On limit theorems for the solutions to multipoint inhomogeneous boundary-value problems with parameters in Sobolev spaces

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The aim of the talk is

to present the results about the character of solvability and continuity in the parameters of solutions to systems of linear differential equations of arbitrary order on a finite interval with the most general inhomogeneous boundary conditions. These boundary-value problems have essential features and require new research methods.

Content

1. Background.
2. Generic boundary conditions.
3. Continuity in a parameter.
4. Applications.

The question about continuous dependence of the solution in a parameter to systems of differential equations has been studied by various mathematicians. The importance of such theorems is particularly related to the fact that they substantiate the well-known M. Bogolyubov averaging principle (1955).

Linear Cauchy matrix problem

$$Y'(t;k) = A(t;k)Y(t;k) + F(t;k), \quad t \in [a,b],$$

$$Y(a;k) = I_m, \quad k \in \mathbb{N} \cup \{0\},$$

where the elements of the matrix-valued functions $A(\cdot;k)$, $F(\cdot;k)$ belong to the Banach space $(L_1)^{m\times m}$, and

$$\left\|A(\cdot,k) - A(\cdot,0)\right\|_1 \to 0, \quad \left\|F(\cdot,k) - F(\cdot,0)\right\|_1 \to 0.$$

Uniform convergence was established by Ya. Tamarkin (1930)

$$\left\|Y(\cdot,k) - Y(\cdot,0)\right\|_\infty \to 0, \quad k \to \infty. \tag{1}$$

In the application to the linear case, M. Krasnosel'skii and S. Krein (1955) gave more general conditions of convergence than condition (1). They consist in

$$\left\|A^{\vee}(\cdot,k)-A^{\vee}(\cdot,0)\right\|_{\infty}\to 0,\quad \left\|F^{\vee}(\cdot,k)-F^{\vee}(\cdot,0)\right\|_{\infty}\to 0, \qquad (2)$$

where

$$A^{\vee}(t;k):=\int_a^t A(s;k)\mathrm{d}s,\quad k\to\infty,$$

and in the existence of an integrated majorant

$$|A(t;k)|\le h(t)\in L_1,\quad t\in[a,b],\quad k\in\mathbb{N}.$$

Then A. Levin (1967) reduced the last inequality to

$$\left\|A(\cdot;k)\right\|_1 \leq const, \quad k \in \mathbb{N}. \tag{3}$$

Moreover, if condition (3) holds, condition (2) is not only sufficient but also necessary for (1).

This result coves the result of W. Reid (1967), in which instead of conditions (2) there was condition

$$A(\cdot,k) \to A(\cdot,0), \quad F(\cdot,k) \to F(\cdot,0), \quad k \to \infty,$$

in the sense of weak convergence in the space $(L_1)^{m\times m}$.
More general sufficient condition for the fulfillment of condition (1) is established by Z. Opial (1967)

$$\left\|A^{\vee}(\cdot,0) - A^{\vee}(\cdot,k)\right\|_{\infty}\left(1 + \left\|A(\cdot;k)\right\|_1\right) \to 0, \quad k \to \infty.$$

These results were later generalized in the papers of Nguen The Hoan (1993), and V. Mikhailets and his followers.

In the case of general boundary-value problems, additional difficulties arise. They are due to the fact that the solution of such problems can not be unique or do not exist.

In the paper of I. Kiguradze (1987), for the first time sufficient conditions were obtained for uniform convergence of solutions to boundary-value problems with general inhomogeneous boundary conditions:

$$B_n y(\cdot,n) = c_n,$$

where linear continuous operators

$$B_n \colon \left(C[a,b];\mathbb{R}\right)^m \to \mathbb{R}^m, \quad n \in \mathbb{N} \cup \{0\}.$$

These boundary conditions include classical boundary conditions, but cannot contain derivatives of the unknown function.

Kiguradze Theorem

Suppose that a homogeneous boundary-value problem, where $n=0$, has only a trivial solution and the following conditions are satisfied:

1) $\sup\limits_n \|A(\cdot;n)\|_1 < \infty$;
2) $\sup\limits_n \|B_n\| < \infty$;
3) $\|A^{\vee}(\cdot,n) - A^{\vee}(\cdot,0)\|_\infty \to 0, \quad n \to \infty$;
4) $\|F^{\vee}(\cdot,n) - F^{\vee}(\cdot,0)\|_\infty \to 0, \quad n \to \infty$;
5) $c_n \to c_0$;
6) $B_n y \to B_0 y, \quad y \in (W_1^1)^m$.

