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meromorphic functions
on foliated manifolds**

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COSIMPLICIAL COHOMOLOGY OF RESTRICTED MEROMORPHIC FUNCTIONS ON FOLIATED MANIFOLDS

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ABSTRACT. Starting from the axiomatic description of meromorphic functions with prescribed analytic properties on complex manifolds, we introduce the cosimplicial cohomology of restricted meromorphic functions defined on foliations of smooth complex manifolds. Spaces for double chain-cochain complex and boundary operator are constructed and examined. Multiplications among pairs of restricted meromorphic functions with non-commutative parameters, as well as for double complex spaces are introduced that their properties are discussed. In particular, we prove that constructions of cosimplicial cohomology of restricted meromorphic functions are non-vanishing, canonical, independent of the choice of the transversal basis for a foliation, and invariant with respect to changes of coordinates on a smooth manifold and on transversal sections. As an application, we provide an example of cohomological invariants generalizing the Gabbillon–Vay invariant for codimension one foliations.

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1. INTRODUCTION

It is natural to consider the cohomology of various structures associated to foliations of a smooth manifold [5, 6, 8, 10, 14, 18, 28, 30, 31]. In this paper we construct explicitly a cohomology theory of meromorphic functions with specified analytical properties. Restricted meromorphic functions depend on non-commutative parameters provided by elements of an infinite-dimensional Lie algebra as well as sets of commutative formal variables which can be associated to local coordinates of certain complex domains. Restricted meromorphic functions are defined subject to several axioms and restrictions on their convergence. A particular example of such functions can be given by bilinear pairing on the algebraic completion of spaces associated to an infinite-dimensional Lie algebras [8, 9, 11, 12, 13, 16, 41]. For a smooth complex manifold M , one needs to identify formal variables of restricted meromorphic functions with local coordinates of domains on M . In the case of a foliated manifold, additional formal parameters can be identified with local coordinates on sections of a transversal basis. Our motivation was to understand further the continuous cohomology [1, 4, 5, 6, 8, 10, 11, 12, 13, 14, 15, 16, 18, 20, 21, 23, 24, 27, 28, 29, 31, 35, 41]

Key words and phrases. Meromorphic functions with prescribed analytic properties, foliations, cohomology.

of structures defined on foliations. One hopes to use properties of restricted meromorphic functions to be able to describe new invariants of foliations. In particular [4], one hopes to relate cohomology of functions with specified behavior originating from infinite-dimensional Lie algebras-valued series on manifolds. In order to construct a cohomology theory on foliations, we define double complex spaces of restricted meromorphic functions on cosimplicial domains introduced in the standard way [41]. Properties of such spaces are then studied. We prove independence of their elements with respect to changes of the transversal basis and coordinates on M and transversal sections. An appropriate coboundary operator is defined. To be able to study examples of cohomology invariants, we introduce a multiplication between pairs of elements of two double complex spaces and determine its properties. An example generalizing the Gabbillon–Vay invariant for codimension one foliations [2, 3, 15, 18, 30, 32, 33, 34] is derived and examined. For further developments, we plan to study applications of restricted meromorphic functions and the cosimplicial construction introduced in this paper for K -theory, description of cohomological invariants of foliations [1, 20, 21, 35, 40, 36], deformation theory [17], Losik’s approach [15, 31], Krichever–Novikov algebras [37, Schl1, 38], current algebras on manifolds [39], factorization algebras [23], and classification of types of leaves for foliations [2, 3, 32, 33, 34].

2. AXIOMATICS OF RESTRICTED MEROMORPHIC FUNCTIONS

Developing ideas of [26], we introduce in this section the notion of restricted meromorphic functions, i.e., meromorphic functions with prescribed analytic behavior on open complex domains. In Sections 3–5 we will see that the space of such functions underlies a cohomology theory. Restricted meromorphic functions depend on an number of non-commutative parameters (provided by an infinite-dimensional Lie algebra elements) as well as commutative formal variables. First, let us set the notations we use. For a tuple of n complex formal variables $\mathbf{z} = (z_1, \dots, z_n)$, we dedicate the notation \mathbf{z}_l for l sets of n values of \mathbf{z} , namely, $\mathbf{z}_l = ((z_{1,1}, \dots, z_{1,n}), \dots, (z_{l,1}, \dots, z_{l,n}))$. For local coordinates on transversal sections of codimension p foliation \mathcal{F} (see Subsection 3.1), the notation \mathbf{z}' for k -sets of p formal parameters is reserved. In general, for a set of m elements (y_1, \dots, y_m) we use the notation \mathbf{y}_m , while for $k_1 \leq k_2 \in \mathbb{Z}$, we denote $\mathbf{y}_{(k_1, k_2)} = (y_{k_1}, y_{k_1+1}, \dots, y_{k_2})$. When m is arbitrary we write \mathbf{y} . In particular, for l sets of n algebra/group elements we denote $\mathbf{g}_l = (g_1, \dots, g_l)$. When combined with l sets of n formal parameters \mathbf{z}_l , we use the specific notation $\mathbf{x}_l = (\mathbf{g}_l, \mathbf{z}_l)$. For a function $A(x)$ we denote the sequence $(A(x_1), \dots, A(x_n))$ by $\bar{A}(\mathbf{x})$, while $\bar{A}(\mathbf{x}_l)$ stands for $(A(x_{1,1}), \dots, A(x_{1,n})), \dots, (A(x_{l,1}), \dots, A(x_{l,n}))$. The notations for products of functions are $A(\mathbf{x}) = (A(x_1) \dots A(x_n))$ and $A(\mathbf{x}_l) = A(x_{1,1}) \dots A(x_{1,n}) \dots A(x_{l,1}) \dots A(x_{l,n})$.

2.1. Meromorphic functions with non-commutative parameters. In this paper we consider functions $f(\mathbf{z}_l)$ of several complex variables defined on sets of open domains extendable to converging meromorphic functions $\mathcal{M}(f(\mathbf{z}_l))$ on larger domains. Denote by $F_{ln}\mathbb{C}$ the configuration space of $l \geq 0$ ordered sets of n complex

coordinates in \mathbb{C}^{ln} , $F_{ln}\mathbb{C} = \{\mathbf{z}_l \in \mathbb{C}^{ln} \mid z_{i,l} \neq z_{j,l'}, i \neq j\}$. Let \mathfrak{g} be an infinite-dimensional Lie algebra generated by $\{\xi_i, i \in \mathbb{Z}\}$, and $\mathcal{G}(\mathbf{x}_l)$ be the space of complex-valued functions depending on l -pairs of elements \mathfrak{g} and \mathbf{z}_l . Abusing notations, for $F \in \mathcal{G}(\mathbf{x}_l)$ we call a linear map with the only possible poles at $z_{i,l} = z_{j,l'}, i \neq j, 1 \leq l, l' \leq n$, $F : \mathbf{x}_l \mapsto \mathcal{M}(F(\mathbf{x}_l))$, the meromorphic function in \mathbf{z}_l .

