{w}_{\varepsilon}|^2 - \nabla_x^{\top} \mathbf{w}_{\varepsilon} \left(\nabla_x \mathbf{v}_{\varepsilon} + \nabla_x^{\top} \mathbf{v}_{\varepsilon} \right) \right) dx \\
&= \varepsilon \int_{\Omega} \varrho_{\varepsilon} \left(|\mathbb{D}(\mathbf{v}_{\varepsilon})|^2 + |\mathbb{A}(\mathbf{v}_{\varepsilon})|^2 + |\nabla_x^{\top} \mathbf{w}_{\varepsilon}|^2 - 2 \nabla_x^{\top} \mathbf{w}_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) \right) dx \\
&= \varepsilon \int_{\Omega} \varrho_{\varepsilon} \left(|\mathbb{A}(\mathbf{v}_{\varepsilon})|^2 + |\mathbb{D}(\mathbf{v}_{\varepsilon}) - \nabla_x^{\top} \mathbf{w}_{\varepsilon}|^2 \right) dx \\
&= \varepsilon \int_{\Omega} \varrho_{\varepsilon} \left(|\mathbb{A}(\mathbf{u}_{\varepsilon})|^2 + |\mathbb{D}(\mathbf{u}_{\varepsilon})|^2 \right) dx.
\end{aligned}$$

Moreover,

$$\int_{\Omega} \nabla p(\varrho_{\varepsilon}) \cdot \mathbf{w}_{\varepsilon} = \varepsilon \int_{\Omega} p'(\varrho_{\varepsilon}) \nabla_x \varrho_{\varepsilon} \cdot \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}}.$$

We consider now the drag term. We can write

$$r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \mathbf{v}_{\varepsilon} = r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}|^3 + r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \mathbf{w}_{\varepsilon}.$$

The second term reads as follows:

$$\varepsilon r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \nabla_x \log \varrho_{\varepsilon} = \varepsilon r_1 \int_{\Omega} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \nabla_x \varrho_{\varepsilon}.$$

By parts integration gives

$$\varepsilon r_1 \int_{\Omega} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \nabla_x \varrho_{\varepsilon} = -\varepsilon r_1 \int_{\Omega} |\mathbf{u}_{\varepsilon}| \operatorname{div}_x \mathbf{u}_{\varepsilon} \varrho_{\varepsilon} - \varepsilon r_1 \int_{\Omega} \varrho_{\varepsilon} \frac{u_k}{|\mathbf{u}_{\varepsilon}|} u_j \partial_j u_k + \varepsilon r_1 \int_{\partial\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} |\mathbf{u}_{\varepsilon}|,$$

where the boundary term is zero thanks to the condition (1.4). We have

$$\begin{aligned}
\left| \varepsilon r_1 \int_{\Omega} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \nabla_x \varrho_{\varepsilon} \right| dx &\leq \varepsilon r_1 \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}| |\mathbb{D}(\mathbf{u}_{\varepsilon})| dx \\
&\leq \varepsilon r_1 \|\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}\|_{L^2(\Omega)} \|\sqrt{\varrho_{\varepsilon}} \mathbb{D}(\mathbf{u}_{\varepsilon})\|_{L^2(\Omega)} \\
&\leq r_1 \left\| \frac{\sqrt{\varrho_{\varepsilon}}}{\varrho_{\varepsilon}^{1/3}} \varrho_{\varepsilon}^{1/3} \mathbf{u}_{\varepsilon} \right\|_{L^2(\Omega)} \varepsilon \|\sqrt{\varrho_{\varepsilon}} \mathbb{D}(\mathbf{u}_{\varepsilon})\|_{L^2(\Omega)} \\
&\leq \frac{r_1}{2} \left\| \frac{\sqrt{\varrho_{\varepsilon}}}{\varrho_{\varepsilon}^{1/3}} \varrho_{\varepsilon}^{1/3} \mathbf{u}_{\varepsilon} \right\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \|\sqrt{\varrho_{\varepsilon}} \mathbb{D}(\mathbf{u}_{\varepsilon})\|_{L^2(\Omega)}^2,
\end{aligned}$$

where the second term can be absorbed. For the first term, we have

$$\frac{r_1}{2} \left\| \frac{\sqrt{\varrho_{\varepsilon}}}{\varrho_{\varepsilon}^{1/3}} \varrho_{\varepsilon}^{1/3} \mathbf{u}_{\varepsilon} \right\|_{L^2(\Omega)}^2 = \frac{r_1}{2} \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho_{\varepsilon}^{2/3}} \varrho_{\varepsilon}^{2/3} \mathbf{u}_{\varepsilon}^2 dx.$$

Namely

$$\frac{r_1}{2} \left\| \frac{\sqrt{\varrho_{\varepsilon}}}{\varrho_{\varepsilon}^{1/3}} \varrho_{\varepsilon}^{1/3} \mathbf{u}_{\varepsilon} \right\|_{L^2(\Omega)}^2 \leq \frac{r_1}{6} \int_{\Omega} \varrho_{\varepsilon} + \frac{r_1}{3} \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}|^3 dx.$$

The second term can be absorbed. The first term is bounded by a constant $c = c(r_1) > 0$ because ϱ_ε is bounded in $L^\infty(0, T; L^\gamma(\Omega))$ with $\gamma > 1$. Consequently, we conclude the derivation of (1.38).

1.4 Main result

We are now in position to state the following theorem

Theorem 1.3. *Let $T > 0$ be given and let $(\varrho^E, \mathbf{u}^E)$ be the strong solution for the compressible Euler system (1.9),(1.10) corresponding to the initial data $(\varrho_0^E, \mathbf{u}_0^E)$ as in Theorem 1.2. For any $\varepsilon \in (0, 1)$, let $(\varrho_{0,\varepsilon}, \mathbf{v}_{0,\varepsilon}, \mathbf{w}_{0,\varepsilon})$ be an initial data such that*

$$\mathbf{v}_{0,\varepsilon} = \mathbf{u}_{0,\varepsilon} + \mathbf{w}_{0,\varepsilon}$$

and

$$\begin{aligned} \varrho_{0,\varepsilon} &\in L^\gamma(\Omega), \quad \varrho_{0,\varepsilon} \geq 0, \quad \nabla_x \sqrt{\varrho_{0,\varepsilon}} \in L^2(\Omega) \\ \varrho_{0,\varepsilon} \mathbf{v}_{0,\varepsilon} &\in L^1(\Omega), \quad \varrho_{0,\varepsilon} \mathbf{v}_{0,\varepsilon} = 0 \text{ if } \varrho_{0,\varepsilon} = 0, \quad \frac{|\varrho_{0,\varepsilon} \mathbf{v}_{0,\varepsilon}|^2}{\varrho_{0,\varepsilon}} \in L^1(\Omega). \end{aligned}$$

Assume that

$$\left[\|\varrho_{0,\varepsilon} - \varrho_0^E\|_{L^\gamma(\Omega)} + \int_\Omega \varrho_{0,\varepsilon} |\mathbf{v}_{0,\varepsilon} - \mathbf{v}_0^E|^2 dx + \int_\Omega \varrho_{0,\varepsilon} |\mathbf{w}_{0,\varepsilon} - \mathbf{w}_0^E|^2 dx \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0 \quad (1.45)$$

and

$$\|\varrho_\varepsilon\|_{L^\gamma([0,T];L^\gamma(\Gamma_\varepsilon))} = o(\varepsilon^{\frac{1}{\gamma}}), \quad \varepsilon^{\frac{\gamma-1}{\gamma}} \int_0^T \int_{\Gamma_\varepsilon} \frac{\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2}{d_\Omega^2(x)} dx dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (1.46)$$

Then

$$\sup_{t \in [0, T]} \left[\|\varrho_\varepsilon - \varrho^E\|_{L^\gamma(\Omega)} + \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon - \mathbf{u}^E|^2 dx \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0. \quad (1.47)$$

Remark 1.7. *Differently from Bardos and Nguyen [5], we require a rate, for the L^γ -norm of the density ϱ_ε , in terms of ε . Moreover, with respect to the requirements of Sueur [44], we assume only the condition on the kinetic energy (1.46)₂. However, our assumption is stronger and implies the analogous introduced in [44]. As remarked by Bardos and Nguyen [5] and Sueur [44], the assumptions (1.46) are implied (when the density is constant), thanks to the Hardy's inequality, by the condition*

$$\varepsilon \int_0^T \int_{\Gamma_\varepsilon} |\nabla_x \mathbf{u}|^2 dx dt \rightarrow 0 \text{ as } \varepsilon \rightarrow 0$$

used by Kato [31] in the incompressible case.

1.5 Preliminary lemma and a priori estimates

The following Lemma holds (see Bresch et al. [8]; Lemma 2)

Lemma 1.1. *Let $(\varrho_{\varepsilon n}, \mathbf{u}_{\varepsilon n})$ be smooth solution of (1.1)–(1.2). Then the following identity holds*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \varrho_{\varepsilon n} |\nabla_x \log \varrho_{\varepsilon n}|^2 + \int \nabla_x \operatorname{div}_x \mathbf{u}_{\varepsilon n} \cdot \nabla_x \varrho_{\varepsilon n} \\ & + \int \varrho_{\varepsilon n} \mathbb{D}(\mathbf{u}_{\varepsilon n}) : \nabla_x \log \varrho_{\varepsilon n} \otimes \nabla_x \log \varrho_{\varepsilon n} = 0. \end{aligned} \quad (1.48)$$

Proof. See Bresch et al. [8]. □

Now, from the continuity equation (1.1) and the energy inequality (1.19) we can deduce the following a priori estimates

$$\begin{aligned} \|\varrho_{\varepsilon n}\|_{L^\infty(0,T;L^1(\Omega) \cap L^\gamma(\Omega))} &\leq C, \quad \|\sqrt{\varrho_{\varepsilon n}} \mathbf{u}_{\varepsilon n}\|_{L^\infty(0,T;L^2(\Omega))} \leq C, \\ \|\sqrt{\varrho_{\varepsilon n}} \mathbb{D}(\mathbf{u}_{\varepsilon n})\|_{L^2(0,T;L^2(\Omega))} &\leq C, \quad \|\varrho_{\varepsilon n}^{1/3} \mathbf{u}_{\varepsilon n}\|_{L^3((0,T) \times \Omega)} \leq C, \end{aligned} \quad (1.49)$$

Moreover, Lemma 1.1 yields

$$\|\nabla \sqrt{\varrho_{\varepsilon n}}\|_{L^\infty(0,T;L^2(\Omega))} \leq C. \quad (1.50)$$

Finally, by (1.50) and the first estimate in (1.49), we can conclude that

$$\|\sqrt{\varrho_{\varepsilon n}}\|_{L^\infty(0,T;W^{1,2}(\Omega))} \leq C. \quad (1.51)$$

2 Convergence

2.1 Relative energy inequality

Inspired by Bresch et al. [14] (see also [11], [12]), we introduce an energy functional of the following type

$$\begin{aligned} & \mathcal{E}(\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon | \varrho^E, \bar{\mathbf{v}}, \bar{\mathbf{w}})(T, \cdot) \\ & = \int_\Omega \left(\frac{1}{2} \varrho_\varepsilon (|\mathbf{v}_\varepsilon - \bar{\mathbf{v}}|^2 + |\mathbf{w}_\varepsilon - \bar{\mathbf{w}}|^2) \right) (T, \cdot) dx \\ & + \int_\Omega (H(\varrho_\varepsilon) - H(\varrho^E) - H'(\varrho^E)(\varrho_\varepsilon - \varrho^E))(T, \cdot) dx \\ & + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \left(\left| \frac{\mathcal{S}(\mathbf{u}_\varepsilon)}{\sqrt{\varrho_\varepsilon}} - \mathbb{D}(\bar{\mathbf{u}}) \right|^2 + \left| \frac{\mathcal{A}(\mathbf{u}_\varepsilon)}{\sqrt{\varrho_\varepsilon}} - \mathbb{A}(\bar{\mathbf{u}}) \right|^2 \right) dx dt \end{aligned} \quad (2.1)$$

where $(\varrho_\varepsilon, \mathbf{v}_\varepsilon, \mathbf{w}_\varepsilon)$ is a weak solution to the system (1.26) - (1.28) and $(\varrho^E, \bar{\mathbf{v}}, \bar{\mathbf{w}})$ are such that

$$\bar{\mathbf{v}} = \bar{\mathbf{u}} + \tilde{\delta}(\varepsilon) \nabla_x \log \varrho^E \quad (2.2)$$

with $\bar{\mathbf{u}}$ smooth velocity field satisfying $\bar{\mathbf{u}}|_{\partial\Omega} = 0$ and $\bar{\mathbf{w}} = \tilde{\delta}(\varepsilon) \nabla_x \log \varrho^E$ such that $\tilde{\delta}(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Remark 2.1. Relation (2.2) can be written in a more general way, defining $\bar{\mathbf{w}} = \tilde{\delta}(\varepsilon)\nabla_x \log r$, with r arbitrary smooth function. However, for technical reasons due to the derivation of the relative energy inequality, and also because r will play the role of the density of the compressible Euler system, we define already here $\bar{\mathbf{w}} = \tilde{\delta}(\varepsilon)\nabla_x \log \varrho^E$. The expression for $\bar{\mathbf{u}}$ will be introduced later in Section 2.3.

In the following we will derive a relative energy inequality satisfied by the weak solution of the ‘‘augmented system’’ (1.26)–(1.28).