Then for sufficiently large n solutions of boundary-value problems exist, are unique, and

$$\left\|y(\cdot,0) - y(\cdot,n)\right\|_\infty \to 0, \quad n \to \infty.$$

All conditions in Kiguradze Theorem are essential. But some of them can be significantly weakened. This was done in the papers of Mikhailets and his Ph.D.-students. In addition, they managed to generalize all these results to systems of linear differential equations of arbitrary order with complex Lebesgue summable coefficients.

These results were used in the analysis of ordinary differential operators with distributions in the coefficients and initiated the study of some new problems.

In the middle of the last century, differential operators with distributions in coefficients appeared in the works of physicists. In particular, Schrödinger operators with potentials containing Dirac δ-measure, or even its derivative. Such operators naturally arise in mathematical models of real physical processes in strongly inhomogeneous media.
The mathematical theory of such operators appeared later and now has several thousand publications, which are listed in the bibliography of

famous monographs:

1. S. Albeverio, F. Gesctezy, R. Hoegh–Krohn, and H. Holden, *Solvable Models in Quantum Mechanics*. Springer, New York (1988).
2. S. Albeverio, P. Kurasov, *Singular Perturbations of Differential Operators*. Cambridge Univ. Press, Cambridge (2000).

For ordinary differential operators, the mathematical theory of a wide class of such operators appeared at the beginning of this century. Its idea is to define such operators as quasi-differential with properly selected quasi–derivatives according to Shin-Zettl. In turn, the analysis of these operators is reduced to the study of systems of linear differential equations of the first order with summable coefficients.

In this connection, the limit theorems for solutions and Green's matrices of linear systems of differential equations have a special interest. They allow us to interpret some classes of quasi-differential operators as limits in the sense of norm or strong resolvent convergence of differential operators with smooth coefficients.

Let the formal differential expression

$$l(y) = -y''(t) + q'(t)y(t), \quad q(\cdot) \in L_2\big([a,b],\mathbb{C}\big) = L_2 \tag{4}$$

be given on a compact interval, where the derivative of a function q is understood in the sense of distributions.
If $q(\cdot) \in BV[a,b]$, then q' is a signed measur on $[a,b]$.

This expression can be defined as the Shin–Zettl quasi-differential expression with following quasi-derivatives:

$$D^{[0]}y := y, \quad D^{[1]}y := y' - qy,$$
$$l(y) = D^{[2]}y := -\big(D^{[1]}y\big)' - qD^{[1]}y - q^2y.$$

If the function q is smooth, then this definition is equivalent to the classical one.

Let us consider the set of quasi-differential expressions $l_\varepsilon(\cdot)$ of the form (4) with functions $q_\varepsilon(\cdot) \in L_2$, $\varepsilon \in [0, \varepsilon_0]$. In the Hilbert space with norm $\|\cdot\|_2$ each of these expressions generates a dense defined closed quasi-differential operator $L_\varepsilon y := l_\varepsilon(y)$.

$$Dom(L_\varepsilon) = \{y \in L_2 : D^{[2]}y \in L_2; \quad \alpha(\varepsilon)Y_a(\varepsilon) + \beta(\varepsilon)Y_b(\varepsilon) = 0\} \subset W_1^1,$$

where matrices $\alpha(\varepsilon)$, $\beta(\varepsilon) \in \mathbb{C}^{2\times 2}$, and vectors

$$Y_a(\varepsilon) := \{y(a), \quad D_\varepsilon^{[1]}(a)\}, Y_b(\varepsilon) := \{y(b), \quad D_\varepsilon^{[1]}(b)\} \in \mathbb{C}^2.$$

Note that the set $Dom(L_\varepsilon)$ may not contain any nontrivial function from C^1.

Theorem (Mikhailets, Goriunov (2010))

Suppose that the resolvent set of the operator L_0 is not empty and

i) $\left\|q_\varepsilon - q_0\right\|_2 \to 0, \quad \varepsilon \to 0+;$

ii) $\alpha(\varepsilon) \to \alpha(0), \quad \beta(\varepsilon) \to \beta(0).$

Then $L_\varepsilon \to L_0$ in the sense of norm resolvent convergence.

Thus, each of the introduced operators is the limit of similar operators with smooth coefficients.

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m, n, r\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{5}$$

$$By = c. \tag{6}$$

Here matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m \times m}$, vector-valued function $f(\cdot) \in (W_p^n)^m$, vector $c \in \mathbb{C}^{rm}$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^{rm} \tag{7}$$

are arbitrarily chosen; vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (5) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (6) with operator (7) is generic condition for this equation.

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order $\geq r$.