Next, let us state further properties of meromorphic functions we require. We define the left action of the permutation group S_{ln} on $F(\mathbf{x}_l)$ by $\sigma(F)(\mathbf{x}_l) = F(\mathbf{g}_l, \mathbf{z}_{\sigma(i)})$, $1 \leq i \leq l$. Denote by $(T_G)_i$ the translation operator $f(z_i) \mapsto f(z_i + z_0)$ acting on the i -th entry. We then define the action of partial derivatives on an element $F(\mathbf{x}_l)$

$$\partial_{z_i} F(\mathbf{x}_l) = F((T_G)_i \mathbf{g}_l, \mathbf{z}_l), \quad \sum_{i \geq 1} \partial_{z_i} F(\mathbf{x}_l) = T_G F(\mathbf{x}_l), \quad (2.1)$$

and call it T_G -derivative property. For $z \in \mathbb{C}$, let

$$e^{zT_G} F(\mathbf{x}_l) = F(\mathbf{g}_l, \mathbf{z}_l + z). \quad (2.2)$$

Let $\text{Ins}_i(A)$ denotes the operator of multiplication by $A \in \mathbb{C}$ at the i -th position. Then we define

$$F(\mathbf{g}_l, \text{Ins}_i(z) \mathbf{z}_l) = F(\text{Ins}_i(e^{zT_G}) \mathbf{g}_l, \mathbf{z}_l), \quad (2.3)$$

equal as equal power series expansions in z , in particular, absolutely convergent on the open disk $|z| < \min_{i \neq j} \{|z_{i,l} - z_{j,l'}|\}$. A meromorphic function has K_G -property if for $z \in \mathbb{C}^\times$ satisfies $z\mathbf{z}_l \in F_{ln}\mathbb{C}$,

$$z^{K_G} F(\mathbf{x}_l) = F(z^{K_G} \mathbf{g}_l, z \mathbf{z}_l). \quad (2.4)$$

Let us recall the notion of a shuffle. For $m \in \mathbb{N}$ and $1 \leq p \leq m-1$, let $J_{m;p}$ be the set of elements of S_m which preserves the order of the first p numbers and the order of the last $(m-p)$ numbers, that is, $J_{m;p} = \{\sigma \in S_m \mid \sigma(1) < \dots < \sigma(p), \sigma(p+1) < \dots < \sigma(m)\}$. Let $J_{m;p}^{-1} = \{\sigma \mid \sigma \in J_{m;p}\}$. For some meromorphic functions we require the property:

$$\sum_{\sigma \in J_{m;p}^{-1}} (-1)^{|\sigma|} \sigma(F(\mathbf{g}_{\sigma(i)}, \mathbf{z}_l)) = 0. \quad (2.5)$$

Let $\mathcal{W}(\mathfrak{g})$ be the space of universal enveloping algebra $U(\mathfrak{g})$ -valued formal series in \mathbf{z}_l . We will consider meromorphic functions $F(\mathbf{x}_l)$ depending on elements $\mathbf{g}_l \in \mathcal{W}(\mathfrak{g})$ and satisfying the above conditions of this subsection. In what follows we concentrate on the case $\mathcal{G}(\mathfrak{g}) = G$ where G is the algebraic completion of $\mathcal{W}(\mathfrak{g})$. In order to define a specific space of restricted meromorphic functions associated to $\mathcal{W}(\mathfrak{g})$ and satisfying certain properties we have to work with the algebraic completion of $\mathcal{W}(\mathfrak{g})$. In particular, for that purpose, we have to consider elements of a $\mathcal{W}(\mathfrak{g})$ with inserted exponentials of the grading operator K_G , i.e., of the form $\sum_{m \in \mathbb{Z}} z^{-m-1} a^{K_G} g_m b^{K_G} g$. For general $a, b, z \in \mathbb{C}^\times$, such elements do not satisfy the properties needed to construct a appropriate cohomology theory of restricted meromorphic functions. Thus we have to extend $\mathcal{W}(\mathfrak{g})$ algebraically and analytically, i.e., to consider $G = \overline{\mathcal{W}}(\mathfrak{g}) = \prod_{m \in \mathbb{Z}} \mathcal{W}_{(m)} = (\mathcal{W}')^*(\mathfrak{g})$, as well as include extra elements to make the structure of $\mathcal{W}(\mathfrak{g})$ compatible with the descending filtration with respect to the grading subspaces, and analytic properties with respect to formal parameters \mathbf{z}_l .

Then G has the structure that is complete in topology determined by the filtration. When a particular set \mathbf{z}_m of formal parameters is specified for G we denote it as $G_{\mathbf{z}_m}$. We assume that G is endowed with non-degenerate bilinear pairing (\cdot, \cdot) (not to be confused with the notation (g, z) for parameters), and denote by \tilde{G} the dual space to G with respect to this pairing. For $\mathbf{g}_l \in G$, denote $\mathbf{x}_l = (\mathbf{g}_l, \mathbf{z}_l)$. In this paper we will deal with meromorphic functions given by $F : \mathbf{x}_l \mapsto \mathcal{M}((\vartheta, F(\mathbf{x}_l)))$, and converging in \mathbf{z}_l on certain complex domains \mathbf{V}_l .

2.2. Restricted meromorphic functions. In this subsection, following [26], we give the definition of meromorphic functions with prescribed analytical behavior on a complex domain. Let us assume that the space G is endowed with a grading $G = \bigcup_{m \in \mathbb{Z}, m > m_0} G^{(m)}$ bounded from below with respect to the grading operator K_G . We denote by $P_m : G \rightarrow G^{(m)}$, the projection of G on $G^{(m)}$. For each element $g \in G$, and $x = (g, z)$, $z \in \mathbb{C}$ let us associate the differential form $\nu_G(x) = \sum_{k \in \mathbb{C}} g_m z^m dz^{\text{wt}(g)}$, where $\text{wt}(g)$ is the weight with respect to the grading operator $K_G g = \text{wt}(g)g$. Finally, we formulate the definition of restricted meromorphic functions with extra sets of parameters. Due to the nature of axiomatics of meromorphic functions described in Subsection 3.2, we have to restrict their analytic behavior to be able to introduce corresponding cohomology theory. In particular, we would like to make $F(\mathbf{x}_l)$ to play the role of cocains. Corresponding coboundary operators are supposed to include insertions into $F(\mathbf{x}_l)$ of extra \mathbf{x} -dependence by means of extra $\bar{\nu}_G(\mathbf{x})$ -forms. The result of such insertions should remain in G . As a formal sum, such map has to be absolutely convergent in \mathbf{z}_l . For this purpose we formulate the following definition representing extra restricting conditions on meromorphic functions introduced in Subsection 3.2.