Now, thanks to the energy inequality (1.38), we can write

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) &\leq \int_{\Omega} \left(\frac{1}{2} \varrho_{\varepsilon} |\bar{\mathbf{v}}|^2 - \varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \cdot \bar{\mathbf{v}} + \frac{1}{2} \varrho_{\varepsilon} |\bar{\mathbf{w}}|^2 - \varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \cdot \bar{\mathbf{w}} \right) (T, \cdot) dx \\
&\quad - \int_{\Omega} \left(\frac{1}{2} \varrho_{\varepsilon} |\bar{\mathbf{v}}|^2 - \varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \cdot \bar{\mathbf{v}} + \frac{1}{2} \varrho_{\varepsilon} |\bar{\mathbf{w}}|^2 - \varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \cdot \bar{\mathbf{w}} \right) (0, \cdot) dx \\
&\quad - \int_{\Omega} (H(\varrho^E) + H'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E)) (T, \cdot) dx \\
&\quad + \int_{\Omega} (H(\varrho^E) + H'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E)) (0, \cdot) dx \\
&\quad - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\bar{\mathbf{u}})) dx dt \\
&\quad + \varepsilon \int_0^T \int_{\Omega} \left(\varrho_{\varepsilon} (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_{\varepsilon})}{\varrho_{\varepsilon}} |\nabla_x \varrho_{\varepsilon}|^2 \right) dx dt,
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
E(0, \cdot) &= \int_{\Omega} \left(\frac{1}{2} \varrho_{0,\varepsilon} (|\mathbf{v}_{0,\varepsilon} - \bar{\mathbf{v}}|^2 + |\mathbf{w}_{0,\varepsilon} - \bar{\mathbf{w}}|^2) \right) (0, \cdot) dx \\
&\quad + \int_{\Omega} (H(\varrho_{0,\varepsilon}) - H(\varrho^E) - H'(\varrho^E)(\varrho_{0,\varepsilon} - \varrho^E)) (0, \cdot) dx
\end{aligned} \tag{2.4}$$

Now, we recall the weak formulation to the system (1.26)–(1.28)

$$\begin{aligned}
- \int_{\Omega} \varrho_{\varepsilon}(T, \cdot) \varphi(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon}(0, \cdot) \varphi(0, \cdot) dx \\
+ \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \varphi dx dt + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla_x \varphi dx dt = 0,
\end{aligned} \tag{2.5}$$

$$\begin{aligned}
- \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon}(T, \cdot) \cdot \boldsymbol{\varphi}(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon}(0, \cdot) \cdot \boldsymbol{\varphi}(0, \cdot) dx + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \cdot \partial_t \boldsymbol{\varphi} dx dt \\
+ \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt \\
- \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \boldsymbol{\varphi} dx dt \\
- r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \boldsymbol{\varphi} + \int_0^T \int_{\Omega} p(\varrho_{\varepsilon}) \operatorname{div}_x \boldsymbol{\varphi} dx dt = 0,
\end{aligned} \tag{2.6}$$

$$\begin{aligned}
& - \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(T, \cdot) \cdot \boldsymbol{\varphi}(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(0, \cdot) \cdot \boldsymbol{\varphi}(0, \cdot) dx + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \cdot \partial_t \boldsymbol{\varphi} dx dt \\
& + \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}) : \nabla_x \boldsymbol{\varphi} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \boldsymbol{\varphi} dx dt = 0
\end{aligned} \tag{2.7}$$

where the viscous terms read as in (1.34)–(1.37).

First, we test the continuity equation (2.5) by $\frac{1}{2}|\bar{\mathbf{v}}|^2$ and $\frac{1}{2}|\bar{\mathbf{w}}|^2$, respectively. We have,

$$\begin{aligned}
& - \int_{\Omega} \frac{1}{2}(\varrho_{\varepsilon} \cdot |\bar{\mathbf{v}}|^2)(T, \cdot) dx + \int_{\Omega} \frac{1}{2}(\varrho_{\varepsilon} \cdot |\bar{\mathbf{v}}|^2)(0, \cdot) dx \\
& + \int_0^T \int_{\Omega} \frac{1}{2} (\varrho_{\varepsilon} \partial_t |\bar{\mathbf{v}}|^2 + \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla |\bar{\mathbf{v}}|^2) dx dt = 0,
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
& - \int_{\Omega} \frac{1}{2}(\varrho_{\varepsilon} \cdot |\bar{\mathbf{w}}|^2)(T, \cdot) dx + \int_{\Omega} \frac{1}{2}(\varrho_{\varepsilon} \cdot |\bar{\mathbf{w}}|^2)(0, \cdot) dx \\
& + \int_0^T \int_{\Omega} \frac{1}{2} (\varrho_{\varepsilon} \partial_t |\bar{\mathbf{w}}|^2 + \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla |\bar{\mathbf{w}}|^2) dx dt = 0.
\end{aligned} \tag{2.9}$$

Now, we test the equation (2.6) by $\bar{\mathbf{v}}$. We have

$$\begin{aligned}
& - \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon}(T, \cdot) \cdot \bar{\mathbf{v}}(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon}(0, \cdot) \cdot \bar{\mathbf{v}}(0, \cdot) dx + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \cdot \partial_t \bar{\mathbf{v}} dx dt \\
& + \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt \\
& - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dx dt \\
& - r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dx dt + \int_0^T \int_{\Omega} p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{v}} dx dt = 0,
\end{aligned} \tag{2.10}$$

Next, we test the equation (2.7) by $\bar{\mathbf{w}}$. We have

$$\begin{aligned}
& - \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(T, \cdot) \cdot \bar{\mathbf{w}}(T, \cdot) dx + \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon}(0, \cdot) \cdot \bar{\mathbf{w}}(0, \cdot) dx + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \cdot \partial_t \bar{\mathbf{w}} dx dt \\
& + \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{w}_{\varepsilon} \otimes \mathbf{u}_{\varepsilon}) : \nabla_x \bar{\mathbf{w}} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \bar{\mathbf{w}} dx dt = 0.
\end{aligned} \tag{2.11}$$

Finally, using the continuity equation (1.26), we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} [\partial_t (H(\varrho^E) + H'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E))] dx dt \\
& = \int_0^T \int_{\Omega} [H'(\varrho^E) \partial_t \varrho^E + \partial_t (H'(\varrho^E))(\varrho_{\varepsilon} - \varrho^E) + H'(\varrho^E) \partial_t \varrho_{\varepsilon} - H'(\varrho^E) \partial_t \varrho^E] dx dt \\
& = \int_0^T \int_{\Omega} [\partial_t (H'(\varrho^E))(\varrho_{\varepsilon} - \varrho^E) - H'(\varrho^E) \operatorname{div}_x (\varrho_{\varepsilon} \mathbf{u}_{\varepsilon})] dx dt \\
& = \int_0^T \int_{\Omega} [\partial_t (H'(\varrho^E))(\varrho_{\varepsilon} - \varrho^E) + \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (H'(\varrho^E))] dx dt.
\end{aligned} \tag{2.12}$$

Plugging (2.8)–(2.12) in (2.3), we obtain

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) &\leq \int_0^T \int_\Omega \frac{1}{2} (\varrho_\varepsilon \partial_t |\bar{\mathbf{v}}|^2 + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla |\bar{\mathbf{v}}|^2) dxdt \\
&+ \int_0^T \int_\Omega \frac{1}{2} (\varrho_\varepsilon \partial_t |\bar{\mathbf{w}}|^2 + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla |\bar{\mathbf{w}}|^2) dxdt \\
&- \int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{v}_\varepsilon \cdot \partial_t \bar{\mathbf{v}} + \varrho_\varepsilon \mathbf{v}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{v}} + p(\varrho_\varepsilon) \operatorname{div}_x \bar{\mathbf{v}}) dxdt \\
&+ \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt \\
&- \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}} dxdt \\
&+ r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \cdot \bar{\mathbf{v}} dxdt \\
&- \int_0^T \int_\Omega (\varrho_\varepsilon \mathbf{w}_\varepsilon \cdot \partial_t \bar{\mathbf{w}} + \varrho_\varepsilon \mathbf{w}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}}) dxdt \\
&- \int_0^T \int_\Omega [\partial_t (H'(\varrho^E)) (\varrho_\varepsilon - \varrho^E) + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x (H'(\varrho^E))] dxdt \\
&- 2\varepsilon \int_0^T \int_\Omega (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\
&+ \varepsilon \int_0^T \int_\Omega \left(\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla_x \varrho_\varepsilon|^2 \right) dxdt.
\end{aligned} \tag{2.13}$$

Rearranging (2.13), we obtain

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) &\leq \int_0^T \int_\Omega \varrho_\varepsilon (\partial_t \bar{\mathbf{v}} \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon) + (\nabla_x \bar{\mathbf{v}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon)) dxdt \\
&+ \int_0^T \int_\Omega \varrho_\varepsilon (\partial_t \bar{\mathbf{w}} \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon) + (\nabla_x \bar{\mathbf{w}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon)) dxdt \\
&+ \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt \\
&- \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_\Omega \varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}} dxdt \\
&+ r_1 \int_0^T \int_\Omega \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \cdot \bar{\mathbf{v}} dxdt \\
&- \int_0^T \int_\Omega [\partial_t (H'(\varrho^E)) (\varrho_\varepsilon - \varrho^E) + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x (H'(\varrho^E)) + p(\varrho_\varepsilon) \operatorname{div}_x \bar{\mathbf{v}}] dxdt \\
&- 2\varepsilon \int_0^T \int_\Omega (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\
&+ \varepsilon \int_0^T \int_\Omega \left(\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla_x \varrho_\varepsilon|^2 \right) dxdt
\end{aligned} \tag{2.14}$$

We multiply (1.9) by $H'(\varrho^E)$

$$\int_0^T \int_{\Omega} [H'(\varrho^E) \partial_t \varrho^E + H'(\varrho^E) \operatorname{div}_x(\varrho^E \mathbf{u}^E)] dxdt = 0.$$

Performing integration by parts, we obtain

$$\int_0^T \int_{\Omega} [H'(\varrho^E) \partial_t \varrho^E - \varrho^E \nabla_x (H'(\varrho^E)) \cdot \mathbf{u}^E] dxdt = 0.$$

From $p(\varrho^E) = H'(\varrho^E) \varrho^E - H(\varrho^E)$, we have $\varrho^E \nabla_x (H'(\varrho^E)) = \nabla_x p(\varrho^E)$. Consequently, from the above relation, we have

$$\int_0^T \int_{\Omega} [H'(\varrho^E) \partial_t \varrho^E - \nabla_x p(\varrho^E) \cdot \mathbf{u}^E] dxdt = 0.$$

Performing again integration by parts, we obtain

$$\int_0^T \int_{\Omega} H'(\varrho^E) \partial_t \varrho^E dxdt = - \int_0^T \int_{\Omega} p(\varrho^E) \operatorname{div}_x \mathbf{u}^E dxdt. \quad (2.15)$$

From (2.14), we have

$$\begin{aligned} \mathcal{E}(T, \cdot) - E(0, \cdot) &\leq \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\partial_t \bar{\mathbf{v}} \cdot (\bar{\mathbf{v}} - \mathbf{v}_{\varepsilon}) + (\nabla_x \bar{\mathbf{v}} \cdot \mathbf{u}_{\varepsilon}) \cdot (\bar{\mathbf{v}} - \mathbf{v}_{\varepsilon})) dxdt \\ &\quad + \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\partial_t \bar{\mathbf{w}} \cdot (\bar{\mathbf{w}} - \mathbf{w}_{\varepsilon}) + (\nabla_x \bar{\mathbf{w}} \cdot \mathbf{u}_{\varepsilon}) \cdot (\bar{\mathbf{w}} - \mathbf{w}_{\varepsilon})) dxdt \\ &\quad + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dxdt \quad (2.16) \\ &\quad + r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dxdt \\ &\quad - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \bar{\mathbf{w}} dxdt \\ &\quad - \int_0^T \int_{\Omega} [\partial_t (H'(\varrho^E)) (\varrho_{\varepsilon} - \varrho^E) + \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (H'(\varrho^E)) + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{v}}] dxdt \\ &\quad - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E - H'(\varrho^E) \partial_t \varrho^E] dxdt \\ &\quad - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\ &\quad + \varepsilon \int_0^T \int_{\Omega} \left(\varrho_{\varepsilon} (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_{\varepsilon})}{\varrho_{\varepsilon}} |\nabla_x \varrho_{\varepsilon}|^2 \right) dxdt \end{aligned}$$

Now, from (1.10) and using the continuity equation (1.9), we deduce

$$\varrho^E (\partial_t \mathbf{u}^E + \mathbf{u}^E \cdot \nabla_x \mathbf{u}^E) + \nabla_x p(\varrho^E) = 0. \quad (2.17)$$

We multiply (2.17) by $\frac{\rho_\varepsilon}{\rho^E}(\bar{\mathbf{u}} - \mathbf{u}_\varepsilon)$. We obtain

$$\rho_\varepsilon \partial_t \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) + \rho_\varepsilon \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) + \frac{\rho_\varepsilon}{\rho^E} \nabla_x p(\rho^E) \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) = 0. \quad (2.18)$$

Now, we consider the first term on the right-hand-side of (2.16). We have

$$\begin{aligned} & \int_0^T \int_\Omega \rho_\varepsilon (\partial_t \bar{\mathbf{v}} \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon) + (\nabla_x \bar{\mathbf{v}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon)) dxdt \\ &= \int_0^T \int_\Omega \rho_\varepsilon (\partial_t \bar{\mathbf{u}} + \varepsilon \partial_t \nabla_x \log \rho^E) \cdot (\bar{\mathbf{u}} + \varepsilon \nabla_x \log \rho^E - \mathbf{u}_\varepsilon - \varepsilon \nabla_x \log \rho_\varepsilon) dxdt \\ &+ \int_0^T \int_\Omega (\nabla_x \bar{\mathbf{u}} + \varepsilon \nabla_x \nabla_x \log \rho^E) \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} + \varepsilon \nabla_x \log \rho^E - \mathbf{u}_\varepsilon - \varepsilon \nabla_x \log \rho_\varepsilon) dxdt \\ &= \int_0^T \int_\Omega \rho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt + \varepsilon \int_0^T \int_\Omega \rho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) dxdt \\ &+ \varepsilon \int_0^T \int_\Omega \rho_\varepsilon \partial_t \nabla_x \log \rho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt + \int_0^T \int_\Omega \rho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt \\ &+ \varepsilon^2 \int_0^T \int_\Omega \rho_\varepsilon \partial_t \nabla_x \log \rho^E \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) dxdt \\ &+ \varepsilon \int_0^T \int_\Omega \rho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) dxdt \\ &+ \varepsilon \int_0^T \int_\Omega \rho_\varepsilon \nabla_x \nabla_x \log \rho^E \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt \\ &+ \varepsilon^2 \int_0^T \int_\Omega \rho_\varepsilon \nabla_x \nabla_x \log \rho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) dxdt, \end{aligned} \quad (2.19)$$

while, the second term on the right-hand-side of (2.16) reads as

$$\begin{aligned} & \int_0^T \int_\Omega \rho_\varepsilon (\partial_t \bar{\mathbf{w}} \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon) + (\nabla_x \bar{\mathbf{w}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon)) dxdt \\ &= \varepsilon^2 \int_0^T \int_\Omega \left(\rho_\varepsilon \partial_t \nabla_x \log \rho^E \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) \right. \\ &\quad \left. + \rho_\varepsilon \nabla_x \nabla_x \log \rho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) \right) dxdt. \end{aligned} \quad (2.20)$$