Complex Sobolev space $W_p^{n+r} := W_p^{n+r}([a,b];\mathbb{C})$

$$W_p^{n+r}([a,b];\mathbb{C}) := \{y \in C^{n+r-1}[a,b] \colon y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\}$$

This space is Banach relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \|y^{(k)}\|_p + \|y^{(n+r)}\|_p,$$

where $\|\cdot\|_p$ is the norm in $L_p([a,b];\mathbb{C})$.
By $\|\cdot\|_{n+r,p}$, we also denote the norms in Banach spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \quad \text{and} \quad (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They consist of the vector-valued functions and matrix-valued functions, respectively, all components of which belong to W_p^{n+r}.

With problem (5), (6), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^{rm}. \tag{8}$$

A linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If this operator is Fredholm, then its range $T(X)$ is closed in Y and the index $\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}$. By $[BY_k]$, we denote the numerical $m \times m$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1.

A block numerical matrix

$$\mathrm{M(L,B)} := ([BY_0], \ldots, [BY_{r-1}]) \in \mathbb{C}^{rm\times rm} \tag{9}$$

is characteristic matrix to problem (5), (6). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m\times m}$.

Theorem 1.

The operator (8) *is invertible if and only if the matrix* $M(L,B)$ *is nondegenerate.*

Parameterized boundary-value problem

Boundary-value problem depending in a parameter $\varepsilon \in [0,\varepsilon_0)$, $\varepsilon_0 > 0$

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \ t \in (a,b), \quad (10)$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon). \quad (11)$$

Here $A_{r-j}(\cdot,\varepsilon) \in (W_p^n)^{m\times m}$, $f(\cdot,\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, linear continuous operator $B(\varepsilon)\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$ are arbitrarily chosen; vector-valued function $y(\cdot,\varepsilon) \in (W_p^{n+r})^m$ is unknown.

Problem (10), (11) is a Fredholm one with zero index for every $\varepsilon \in [0,\varepsilon_0)$.

Definition 2.

The solution to the problem (10), (11) depends continuously in a parameter ε at $\varepsilon = 0$ if the conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0,\varepsilon_1)$ and arbitrary chosen $f(\cdot;\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;

($**$) the convergence of right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ and $c(\varepsilon) \to c(0)$ implies the convergence of solutions

Consider the following conditions:

(0) the homogeneous boundary-value problem

$$L(0)y(t,0)=0, \quad t\in(a,b), \quad B(0)y(\cdot,0)=0$$

has only the trivial solution;

(I) $A_{r-j}(\cdot;\varepsilon)\to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$ for every $j\in\{1,\dots,r\}$;

(II) $B(\varepsilon)y\to B(0)y$ in $\mathbb{C}^{rm}$ for every $y\in(W_p^{n+r})^m$.

Theorem 2.

The solution to the problem (10), (11) depends continuously in the parameter ε at $\varepsilon=0$ if and only if this problem satisfies Conditions (0), (I), and (II).

Gnyp, Mikhailets, and Murach (2016) gave a constructive criterion of continuous dependence in a parameter in Sobolev spaces W_p^{n+r}, where $1 \leq p < \infty$. The proof of criterion is based on the fact that the linear continuous operator $B\colon (W_p^{n+r})^m \to \mathbb{C}^{rm}$ admits the unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\,\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m. \tag{12}$$

Here, the matrices $\alpha_k \in \mathbb{C}^{rm\times m}$, and the matrix-valued function $\Phi(\cdot) \in L_{p'}\left([a,b];\mathbb{C}^{rm\times m}\right)$, $1/p + 1/p' = 1$.

Our method of proof allows to investigate such problems in Sobolev spaces W_p^{n+r}, where $1 \leq p \leq \infty$, and some others function spaces.

We supplement our result with a two-sided estimate of the error $\left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p}$ of solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \left\|L(\varepsilon)y(\cdot;0)-f(\cdot;\varepsilon)\right\|_{n,p} + \left\|B(\varepsilon)y(\cdot;0)-c(\varepsilon)\right\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to problem (10), (11).

Theorem 3.

Let the problem (10), (11) satisfies Conditions (0), (I), and (II). Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$, γ_1, and γ_2, such that

$$\gamma_1\,\widetilde{d}_{n,p}(\varepsilon) \le \left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p} \le \gamma_2\,\widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in (0,\varepsilon_2)$. Here, the numbers ε_2, γ_1, and γ_2 do not depend on $y(\cdot;0)$, and $y(\cdot;\varepsilon)$.

Thus, the error and discrepancy of the solution to problem (10), (11) are of the same degree of smallness [2, 5].

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (10)

multipoint Fredholm boundary condition

$$B(\varepsilon)y(\cdot,\varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon), \varepsilon) = q(\varepsilon), \qquad (13)$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m\times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a,b]$ are arbitrarily given.