Definition 1. We assume that there exist positive integers $\beta(g_{l',i}, g_{l'',j})$ depending only on $g_{l',i}, g_{l'',j} \in G$ for $i, j = 1, \dots, (l+k)n$, $k \geq 0$, $i \neq j$, $1 \leq l', l'' \leq n$. Let \mathbf{l}_n be a partition of $(l+k)n = \sum_{i \geq 1} l_i$, and $k_i = l_1 + \dots + l_{i-1}$. For $\zeta_i \in \mathbb{C}$, define for $\tilde{k}_i = k_i + l_i$, $f_i = F(\bar{\nu}_G(\mathbf{g}_{\tilde{k}_i}, \mathbf{z}_{\tilde{k}_i} - \zeta_i))$, for $i = 1, \dots, ln$. We then call a meromorphic function $F(\mathbf{x}_l)$ satisfying properties (2.1)–(2.4), a restricted meromorphic function if under the following conditions on domains, $|z_{k_i+p} - \zeta_i| + |z_{k_j+q} - \zeta_j| < |\zeta_i - \zeta_j|$, for $i, j = 1, \dots, k$, $i \neq j$, and for $p = 1, \dots, l_i$, $q = 1, \dots, l_j$, the function $\sum_{\mathbf{r}_n \in \mathbb{Z}^n} F(\overline{P_{r_i} f_i}, \zeta_l)$, is absolutely convergent to an analytic extension in \mathbf{z}_{l+k} , independently of complex parameters ζ_l , with the only possible poles on the diagonal of \mathbf{z}_{l+k} of order less than or equal to $\beta(g_{l',i}, g_{l'',j})$. In addition to that, for $\mathbf{g}_{l+k} \in G$, the series $\sum_{q \in \mathbb{C}} F(\bar{\nu}_G(x_k) \mathbf{P}_q(F(\mathbf{x}_{(k,k+l)})))$, is absolutely convergent when $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_s| > 0$, for $i = 1, \dots, k$ and $s = k+1, \dots, l+k$, and the sum can be analytically extended to a meromorphic function in \mathbf{z}_{l+k} with the only possible poles at $z_i = z_j$ of orders less than or equal to $\beta(g_{l',i}, g_{l'',j})$.

For an arbitrary $\theta \in \tilde{G}$ and $l \geq 0$ complex variables \mathbf{z}_l defined in domains \mathbf{V}_l , let us introduce the following vector $\bar{F}(\mathbf{x}_l) = [F(\mathbf{g}_l, \mathbf{z}_l \mathbf{d}\mathbf{z}_{i(l_n)})]$, containing meromorphic functions given by matrix elements with respect to the pairing (\cdot, \cdot) , where $i(j)$, $j = 1, \dots, ln$, are cycling permutations of $(1, \dots, ln)$ starting with j . For k elements \mathbf{g}'_k ,

$k \geq 0$, we call the space of all vectors $\overline{F}(\mathbf{x}_l)$ combined with k -sets of forms $\overline{v}_G(\mathbf{x}'_k)$ depending on p complex variables \mathbf{z}'_k defined in domains \mathbf{U}_k satisfying T_G - and K_G -properties (2.1), (2.4), and (2.5), the space $\Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ of restricted meromorphic functions.

2.3. Coordinate change invariance. In this subsection we show that vectors $\overline{F}(\mathbf{x}_l) \in \Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ containing restricted meromorphic functions are canonical with respect to changes of sets of formal parameters. Let $\text{Aut } \mathcal{O}^{(ln)} = \text{Aut}_{\mathbb{C}}[[\mathbf{z}_l]]$ be the group of formal automorphisms of ln -dimensional formal power series algebra $\mathbb{C}[[\mathbf{z}_l]]$. In what follows, we assume that G is equipped with the action of operators $(z^{m+1}\partial_z)$, $m \in \mathbb{N}$, and its Lie subalgebra $\text{Der}_0 \mathcal{O}^{(ln)}$ of \mathfrak{g} is given by the Lie algebra of $\text{Aut } \mathcal{O}^{(ln)}$. Since the vector fields $(z^{m+1}\partial_z)$ act on G as operators of degree $(-m)$, the action of the Lie subalgebra $\text{Der}_+ \mathcal{O}^{(ln)}$ is locally nilpotent. The operator $(z\partial_z)$ acts as the grading operator K_G , which is diagonalizable with integral eigenvalues. Thus, the action of $\text{Der } \mathcal{O}^{(ln)}$ on G can be exponentiated to an action of $\text{Aut } \mathcal{O}^{(ln)}$. We write an element of $\text{Aut } \mathcal{O}^{(ln)}$ as $\mathbf{z} \rightarrow \boldsymbol{\rho}_n(\mathbf{z})$, where elements of $\boldsymbol{\rho}_n$ are $\rho_i(\mathbf{z}) = \sum_{i_1 \geq 0, \dots, i_n \geq 0, \sum_{j=1}^n i_j \geq 1} a_{i_k} \mathbf{z}_k^{i_k}$, where $a_{i_k} \in \mathbb{C}$, and are the images of ρ_i , $i = 1, \dots, l$ in the finite dimensional \mathbb{C} -vector space. In order to represent the action of the group $\text{Aut } \mathcal{O}^{(ln)}$ on the variables \mathbf{z} of \overline{F} in terms of an action on elements \mathfrak{g}_l , we have to transfer (as in $n = 1$ case of [7]) to an exponential form of the transformations $\rho_i(\mathbf{z})$ with corresponding coefficients $\beta_{\mathbf{r}_i}^{(j)} \in \mathbb{C}$ recursively found [19] in terms of coefficients $a_{\mathbf{r}_i}^{(i)}$.