Consequently, using (2.18), we have

$$\begin{aligned} & \int_0^T \int_\Omega \rho_\varepsilon (\partial_t \bar{\mathbf{v}} \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon) + (\nabla_x \bar{\mathbf{v}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon)) dxdt \\ &\quad + \int_0^T \int_\Omega \rho_\varepsilon (\partial_t \bar{\mathbf{w}} \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon) + (\nabla_x \bar{\mathbf{w}} \cdot \mathbf{u}_\varepsilon) \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon)) dxdt \\ &= \int_0^T \int_\Omega \rho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) + \varepsilon \int \int \rho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \rho^E - \nabla_x \log \rho_\varepsilon) dxdt \\ &\quad + \varepsilon \int_0^T \int_\Omega \rho_\varepsilon \partial_t \nabla_x \log \rho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt \end{aligned} \quad (2.21)$$

$$\begin{aligned}
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& + \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& - \int_0^T \int_{\Omega} \left(\varrho_\varepsilon \partial_t \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) + \varrho_\varepsilon \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) \right. \\
& \quad \left. + \frac{\varrho_\varepsilon}{\varrho^E} \nabla_x p(\varrho^E) \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) \right) dx dt.
\end{aligned}$$

Plugging (2.21) in (2.16), we have

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) & \leq \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \nabla_x \log \varrho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& + \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dx dt \\
& - \int_0^T \int_{\Omega} \left(\varrho_\varepsilon \partial_t \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) + \varrho_\varepsilon \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) \right. \\
& \quad \left. + \frac{\varrho_\varepsilon}{\varrho^E} \nabla_x p(\varrho^E) \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) \right) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}} dx dt
\end{aligned} \tag{2.22}$$

$$\begin{aligned}
& -\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dx dt + r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dx dt \\
& - \int_0^T \int_{\Omega} [\partial_t (H'(\varrho^E)) (\varrho_{\varepsilon} - \varrho^E) + \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (H'(\varrho^E)) + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{v}}] dx dt \\
& - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E - H'(\varrho^E) \partial_t \varrho^E] dx dt \\
& - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\bar{\mathbf{u}})) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \left(\varrho_{\varepsilon} (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_{\varepsilon})}{\varrho_{\varepsilon}} |\nabla_x \varrho_{\varepsilon}|^2 \right) dx dt
\end{aligned}$$

Now, we compute

$$\begin{aligned}
\partial_t (H'(\varrho^E)) &= -p'(\varrho^E) \operatorname{div} \mathbf{u}^E - H''(\varrho^E) \nabla_x \varrho^E \cdot \mathbf{u}^E \\
&= -p'(\varrho^E) \operatorname{div}_x n \mathbf{u}^E - \nabla_x (H'(\varrho^E)) \cdot \mathbf{u}^E.
\end{aligned} \tag{2.23}$$

Consequently,

$$\begin{aligned}
& - \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} \nabla_x p(\varrho^E) \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt - \int_0^T \int_{\Omega} \partial_t (H'(\varrho^E)) (\varrho_{\varepsilon} - \varrho^E) dx dt \\
& - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (H'(\varrho^E)) dx dt + \int_0^T \int_{\Omega} H'(\varrho^E) \partial_t \varrho^E dx dt \\
& = - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x (H'(\varrho^E)) (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt + \int_0^T \int_{\Omega} p'(\varrho^E) (\varrho_{\varepsilon} - \varrho^E) \operatorname{div} \mathbf{u}^E dx dt \\
& + \int_0^T \int_{\Omega} \nabla (H'(\varrho^E)) (\varrho_{\varepsilon} - \varrho^E) \cdot \mathbf{u}^E dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (H'(\varrho^E)) dx dt \\
& + \int_0^T \int_{\Omega} \varrho^E \nabla_x (H'(\varrho^E)) \cdot \mathbf{u}^E dx dt \\
& = \int_0^T \int_{\Omega} p'(\varrho^E) (\varrho_{\varepsilon} - \varrho^E) \operatorname{div} \mathbf{u}^E dx dt.
\end{aligned} \tag{2.24}$$

Plugging (2.24) in (2.22), we obtain

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) &\leq \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \bar{\mathbf{u}} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) + \varepsilon \int \int \varrho_{\varepsilon} \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt
\end{aligned}$$

$$\begin{aligned}
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) + \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt \\
& + r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dx dt \tag{2.25} \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \bar{\mathbf{w}} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dx dt \\
& - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{v}} - p'(\varrho^E) (\varrho_{\varepsilon} - \varrho^E) \operatorname{div}_x \mathbf{u}^E] dx dt \\
& - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\bar{\mathbf{u}})) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \left(\varrho_{\varepsilon} (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_{\varepsilon})}{\varrho_{\varepsilon}} |\nabla_x \varrho_{\varepsilon}|^2 \right) dx dt
\end{aligned}$$

After some calculations, we can rewrite (2.25) as follows

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) & \leq - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \mathbf{v}_{bl} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
& + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
& + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt \\
& - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \bar{\mathbf{w}} dx dt \\
& + r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dx dt
\end{aligned}$$

$$\begin{aligned}
& - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_\varepsilon) \operatorname{div}_x \bar{\mathbf{v}} - p'(\varrho^E)(\varrho_\varepsilon - \varrho^E) \operatorname{div}_x \mathbf{u}^E] dxdt \\
& - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\
& + \varepsilon \int_0^T \int_{\Omega} \left(\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2) - \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla_x \varrho_\varepsilon|^2 \right) dxdt
\end{aligned} \tag{2.26}$$

Now, we observe that

$$\begin{aligned}
& -\varepsilon \int_0^T \int_{\Omega} \frac{p'(\varrho_\varepsilon)}{\varrho_\varepsilon} |\nabla_x \varrho_\varepsilon|^2 dxdt \\
& = -\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon (p'(\varrho_\varepsilon) \nabla_x \log \varrho_\varepsilon - p'(\varrho^E) \nabla_x \log \varrho^E) (\nabla_x \log \varrho_\varepsilon - \nabla_x \log \varrho^E) dxdt \\
& + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_\varepsilon}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_\varepsilon}{\varrho_\varepsilon} \right) dxdt \\
& - \varepsilon \int_0^T \int_{\Omega} p'(\varrho_\varepsilon) \nabla_x \varrho_\varepsilon \frac{\nabla_x \varrho^E}{\varrho^E} dxdt.
\end{aligned}$$

Consequently, rewriting (2.26), we derive the relative energy in its final version as follows

$$\begin{aligned}
\mathcal{E}(T, \cdot) - E(0, \cdot) + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon (p'(\varrho_\varepsilon) \nabla_x \log \varrho_\varepsilon - p'(\varrho^E) \nabla_x \log \varrho^E) \\
\times (\nabla_x \log \varrho_\varepsilon - \nabla_x \log \varrho^E) dxdt \leq \sum_{i=1}^{11} R_i,
\end{aligned} \tag{2.27}$$

where

$$\begin{aligned}
R_1 &= \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \mathbf{v}_{bl} \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt, \\
R_2 &= \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dxdt, \\
R_3 &= \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \nabla_x \log \varrho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt, \\
R_4 &= 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dxdt, \\
R_5 &= \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt - \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt, \\
R_6 &= \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dxdt, \\
R_7 &= \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dxdt, \\
R_8 &= 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dxdt,
\end{aligned}$$

$$\begin{aligned}
R_9 &= \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon : \nabla_x \bar{\mathbf{v}} dx dt, \\
&+ \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}} dx dt - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dx dt \\
&+ \varepsilon \int_0^T \int_{\Omega} (\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2)) dx dt, \\
R_{10} &= - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_\varepsilon) \operatorname{div}_x \bar{\mathbf{v}} - p'(\varrho^E)(\varrho_\varepsilon - \varrho^E) \operatorname{div}_x \mathbf{u}^E] dx dt \\
&+ \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_\varepsilon}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_\varepsilon}{\varrho_\varepsilon} \right) dx dt \\
&- \varepsilon \int_0^T \int_{\Omega} p'(\varrho_\varepsilon) \nabla_x \varrho_\varepsilon \frac{\nabla_x \varrho^E}{\varrho^E} dx dt, \\
R_{11} &= r_1 \int_0^T \int_{\Omega} \varrho_\varepsilon |\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon| (\mathbf{v}_\varepsilon - \mathbf{w}_\varepsilon) \cdot \bar{\mathbf{v}} dx dt.
\end{aligned}$$

2.2 Kato type “fake” boundary layer

In the spirit of [31] (see also [5], [44] and [46]), we introduce a Kato type “fake” boundary layer. We consider $(\varrho^E, \mathbf{u}^E)$ the strong solution of (1.9)–(1.12), and we define

$$\mathbf{v}_{bl} \triangleq \xi \left(\frac{d_\Omega(x)}{\delta} \right) \mathbf{u}^E, \quad (2.28)$$

with $\delta = \delta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ with the rate $\delta(\varepsilon)$ given below, and $\xi : [0, +\infty) \rightarrow [0, +\infty)$ is a smooth cut-off function, such that

$$\xi(0) = 1, \quad \operatorname{supp} \xi \subseteq [0, 1), \quad \|\xi\|_{L^\infty} < \infty, \quad \|\xi'\|_{L^\infty} < \infty.$$

It follows that $\mathbf{v}_{bl} = \mathbf{u}^E$ on the boundary $[0, T] \times \partial\Omega$ and it has support in the domain $[0, T] \times \Gamma_\varepsilon$ and recall that $\Gamma_{c\varepsilon} = \{x \in \Omega : d_\Omega(x) \leq c\varepsilon\}$. Moreover, the quantity \mathbf{v}_{bl} satisfies the following properties (see [5], [31], [44]) with $\delta = c\varepsilon$, $c > 0$: for $1 \leq p < +\infty$, and for $\varepsilon \rightarrow 0$, we have

$$\begin{aligned}
\|\mathbf{v}_{bl}\|_{C([0, T] \times \Omega)} &\leq C, \quad \|\mathbf{v}_{bl}\|_{C([0, T] \times L^p(\Omega))} \leq C\varepsilon^{\frac{1}{p}}, \quad \|\partial_t \mathbf{v}_{bl}\|_{C([0, T] \times L^p(\Omega))} \leq C\varepsilon^{\frac{1}{p}}, \\
\|\partial_t \mathbf{v}_{bl}\|_{L^\infty([0, T] \times \Omega)} &\leq C, \quad \|\nabla_x \mathbf{v}_{bl}\|_{L^\infty([0, T] \times \Omega)} = O(\varepsilon^{-1}), \\
\|\operatorname{div}_x \mathbf{v}_{bl}\|_{L^\infty([0, T] \times \Omega)} &\leq C, \quad \|\operatorname{div}_x \mathbf{v}_{bl}\|_{C([0, T] \times L^p(\Omega))} \leq C\varepsilon^{\frac{1}{p}}, \\
\|d_\Omega(x) \nabla_x \mathbf{v}_{bl}\|_{L^\infty([0, T] \times \Omega)} &\leq C, \quad \|d_\Omega^2(x) \nabla_x \mathbf{v}_{bl}\|_{L^\infty([0, T] \times \Omega)} \leq C\varepsilon, \\
\|d_\Omega(x) \nabla_x \mathbf{v}_{bl}\|_{L^\infty([0, T]; L^2(\Omega))} &= O(\varepsilon^{\frac{1}{2}}), \quad \|d_\Omega(x) \nabla_x \mathbf{v}_{bl}\|_{L^\infty([0, T]; L^p(\Omega))} = O(\varepsilon^{\frac{1}{p}}).
\end{aligned} \quad (2.29)$$

Remark 2.2. To prove the estimates (2.29), the following quantities have been introduced

$$\begin{aligned} z(x) &= \xi \left(\frac{d_\Omega(x)}{\delta} \right), & \tilde{\xi}(r) &= r\xi'(r), & \tilde{z}(x) &= \tilde{\xi} \left(\frac{d_\Omega(x)}{\delta} \right), \\ \widehat{\xi}(r) &= r^2\xi'(r), & \widehat{z}(x) &= \widehat{\xi} \left(\frac{d_\Omega(x)}{\delta} \right), & \mathbf{v}_{bl} &= z\mathbf{u}^E \end{aligned} \quad (2.30)$$

and recall that $\delta = \delta(\varepsilon) = c\varepsilon, c > 0$. Relations (2.30) will be used in the subsequent analysis of the inviscid limit in order to handle some of the viscous terms (see Subsection 2.3.3 below). Moreover, with the above notation, we have (see [44, p. 170])

$$\begin{aligned} \mathbf{n} \cdot \nabla_x \mathbf{v}_{bl} &= z\mathbf{n} \cdot \nabla_x \mathbf{u}^E + \frac{1}{c\varepsilon} \xi' \left(\frac{d_\Omega(x)}{c\varepsilon} \right) \mathbf{u}^E \\ &= z\mathbf{n} \cdot \nabla_x \mathbf{u}^E + \frac{1}{d_\Omega} \tilde{z} \mathbf{n} \cdot \mathbf{n} \otimes \mathbf{u}^E \end{aligned} \quad (2.31)$$

and

$$\begin{aligned} \operatorname{div} \mathbf{v}_{bl} &= z \operatorname{div} \mathbf{u}^E + \mathbf{u}^E \cdot \nabla_x z \\ &= z \operatorname{div} \mathbf{u}^E + \frac{\mathbf{u}^E \cdot \mathbf{n}}{d_\Omega} \tilde{z}. \end{aligned} \quad (2.32)$$

2.3 Inviscid limit

We now consider $\bar{\mathbf{u}} = \mathbf{u}^E - \mathbf{v}_{bl}$ in (2.27). In what follows we provide suitable bounds, and we pass to the limit as $\varepsilon \rightarrow 0$, for the terms $R_i, i = 1, \dots, 11$ involved in (2.27). For simplicity, and without loss of generality, we set $\tilde{\delta}(\varepsilon) = \varepsilon$.

