Oscillatory solutions to problems in fluid mechanics: Analysis and numerics

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Euler system of gas dynamics

Leonhard Paul Euler 1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x \left(\varrho \mathbf{u} \otimes \mathbf{u} \right) + \nabla_x p(\varrho) = 0, \ p(\varrho) = a \varrho^\gamma$$

$$a > 0, \ \gamma > 1$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial \Omega} = 0, \ \Omega \subset R^d \ \text{(bounded)}, \ d = 2, 3$$

Initial state (data)

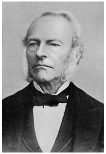
$$\varrho(0, \cdot) = \varrho_0, \ (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

Navier–Stokes system, real fluids

Claude-Louis Navier
1785–1836

Equation of continuity – Mass conservation

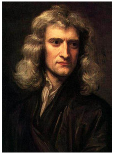
$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$



George Gabriel Stokes
1819–1903

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x \left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$



Isaac Newton
1642–1727

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{d} \mathrm{div}_x \mathbf{u} \right) + \lambda \mathrm{div}_x \mathbf{u} \mathbb{I}$$

no slip condition: $\mathbf{u}|_{\partial \Omega} = 0$

Admissibility, energy balance

Energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma$$

Euler system (conservative)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho, \mathbf{u})\ \mathrm{d}x = \boxed{(\leq)}\,0$$

Navier–Stokes system (dissipative)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho, \mathbf{u})\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x = \boxed{(\leq)}\,0$$

Consistent (stable) approximation, Euler system

Approximate equation of continuity

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \varrho_n \mathbf{u}_n \cdot \nabla_x \varphi] \, \mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x + e_{1,n}[\varphi]$$

Approximate momentum equation

$$\int_0^T \int_\Omega [\varrho_n \mathbf{u}_n \cdot \partial_t \boldsymbol{\varphi} + \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi}] \, \mathrm{d}x\mathrm{d}t$$

$$= - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

Stability - approximate energy inequality

$$\int_\Omega \left[\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + P(\varrho_n) \right] \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \mathrm{d}x + e_{3,n}$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0, \ \ e_{3,n} \to 0 \text{ as } n \to \infty$$

Consistent approximation, Navier–Stokes system

Approximate equation of continuity

$$\int_0^T \int_\Omega [\varrho_n \partial_t \varphi + \varrho_n \mathbf{u}_n \cdot \nabla_x \varphi] \,\mathrm{d}x\mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0,\cdot) \ \mathrm{d}x + e_{1,n}[\varphi]$$

Approximate momentum equation

$$\int_0^T \int_\Omega [\varrho_n \mathbf{u}_n \cdot \partial_t \boldsymbol{\varphi} + \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \boldsymbol{\varphi} + p(\varrho_n) \mathrm{div}_x \boldsymbol{\varphi}] \,\mathrm{d}x\mathrm{d}t$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \ \mathrm{d}x\mathrm{d}t - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0,\cdot) \ \mathrm{d}x + e_{2,n}[\boldsymbol{\varphi}]$$

Stability - approximate energy inequality

$$\int_\Omega \left[\frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + P(\varrho_n) \right] (\tau,\cdot) \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x\mathrm{d}t$$

$$\leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \mathrm{d}x + e_{3,n}$$

Consistency

$$e_{1,n}[\varphi] \to 0, \ \ e_{2,n}[\boldsymbol{\varphi}] \to 0, \ \ e_{3,n} \to 0 \text{ as } n \to \infty$$

Examples of consistent approximations of Euler system

- **Zero viscosity limit:**

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = 0$$
$$\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) = \varepsilon_n \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n),\ \varepsilon_n \to 0$$

- **Artificial viscosity limit:**

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \varepsilon_n \Delta_x \varrho_n$$
$$\partial_t(\varrho_n \mathbf{u}_n) + \mathrm{div}_x(\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) + \nabla_x p(\varrho_n) = \varepsilon_n \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_n),\ \varepsilon_n \to 0$$

- **Limits of certain numerical schemes:** Lax–Friedrichs scheme, MAC scheme, Godunov scheme, general finite volume scheme etc.