Next, we recall the general definition of a torsor [7]. Let \mathfrak{G} be a group, and S a non-empty set. Then S is called a \mathfrak{G} -torsor if it is equipped with a simply transitive right action of \mathfrak{G} , i.e., given $s_1, s_2 \in S$, there exists a unique $\mu \in \mathfrak{G}$ such that $s_1 \cdot \mu = s_2$, where the right action is given by $s_1 \cdot (\mu\mu') = (s_1 \cdot \mu) \cdot \mu'$. The choice of any $s_1 \in S$ allows us to identify S with \mathfrak{G} by sending $s_1 \cdot \mu$ to μ . Finally, we state

Lemma 1. *The vector $\overline{F}(\mathbf{x}_l) \in \Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ is invariant with respect to changes of formal variables.*

Proof. Consider the vector $\overline{F}(\tilde{\mathbf{x}}_l) = [F(\mathbf{g}_l, \tilde{\mathbf{z}}_l \mathbf{d}\tilde{\mathbf{z}}_{i(ln)})]$. Note that $d\tilde{z}_j = \sum_{i=1}^{ln} dz_i \partial_{z_i} \rho_j$, $\partial_{z_i} \rho_j = \partial \rho_j / \partial z_i$. By definition of the action of $\text{Aut}_n \mathcal{O}_{ln}^{(1)}$, for $d\tilde{\mathbf{z}}_i$, we have $\overline{F}(\tilde{\mathbf{x}}_l) = \mathbf{R}(\boldsymbol{\rho}_{ln}) [F(\mathbf{g}_l, \mathbf{z}_l \mathbf{d}\tilde{\mathbf{z}}_{i(ln)})] = \mathbf{R}(\boldsymbol{\rho}_{ln}) [F(\mathbf{g}_l, \mathbf{z}_l \sum_{j=1}^{ln} \partial_j \rho_{i(ln)} dz_j)]$, with $\mathbf{R}(\boldsymbol{\rho}_{ln}) = [\hat{\partial}_J \rho_{i(I)}] = [\hat{\partial}_J \rho_{i_1(I)}, \hat{\partial}_J \rho_{i_2(I)}, \dots, \hat{\partial}_J \rho_{i_n(I)}]^T$. The index operator J takes the value of index z_j of arguments in the vector $\overline{F}(\tilde{\mathbf{x}}_l)$ while the index operator I takes values of index of differentials dz_i in each entry of the vector \overline{F} . The index operator $i(I) = (i_1(I), \dots, i_n(I))$ is given by consequent cycling permutations of I . Taking into account the property (2.2), we define the operator

$$\hat{\partial}_J \rho_a = \exp \left(- \sum_{\mathbf{r}_{ln}, \sum_{i=1}^{ln} r_i \geq 1, 1 \leq J \leq ln} r_J \beta_{\mathbf{r}_n}^{(a)}(\zeta)_{ln} \partial_{z_J} \right), \quad (2.6)$$

which contains index operators J as index of a dummy variable ζ_J turning into z_j , $j = 1, \dots, ln$. (2.6) acts on each argument of maps F in the vector \bar{F} . Due to properties of G required above, the action of operators $R(\rho_{ln})$ on $\mathbf{g}_l \in G$ results in a sum of finitely many terms. By using (2.2) and linearity of the mapping \bar{F} , we obtain $\bar{F}(\tilde{\mathbf{x}}_l) = \bar{F}(\mathbf{g}_l, \tilde{\mathbf{z}}_l \mathbf{d}\tilde{z}_l) = [F(\mathbf{g}_l, \mathbf{z}_l \mathbf{d}z_{i(ln)})]$. We then conclude that the vector \bar{F} is invariant, i.e., $\bar{F}(\tilde{\mathbf{x}}_l) = \bar{F}(\mathbf{x}_l)$. Definition 1 of restricted meromorphic functions $\bar{F}(\mathbf{x}_l) \in \Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ consists of two conditions on $\bar{F}(\mathbf{x}_l)$ and $\bar{\nu}_G(\mathbf{z}'_k)$. The first requires existence of positive integers $\beta_m^n(g_i, g_j)$ depending on g_i, g_j only, and the second restricts orders of poles of corresponding sums. The insertions of a sequence of k forms $\bar{\nu}_G(\mathbf{x}')$ which are present in Definition 1 of prescribed rational functions keep elements \bar{F} invariant with respect to coordinate changes. Thus, elements of $\Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ -spaces are invariant under the action of the group $\text{Aut}_{ln+kp} \mathcal{O}_{ln+kp}^{(1)}$. \square

3. COSIMPLICIAL DOUBLE COMPLEX DEFINED FOR A FOLIATION

In this section we introduce spaces for double complexes used to define the restricted meromorphic functions cohomology of a foliation \mathcal{F} of codimension p on a smooth complex manifold M of dimension n .

3.1. Foliation holonomy embeddings and transversal basis. Let us first recall [10] some definitions concerning transversal basis and holonomy embeddings for a foliation \mathcal{F} . A transversal section is an embedded p -dimensional submanifold $U \subset M$ everywhere transverse to all leaves of \mathcal{F} . Suppose α is a path between two points p and \tilde{p} on the same leaf of \mathcal{F} . Let U and \tilde{U} be transversal sections coming through points p and \tilde{p} . Then α determines a transport along the leaves from a neighborhood of p in U to a neighborhood of \tilde{p} in \tilde{U} . Thus it is assumed that there exists a germ of a diffeomorphism $\text{hol}(\alpha) : (U, p) \rightarrow (\tilde{U}, \tilde{p})$ which is called the holonomy of the path α . In the case that the transport above is defined in all of U and embeds U into \tilde{U} , this embedding $h : U \hookrightarrow \tilde{U}$ is called the holonomy embeddings. Now recall the definition of the transversal basis for a foliation \mathcal{F} . A transversal basis for \mathcal{F} is a family \mathcal{U} of transversal sections $U \subset M$ with the property that, if \tilde{U} is any transversal section through a given point $p \in M$, there exists a holonomy embedding $h : U \hookrightarrow \tilde{U}$ with $U \in \mathcal{U}$ and $p \in h(U)$.