2.3.1 Time dependent term

For R_1 , we have

$$\begin{aligned} &\int_0^T \int_\Omega \varrho_\varepsilon \partial_t \mathbf{v}_{bl} \cdot (\bar{\mathbf{u}} - \mathbf{u}_\varepsilon) dx dt \\ &= \int_0^T \int_\Omega \varrho_\varepsilon \partial_t \mathbf{v}_{bl} \cdot [\bar{\mathbf{v}} - \mathbf{v}_\varepsilon - (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon)] dx dt \\ &= \int_0^T \int_\Omega \varrho_\varepsilon \partial_t \mathbf{v}_{bl} \cdot (\bar{\mathbf{v}} - \mathbf{v}_\varepsilon) dx dt - \int_0^T \int_\Omega \varrho_\varepsilon \partial_t \mathbf{v}_{bl} \cdot (\bar{\mathbf{w}} - \mathbf{w}_\varepsilon) dx dt \\ &\leq \int_0^T \left(\int_\Omega \varrho_\varepsilon |\partial_t \mathbf{v}_{bl}|^2 dx \right)^{1/2} \left(\int_\Omega \varrho_\varepsilon |\bar{\mathbf{v}} - \mathbf{v}_\varepsilon|^2 dx \right)^{1/2} dt \\ &\quad + \int_0^T \left(\int_\Omega \varrho_\varepsilon |\partial_t \mathbf{v}_{bl}|^2 dx \right)^{1/2} \left(\int_\Omega \varrho_\varepsilon |\bar{\mathbf{w}} - \mathbf{w}_\varepsilon|^2 dx \right)^{1/2} dt \\ &\leq C\varepsilon^{\frac{1}{p}} + C \int_0^T \mathcal{E}(t, \cdot) dt. \end{aligned}$$

2.3.2 Convective terms

For R_2 , we have

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \bar{\mathbf{u}} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t (\mathbf{u}^E - \mathbf{v}_{bl}) \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \partial_t \mathbf{u}^E \cdot \nabla_x \log \varrho^E - \varrho_{\varepsilon} \partial_t \mathbf{u}^E \cdot \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&\quad - \varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \partial_t \mathbf{v}_{bl} \cdot \nabla_x \log \varrho^E - \varrho_{\varepsilon} \partial_t \mathbf{v}_{bl} \cdot \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \left[\varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \partial_t \mathbf{u}^E \cdot \nabla_x \log \varrho^E - \partial_t \mathbf{u}^E \cdot \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}}) dx dt \right. \\
&\quad \left. - \varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \partial_t \mathbf{v}_{bl} \cdot \nabla_x \log \varrho^E - \partial_t \mathbf{v}_{bl} \cdot \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}}) dx dt \right] \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

For the term R_3 , we have

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}^E \partial_t \nabla_x \log \varrho^E dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{bl} \partial_t \nabla_x \log \varrho^E dx dt \\
&\quad - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \partial_t \nabla_x \log \varrho^E dx dt \\
&= \left[\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}^E \partial_t \nabla_x \log \varrho^E dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{bl} \partial_t \nabla_x \log \varrho^E dx dt \right. \\
&\quad \left. - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \partial_t \nabla_x \log \varrho^E dx dt \right] \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

Again, for R_4 , we have

$$\begin{aligned}
& 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \left[2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \partial_t \nabla_x \log \varrho^E \cdot \nabla_x \log \varrho^E dx dt \right. \\
&\quad \left. - 2\varepsilon^2 \int_0^T \int_{\Omega} \partial_t \nabla_x \log \varrho^E \cdot \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} dx dt \right] \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0.
\end{aligned}$$

Let us now consider the term R_5 , to get

$$\begin{aligned}
& \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}^E \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) (\nabla_x \bar{\mathbf{u}} \mathbf{u}_{\varepsilon} - \nabla_x \mathbf{u}^E \mathbf{u}^E) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) (\nabla_x \bar{\mathbf{u}} \mathbf{u}_{\varepsilon} + \nabla_x \bar{\mathbf{u}} \mathbf{u}^E - \nabla_x \bar{\mathbf{u}} \mathbf{u}^E - \nabla_x \mathbf{u}^E \mathbf{u}^E) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) [\nabla_x \bar{\mathbf{u}} (\mathbf{u}_{\varepsilon} - \mathbf{u}^E) - \mathbf{u}^E \nabla_x \mathbf{v}_{bl}] dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) [\nabla_x \bar{\mathbf{u}} (\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}} - \mathbf{v}_{bl}) - \mathbf{u}^E \nabla_x \mathbf{v}_{bl}] dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) (\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}}) dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \nabla_x \bar{\mathbf{u}} \mathbf{v}_{bl} dx dt \\
&\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \mathbf{u}^E \nabla_x \mathbf{v}_{bl} dx dt \\
&= \widetilde{R}_5 - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \nabla_x \bar{\mathbf{u}} \mathbf{v}_{bl} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \mathbf{u}^E \nabla_x \mathbf{v}_{bl} dx dt.
\end{aligned} \tag{2.33}$$

For \widetilde{R}_5 , we have

$$\begin{aligned}
\widetilde{R}_5 &= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot (\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}}) \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\nabla_x \mathbf{u}^E - \nabla_x \mathbf{v}_{bl}) (\mathbf{u}_{\varepsilon} - \mathbf{u}^E + \mathbf{v}_{bl}) \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (\mathbf{u}_{\varepsilon} - \mathbf{u}^E) \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt \\
&\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} (\mathbf{v}_{bl} - \mathbf{u}^E) \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \widetilde{R}_{5,1} + \widetilde{R}_{5,2} + \widetilde{R}_{5,3} + \widetilde{R}_{5,4}.
\end{aligned}$$

For $\widetilde{R}_{5,1}$, integrating by parts, we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} \mathbf{u}_{\varepsilon} dx dt \\
&= - \int_0^T \int_{\Omega} \nabla_x \varrho_{\varepsilon} \mathbf{u}^E (z \mathbf{u}^E) \bar{\mathbf{u}} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}^E \operatorname{div}_x \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt \\
&\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}^E (z \mathbf{u}^E) \operatorname{div}_x \bar{\mathbf{u}} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (z \mathbf{u}^E) \mathbf{u}_{\varepsilon} dx dt,
\end{aligned}$$

and hence

$$\begin{aligned}
& \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= -2 \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \nabla_x \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt - \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E \operatorname{div}_x \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt \\
&\quad - \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E \mathbf{v}_{bl} \operatorname{div}_x \bar{\mathbf{u}} dx dt - \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \nabla_x \mathbf{u}^E \mathbf{v}_{bl} \mathbf{u}_{\varepsilon} dx dt \\
&\leq \left[C \|\mathbf{v}_{bl}\|_{L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} + C \|\operatorname{div}_x \mathbf{v}_{bl}\|_{L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} \right] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned} \tag{2.34}$$

Indeed, the first term on the right-hand side can be bounded as follows

$$\begin{aligned}
\left| 2 \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \nabla_x \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt \right| &\leq 2 \int_0^T \left(\|\sqrt{\varrho_{\varepsilon}}\|_{L^{2\gamma}(\Omega)} \|\nabla_x \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E\|_{L^2(\Omega)} \right. \\
&\quad \left. \times \|\mathbf{v}_{bl}\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)} \|\bar{\mathbf{u}}\|_{L^{\infty}(\Omega)} \right) dt,
\end{aligned}$$

and we pass to the limit, thanks to (2.29), recalling that $\|\mathbf{v}_{bl}\|_{L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} = O(\varepsilon^{\frac{\gamma-1}{2\gamma}})$, as $\varepsilon \rightarrow 0$. The second term can be bounded still using Hölder's inequality along with (2.29), i.e.

$$\begin{aligned}
\left| \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}^E \operatorname{div}_x \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt \right| &\leq \int_0^T \left(\|\sqrt{\varrho_{\varepsilon}}\|_{L^{2\gamma}(\Omega)} \|\sqrt{\varrho_{\varepsilon}} \mathbf{u}^E\|_{L^2(\Omega)} \right. \\
&\quad \left. \times \|\operatorname{div}_x \mathbf{v}_{bl}\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)} \|\bar{\mathbf{u}}\|_{L^{\infty}(\Omega)} \right) dt,
\end{aligned}$$

and to pass to the limit we exploit the fact that $\|\operatorname{div}_x \mathbf{v}_{bl}\|_{L^{\infty}(0,T;L^{\frac{2\gamma}{\gamma-1}}(\Omega))} = O(\varepsilon^{\frac{\gamma-1}{2\gamma}})$, as $\varepsilon \rightarrow 0$. The other terms can be treated in a similar way. This fully justifies the passage to the limit performed in the last line of (2.34).

Let us now consider $\widetilde{R}_{5,2}$. We have

$$\begin{aligned}
\int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (\mathbf{u}_{\varepsilon} - \mathbf{u}^E) (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt &= - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (\mathbf{u}_{\varepsilon} - \mathbf{u}^E)^2 dx dt \\
&\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (\mathbf{u}_{\varepsilon} - \mathbf{u}^E) \mathbf{v}_{bl} dx dt,
\end{aligned}$$

where the second term can be treated similarly to $\widetilde{R}_{5,1}$ and it goes to zero as $\varepsilon \rightarrow 0$.

For the first term on the right-hand side, we get

$$\int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E (\mathbf{u}_{\varepsilon} - \mathbf{u}^E)^2 dx dt = \int_0^T \int_{\Omega} (\varrho_{\varepsilon} (\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}})^2 + \varrho_{\varepsilon} |\mathbf{v}_{bl}|^2 - 2\varrho_{\varepsilon} (\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}}) \mathbf{v}_{bl}) dx dt,$$

and the last two terms go to zero as $\varepsilon \rightarrow 0$.

Moreover, we have

$$\begin{aligned} \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{u}_{\varepsilon} - \bar{\mathbf{u}})^2 dxdt &= \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{v}_{\varepsilon} - \bar{\mathbf{v}})^2 dxdt + \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{w}_{\varepsilon} - \bar{\mathbf{w}})^2 dxdt \\ &\quad + 2 \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{v}_{\varepsilon} - \bar{\mathbf{v}})(\bar{\mathbf{w}} - \mathbf{w}_{\varepsilon}) dxdt, \end{aligned} \quad (2.35)$$

and for the first two addends in the right-hand side, we observe that

$$\int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{v}_{\varepsilon} - \bar{\mathbf{v}})^2 dxdt + \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{w}_{\varepsilon} - \bar{\mathbf{w}})^2 dxdt \leq C \int_0^T \mathcal{E}(t, \cdot) dt,$$

while, for the last term on the right-hand side of (2.35), we have

$$\begin{aligned} 2 \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{v}_{\varepsilon} - \bar{\mathbf{v}})(\bar{\mathbf{w}} - \mathbf{w}_{\varepsilon}) dxdt &= 2 \int_0^T \int_{\Omega} (\varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \bar{\mathbf{w}} - \varrho_{\varepsilon} \mathbf{v}_{\varepsilon} \mathbf{w}_{\varepsilon} - \varrho_{\varepsilon} \bar{\mathbf{v}} \bar{\mathbf{w}} + \varrho_{\varepsilon} \bar{\mathbf{v}} \mathbf{w}_{\varepsilon}) dxdt \\ &= 2 \int_0^T \int_{\Omega} [\varrho_{\varepsilon}(\mathbf{u}_{\varepsilon} + \varepsilon \nabla_x \log \varrho_{\varepsilon}) \varepsilon \nabla_x \log \varrho^E - \varrho_{\varepsilon}(\mathbf{u}_{\varepsilon} + \varepsilon \nabla_x \log \varrho_{\varepsilon}) \varepsilon \nabla \log \varrho_{\varepsilon}] dxdt \\ &\quad - 2 \int_0^T \int_{\Omega} [\varrho_{\varepsilon}(\bar{\mathbf{u}} + \varepsilon \nabla_x \log \varrho^E) \varepsilon \nabla_x \log \varrho^E - \varrho_{\varepsilon}(\bar{\mathbf{u}} + \varepsilon \nabla_x \log \varrho^E) \varepsilon \nabla_x \log \varrho_{\varepsilon}] dxdt \\ &= 2\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \log \varrho^E dxdt + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \log \varrho_{\varepsilon} \nabla_x \log \varrho^E dxdt \\ &\quad - 2\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \log \varrho_{\varepsilon} dxdt - 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\nabla_x \log \varrho_{\varepsilon}|^2 dxdt \\ &\quad - 2\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \bar{\mathbf{u}} \nabla_x \log \varrho^E dxdt - 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\nabla_x \log \varrho^E|^2 dxdt \\ &\quad + 2\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \bar{\mathbf{u}} \nabla_x \log \varrho_{\varepsilon} dxdt + 2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \log \varrho^E \nabla_x \log \varrho_{\varepsilon} dxdt, \end{aligned}$$

and hence

$$\begin{aligned} 2 \int_0^T \int_{\Omega} \varrho_{\varepsilon}(\mathbf{v}_{\varepsilon} - \bar{\mathbf{v}})(\bar{\mathbf{w}} - \mathbf{w}_{\varepsilon}) dxdt &= \left[2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \nabla_x \log \varrho^E dxdt + 2\varepsilon^2 \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \nabla_x \log \varrho^E dxdt \right. \\ &\quad - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \nabla_x \sqrt{\varrho_{\varepsilon}} dxdt - 8\varepsilon^2 \int_0^T \int_{\Omega} |\nabla_x \sqrt{\varrho_{\varepsilon}}|^2 dxdt \\ &\quad - 2\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \bar{\mathbf{u}} \nabla_x \log \varrho^E - 2\varepsilon^2 dxdt \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\nabla_x \log \varrho^E|^2 dxdt \\ &\quad \left. + 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \bar{\mathbf{u}} \nabla_x \sqrt{\varrho_{\varepsilon}} + 2\varepsilon^2 dxdt \int_0^T \int_{\Omega} \nabla_x \log \varrho^E \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} dxdt \right] \longrightarrow 0 \text{ as } \varepsilon \rightarrow 0. \end{aligned}$$