Euler as ill–posed system

Initial state

$$\varrho(0,\cdot) = \varrho_0, \ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0$$

The initial data are *wild* if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0,\tau]$, $0 < \tau < T$

E. Chiodaroli (Pisa)

Theorem (E. Chiodaroli, EF 2022) The set of wild data is dense in $L^2 \times L^2$

Strong vs. weak convergence

Uniform bounds (stability):

$$(\varrho_n)_{n\geq 1} \text{ bounded in } L^\infty(0,T;L^\gamma(\Omega))$$

$$\mathbf{m}_n \equiv \varrho_n \mathbf{u}_n,\ (\mathbf{m}_n)_{n\geq 0}\text{bounded in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Weak convergence (up to a subsequence):

$$\varrho_n \to \varrho \text{ weakly - (*) in } L^\infty(0,T;L^\gamma(\Omega))$$

$$\mathbf{m}_n \to \mathbf{m} \text{ weakly - (*) in } L^\infty(0,T;L^{\frac{2\gamma}{\gamma+1}}(\Omega;R^d))$$

Weak convergence $\approx$ convergence of integral averages:

$$v_n \to v \text{ weakly } \Leftrightarrow \int_B v_n \to \int_B v \text{ for any Borel } B \Leftrightarrow \int v_n \phi \to \int v\phi$$

When weak $\Rightarrow$ strong (pointwise a.a.)

Suppose that at least one of the following holds:

- The (limit) Euler system admits a regular solution $(\varrho, \mathbf{m})$ in $(0, T) \times \Omega$
- The weak limit $(\varrho, \mathbf{m})$ belongs to the class C^1 - it is continuously differentiable in $[0, T] \times \Omega$
- (*) The limit $(\varrho, \mathbf{m})$ is a *weak* solution of the Euler system

$$\Rightarrow$$

$$\varrho_n \to \varrho \text{ (strongly) in } L^1((0,T) \times \Omega)$$

$$\mathbf{m}_n \to \mathbf{m} \text{ (strongly) in } L^1((0,T) \times \Omega; R^d)$$

in particular (up to a subsequence)

$$\varrho_n \to \varrho, \ \mathbf{m}_n \to \mathbf{m} \text{ a.a. in } (0,T) \times \Omega$$

Strong convergence to weak solution

Exterior domain (convex obstacle):

$$\Omega = R^d \setminus C, \ \ C - \text{compact convex}$$

Far field conditions:

$$\varrho_n \to \varrho_\infty \geq 0, \ \ \mathbf{m}_n \to \mathbf{m}_\infty \text{ as } |x| \to \infty$$

EF, M. Hofmanová:
The following is equivalent:

- $$\varrho, \mathbf{m} \text{ weak solution to the Euler system}$$
- $$\varrho_n \to \varrho, \ \ \mathbf{m}_n \to \mathbf{m} \text{ strongly (pointwise) in } \Omega$$

Martina Hofmanová (Bielefeld)

Conclusion:
If the convergence is NOT strong, then the limit is NOT a solution of the Euler system

Weak convergence of consistent approximations

Weak convergence:

If consistent approximations DO NOT converge strongly, the following must be satisfied:

- the limit Euler system does not admit a strong solution
- the limit $(\varrho, \mathbf{m})$ is not C^1 smooth
- the limit $(\varrho, \mathbf{m})$ is not a weak solution of the Euler system

Visualization of weak convergence?

- **Oscillations.** Weakly converging sequence may develop oscillations. Example:

$$\sin(nx) \to 0 \text{ weakly as } n \to \infty$$

- **Concentrations.**

$$n\theta(nx) \to \delta_0 \text{ weakly-(*) in } \mathcal{M}(R)$$

if

$$\theta \in C_c^\infty(R),\ \theta \geq 0, \int_R \theta = 1$$

Statistical description – Young measure

L. C. Young

Young measure:

$$b(\varrho_n, \mathbf{m}_n) \to \overline{b(\varrho, \mathbf{m})} \text{ weakly-(*) in } L^\infty((0,T) \times \Omega)$$

(up to a subsequence) for any $b \in C_c(R^{d+2})$

Young measure:
$\mathcal{V}$ – a parametrized family of probability measures $\{\mathcal{V}_{t,x}\}_{(t,x)\in(0,T)\times\Omega}$ on the phase space R^{d+2}:

$$\overline{b(\varrho, \mathbf{m})}(t,x) = \langle \mathcal{V}_{t,x}; b(\widetilde{\varrho}, \widetilde{\mathbf{m}}) \rangle \text{ for a.a. } (t,x)$$

Limit problem – measure valued solutions

Equation of continuity

$$\int_0^T \int_\Omega [\varrho \partial_t \varphi + \mathbf{m} \cdot \nabla_x \varphi] \, \mathrm{d}x \mathrm{d}t = - \int_\Omega \varrho_0 \varphi(0, \cdot) \ \mathrm{d}x$$