3.2. Restricted meromorphic functions associated to a foliation. Suppose M is endowed with a coordinate chart $\mathcal{V} = \{\mathbf{V}_m, m \in \mathbb{Z}\}$. Let \mathbf{p}_l be a set of l points in \mathbf{V}_l . Let us identify l -sets of n -tuples of formal variables \mathbf{z}_l with l sets of n local coordinates on \mathbf{V}_l . Note that according to our construction, M can be infinite-dimensional. Thus, in that case, we consider l infinite sets of complex coordinates. For another k sets of p -tuples of points \mathbf{p}'_k on transversal sections \mathbf{U}_k of the transversal basis \mathcal{U} of foliation \mathcal{F} , we take k sets of p -tuples of $\mathbf{x}'_k = (\mathbf{g}'_k, \mathbf{z}'_k)$, identify formal parameters \mathbf{z}'_k with local coordinates of k points \mathbf{p}'_k , and consider corresponding series $\bar{\nu}_G(\mathbf{x}'_k)$. While the choice of the initial point p'_0 is arbitrary, we assume that other points are related by the holonomy embeddings $\mathbf{h}_k, p'_0 \xrightarrow{h_1} p'_1 \xrightarrow{h_1} \dots \xrightarrow{h_k} p'_k$. It may happened that $z_{1,j}$ coincides with some $z_{p,j}$. In order to work with objects having

coordinate invariant formulation on M we consider restricted meromorphic functions $\bar{F}(\mathbf{x}_l) \in \Theta(G, \mathcal{V}, \mathcal{U})$. In [7], they proved that $\nu_G(x)$ is an invariant object with respect to changes of coordinate in one-dimensional complex case. In Section 6.2 we proved that elements of the space $\Theta(G, \mathbf{V}_l, \mathbf{U}_k)$ and sets of forms $\bar{\nu}_G(\mathbf{x}'_k)$ are invariant with respect to change of coordinates, i.e., to the group of coordinate transformations $\text{Aut } \mathcal{O}^{(ln)}, \tilde{\mathbf{z}}_l \mapsto \mathbf{z}_l$, and corresponding differentials.

3.3. Spaces for cosimplicial double complexes. In [16] the original approach to cohomology of vector fields of manifolds was initiated. We find another approach to cohomology of Lie algebra of vector fields on a manifold in the cosimplicial setup in [12, 41]. Taking into account the standard methods of defining canonical (i.e., independent of the choice of covering \mathcal{U}) cosimplicial object [12, 41] as well as the Čech-de Rham cohomology construction [10] for a foliation, we consider restricted meromorphic functions $F(\mathbf{x}_l)$ with k sets of forms $\bar{\nu}_G(\mathbf{x}'_k)$ and give the following definition of a cosimplicial double complex for G .

For \mathbf{x}_l with local coordinates $\mathbf{z}_l \in \mathcal{V}$, and a transversal basis $\mathcal{U} = \{U_i, i \in I\}$ for \mathcal{F} on M , consider vectors $\bar{F}(\mathbf{x}_l)$ defined in Subsection 2.2. Then let us associate to any subset $\{i_1 < \dots < i_k\}$ of I , the space of restricted meromorphic functions with k sets of $\bar{\nu}_G(\mathbf{x}')$ -forms with local coordinates $\mathbf{z}' \in \mathbf{U}_l$, satisfying Definition 1, and defined on the intersection of transversal sections

$$C_k^l(G, \mathcal{V}, \mathcal{U}, \mathcal{F}) = \Theta \left(G, \mathcal{V}, \bigcap_{U_{i_0} \xrightarrow{h_{i_0}} \dots \xrightarrow{h_{i_{k-1}}} U_{i_k}, i_1 \leq \dots \leq i_k, k \geq 0} U_{i_k} \right), \quad (3.1)$$

where the intersection ranges over all k -tuples of holonomy embeddings \mathbf{h}_k , among transversal sections of foliation in \mathcal{U} . Since we assume that the points \mathbf{p}'_k , are related by holonomies, each consequent arrow h_j in the holonomy sequence in the intersection (3.1) introduces p ν_G -forms in addition to the initial $(j-1)$ sets of forms with formal parameters.

4. PROPERTIES OF THE DOUBLE COMPLEX SPACES $C_k^l(G, \mathcal{V}, \mathcal{U}, \mathcal{F})$

In this section we fix G and \mathcal{F} and omit them and \mathcal{V} from notations where it is possible, and study properties of $C_k^l(\mathcal{U})$ spaces. Let us set $C_k^0(\mathcal{U}) = G$. Then we have

Lemma 2. *The spaces (3.1) are non-zero, and $C_k^l(\mathcal{U}) \subset C_{k-1}^l(\mathcal{U})$.*

Proof. Recall the conditions on k $\bar{\nu}_G(\mathbf{x})$ -forms given in Definition (1). Since exists the lower limit on domain of absolute convergence given in Definition 1, the extension of the tuple of k -homology embeddings in (3.1) by another embedding preserves the conditions applied to the mappings $\bar{F}(\mathbf{x})$ which belong to the spaces (3.1). Thus, (3.1) is non-zero. \square

Lemma 3. *For any $l, p \geq 0$, the construction (3.1) of double complex spaces $C_k^l(\mathcal{U})$ does not depend of the choice of transversal basis \mathcal{U} .*

Proof. Suppose we consider another transversal basis \mathcal{U}' for \mathcal{F} . The of the double complex spaces then assumes that vectors \bar{F} on U'_j are defined on all transversal sections of \mathcal{U}' . According to the definition of the transversal basis given above, for each transversal section U_i which belongs to the original basis \mathcal{U} in (3.1) there exists a holonomy embedding $\tilde{h}_i : U_i \hookrightarrow U'_j$, i.e., it embeds U_i into a section U'_j of our new transversal basis \mathcal{U}' . Then then statement of the proposition follows. \square

In what follows, we omit \mathcal{U} from notations of (3.1). Next, we prove that the construction of spaces (3.1) for the chain-cochain double complex is independent of the choice of coordinates on \mathcal{V} and \mathcal{U} .

Proposition 1. *Elements $\bar{F}(\mathbf{x}_l) \in C_k^l$ with $\mathbf{x}_l \in \mathcal{V}$ and the forms $\bar{\nu}_G(\mathbf{x}'_k)$, with $\mathbf{x}'_k \in \mathcal{U}_k$ of \mathcal{F} are canonical, i.e., independent on changes $(\mathbf{z}_l, \mathbf{z}'_k) \mapsto (\tilde{\mathbf{z}}_l, \tilde{\mathbf{z}}'_k) = \rho_{l+k}(\mathbf{z}_l, \mathbf{z}'_k)$ of local coordinates on \mathcal{V} and \mathcal{U} .*