Next we turn to $\widetilde{R}_{5,3}$, to get

$$\begin{aligned} \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt &= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt \\ &\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} \mathbf{u}_{\varepsilon} dx dt. \end{aligned} \quad (2.36)$$

For the first term on the right-hand side, we have

$$\begin{aligned} \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} dx dt &= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x (z \mathbf{u}^E) \bar{\mathbf{u}} dx dt \\ &= \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} z \nabla_x \mathbf{u}^E \bar{\mathbf{u}} dx dt + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \nabla_x z \otimes \mathbf{u}^E \bar{\mathbf{u}} dx dt \\ &= \left[\int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} z \nabla_x \mathbf{u}^E \bar{\mathbf{u}} dx dt \right. \\ &\quad \left. + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \frac{1}{d_{\Omega}(x)} \tilde{z} n \otimes \mathbf{u}^E \bar{\mathbf{u}} dx dt \right] \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned} \quad (2.37)$$

Here, the first term on the right-hand side of (2.37) can be easily bounded as done (2.34), i.e.

$$\begin{aligned} \left| \int_0^T \int_{\Omega} z \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{u}^E \bar{\mathbf{u}} dx dt \right| &\leq \int_0^T \left(\|\sqrt{\varrho_{\varepsilon}}\|_{L^{2\gamma}(\Omega)} \|\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}\|_{L^2(\Omega)} \|z\|_{L^{\frac{2\gamma}{\gamma-1}}(\Omega)} \right. \\ &\quad \left. \times \|\bar{\mathbf{u}}\|_{L^{\infty}(\Omega)} \|\nabla_x \mathbf{u}^E\|_{L^{\infty}(\Omega)} \right) dt, \end{aligned}$$

where we used (1.15) and, in particular, relations (2.29) for passing to the limit. The second term, instead, can be controlled as follows

$$\left| \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \cdot \frac{1}{d_{\Omega}(x)} \tilde{z} n \otimes \mathbf{u}^E \bar{\mathbf{u}} dx dt \right| \leq \hat{C} \int_0^T \|\sqrt{\varrho_{\varepsilon}}\|_{L^{2\gamma}(\Omega)} \left\| \frac{\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}}{d_{\Omega}} \right\|_{L^2(\Omega)} \|\tilde{z}\|_{L^{\infty}(\Omega)} \|1\|_{L^{\frac{2\gamma}{\gamma-1}}(\Gamma_{\varepsilon})} dt$$

with $\hat{C} = \hat{C}(\|\mathbf{u}^E\|_{L^{\infty}([0,T] \times \Omega)}, \|\bar{\mathbf{u}}\|_{L^{\infty}([0,T] \times \Omega)})$, under the assumption (1.46)₂ and exploiting that $\text{supp } \mathbf{v}_{bl} \subseteq [0, T] \times \Gamma_{\varepsilon}$.

For the second term on the right-hand side of (2.36), we have

$$\begin{aligned} \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \nabla_x \mathbf{v}_{bl} \mathbf{u}_{\varepsilon} dx dt &= \int_0^T \int_{\Omega} \frac{\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}}{d_{\Omega}(x)} \frac{\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}}{d_{\Omega}(x)} d_{\Omega}^2(x) \nabla_x \mathbf{v}_{bl} \\ &\leq \|d_{\Omega}^2(x) \nabla_x \mathbf{v}_{bl}\|_{L^{\infty}([0,T] \times \Omega)} \int_0^T \int_{\Gamma_{\varepsilon}} \frac{\varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}|^2}{d_{\Omega}^2(x)} dx dt \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

still using that the support of \mathbf{v}_{bl} in contained in $[0, T] \times \Gamma_{\varepsilon}$, and thanks to the fact that

$$\varepsilon \int_0^T \int_{\Gamma_{\varepsilon}} \frac{\varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}|^2}{d_{\Omega}^2(x)} dx dt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad (2.38)$$

which is a direct consequence of (1.46)₂.

Finally, for $\widetilde{R}_{5,4}$, we have

$$\begin{aligned}
& \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} (\mathbf{v}_{bl} - \mathbf{u}^E) (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dx dt \\
&= - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dx dt \\
&= - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} \bar{\mathbf{u}} dx dt \\
&\quad + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} \mathbf{u}_{\varepsilon} dx dt,
\end{aligned}$$

where for the first term we have

$$\begin{aligned}
\int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \bar{\mathbf{u}} \bar{\mathbf{u}} dx dt &\leq \|\bar{\mathbf{u}}\|_{L^{\infty}([0,T];L^{\infty}(\Omega))}^2 \int_0^T \|\varrho_{\varepsilon}\|_{L^{\gamma}(\Omega)} \|\nabla_x \mathbf{v}_{bl}\|_{L^{\infty}(\Omega)} \|1\|_{L^{\frac{\gamma}{\gamma-1}}(\Gamma_{\varepsilon})} dt \\
&\leq C \varepsilon^{-\frac{1}{\gamma}} \|\varrho_{\varepsilon}\|_{L^{\gamma}([0,T];L^{\gamma}(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0
\end{aligned}$$

under the assumptions (1.46)₁, while the second term can be handle similarly to $\widetilde{R}_{5,3}$ and goes to zero as $\varepsilon \rightarrow 0$.

Finally, we consider the second and third term in (2.33). We have,

$$\begin{aligned}
& \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \nabla_x \bar{\mathbf{u}} \mathbf{v}_{bl} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \mathbf{u}^E \nabla_x \mathbf{v}_{bl} dx dt \\
&= \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \nabla_x \mathbf{u}^E \mathbf{v}_{bl} dx dt - \int_0^T \int_{\Omega} \varrho_{\varepsilon} (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) \nabla_x \mathbf{v}_{bl} \mathbf{v}_{bl} dx dt \\
&\quad - \int_0^T \int_{\Omega} \varrho_{\varepsilon} \bar{\mathbf{u}} \mathbf{u}^E \nabla_x \mathbf{v}_{bl} dx dt + \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{u}_{\varepsilon} \mathbf{u}^E \nabla_x \mathbf{v}_{bl} dx dt,
\end{aligned}$$

where the first term is the same as in $\widetilde{R}_{5,1}$ while the other terms could be handled with similar argument as in $\widetilde{R}_{5,4}$. In conclusions, we have $R_5 \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For R_6 , we have

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x (\mathbf{u}^E - \mathbf{v}_{bl}) \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&\quad - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dx dt \\
&= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{u}^E \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho_{\varepsilon} dx dt
\end{aligned}$$

$$- \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho_{\varepsilon} dxdt$$

and so

$$\begin{aligned} & \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \bar{\mathbf{u}} \cdot \mathbf{u}_{\varepsilon} \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_{\varepsilon}) dxdt \\ &= \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \mathbf{u}^E \cdot \nabla_x \log \varrho^E dxdt - \varepsilon \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_{\varepsilon}} \cdot \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \mathbf{u}^E dxdt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \nabla_x \varrho_{\varepsilon} \cdot \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{bl} \cdot \nabla_x \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \nabla_x \log \varrho^E dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho_{\varepsilon} dxdt \\ &= \left[\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \mathbf{u}^E \cdot \nabla_x \log \varrho^E dxdt - \varepsilon \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_{\varepsilon}} \cdot \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \mathbf{u}^E dxdt \right. \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \mathbf{v}_{bl} \cdot \nabla_x \log \varrho^E dxdt + \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \nabla_x \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt \\ & \quad \left. + \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \mathbf{v}_{bl} \cdot \nabla_x \nabla_x \log \varrho^E dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho_{\varepsilon} dxdt \right] \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned} \tag{2.39}$$

Observe that, the term with integrating function $f(x, t) = \sqrt{\varrho_{\varepsilon}} \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \nabla_x \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E$ can be rewritten in terms of those already present in brackets. Indeed, integrating by parts, we have

$$\begin{aligned} & \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \nabla_x \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt \\ &= -2\varepsilon \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_{\varepsilon}} \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \nabla_x \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho^E dxdt \\ & \quad - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \mathbf{v}_{bl} \cdot \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \cdot \nabla_x \nabla_x \log \varrho^E dxdt. \end{aligned}$$

In particular, for the last term on the right-hand side of (2.39), we have that

$$\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{v}_{bl} \cdot \mathbf{u}_{\varepsilon} \cdot \nabla_x \log \varrho_{\varepsilon} dxdt \leq \varepsilon \int_0^T \int_{\Omega} \varepsilon \nabla_x \mathbf{v}_{bl} \cdot \frac{\sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon}}{d_{\Omega}(x)} \cdot \nabla_x \sqrt{\varrho_{\varepsilon}} dxdt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0$$

thanks to (2.38).

For R_7 , we have

$$\begin{aligned} & \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} - \mathbf{u}_{\varepsilon}) dxdt \\ &= \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_{\varepsilon} \cdot (\mathbf{u}^E - \mathbf{v}_{bl} - \mathbf{u}_{\varepsilon}) dxdt \end{aligned}$$

$$\begin{aligned}
&= \left[\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}^E dxdt - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{v}_{bl} dxdt \right. \\
&\quad \left. - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \nabla_x \log \varrho^E dxdt \right] \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

For R_8 , we have

$$\begin{aligned}
&2\varepsilon^2 \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \nabla_x \log \varrho^E \cdot \mathbf{u}_\varepsilon \cdot (\nabla_x \log \varrho^E - \nabla_x \log \varrho_\varepsilon) dxdt \\
&= \left[2\varepsilon^2 \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \nabla_x \log \varrho^E \cdot \nabla_x \log \varrho^E dxdt \right. \\
&\quad \left. - 2\varepsilon^2 \int_0^T \int_{\Omega} \nabla_x \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \cdot \nabla_x \nabla_x \log \varrho^E dxdt \right] \longrightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.
\end{aligned}$$

2.3.3 Viscous and pressure terms

For R_9 , we have

$$\begin{aligned}
&\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{A}(\mathbf{v}_\varepsilon) : \nabla_x \bar{\mathbf{v}} dxdt - \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x \mathbf{w}_\varepsilon : \nabla_x \bar{\mathbf{v}} dxdt \\
&\quad + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \nabla_x^\top \mathbf{u}_\varepsilon : \nabla_x \bar{\mathbf{w}} dxdt - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\
&\quad + \varepsilon \int_0^T \int_{\Omega} (\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2)) dxdt. \\
&= \left[-\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_j \partial_{ii} (\bar{\mathbf{v}})_j - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{v}})_j \right] \\
&\quad \left\{ -\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_i \partial_{ji} (\bar{\mathbf{v}})_j - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_i \partial_j \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{v}})_j \right\} \\
&\quad \left[-\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_j \partial_{ii} (\bar{\mathbf{v}})_j - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{v}})_j \right] \\
&\quad \left\{ +\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_i \partial_{ji} (\bar{\mathbf{v}})_j + 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_i \partial_j \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{v}})_j \right\} \\
&\quad - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{w}_\varepsilon)_j \partial_{ii} (\bar{\mathbf{v}})_j - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{w}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{v}})_j \\
&\quad - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_i \partial_{ji} (\bar{\mathbf{w}})_j - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_i \partial_j \sqrt{\varrho_\varepsilon} \partial_i (\bar{\mathbf{w}})_j \\
&\quad - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\bar{\mathbf{u}})) dxdt \\
&\quad + \varepsilon \int_0^T \int_{\Omega} (\varrho_\varepsilon (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2)) dxdt
\end{aligned}$$

where the sum of the terms in $\{\dots\}$ is zero, and we add the term in [...]. Hence,

$$\begin{aligned}
& \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{D}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \mathbb{A}(\mathbf{v}_{\varepsilon}) : \nabla_x \bar{\mathbf{v}} dx dt - \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x \mathbf{w}_{\varepsilon} : \nabla_x \bar{\mathbf{v}} dx dt \\
& \quad + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} \nabla_x^{\top} \mathbf{u}_{\varepsilon} : \nabla_x \bar{\mathbf{w}} dx dt - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\bar{\mathbf{u}}) + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\bar{\mathbf{u}})) dx dt \\
& \quad + \varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} (|\mathbb{D}(\bar{\mathbf{u}})|^2 + |\mathbb{A}(\bar{\mathbf{u}})|^2)) dx dt \\
& = -2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_{ii} (\mathbf{u}^E - \mathbf{v}_{bl} + \varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - 4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i (\mathbf{u}^E - \mathbf{v}_{bl} + \varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_{ii} (\mathbf{u}^E - \mathbf{v}_{bl} + \varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i (\mathbf{u}^E - \mathbf{v}_{bl} + \varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_i \partial_{ji} (\varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_i \partial_j \sqrt{\varrho_{\varepsilon}} \partial_i (\varepsilon \nabla_x \log \varrho^E)_j dx dt \\
& \quad - 2\varepsilon \int_0^T \int_{\Omega} (\sqrt{\varrho_{\varepsilon}} \mathcal{S}(\mathbf{u}_{\varepsilon}) \mathbb{D}(\mathbf{u}^E - \mathbf{v}_{bl}) dx dt + \sqrt{\varrho_{\varepsilon}} \mathcal{A}(\mathbf{u}_{\varepsilon}) \mathbb{A}(\mathbf{u}^E - \mathbf{v}_{bl})) dx dt \\
& \quad + \varepsilon \int_0^T \int_{\Omega} (\varrho_{\varepsilon} (|\mathbb{D}(\mathbf{u}^E - \mathbf{v}_{bl})|^2 + |\mathbb{A}(\mathbf{u}^E - \mathbf{v}_{bl})|^2)) dx dt \triangleq \sum_{i=1}^8 I_i.
\end{aligned}$$

The terms $I_1 \dots I_4$ that do not contain the boundary layer velocity \mathbf{v}_{bl} go easily to zero as $\varepsilon \rightarrow 0$. Consequently, we concentrate on the terms containing \mathbf{v}_{bl} . We consider I_1 and I_2 , namely

$$\begin{aligned}
I_1 & = 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_{ii} (\mathbf{v}_{bl})_j dx dt \\
& = 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_j \partial_{ii} (\mathbf{v}_{bl})_j dx dt + 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_{ii} (\mathbf{v}_{bl})_j dx dt
\end{aligned} \tag{2.40}$$

and

$$\begin{aligned}
I_2 & = 4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{v}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i (\mathbf{v}_{bl})_j dx dt \\
& = 4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{u}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i (\mathbf{v}_{bl})_j dx dt + 4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} (\mathbf{w}_{\varepsilon})_j \partial_i \sqrt{\varrho_{\varepsilon}} \partial_i (\mathbf{v}_{bl})_j dx dt.
\end{aligned} \tag{2.41}$$

Now, setting $\check{\xi}(r) = r^2 \xi''(r)$ and $\check{z}(x) = \check{\xi}(\frac{d\Omega}{\varepsilon})$, we have

$$\partial_{ii} \mathbf{v}_{bl,j} = \frac{1}{d_{\Omega}} \check{z} n_i \partial_i \mathbf{u}_j^E + z \partial_i^2 \mathbf{u}_j^E + \frac{1}{d_{\Omega}} \check{z} \partial_i (n_i \mathbf{u}_j^E) + \frac{1}{d_{\Omega}^2} \check{z} n_i^2 \mathbf{u}_j^E \tag{2.42}$$

with z and \tilde{z} defined in (2.30). Consequently, noticing that

$$\frac{\tilde{z}}{d_\Omega(r)} = \frac{\xi'(r)}{c\varepsilon}, \quad \tilde{z} = \frac{\xi''(r)}{(c\varepsilon)^2},$$

we can rewrite (2.42) as follows

$$\partial_{ii} \mathbf{v}_{bl,j} = \frac{\xi'(r)}{c\varepsilon} n_i \partial_i \mathbf{u}_j^E + z \partial_i^2 \mathbf{u}_j^E + \frac{\xi'(r)}{c\varepsilon} \partial_i (n_i \mathbf{u}_j^E) + \frac{\xi''(r)}{(c\varepsilon)^2} n_i^2 \mathbf{u}_j^E.$$

The worst term in this relation is the last one which is of leading order with respect to $\varepsilon \rightarrow 0$. Now, for I_1 and I_2 , we focus and deal with such a term, then the remaining ones can be handled in a similar way.