Momentum equation

$$\int_0^T \int_\Omega \left[\mathbf{m} \cdot \partial_t \boldsymbol{\varphi} + \overline{\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho}} : \nabla_x \boldsymbol{\varphi} + \overline{p(\varrho)} \mathrm{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \mathrm{d}t$$

$$= - \int_\Omega \varrho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot) \ \mathrm{d}x$$

Admisibility - energy inequality

$$\int_\Omega \left[\frac{1}{2} \overline{\frac{|\mathbf{m}|^2}{\varrho}} + \overline{P(\varrho)} \right] \mathrm{d}x \leq \int_\Omega \left[\frac{1}{2} \varrho_0 |\mathbf{u}_0|^2 + P(\varrho_0) \right] \mathrm{d}x$$

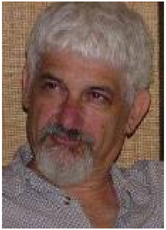
Visualising weak convergence – computing Young measure

visualizing Young measure $\Leftrightarrow$ computing $\overline{b(\varrho, \mathbf{m})}$

Problems:

- $b(\varrho_n, \mathbf{m}_n)$ converge only weakly
- extracting subsequences
- only statistical properties relevant $\Rightarrow$ knowledge of the “tail” of the sequence of approximate solutions absolutely necessary

Strong instead of weak



Janos Komlos (Ruthers Univ.)

Komlos theorem (a variant of Strong Law of Large Numbers):

$$(U_n)_{n \geq 1} \text{ bounded in } L^1(Q)$$

$$\Rightarrow$$

$$\frac{1}{N} \sum_{k=1}^{N} U_{n_k} \to \overline{U} \text{ a.a. in } Q \text{ as } N \to \infty$$

Generating Young measure:

$$\mathbf{U}_n = (\varrho_n, \mathbf{m}_n) \in R^{d+1} \text{ phase space}$$

$$(\mathbf{U}_n)_{n \geq 1} \text{ bounded in } L^1((0,T) \times \Omega; R^d) \approx \mathcal{V}^n_{t,x} = \delta_{\mathbf{U}_n(t,x)}$$

$$\Rightarrow$$

$$\frac{1}{N} \sum_{k=1}^{N} \mathcal{V}^{n_k}_{t,x} \to \mathcal{V}_{t,x} \text{ narrowly } \boxed{a.a.} \text{ in } ((0,T) \times \Omega) \text{ as } N \to \infty$$

Erich J. Balder (Utrecht)

(S) - convergence, basic idea

Trivial example of oscillatory sequence:

$$U_n = \begin{cases} 1 \text{ for } n \text{ odd} \\ -1 \text{ for } n \text{ even} \end{cases}$$

Convergence via Young measure approach:

Convergence up to a *subsequence***:**

$$U_n \approx \delta_{U_n}, \ U_{n_k} \to \begin{cases} \delta_1 \text{ as } k \to \infty, \ n_k \text{ odd} \\ \delta_{-1} \text{ as } k \to \infty, \ n_k \text{ even} \end{cases}$$

Convergence via averaging:

$$U_n \approx \delta_{U_n}, \ \frac{1}{N} \sum_{n=1}^{N} U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$$

$$\frac{1}{w_N} \sum_{n=1}^{N} w\left(\frac{n}{N}\right) U_n \to \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1, \ w_N \equiv \sum_{n=1}^{N} w\left(\frac{n}{N}\right)$$

(S)–convergence

(S)–convergent approximate sequence:

An approximate sequence $(\mathbf{U}_n)_{n\geq 1}$ is $\boxed{(S)-convergent}$ if for any $b \in C_c(R^D)$:

- **Correlation limit**

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m) \mathrm{d}y \text{ exists for any fixed } m$$

- **Correlation disintegration**

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{n,m=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y$$
$$= \lim_{M\to\infty} \frac{1}{M} \sum_{m=1}^{N} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_Q b(\mathbf{U}_n) b(\mathbf{U}_m)\ \mathrm{d}y \right)$$

Basic properties of (S)–convergence

Equivalence to convergence of ergodic (Cesàro means):

$$(\mathbf{U}_n)_{n\geq 1} \text{ (S)–convergent } \Leftrightarrow \frac{1}{N}\sum_{n=1}^{N} b(\mathbf{U}_n) \to \overline{b(\mathbf{U})} \ \boxed{\textit{strongly}} \text{ in } L^1(Q)$$

(S)– limit (parametrized measure):

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V},\ \{\mathcal{V}_y\}_{y\in Q},\ \mathcal{V}_y \in \mathfrak{P}(R^D),\ \left\langle \mathcal{V}_y; b(\widetilde{U}) \right\rangle = \overline{b(\mathbf{U})}(y)$$