Proof. In Lemma 1 we proved that elements of $\bar{F}(\mathbf{x}_l) \in \Theta(G, \mathcal{V}, \mathcal{U})$ are coordinate change-invariant. The construction of the double complex spaces (3.1) assumes that $\bar{F}(\mathbf{x}_l) \in C_k^l$ satisfies conditions of definition of a Θ -space and with extra conditions of Definition 1 on k sets of p $\bar{\nu}_G(\mathbf{x}_k)$ -forms. In [7] they proved in one-dimensional complex case, that the form $\nu_g(x)$ containing the wt (g_i) -power of the differential dz is invariant with respect to the action of the group $\text{Aut } \mathcal{O}^{(1)}$. Here we prove that $\bar{\nu}_G(\mathbf{x}'_k)$ are invariant with respect to the change of k sets of p local coordinates $\mathbf{z}'_p \mapsto \tilde{\mathbf{z}}'_p(\mathbf{z}'_p)$ on a transversal section of \mathcal{F} . Let \mathbf{z}' be coordinates on a coordinate chart around a point p' on a transversal section $U \in \mathcal{U}$. Define a wt (g'_i) -differential on coordinate chart around p' with values in $\text{End}(G)_{\mathbf{z}'}$ as follows: identify $\text{End } G_{\mathbf{z}'}$ with $\text{End } G_{\mathbf{z}'}$ using the coordinates \mathbf{z}' . Let $\tilde{\mathbf{z}}' = (\rho)_p(\mathbf{z}')$, be another k -set of coordinates on a p -dimensional coordinates on transversal sections. Let us express the set of wt (g'_i) -differentials on $D_{p'}^{(p') \times} \nu_G(g'_i, \tilde{z}'_i)$, $i = 1, \dots, p$, in terms of of the coordinates \mathbf{z}' . We would like to show that it coincides with the set of wt (g'_i) -differentials $\bar{\nu}_G(\mathbf{z}')$. We will use the notion of torsors in order to prove the independence of formal series operators multiplied by some power of differentials for for elements $g \in G$ of wt $(g) \in \mathbb{Z}_+$ such that $\mathbf{z}'^{m+1} \partial_{\mathbf{z}'} g = 0$, for $m > 0$. The general case then follows. Consider a vector (g'_i, \mathbf{z}') $\in G_{\mathbf{z}'}$ with $g'_i \in G$. Then the same vector equals $(R_i^{-1}(\rho_p) g'_i, \tilde{\mathbf{z}}')$, i.e., it is identified with $R_i^{-1}(\rho_p) g'_i \in G$, using the coordinates $\tilde{\mathbf{z}}'$. Here $R_i(\rho_p)$ is an operator representing transformation of $\mathbf{z}' \rightarrow \tilde{\mathbf{z}}'$, as an action on G . Therefore if we have an operator on $G_{\mathbf{z}'}$ which is equal to a $\text{Aut } \mathcal{O}^{(p)}$ -torsor S under the identification $\text{End } G_{\mathbf{z}'} \in \text{End } G$ using the coordinates $\tilde{\mathbf{z}}'$, then this operator equals $R_i(\rho_p) S R_i^{-1}(\rho_p)$, under the identification $\text{End } G_{\mathbf{z}'} \in \text{End } G^{(i)}$ using the coordinates (g'_i, \mathbf{z}') . Thus, in terms of the coordinates (g'_i, \mathbf{z}') , the differential $\nu_G(g'_i, \tilde{z}'_i)$ becomes $\nu_G(g'_i, z'_i) = R_i(\rho) \nu_G(g'_i, \rho(\mathbf{z}')) R_i^{-1}(\rho)$. According to Definition (3.1), elements $\mathcal{F}(\mathbf{x}_l)$ satisfy conditions of Definition 1 with the number kp of $\bar{\nu}_g(\mathbf{z}'_k)$ forms. Thus we see that \mathcal{F} is a canonical object suitable for C_k^l . \square

5. CHAIN-COCHAIN OPERATORS

Let us fix G and \mathcal{F} and skip them from notations. Recall notations provided in Section 2. Denote by $\mathbf{x}_{l, \widehat{i_1}, \dots, \widehat{i_m}}$, $1 \leq i_1 \leq \dots \leq i_m \leq l$, the set \mathbf{x} with $(\mathbf{x}_{i_1}, \dots, \mathbf{x}_{i_m})$ tuples of \mathbf{x} being omitted. For $\overline{F} \in C_k^l$, let us define the operator D_k^l by

$$\begin{aligned} D_k^l \overline{F}(\mathbf{x}_l) &= T_1(\overline{\nu}_G(\mathbf{x}_1)) \cdot \overline{F}(\mathbf{x}_{l+1, \widehat{1}}) \\ &+ \sum_{i=1}^l (-1)^i T_i(\overline{\nu}_G(\mathbf{x}_i)) T_{i+1}(\overline{\nu}_G(\mathbf{x}_{i+1})) \cdot \overline{F}(\mathbf{x}_{l+1, \widehat{i}, \widehat{i+1}}) \\ &+ (-1)^{l+1} T_1(\overline{\nu}_G(\mathbf{x}_{l+1})) \cdot \overline{F}(\mathbf{x}_{\widehat{l+1}}), \end{aligned} \quad (5.1)$$

where $\overline{\nu}_G(\mathbf{x}_m) = [\nu_G(x_{m,1}), \dots, \nu_G(x_{m,n})]^T$, and $T_i(\gamma) \cdot \overline{F}(\mathbf{x}_{l+1})$ denotes insertion of γ at i -th position of $\overline{F}(\mathbf{x}_{l+1})$. Next, we have

Proposition 2. *The operator (5.1) forms the double complex $D_k^l : C_k^l \rightarrow C_{k-1}^{l+1}$ on the spaces (3.1) (when lower index is zero the sequence terminates) and $D_{k-1}^{l+1} \circ D_k^l = 0$.*

Proof. Note that $(l+1)n$ formal variables \mathbf{z}_{l+1} in the Definition (3.1) are identified with coordinates of $l+1$ arbitrary points on $\mathcal{V} \subset M$ not related to coordinates on transversal sections. By Proposition 2.8 of [26], $D_k^l \overline{F}(\mathbf{x}_l)$ satisfies Definition (1) for $(k-1)$ $\overline{\nu}_G(\mathbf{x}'_{k-1})$ -forms and has the T_G -derivative (2.1) property and the K_G -conjugation (2.2) properties according of Subsection 3.2. So $D_k^l \overline{F}(\mathbf{x}_l) \in C_{k-1}^{l+1}$ and D_k^l is indeed a map whose image is in C_{k-1}^{l+1} . In [26] we find the construction of double chain-cochain complex for $n=1$ case and various $l \geq 0$. In particular, (c.f. Proposition 4.1), the chain condition for such double complex was proven. Consider now $D_k^l \overline{F}(\mathbf{x}_{l+1})$. By construction of the coboundary operator in each component of the \overline{F} a $n=1$ case of the action of D_k^l is realized. Thus, according to Proposition 4.1 of [26], each component of $D_{k-1}^{l+1} \circ D_k^l$ vanishes. \square

According to Proposition 2 one defines the (l, k) -th restricted meromorphic function cosimplicial cohomology $H_{k, \text{cos}}^l(G, \mathcal{F})$ of a foliation \mathcal{F} of M to be $H_{k, \text{cos}}^l(G, \mathcal{F}) = \text{Ker } D_k^l / \text{Im } D_{k+1}^{l-1}$.