Exploiting Hölder's inequality we reach

$$\begin{aligned} & 2\varepsilon \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\mathbf{v}_\varepsilon)_j \frac{\xi''(r)}{(c\varepsilon)^2} n_i^2 \mathbf{u}_j^E dx dt \\ & \leq \varepsilon^2 C \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} \frac{\sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j}{d_\Omega(x)} \frac{\xi''(r)}{(c\varepsilon)^2} n_i^2 \mathbf{u}_j^E dx dt \\ & \quad + 2\varepsilon^2 \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} \sqrt{\varrho_\varepsilon} (\nabla_x \log \varrho_\varepsilon)_j \frac{\xi''(r)}{(c\varepsilon)^2} n_i^2 \mathbf{u}_j^E dx dt \\ & \leq \varepsilon^2 C \int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)} \left\| \frac{\sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j}{d_\Omega(x)} \right\|_{L^2(\Omega)} \left\| \frac{\xi''(r)}{(c\varepsilon)^2} n_i^2 \mathbf{u}_j^E \right\|_{L^\infty(\Omega)} \|1\|_{L^q(\Gamma_\varepsilon)} dt \\ & \quad + \frac{2}{c} \int_0^T \int_\Omega \sqrt{\varrho_\varepsilon} (\nabla_x \sqrt{\varrho_\varepsilon})_j \xi''(r) n_i^2 \mathbf{u}_j^E dx dt \\ & \leq \bar{c} \varepsilon^{\frac{1}{q}} \|\xi''(r) n_i^2 \mathbf{u}_j^E\|_{L^\infty(0,T;L^\infty(\Omega))} \int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)} \left\| \frac{\sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j}{d_\Omega(x)} \right\|_{L^2(\Omega)} dt \\ & \quad + \hat{c} \varepsilon^{\frac{1}{q}} \|\xi''(r) n_i^2 \mathbf{u}_j^E\|_{L^\infty(0,T;L^\infty(\Omega))} \int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)} \|\nabla_x \sqrt{\varrho_\varepsilon}\|_{L^2(\Omega)} dt \triangleq I_{11} + I_{12} \end{aligned}$$

where $\frac{1}{q} + \frac{1}{2\gamma} = \frac{1}{2}$ and

$$\|1\|_{L^q(\Gamma_\varepsilon)} = \varepsilon^{1/q}, \quad \text{with } q = \frac{2\gamma}{\gamma-1}.$$

By using the smoothness of $\xi''(r) n_i^2 \mathbf{u}_j^E$ and applying Hölder's inequality in time, we infer

$$\begin{aligned} I_{11} & \leq c' \varepsilon^{\frac{1}{q}} \|\xi''(r) n_i^2 \mathbf{u}_j^E\|_{L^\infty(0,T;L^\infty(\Omega))} \left(\int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)}^2 dt \right)^{\frac{1}{2}} \left(\int_0^T \left\| \frac{\sqrt{\varrho_\varepsilon} (\mathbf{u}_\varepsilon)_j}{d_\Omega(x)} \right\|_{L^2(\Omega)}^2 dt \right)^{\frac{1}{2}} \\ & \leq c'' \varepsilon^{\frac{1}{q}} \left(\int_0^T \int_\Omega \frac{\varrho_\varepsilon |(\mathbf{u}_\varepsilon)_j|^2}{d_\Omega^2(x)} dx dt \right)^{1/2} = c'' \left(\varepsilon^{\frac{\gamma-1}{\gamma}} \int_0^T \int_\Omega \frac{\varrho_\varepsilon |(\mathbf{u}_\varepsilon)_j|^2}{d_\Omega^2(x)} dx dt \right)^{1/2} \end{aligned}$$

Then, for I_{11} , we conclude

$$I_{11}^2 \leq \varepsilon^{\frac{\gamma-1}{\gamma}} C \int_0^T \int_\Omega \frac{\varrho_\varepsilon |(\mathbf{u}_\varepsilon)_j|^2}{d_\Omega^2(x)} dx dt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

under the assumption (1.46)₂. Moreover, for I_{12} , with analogous calculations, we have

$$I_{12}^2 \leq C\varepsilon^{\frac{\gamma-1}{\gamma}} \int_0^T \int_{\Omega} |(\nabla_x \sqrt{\varrho_\varepsilon})_j|^2 dx dt \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \quad (2.43)$$

Similarly, as in R_6 , the first term on the right-hand-side of (2.41) is such that

$$4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon}(\mathbf{u}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i (\mathbf{v}_{bl})_j \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

thanks again to (1.46)₂.

Now, we consider the second term in the right-hand side of (2.41). We have

$$4\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon}(\mathbf{w}_\varepsilon)_j \partial_i \sqrt{\varrho_\varepsilon} \partial_i (\mathbf{v}_{bl})_j dx dt = 8\varepsilon^2 \int_0^T \int_{\Omega} (\nabla_x \sqrt{\varrho_\varepsilon})_j (\nabla_x \sqrt{\varrho_\varepsilon})_i \partial_i (\mathbf{v}_{bl})_j dx dt$$

that goes to zero as $\varepsilon \rightarrow 0$. Similarly we can handle the terms I_3 and I_4 .

The terms I_5 and I_6 go to zero as $\varepsilon \rightarrow 0$. For I_7 , we have

$$\left[-2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\mathbf{u}^E) dx dt + 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\mathbf{v}_{bl}) dx dt \right] \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0,$$

where, in particular, for the second term we have

$$\begin{aligned} & 2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \mathcal{S}(\mathbf{u}_\varepsilon) \mathbb{D}(\mathbf{v}_{bl}) dx dt \\ & \leq \varepsilon \|\mathbb{D}(\mathbf{v}_{bl})\|_{L^\infty([0,T];L^\infty(\Omega))} \int_0^T \|\sqrt{\varrho_\varepsilon}\|_{L^{2\gamma}(\Omega)} \|\mathcal{S}(\mathbf{u}_\varepsilon)\|_{L^2(\Omega)} \|1\|_{L^{\frac{2\gamma}{\gamma-1}}(\Gamma_\varepsilon)} dt \\ & \leq C\varepsilon^{\frac{\gamma-1}{2\gamma}} \left(\int_0^T \|\varrho_\varepsilon\|_{L^\gamma(\Omega)}^\gamma \right)^{\frac{1}{2\gamma}} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

thanks to (1.46)₁. Similar analysis could be done for the term

$$-2\varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_\varepsilon} \mathcal{A}(\mathbf{u}_\varepsilon) \mathbb{A}(\mathbf{u}^E - \mathbf{v}_{bl}) dx dt.$$

For I_8 , we have

$$\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon |\mathbb{D}(\mathbf{u}^E)|^2 dx dt - 2\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{u}^E) \mathbb{D}(\mathbf{v}_{bl}) dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon |\mathbb{D}(\mathbf{v}_{bl})|^2 dx dt,$$

where the first term goes to zero as $\varepsilon \rightarrow 0$, while the second and the third term could be handled similar as above, namely

$$\begin{aligned} & -2\varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{u}^E) : \mathbb{D}(\mathbf{v}_{bl}) dx dt + \varepsilon \int_0^T \int_{\Omega} \varrho_\varepsilon \mathbb{D}(\mathbf{v}_{bl}) : \mathbb{D}(\mathbf{v}_{bl}) dx dt \\ & \leq C\varepsilon^{-\frac{1}{\gamma}} \|\varrho_\varepsilon\|_{L^\gamma([0,T];L^\gamma(\Omega))} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \end{aligned}$$

thanks again to (1.46)₁. With a similar we can estimate the term

$$\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbb{A}(\mathbf{u}^E - \mathbf{v}_{bl})|^2 dx dt.$$

In conclusions, we have $R_9 \rightarrow 0$ as $\varepsilon \rightarrow 0$.

For the term R_{10} , we have

$$\begin{aligned} & - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{v}} - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E) \operatorname{div}_x \mathbf{u}^E] dx dt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} p'(r) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}} \right) dx dt \\ & \quad - \varepsilon \int_0^T \int_{\Omega} p'(\varrho_{\varepsilon}) \nabla_x \varrho_{\varepsilon} \frac{\nabla_x \varrho^E}{\varrho^E} dx dt \\ & = - \int_0^T \int_{\Omega} \left[-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{u}} \right. \\ & \quad \left. + p(\varrho_{\varepsilon}) \operatorname{div}_x \bar{\mathbf{w}} - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E) \operatorname{div}_x \mathbf{u}^E \right] dx dt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}} \right) dx dt \\ & \quad - \varepsilon \int_0^T \int_{\Omega} p'(\varrho_{\varepsilon}) \nabla_x \varrho_{\varepsilon} \frac{\nabla_x \varrho^E}{\varrho^E} dx dt \\ & = - \int_0^T \int_{\Omega} \left[-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_{\varepsilon}) \operatorname{div}_x (\mathbf{u}^E - \mathbf{v}_{bl}) \right. \\ & \quad \left. + \varepsilon p(\varrho_{\varepsilon}) \operatorname{div}_x \nabla_x \log \mathbf{u}^E - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E) \operatorname{div}_x \mathbf{u}^E \right] dx dt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}} \right) dx dt \\ & \quad - \varepsilon \int_0^T \int_{\Omega} p'(\varrho_{\varepsilon}) \nabla_x \varrho_{\varepsilon} \frac{\nabla_x \varrho^E}{\varrho^E} dx dt, \end{aligned}$$

where

$$-\varepsilon \int_0^T \int_{\Omega} p(\varrho_{\varepsilon}) \operatorname{div}_x \nabla_x \log \mathbf{u}^E dx dt - \varepsilon \int_0^T \int_{\Omega} p'(\varrho_{\varepsilon}) \nabla_x \varrho_{\varepsilon} \frac{\nabla_x \varrho^E}{\varrho^E} dx dt = 0.$$

Consequently, we consider the following quantity

$$\begin{aligned} & - \int_0^T \int_{\Omega} [-p(\varrho^E) \operatorname{div}_x \mathbf{u}^E + p(\varrho_{\varepsilon}) \operatorname{div}_x (\mathbf{u}^E - \mathbf{v}_{bl}) - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E) \operatorname{div}_x \mathbf{u}^E] dx dt \\ & \quad + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}} \right) dx dt \\ & = - \int_0^T \int_{\Omega} [-p(\varrho^E) + p(\varrho_{\varepsilon}) - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E)] \operatorname{div}_x \mathbf{u}^E dx dt \\ & \quad + \int_0^T \int_{\Omega} p(\varrho_{\varepsilon}) \operatorname{div}_x \mathbf{v}_{bl} dx dt + \varepsilon \int_0^T \int_{\Omega} \frac{\varrho_{\varepsilon}}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_{\varepsilon}}{\varrho_{\varepsilon}} \right) dx dt, \end{aligned}$$

where

$$\left| \int_0^T \int_{\Omega} [-p(\varrho^E) + p(\varrho_\varepsilon) - p'(\varrho^E)(\varrho_\varepsilon - \varrho^E)] \operatorname{div}_x \mathbf{u}^E dx dt \right| \leq C \int_0^T \mathcal{E}(t, \cdot) dt$$

and

$$\left[\varepsilon \int_0^T \int_{\Omega} \frac{\varrho_\varepsilon}{\varrho^E} p'(\varrho^E) \nabla_x \varrho^E \left(\frac{\nabla_x \varrho^E}{\varrho^E} - \frac{\nabla_x \varrho_\varepsilon}{\varrho_\varepsilon} \right) dx dt \right] \rightarrow 0 \text{ as } \varepsilon \rightarrow 0.$$