Convergence in Wasserstein distance:

$$\int_Q |\mathbf{U}_n|^p \ \mathrm{d}y \leq c \text{ uniformly for } n = 1, 2, \ldots,\ p > 1$$

$$\mathbf{U}_n \overset{(S)}{\to} \mathcal{V} \ \Rightarrow \ \int_Q \left| d_{W_s}\left[\frac{1}{N}\sum_{n=1}^{N} \delta_{\mathbf{U}_n(y)}; \mathcal{V}_y \right] \right|^s \ \mathrm{d}y \to 0 \text{ as } N \to \infty,\ s < p$$

Computing defect numerically -EF, M.Lukáčová, B.She

$\mathbf{U}_n = (\varrho_n, \mathbf{m}_n)$ consistent approximation of the Euler system

Monge–Kantorowich (Wasserstein) distance:

$$\left\| \mathrm{dist} \left(\frac{1}{N} \sum_{k=1}^{N} \mathcal{V}^{n_k}_{t,x}; \mathcal{V}_{t,x} \right) \right\|_{L^q((0,T)\times\Omega)} \to 0$$

for some $q > 1$

Convergence in the first variation:

$$\frac{1}{N} \sum_{k=1}^{N} \left\langle \mathcal{V}^{n_k}_{t,x}; \left| \widetilde{\mathbf{U}} - \frac{1}{N} \sum_{k=1}^{N} \mathbf{U}_n \right| \right\rangle \to \left\langle \mathcal{V}_{t,x}; \left| \widetilde{\mathbf{U}} - \mathbf{U} \right| \right\rangle$$

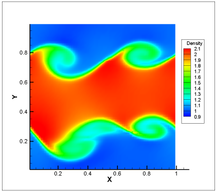
in $L^1((0, T) \times \Omega)$

Mária Lukáčová (Mainz)

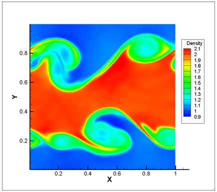
Bangwei She (CAS Praha)

Experiment, density for
Kelvin–Helmholtz problem (M. Lukáčová, Yue Wang)

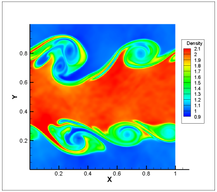
density ϱ
$n = 128,\ T = 2$



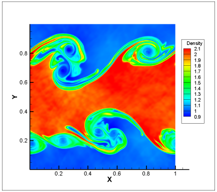
density ϱ
$n = 256,\ T = 2$



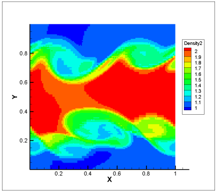
density ϱ
$n = 512,\ T = 2$



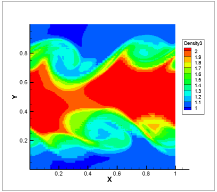
density ϱ
$n = 1024,\ T = 2$



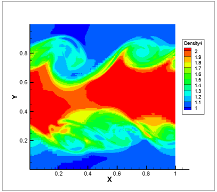
Cèsaro averages density ϱ
$n = 128,\ T = 2$



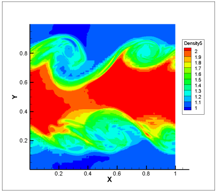
Cèsaro averages density ϱ
$n = 256,\ T = 2$



Cèsaro averages density ϱ
$n = 512,\ T = 2$



Cèsaro averages density ϱ
$n = 1024,\ T = 2$



Consistent approximation of Navier–Stokes system

Bounded consistent approximations

- $$(\varrho_0, \mathbf{u}_0) \text{ smooth initial data satisfying compatibility conditions}$$

- $$(\varrho_n, \mathbf{u}_n)_{n \geq 0} \text{ consistent approximation of Navier–Stokes system}$$

- $$\sup_{n \geq 1} \|(\varrho_n, \mathbf{u}_n)\|_{L^\infty} \leq c \text{ uniformly for } n \to \infty$$

$$\Rightarrow$$

$$\varrho_n \to \varrho \text{ in } L^1((0,T) \times \Omega),\ \mathbf{u}_n \to \mathbf{u} \text{ in } L^1((0,T) \times \Omega; R^d$$

$$(\varrho, \mathbf{u}) \text{ a regular solution of Navier–Stokes system}$$

Proof based on (i) the local regularity result of Valli, Zajaczkowski, (ii) weak strong uniqueness by EF, Novotny, Gwiazda, Swierczewska-Gwiazda, Wiedemann, and (iii) conditional regularity by Sun, Wang, Zhang