 6. THE MULTIPLICATION ON PAIRS OF DOUBLE COMPLEX SPACES $C_k^l(G, \mathcal{V}, \mathcal{F})$

In order to introduce and study simplest invariants associated to the cohomology of a foliation of codimension p , we first have to define a multiplication among elements of double complex spaces $C_k^l(G, \mathcal{V}, \mathcal{F})$ for various l and k . In this Section we fix $G, \mathcal{F}, \mathcal{V}$ and \mathcal{U} , and skip them from further notations. The multiplication of two $\Theta(G, \mathcal{V}, \mathcal{U})$ spaces is associated with a sum of multiplications of restricted meromorphic functions over a basis in G , and powers of a complex parameter for the multiplication. The formal parameters \mathbf{z}_{l_i} , $i=1, 2$ are identified with local coordinates of l_i points \mathbf{p}_{l_i} on M . Some r points among \mathbf{p}_{l_1} may coincide with points among \mathbf{p}_{l_2} . Similarly, on transversal section, some t points may coincide \mathbf{p}'_{l_i} , $i=1, 2$. In that case we keep only one from each pair of coinciding parameters.

6.1. The multiplication of restricted meromorphic functions. The structure of the space of restricted meromorphic functions $\Theta(G, \mathcal{V}, \mathcal{U})$ is quite complicated and it is difficult to introduce algebraically a multiplication among two such spaces. As it follows from definition of the configuration space $F_{ln}\mathbb{C}$ in Subsection 3.2, in the case of coincidence of two formal parameters they are excluded from $F_{ln}\mathbb{C}$. Thus, we require that the set of formal parameters $(\tilde{\mathbf{z}}_{l_1+l_2-r})$ would belong to $F_{l_1+l_2-r}\mathbb{C}$. This leads to the fall off of the total number of formal parameters for $\Theta(l_1 + l_2 - r, k_1 + k_2 - t)$. For elements $A, B \in G$ the multiplication is associated to the sum $\Omega(A, B) = \sum_{m \in \mathbb{Z}} \lambda^m \sum_{u \in G_m} (\vartheta, Au) (\bar{u}, B)$, where $\bar{u} \in \tilde{G}$. The sum is taken over any $G_{(m)}$ -basis $\{u\}_m$, where \bar{u} is the dual of u with respect to a non-degenerate bilinear pairing (\cdot, \cdot) on G . By the standard reasoning $\Omega(A, B)$ does not depend on the choice of a basis of $\{u\}_m$ of $G_{(m)}$, $m \in \mathbb{Z}$.

For $\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2} \in G$, $\mathbf{z}_{l_1}, \tilde{\mathbf{z}}_{l_2} \in F_{(l_1+l_2-r)n}\mathbb{C}$, $i = 1, 2$, and $\bar{F}(\mathbf{x}_{l_1}) \in \Theta(l_1, k_1)$, $\bar{F}(\tilde{\mathbf{x}}_{l_2}) \in \Theta(l_2, k_2)$, introduce the multiplication

$$* : \Theta(l_1, k_1) \times \Theta(l_2, k_2) \rightarrow \Theta(l_1 + l_2 - r, k_1 + k_2 - t), \quad (6.1)$$

$$\bar{F}(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2}; \lambda) = \Omega(\nu_G(\bar{F}(\mathbf{x}_{l_1}), \zeta_1), \nu_G(\bar{F}(\tilde{\mathbf{x}}_{l_2}), \zeta_2)), \quad (6.2)$$

as an absolutely converging function in λ on the configuration space $F\mathbb{C}_{(l_1+l_2-r)n}$ of $(\mathbf{z}_{l_1}, \tilde{\mathbf{z}}_{l_2})$ with only possible poles at $z_i = z_{i'}$, $\tilde{z}_j = \tilde{z}_{j'}$, and $z_i = \tilde{z}_j$, $1 \leq i, i' \leq l_1$, $1 \leq j, j' \leq l_2 - r$, with excluded \hat{z}_{l_j} , and parametrized by $\zeta_1, \zeta_2 \in \mathbb{C}$, with all monomials $(z_{i_l} - z_{j_l})$, $1 \leq l \leq r$, excluded from (6.2). We will omit λ from further notations for $\bar{F}(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2}; \lambda)$.

6.2. Properties of *-multiplication of Θ -spaces. In this subsection we study properties of the multiplication $F(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2})$ given by (6.2). We define the action of differentiation $\partial_l = \partial_{z_l} = \partial/\partial z_l$, $1 \leq l \leq l_1 + l_2 - r$, of $\bar{F}(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2})$ with respect to the l -th entry of $(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2})$ as follows $\partial_l \bar{F}(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2}) = \Omega(\partial_{x_i}^{\delta_{l,i}} \nu_G(\bar{F}(\mathbf{x}_{l_1}), \zeta_1), \partial_{\tilde{x}_j}^{\delta_{l,j}} \nu_G(\bar{F}(\tilde{\mathbf{x}}_{l_2}), \zeta_2))$, for $j \in [1, \dots, l_i, \dots, l_j, \dots, n]$. It is elementary to check the following

Lemma 4. *The multiplication (6.2) satisfies the T_G -derivative (2.2) and K_G -conjugation (2.4). \square*

We then define the action of an element $\sigma \in S_{k+n}$ on the multiplication of $\bar{F}(\mathbf{x}_{l_i}) \in \Theta(l_i, k_i)$, $i = 1, 2$, as

$$\sigma(\bar{F})(\mathbf{x}_{l_1}, \tilde{\mathbf{x}}_{l_2}) = \bar{F}(\mathbf{x}_{(\sigma(l_1))}, \tilde{\mathbf{x}}_{(\sigma(l_2))}) = \Omega(\nu_G(\bar{F}(\mathbf{x}_{(\sigma(l_1))}), \zeta_1), \nu_G(\bar{F}(\tilde{\mathbf{x}}_{(\sigma(l_2))}), \zeta_2)). \quad (6.3)$$

6.3. The multiplication on pairs of double complex spaces C_k^l . Here we consider an application of the material of subsection 6.1 to the double complex spaces C_k^l (3.1) described in previous subsections. Let us introduce the multiplication of elements of two double complex spaces $C_{k_i}^{l_i}$, $i = 1, 2$, associated with the same foliation of codimension p , with the image in another double complex space $C_{k_1+k_2-t}^{l_1+l_2-r}$. We assume the same \mathcal{V} and \mathcal{U} for these spaces. This multiplication is coherent with

respect to the original differential (5.1), and the symmetry property (2.5). We prove the canonicity of the multiplication, and derive an analogue of Leibniz formula.