For the remaining term, we have

$$\int_0^T \int_{\Omega} p(\varrho_\varepsilon) \operatorname{div}_x \mathbf{v}_{bl} dx dt = \int_0^T \int_{\Omega} (p(\varrho_\varepsilon) - p(\varrho^E)) \operatorname{div}_x \mathbf{v}_{bl} dx dt + \int_0^T \int_{\Omega} p(\varrho^E) \operatorname{div}_x \mathbf{v}_{bl} dx dt,$$

where the second term goes to zero as $\varepsilon \rightarrow 0$. In order to handle the first term, we observe that since $p(\varrho_\varepsilon)$ is strictly convex, the quantity

$$\varrho_\varepsilon \rightarrow H(\varrho_\varepsilon | \varrho^E) = H(\varrho_\varepsilon) - H(\varrho^E) - H'(\varrho^E)(\varrho_\varepsilon - \varrho^E)$$

is non-negative strictly convex function on $(0, \infty)$ equal to zero when $\varrho_\varepsilon = \varrho^E$ and growing at infinity with the rate $\varrho_\varepsilon^\gamma$. Consequently, the integral $\int_{\Omega} H(\varrho_\varepsilon | \varrho^E)(t, \cdot) dx$ provides a control of $(\varrho_\varepsilon - \varrho^E)(t, \cdot)$ in L^2 over the sets $\{x : |\varrho_\varepsilon - \varrho^E|(t, x) < 1\}$ and in L^γ over the sets $\{x : |\varrho_\varepsilon - \varrho^E|(t, x) \geq 1\}$, such that

$$H(\varrho_\varepsilon | \varrho^E) \approx |\varrho_\varepsilon - \varrho^E|^2 1_{\{|\varrho_\varepsilon - \varrho^E| < 1\}} + |\varrho_\varepsilon - \varrho^E|^\gamma 1_{\{|\varrho_\varepsilon - \varrho^E| \geq 1\}}, \quad \forall \varrho_\varepsilon \geq 0, \quad (2.44)$$

in the sense that $H(\varrho_\varepsilon | \varrho^E)$ gives an upper and lower bound in term of the right-hand side quantity (see, for example, Sueur [44] Section 2.1 relations (18)–(20)). In particular, since Ω is bounded, there exists a constant $c > 0$ such that (see Sueur [44] Section 2.1, relation (20))

$$\begin{aligned} c \|\varrho_\varepsilon - \varrho^E\|_{L^\gamma(\Omega)} &\leq \left(\int_{\Omega} H(\varrho_\varepsilon | \varrho^E) dx \right)^\gamma + \int_{\Omega} H(\varrho_\varepsilon | \varrho^E) dx, \\ c \int_{\Omega} H(\varrho_\varepsilon | \varrho^E) dx &\leq \|\varrho_\varepsilon - \varrho^E\|_{L^\gamma(\Omega)}^\gamma + \|\varrho_\varepsilon - \varrho^E\|_{L^\gamma(\Omega)}^2. \end{aligned} \quad (2.45)$$

Consequently, we can write

$$\begin{aligned} &\int_0^T \int_{\Omega} (p(\varrho_\varepsilon) - p(\varrho^E)) \operatorname{div}_x \mathbf{v}_{bl} dx dt \\ &= \int_0^T \int_{\Omega} (p(\varrho_\varepsilon) - p'(\varrho^E)(\varrho_\varepsilon - \varrho^E) - p(\varrho^E)) \operatorname{div}_x \mathbf{v}_{bl} dx dt \\ &\quad + \int_0^T \int_{\Omega \cap \{|\varrho_\varepsilon - \varrho^E| < 1\}} p'(\varrho^E)(\varrho_\varepsilon - \varrho^E) \operatorname{div}_x \mathbf{v}_{bl} dx dt \\ &\quad + \int_0^T \int_{\Omega \cap \{|\varrho_\varepsilon - \varrho^E| \geq 1\}} p'(\varrho^E)(\varrho_\varepsilon - \varrho^E) \operatorname{div}_x \mathbf{v}_{bl} dx dt \end{aligned}$$

and

$$\left| \int_0^T \int_{\Omega} (p(\varrho_\varepsilon) - p(\varrho^E)) \operatorname{div}_x \mathbf{v}_{bl} dx dt \right| \leq C\varepsilon + C \int_0^T \mathcal{E}(t, \cdot) dt.$$

2.3.4 Damping term

For the term R_{11} , we have

$$\begin{aligned}
& r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}| (\mathbf{v}_{\varepsilon} - \mathbf{w}_{\varepsilon}) \cdot \bar{\mathbf{v}} dx dt \\
&= r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \cdot (\bar{\mathbf{u}} + \bar{\mathbf{w}}) dx dt \\
&= r_1 \int_0^T \int_{\Omega} \varrho_{\varepsilon} |\mathbf{u}_{\varepsilon}| \mathbf{u}_{\varepsilon} \cdot (\mathbf{u}^E - \mathbf{v}_{bl} + \varepsilon \nabla_x \log \varrho^E) dx dt \\
&= \left[r_1 \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} |\mathbf{u}_{\varepsilon}| \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \mathbf{u}^E dx dt - r_1 \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} |\mathbf{u}_{\varepsilon}| \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \mathbf{v}_{bl} dx dt \right. \\
&\quad \left. + r_1 \varepsilon \int_0^T \int_{\Omega} \sqrt{\varrho_{\varepsilon}} |\mathbf{u}_{\varepsilon}| \sqrt{\varrho_{\varepsilon}} \mathbf{u}_{\varepsilon} \nabla_x \log \varrho^E dx dt \right] \longrightarrow 0 \text{ as } (r_1, \varepsilon) \rightarrow 0.
\end{aligned}$$

2.3.5 Proof of Theorem 1.3

From the previous estimates, back to (2.27), we end up with

$$\begin{aligned}
& \mathcal{E}(T, \cdot) - E(0, \cdot) + \varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} (p'(\varrho_{\varepsilon}) \nabla_x \log \varrho_{\varepsilon} - p'(\varrho^E) \nabla_x \log \varrho^E) (\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E) dx dt \\
&\leq C \varepsilon^{\frac{1}{p}} + C \varepsilon + C \eta(\varepsilon) + C \int_0^T \mathcal{E}(t, \cdot) dt
\end{aligned} \tag{2.46}$$

where $\eta(\varepsilon)$ is such that $\eta(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$ and represents the terms analyzed above that go to zero as $\varepsilon \rightarrow 0$.

In order to conclude the proof of Theorem 1.3 we need to handle the term

$$\varepsilon \int_0^T \int_{\Omega} \varrho_{\varepsilon} (p'(\varrho_{\varepsilon}) \nabla_x \log \varrho_{\varepsilon} - p'(\varrho^E) \nabla_x \log \varrho^E) (\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E) dx dt.$$

This can be as in the same spirit of Bresch et al. [12]. We have

$$\begin{aligned}
& \varrho_{\varepsilon} (p'(\varrho_{\varepsilon}) \nabla_x \log \varrho_{\varepsilon} - p'(\varrho^E) \nabla_x \log \varrho^E) (\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E) \\
&= \varrho_{\varepsilon} p'(\varrho_{\varepsilon}) |\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E|^2 + \varrho_{\varepsilon} (p'(\varrho_{\varepsilon}) \\
&\quad - p'(\varrho^E)) \nabla_x \log \varrho^E (\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E) \\
&= \varrho_{\varepsilon} p'(\varrho_{\varepsilon}) |\nabla_x \log \varrho_{\varepsilon} - \nabla_x \log \varrho^E|^2 + \nabla_x [p(\varrho_{\varepsilon}) - p(\varrho^E) \\
&\quad - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E)] \nabla_x \log \varrho^E \\
&\quad - [\varrho_{\varepsilon} (p'(\varrho_{\varepsilon}) - p'(\varrho^E)) - p''(\varrho^E)(\varrho_{\varepsilon} - \varrho^E) \varrho^E] |\nabla_x \log \varrho^E|^2.
\end{aligned} \tag{2.47}$$

The first term on the right hand side of (2.47) is positive and thus can be neglected in (2.46). For the remaining terms, integrating by parts, we have

$$\int_0^T \int_{\Omega} \nabla_x [p(\varrho_{\varepsilon}) - p(\varrho^E) - p'(\varrho^E)(\varrho_{\varepsilon} - \varrho^E)] \nabla_x \log \varrho^E$$

$$\begin{aligned}
& - [\varrho_\varepsilon(p'(\varrho_\varepsilon) - p'(\varrho^E)) - p''(\varrho^E)(\varrho_\varepsilon - \varrho^E)\varrho^E] |\nabla_x \log \varrho^E|^2 dxdt \\
= & - \int_0^T \int_\Omega |(p(\varrho_\varepsilon) - p(\varrho^E) - p'(\varrho^E)(\varrho_\varepsilon - \varrho^E))| |\Delta \log \varrho^E| dxdt \\
& - \int_0^T \int_\Omega [\varrho_\varepsilon(p'(\varrho_\varepsilon) - p'(\varrho^E)) - p''(\varrho^E)(\varrho_\varepsilon - \varrho^E)\varrho^E] |\nabla_x \log \varrho^E|^2 dxdt.
\end{aligned}$$

Now, from Lemma 2.2 in [5], we observe that

$$[\varrho_\varepsilon(p'(\varrho_\varepsilon) - p'(\varrho^E)) - p''(\varrho^E)(\varrho_\varepsilon - \varrho^E)\varrho^E] \approx H(\varrho_\varepsilon|\varrho^E).$$

Consequently, we have

$$\begin{aligned}
\varepsilon \int_0^T \int_\Omega \varrho_\varepsilon(p'(\varrho_\varepsilon)\nabla_x \log \varrho_\varepsilon - p'(\varrho^E)\nabla_x \log \varrho^E)(\nabla_x \log \varrho_\varepsilon - \nabla_x \log \varrho^E) dxdt \\
\leq C\varepsilon \int_0^T \mathcal{E}(t, \cdot) dt.
\end{aligned}$$

Now, from the assumption (1.45) we have $E(0, \cdot) \rightarrow 0$ as $\varepsilon \rightarrow 0$. Consequently, from (2.46), letting $\varepsilon \rightarrow 0$ we end up with

$$\mathcal{E}(T, \cdot) \leq C \int_0^T \mathcal{E}(t, \cdot) dt \tag{2.48}$$

and applying Gronwall's inequality we end the proof of Theorem 1.3.

Acknowledgment

M. Caggio has been supported by the Praemium Academiae of Š. Nečasová, and by the Czech Science Foundation under the grant GAČR 22-01591S. L. Bisconti is member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM).

Appendix

In the following we provide some considerations (and details) on the additional boundary condition introduced in (1.29)₂, i.e.

$$[\varrho_\varepsilon \mathbf{w}_\varepsilon] \times \mathbf{n}|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega \iff [\varrho_\varepsilon \nabla \log \varrho_\varepsilon] \times \mathbf{n}|_{\partial\Omega} = 0, \quad \text{on } \partial\Omega, \tag{2.49}$$

which is considered in the sense of distribution:

$$C_0^\infty(\partial\Omega) \ni \phi \longmapsto \left([\varrho_\varepsilon \nabla \log \varrho_\varepsilon] \times \mathbf{n}|_{\partial\Omega} \right) (\phi) \triangleq \int_{\partial\Omega} \nabla \varrho_\varepsilon \times \mathbf{n} \cdot \phi \, ds, \tag{2.50}$$

for a.a. $t \in [0, T]$, $T > 0$. Next, to keep the notation concise, we omit the square brackets around $\varrho_\varepsilon \nabla \log \varrho_\varepsilon$.

Let us consider $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\varphi \in C_c^\infty(\mathbb{R}^3)$, with $\text{supp } \varphi \subseteq \overline{B(0,1)}$ and $\int_{B(0,1)} \varphi \, dx = 1$. For $n \in \mathbb{N}$, set $\varphi_n = (1/n)^3 \varphi(x/n)$, with $\text{supp } \varphi_n \subseteq \overline{B(0,1/n)}$. Then, as a standard property of convolutions, we have that $\text{supp } \varphi_n * f \subseteq \Omega + \overline{B(0,1/n)}$ and that

$$\|\varphi_n * f - f\|_{L^p(\Omega)} \rightarrow 0 \text{ as } n \rightarrow +\infty \quad (2.51)$$

for $f \in L^p(\Omega)$, $1 \leq p < +\infty$. Define $\varrho_{\varepsilon,n} = \varphi_n * \varrho_\varepsilon$ with $\text{supp } \varrho_{\varepsilon,n} \subseteq \Omega + \overline{B(0,1/n)} \triangleq \Omega_n$. Clearly we have that

$$\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n} = \nabla \varrho_{\varepsilon,n} \Rightarrow \nabla \times (\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n}) = 0, \text{ in } \Omega, \text{ for a.a. } t \in [0, T], \quad (2.52)$$

and hence, for any $\phi \in C_0^\infty(\Omega_n)$, integrating by parts we have that

$$\int_{\Omega} (\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n}) \cdot \nabla \times \phi \, dx = \int_{\Omega} \underbrace{\nabla \times (\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n})}_{=0, \text{ thanks to (2.52)}} \cdot \phi \, dx + \int_{\partial\Omega} (\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n}) \times \mathbf{n} \cdot \phi \, ds,$$

for a.a. $t \in [0, T]$. On the other hand, we have that $\varrho_{\varepsilon,n} \nabla \log \varrho_{\varepsilon,n} = 2\sqrt{\varrho_{\varepsilon,n}} \nabla \sqrt{\varrho_{\varepsilon,n}}$, and so the above relation can be equivalently rewritten as

$$2 \int_{\Omega} (\sqrt{\varrho_{\varepsilon,n}} \nabla \sqrt{\varrho_{\varepsilon,n}}) \cdot \nabla \times \phi \, dx = \int_{\partial\Omega} (\nabla \varrho_{\varepsilon,n}) \times \mathbf{n} \cdot \phi \, ds, \text{ for a.a. } t \in [0, T].$$

By using the fact that $\sqrt{\varrho_{\varepsilon,n}} \in L^2(0, T; W^{1,2}(\Omega))$, and $\sqrt{\varrho_{\varepsilon,n}(t)} \rightarrow \sqrt{\varrho_\varepsilon(t)}$ in $W^{1,2}(\Omega)$ strongly, for a.a. $t \in [0, T]$, then $\nabla \varrho_{\varepsilon,n} = 2\sqrt{\varrho_{\varepsilon,n}} \nabla \sqrt{\varrho_{\varepsilon,n}}$ is uniformly bounded, with respect to n , in $L^2(0, T; L^{3/2}(\Omega))$ (see Remark 2.3 below), and

$$\nabla \varrho_{\varepsilon,n} \rightharpoonup \mathcal{B} \text{ in } L^2(0, T; L^{3/2}(\Omega)) \text{ weakly,} \quad (2.53)$$

and

$$\nabla \varrho_{\varepsilon,n}(t) = 2\sqrt{\varrho_{\varepsilon,n}(t)} \nabla \sqrt{\varrho_{\varepsilon,n}(t)} \rightarrow 2\sqrt{\varrho_\varepsilon(t)} \nabla \sqrt{\varrho_\varepsilon(t)} \text{ in } L^{3/2}(\Omega) \text{ strongly,} \quad (2.54)$$

for a.a. $t \in [0, T]$. Also, since $\varrho_\varepsilon \in L^\infty(0, T; L^1(\Omega) \cap L^\gamma(\Omega))$, we have that $\varrho_{\varepsilon,n}(t) \rightarrow \varrho_\varepsilon(t)$ in $L^1(\Omega) \cap L^\gamma(\Omega)$ strongly, for a.a. $t \in [0, T]$. Due to the uniqueness of the limit in the above convergence types, we have that $\nabla \varrho_\varepsilon(t) \triangleq \mathcal{B}(t) = 2\sqrt{\varrho_\varepsilon(t)} \nabla \sqrt{\varrho_\varepsilon(t)} \in L^{3/2}(\Omega)$, for a.a. $t \in [0, T]$. In particular, as a consequence of (2.53), we have that $\nabla \varrho_\varepsilon \in L^2(0, T; L^{3/2}(\Omega))$.