Random (uncertain) data – framework

Initial data (conservative variables):

$$\varrho_0, \ \mathbf{m}_0 = \varrho_0 \mathbf{u}_0$$

Data (phase) space

$$\mathcal{D} = \left\{ (\varrho_0, \mathbf{m}_0) \ \middle| \varrho_0 \in L^\gamma(\Omega), \ \mathbf{m}_0 \in L^{\frac{2\gamma}{\gamma+1}}(\Omega; R^d)) \int_Q E(\varrho_0, \mathbf{m}_0) \ \mathrm{d}x < \infty \right\}$$

$$\subset X_{\mathcal{D}} = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; R^d) \ \text{ – Polish space}$$

Probability measures

$\mathfrak{P}[\mathcal{D}]$ – the set of probability measures on $X_{\mathcal{D}}$ supported by $\mathcal{D}$

Random data, weak approach

$$\varrho_0, \mathbf{u}_0 \in \mathcal{D} \subset X_{\mathcal{D}}$$

weak approach $\Leftrightarrow$ determining distribution (law) of solutions

Generating sequences of random data

$$(\varrho_0^n, \mathbf{m}_0^n) \in \mathcal{D}$$

$$\frac{1}{N} \sum_{n=1}^{N} F\left(\varrho_0^n, \mathbf{m}_0^n\right) \to \mathbb{E}\left[F[\varrho_0, \mathbf{m}_0]\right] \text{ as } N \to \infty$$

for any $F \in BC(X_{\mathcal{D}})$

Expected value

$$\mathbb{E}\left[F(\varrho_0, \mathbf{m}_0)\right] = \int_{X_D} F\left(\hat{\varrho}, \hat{\mathbf{u}}\right) \ \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{m}_0]$$

Distribution of the initial data

$$\mathcal{L}[\varrho_0, \mathbf{m}_0] \in \mathfrak{P}[\mathcal{D}] - \text{probability measure on the space of data}$$

Main goal, convergence

$(\varrho_0^n, \mathbf{m}_0^n) \in \mathcal{D} \ \rightarrow \ (\varrho^{h,n}, \mathbf{m}^{h,n})$ consistent (numerical) approximation

Sequence of empirical measures:

$$\frac{1}{N} \sum_{n=1}^{N} \delta_{\varrho^{h,n}, \mathbf{m}^{h,n}}$$

Convergence in law:

$$\frac{1}{N} \sum_{n=1}^{N} F[\varrho^{h,n}, \mathbf{m}^{h,n}] \rightarrow \mathbb{E}\left[F[\varrho, \mathbf{m}] \right] \ \text{as } h \rightarrow 0, \ N \rightarrow \infty$$

for any $F \in BC\Big(W^{-m,2}((0,T) \times \Omega) \times W^{-m,2}((0,T) \times \Omega; R^d) \Big)$
Limit solution:

$$\mathbb{E}\left[F[\varrho, \mathbf{m}] \right] = \int_{X_D} F\left[(\varrho, \mathbf{m})[\hat{\varrho}, \hat{\mathbf{m}}] \right] \mathrm{d}\mathcal{L}[\varrho_0, \mathbf{m}_0]$$

$(\varrho, \mathbf{m})[\hat{\varrho}, \hat{\mathbf{m}}]$ - smooth (whence unique) solution of the Navier-Stokes system with the initial data $[\hat{\varrho}, \hat{\mathbf{m}}]$

Boundedness in probability

Consistent (numerical) approximation

$$h = h(\ell),\ N = N(\ell),\ h(\ell) \searrow 0,\ N(\ell) \nearrow \infty \text{ as } \ell \to \infty.$$

$$\frac{1}{N}\sum_{n=1}^{N} \delta_{[\varrho^{h,n},\mathbf{m}^{h,n}]},\ \mathbf{m}^{h,n} = \varrho^{h,n}\mathbf{u}^{h,n}$$

Boundedness in probability:
For any $\varepsilon > 0$, there is $M = M(\varepsilon)$ such that

$$\frac{\#\left\{\|\varrho^{h,n}, \mathbf{u}^{h,n}\|_{L^\infty((0,T)\times\Omega;R^{d+1})} > M,\ n \leq N\right\}}{N} < \varepsilon \text{ for any } \ell = 1, 2, \dots$$

Convergence (EF, M. Lukáčová):
Any sequence of consistent approximations that is bounded in probability converges in law to a (statistical) solution of the Navier–Stokes system