It is not assumed that the coefficients $A_{r-j}(\cdot,\varepsilon)$, $\beta_{j,k}^{(l)}(\varepsilon)$ or points $t_{j,k}(\varepsilon)$ have a certain regularity on the parameter ε as $\varepsilon > 0$. It will be required that for each fixed $j \in \{1,\dots,N\}$ all the points $t_{j,k}(\varepsilon)$ have a common limit as $\varepsilon \to 0+$, but for the zero-point series $t_{0,k}(\varepsilon)$ this requirement will not be necessary. We consider the case where the points of the interval $[a,b]$ appearing in boundary conditions are not fixed and depend on a numerical parameter and the number of these points may change.

The solution $y(\cdot,\varepsilon)$ to problem (10), (13) is continuous in the parameter ε if it exists, is unique, and satisfies the limit relation

$$\left\| y(\cdot,\varepsilon) - y(\cdot,0) \right\|_{n+r,p} \to 0 \quad \text{as} \quad \varepsilon \to 0+. \qquad (14)$$

Assumptions as $\varepsilon \to 0+$:

(α) $t_{j,k}(\varepsilon) \to t_j$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(β) $\sum_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \to \beta_j^{(l)}$ for all $j \in \{1,\dots,N\}$, and $l \in \{0,\dots,n+r-1\}$;

(γ) $\sum_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(l)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$;

(δ) $\sum_{k=1}^{\omega_0(\varepsilon)} \left\|\beta_{0,k}^{(l)}(\varepsilon)\right\| \to 0$ for all $k \in \{1,\dots,\omega_0(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$.

Assumptions (β) and (γ) imply that the norms of the coefficients $\beta_{j,k}^{(l)}(\varepsilon)$ can increase as $\varepsilon \to 0+$, but not too fast.

Theorem 4.

Let the boundary-value problem (10), (13) for $p = \infty$ satisfies the assumptions (α), (β), (γ), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (14).

Assumptions as $\varepsilon \to 0+$:

(γ_p) $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(n+r-1)}(\varepsilon)\| |t_{j,k}(\varepsilon) - t_j|^{1/p'} = O(1)$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(γ') $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\| |t_{j,k}(\varepsilon) - t_j| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-2\}$.

Theorem 5.

Let the boundary-value problem (10), (13) for $1 \le p < \infty$ satisfies the assumptions (α), (β), (γ_p), (γ'), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (14) [4, 6].

Remark 2.

The systems of conditions (α), (β), (γ), (δ) and (α), (β), (γ_p), (γ'), (δ) do not guarantee uniform convergence of continuous operators $B(\varepsilon)$ to $B(0)$ as $\varepsilon \to 0+$.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{15}$$

$$By = c, \tag{16}$$

where $1 \leq p < \infty$, $A_{r-j}(\cdot)$, $f(\cdot)$, c, and linear continuous operator B satisfy the above conditions to problem (5), (6).

A sequence of multipoint boundary-value problems

$$(L_k y_k)(t) := y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y_k^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{17}$$

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c. \tag{18}$$

Suppose that for boundary-value problem (15), (16), the corresponding homogeneous boundary-value problem has only a trivial solution. Then the inhomogeneous problem has a unique solution for arbitrary right-hand sides.

Theorem 6.

For the boundary-value problem (15), (16) there is a sequence of multipoint boundary-value problems of the form (17), (18) such that they are well-posedness for sufficiently large k and the asymptotic property is fulfilled

$$y_k \to y \quad \text{in} \quad (W_p^{n+r})^m \quad \text{for} \quad k \to \infty.$$

The sequence can be chosen independently of f and c, and constructed explicitly.

1. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 10, 1526–1537. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01588-w*
2. *Atlasiuk, O. M.; Mikhailets, V. A.* Fredholm one-dimensional boundary-value problems with parameter in Sobolev spaces. Ukrainian Math. J. 70 (2019), no. 11, 1677–1687. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-019-01599-7*
3. *Atlasiuk, O. M.; Mikhailets, V. A.* On solvability of inhomogeneous boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2019), no. 11, 3–7. *DOI: 10.15407/dopovidi2019.11.003*

4. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems in Sobolev spaces. J. Math. Sci. (Scopus, SJR Q3) 247 (2020), no. 2, 238–247. *DOI: 10.1007/s10958-020-04799-w*
5. *Atlasiuk, O. M.; Mikhailets, V. A.* On Fredholm parameter-dependent boundary-value problems in Sobolev spaces. Dopov. Nac. akad. nauk Ukr. (2020), no. 6, 3–6. *DOI: 10.15407/dopovidi2020.06.003*
6. *Atlasiuk, O. M.* Limit theorems for solutions of multipoint boundary-value problems with a parameter in Sobolev spaces. Ukrainian Math. J. 72 (2021), no. 8, 1175–1184. (Scopus, WoS, SJR Q2) *DOI: 10.1007/s11253-020-01859-x*

Thank you for your attention!