Then $\varrho_\varepsilon(t) \in W^{1, \frac{3}{2}}(\Omega)$ and $\varrho_\varepsilon(t) \in W^{\frac{1}{3}, \frac{3}{2}}(\partial\Omega)$, for a.a. $t \in [0, T]$. Thus, in (2.50), we have that $(\nabla \varrho_\varepsilon \times \mathbf{n})(t) \in (W^{\frac{2}{3}, 3}(\partial\Omega))^* = W^{-\frac{2}{3}, \frac{3}{2}}(\partial\Omega)$, for a.a. $t \in [0, T]$.

Remark 2.3. For any n , the vector fields φ_n and ϱ_ε are extended to zero outside Ω_n . We have that

$$\begin{aligned} \int_0^T \|\nabla \varrho_{\varepsilon,n}(t)\|_{L^{3/2}(\Omega)}^2 dt &\leq \int_0^T \|\nabla \varphi_n * \varrho_\varepsilon(t)\|_{L^{3/2}(\Omega_n)}^2 dt \\ &= \int_0^T \|\nabla \varphi_n * \varrho_\varepsilon(t)\|_{L^{3/2}(\mathbb{R}^3)}^2 dt \\ &\leq \|\nabla \varphi_n\|_{L^q(B(0,1/n))}^2 \int_0^T \|\varrho_\varepsilon(t)\|_{L^\gamma(\Omega)}^2 dt \\ &\leq CT \|\varrho_\varepsilon(t)\|_{L^\infty(0, T; L^\gamma(\Omega))}^2 \leq CT, \end{aligned}$$

where we used the Young inequality for convolutions with $p = \gamma$, $q = 3\gamma/(5\gamma - 3)$, $r = 3/2$ and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$, and $q \geq 1$.

References

- [1] Agemi, Rentaro, The initial-boundary value problem for inviscid barotropic fluid motion. *Hokkaido Math. J.* **10(1)**, 156-182, 1981.
- [2] Antonelli, Paolo; Spirito, Stefano, On the compactness of weak solutions to the Navier-Stokes-Korteweg equations for capillary fluids. *Nonlinear Anal.* **187**, 110-124, 2019.
- [3] Antonelli, P., Hientzsch, L. E., Marcati, P., On the low Mach number limit for quantum Navier-Stokes equations, *SIAM J. Math. Anal.*, **52(6)**, 6105-6139, 2020.
- [4] Antonelli, P., Hientzsch, L. E., Spirito, S., Global existence of finite energy weak solutions to the quantum Navier-Stokes equations with non-trivial far-field behavior, *J. Differential Equations*, **290**, 147-177, 2021.
- [5] Bardos, Claude; Nguyen, Toan T., Remarks in the inviscid limit for the compressible flows. *Recent advances in partial differential equations and applications*, 55–67, *Contemp. Math.*, 666, Amer. Math. Soc., Providence, RI, 2016.
- [6] Beirão da Veiga, Hugo, On the barotropic motion of compressible perfect fluids. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **8(2)**, 317–351, 1981.
- [7] Bresch, D., Desjardins, B., Sur un modèle de Saint-Venant visqueux et sa limite quasi-géostrophique, *C. R. Math. Acad. Sci. Paris*, **35(12)**, 1079–1084, 2002.
- [8] Bresch D., Desjardins B. and Lin C. H., On some compressible fluid models: Korteweg, lubrication, and shallow water systems, *Comm. Partial Differential Equations*, **28(3-4)**, 843–886, 2003.
- [9] Bresch, D., Desjardins, B., Quelques modèles diffusifs capillaires de type Korteweg, *C. R. Acad. Sci. Paris, Méch.*, **332(11)**, 881–886, 2004.
- [10] Bresch, D., Desjardins, B., Gérard-Varet, D., On compressible Navier-Stokes equations with density dependent viscosities in bounded domains. *J. Math. Pures Appl. (9)*, **87(2)**, 227-235, 2007
- [11] Bresch, D., Pascal, N., Vila, J.-P., Relative entropy for compressible Navier-Stokes equations with density-dependent viscosities and applications, *C. R. Math. Acad. Sci. Paris*, **354(1)**, 45-49, 2016.
- [12] Bresch, D., Pascal, N., Vila, J.-P., Relative entropy for compressible Navier-Stokes equations with density dependent viscosities and various applications, *LMLFN 2015–low velocity flows–application to low Mach and low Froude regimes*, 40–57, *ESAIM Proc. Surveys*, 58, *EDP Sci., Les Ulis*, 2017.

- [13] Bresch, D., Desjardins, B., Zatorska, E., Two-velocity hydrodynamics in fluid mechanics: Part II. Existence of global κ -entropy solutions to the compressible Navier-Stokes systems with degenerate viscosities, *J. Math. Pure. Appl.*, **104**, 801–836, 2015.
- [14] Bresch, D., Gisclon, M., Lacroix-Violet, I., On Navier-Stokes-Korteweg and Euler-Korteweg Systems: Application to Quantum Fluids Models, *Arch. Rational Mech. Anal.*, **233(3)**, 975–1025, 2019.
- [15] Bresch, D., Vasseur, A. F., Yu, C., Global existence of entropy-weak solutions to the compressible Navier-Stokes equations with non-linear density dependent viscosities, *J. Eur. Math. Soc. (JEMS)*, **24(5)**, 1791–1837, 2022.
- [16] Caggio, M., Donatelli, D., High Mach number limit for Korteweg fluids with density dependent viscosity, *J. Differential Equations*, **277**, 1–37, 2021.
- [17] Cianfarani Carnevale, G., Lattanzio, C., High friction limit for Euler-Korteweg and Navier-Stokes-Korteweg models via relative entropy approach, *J. Differential Equations* **269(12)**, 10495-10526, 2020.
- [18] Cianfarani Carnevale, G., Lattanzio, C., High friction limits of Euler-Navier-Stokes-Korteweg equations for multicomponent models, arXiv:2203.02707.
- [19] Constantin, P., Kukavica, I., Vicol, V., On the inviscid limit of the Navier-Stokes equations, *Proc. Amer. Math. Soc.*, **143(7)**, 3075-3090, 2015.
- [20] Dafermos, C.M., The second law of thermodynamics and stability, *Arch. Rational Mech. Anal.* **70(2)**, 167–179, 1979.
- [21] Ebin, David G., The initial-boundary value problem for subsonic fluid motion. *Comm. Pure Appl. Math.* **32(1)**, 1-19, 1979.
- [22] Ebin, David G., Motion of slightly compressible fluids in a bounded domain. I. *Comm. Pure Appl. Math.* **35(4)**, 451-485, 1982.
- [23] Feireisl, E., Novotný, A., Sun, Y., Suitable weak solutions to the Navier-Stokes equations of compressible viscous fluids, *Indiana Univ. Math. J.*, **60(2)**, 611–631, 2011
- [24] Feireisl, E., Jin, B. J., Novotný, A., Relative Entropies, Suitable Weak Solutions, and Weak-Strong Uniqueness for the Compressible Navier-Stokes System, *J. Math. Fluid Mech.*, **14**, 717-730, 2012.
- [25] Geng, Y., Li, Y., Zhu, S., Vanishing viscosity limit of the Navier-Stokes equations to the Euler equations for compressible fluid flow with vacuum, *Arch. Ration. Mech. Anal.*, **234(2)**, 727-775, 2019.
- [26] Gisclon, M., Lacroix-Violet, I., About the barotropic compressible quantum Navier-Stokes equations, *Nonlinear Analysis*, **128**, 106–121, 2015.
- [27] Grenier, E., Boundary layers, *Handbook of mathematical fluid dynamics. Vol. III*, North-Holland, Amsterdam, pp. 245-309, 2004.

- [28] Guès, O., Métivier, G., Williams, M., Zumbrun, K., Existence and stability of non-characteristic boundary layers for the compressible Navier-Stokes and viscous MHD equations. *Arch. Ration. Mech. Anal.*, **197**(1), 1-87, 2010.
- [29] Guo, Y., Nguyen, T. T., Prandtl boundary layer expansions of steady Navier-Stokes flows over a moving plate, *Ann. PDE*, **3**(1), 2017.
- [30] Jüngel, A., Global weak solutions to compressible Navier-Stokes equations for quantum fluids, *SIAM J. Math. Anal.*, **42**, 1025–1045, 2010.
- [31] Kato, T., Remarks on zero viscosity limit for nonstationary Navier–Stokes flows with boundary. In: *Seminar on nonlinear partial differential equations*, vol. 2, pp. 85–98. Math. Sci. Res. Inst. Publ., Berkeley (1984).
- [32] Kelliher, J. P., On Kato’s conditions for vanishing viscosity, *Indiana Univ. Math. J.*, **56**(4), 1711–1721, 2007.
- [33] Li, J., Xin, Z., Global existence of weak solutions to the barotropic compressible Navier-Stokes flows with degenerate viscosities (<http://arxiv.org/abs/1504.06826v2>).
- [34] Liu, C-J., and Wang, Y-G., Stability of boundary layers for the nonisentropic compressible circularly symmetric 2D flow, *SIAM J. Math. Anal.*, **46**(1), 256-309, 2014.
- [35] Lopes Filho, M. C., Mazzucato, A. L., Nussenzveig Lopes, H. J., Taylor, M., Vanishing viscosity limits and boundary layers for circularly symmetric 2D flows, *Bull. Braz. Math. Soc.*, **39**(4), 471–513, 2008.
- [36] Maekawa, Y., On the inviscid limit problem of the vorticity equations for viscous incompressible flows in the half-plane, *Comm. Pure Appl. Math.*, **67**(7), 1045-1128, 2014.
- [37] Oleinik, O. A., Samokhin, V. N., *Mathematical models in boundary layer theory*, Applied Mathematics and Mathematical Computation, vol. 15, Chapman & Hall/CRC, Boca Raton, FL, 1999.
- [38] Prandtl, L., Über Flüssigkeitsbewegungen bei sehr kleiner Reibung. In "Verh. Int. Math. Kongr., Heidelberg 1904", Teubner 1905, 484-494.
- [39] Rousset, F., Characteristic boundary layers in real vanishing viscosity limits, *J. Differ. Equ.*, **210**(1), 25-64, 2005.
- [40] Sammartino, M., Caffisch, R. E., Zero viscosity limit for analytic solutions, of the Navier-Stokes equation on a half-space. I. Existence for Euler and Prandtl equations, *Comm. Math. Phys.*, **192**(2), 433-461, 1998.
- [41] Sammartino, M., Caffisch, R. E., Zero viscosity limit for analytic solutions of the Navier-Stokes equation on a half-space. II. Construction of the Navier-Stokes solution, *Comm. Math. Phys.*, **192**(2), 463-491, 1998.

- [42] Rauch, J., Massey, F., Differentiability of solutions to hyperbolic initial-boundary value problems. *Trans. Am. Math. Soc.* **189**, 303-318, 1974.
- [43] Schochet, S., The compressible Euler equations in a bounded domain: existence of solutions and the incompressible limit. *Comm. Math. Phys.* **104(1)**, 49-75, 1986.
- [44] Sueur, F., On the inviscid limit for the compressible Navier-Stokes system in an impermeable bounded domain, *J. Math. Fluid Mech.*, **16(1)**, 163–178, 2014.
- [45] Vasseur, A., Yu, C., Existence of global weak solutions for 3D degenerate compressible Navier-Stokes equations, *Invent. Math.*, **206**, 935–974, 2016.
- [46] Wang, Ya-Guang, Zhu, Shi-Yong, On the inviscid limit for the compressible Navier-Stokes system with no-slip boundary condition, *Quart. Appl. Math.*, **76(3)**, 499–514, 2018.
- [47] Wang, X., A Kato type theorem on zero viscosity limit of Navier-Stokes flows, *Indiana Univ. Math. J.*, **50**, 223-241, 2001.
- [48] Wang, X-P., Wang, Y-G., Xin, Z., Boundary layers in incompressible Navier-Stokes equations with Navier boundary conditions for the vanishing viscosity limit, *Commun. Math. Sci.*, **8(4)**, 965-998, 2010.
- [49] Wang, Y-G., Williams, M., The inviscid limit and stability of characteristic boundary layers for the compressible Navier-Stokes equations with Navier-friction boundary conditions, *Ann. Inst. Fourier*, **62(6)**, 2257-2314, 2012.
- [50] On the inviscid limit for the compressible Navier-Stokes system with no-slip boundary condition, *Quart. Appl. Math.*, **76(3)**, 499-514, 2018.
- [51] Xiao, Y., Xin, Z., On the vanishing viscosity limit for the 3D Navier-Stokes equations with a slip boundary condition, *Comm. Pure Appl. Math.*, **60(7)**, 1027-1055, 2007.
- [52] Xin, Z., Yanagisawa, T., Zero-viscosity limit of the linearized Navier-Stokes equations for a compressible viscous fluid in the half-plane, *Comm. Pure Appl. Math.*, **52(4)**, 479-541, 1999.