For $\bar{F}(\mathbf{x}_{l_1}) \in C_{k_1}^{l_1}$, and $\bar{F}(\tilde{\mathbf{x}}_{l_2}) \in C_{k_2}^{l_2}$, the multiplication $\bar{F}(\mathbf{x}_{l_1}) * \bar{F}(\tilde{\mathbf{x}}_{l_2}) \mapsto \bar{F}(\tilde{\mathbf{x}}_{l_1+l_2-r})$, is a $\Theta(l_1 + l_2 - r, k_1 + k_1 - t)$ -valued restricted meromorphic function

$$\bar{F}(\tilde{\mathbf{x}}_{l_1+l_2-r}) = \bar{F}(\mathbf{x}_{l_1}) * \bar{F}(\tilde{\mathbf{x}}_{l_2}) = \Omega(\nu_G(\bar{F}(\mathbf{x}_{l_1}), \zeta_1), \nu_G(\bar{F}(\tilde{\mathbf{x}}_{l_2}), \zeta_2)), \quad (6.4)$$

induced by (6.2). The action of $\sigma \in S_{l_1+l_2}$ on the multiplication $\bar{F}(\mathbf{x}_{l_2}, \tilde{\mathbf{x}}_{l_2})$ (6.3) is given as in Section 2. Since the multiplication (6.2) defined for two Θ -spaces induces the multiplication (6.4) with increased number of \mathbf{x} -variables and $\bar{\nu}_G(\mathbf{x}')$ -forms, we then obtain the main proposition of this section

Proposition 3. *For $\bar{F}(\mathbf{x}_{l_1}) \in C_{k_1}^{l_1}$ and $\bar{F}(\tilde{\mathbf{x}}_{l_2}) \in C_{k_2}^{l_2}$, the multiplication $\bar{F}(\mathbf{x}_{l_2}, \tilde{\mathbf{x}}_{l_2})$ (6.3) belongs to the space $C_{k_1+k_2-t}^{l_1+l_2-r}$, i.e., $*$: $C_{k_1}^{l_1} \times C_{k_2}^{l_2} \rightarrow C_{k_1+k_2-t}^{l_1+l_2-r}$.*

6.4. Coboundary operator acting on the multiplication space. Since the result of multiplication (6.3) of elements of the spaces $C_{k_i}^{l_i}$, $i = 1, 2$, belongs to $C_{k_1+k_2-t}^{l_1+l_2-r}$, thus the multiplication admits the action of $D_{k_1+k_2-t}^{l_1+l_2-r}$ defined in (5.1). The coboundary operator (5.1) possesses a version of Leibniz law with respect to the multiplication (6.3). Indeed, by elementary computation we get

Lemma 5. *For $\bar{F}(\mathbf{x}_{l_i}) \in C_{k_i}^{l_i}$, $i = 1, 2$, the action of $D_{k_1+k_2}^{l_1+l_2}$ on the multiplication is given by $D_{k_1+k_2}^{l_1+l_2}(\bar{F}(\mathbf{x}_{l_1}) * \bar{F}(\tilde{\mathbf{x}}_{l_2})) = (D_{k_1}^{l_1} \bar{F}(\mathbf{x}_{l_1})) * \bar{F}(\tilde{\mathbf{x}}_{l_2}) + (-1)^{l_1} \bar{F}(\mathbf{x}_{l_1}) * (D_{k_2}^{l_2} \bar{F}(\tilde{\mathbf{x}}_{l_2}))$. \square*

Remark 1. Checking (5.1) we see that an extra arbitrary element $g_{l_2+1} \in G$, as well as corresponding extra arbitrary formal parameter z_{l_2+1} appear as a result of the action of $D_{k_1}^{l_2}$ on $\bar{F}(\mathbf{x}_{l_2}) \in C_{k_2}^{l_2}$ mapping it to $C_{k_1-1}^{l_2+1}$.

6.5. Relations to Čech-de Rham cohomology in Crainic–Moerdijk construction. Recall the construction of the Čech-de Rham cohomology by Crainic and Moerdijk [10] for a foliation \mathcal{F} of codimension p on a smooth manifold M . Let \mathcal{U} be a transversal basis for \mathcal{F} . Consider the double complex $C^{k,l} = \prod_{U_0 \xrightarrow{h_1} \dots \xrightarrow{h_p} U_k} \omega^l(U_0)$, where the multiplication ranges over all k -tuples of holonomy embeddings between transversal sections from a fixed transversal basis \mathcal{U} . The vertical differential is defined as $(-1)^k d : C^{k,l} \rightarrow C^{k,l+1}$, where d is the ordinary de Rham differential. The horizontal differential $\delta : C^{k,l} \rightarrow C^{k+1,l}$, is given by $\delta = \sum_{i=1}^k (-1)^i \delta_i$, where $\delta_i \omega(\mathbf{h}_{k+1}) = \delta_{i,0} h_1^* \omega(\mathbf{h}_{(2,k+1)}) + (1 - \delta_{i,0} - \delta_{i,k+1}) \omega(\mathbf{h}_{(1,i-1)}, h_{i+1} h_i, \mathbf{h}_{(i+2,k+1)}) + \delta_{i,k+1} \omega(\mathbf{h}_k)$, expressed in terms of differential forms. This double complex constitutes a bigraded differential algebra endowed with a natural multiplication $(\omega \eta)(\mathbf{h}_{k+k'}) = (-1)^{kk'} \omega(\mathbf{h}_k) (\mathbf{h}_k^* \cdot \eta(\mathbf{h}_{(k+1,k+k')}))$, for $\omega \in C^{k,l}$ and $\eta \in C^{k',l'}$, thus $(\omega \cdot \eta)(\mathbf{h}_{k+k'}) \in C^{k+k',l+l'}$. The cohomology of this complex is called the Čech-de Rham cohomology of the leaf space M/\mathcal{F} with respect to the transversal basis \mathcal{U} . The Čern-de

Rham cohomology of a foliated smooth manifold introduced in [10] results from restricted meromorphic function cohomology introduced in this paper. Indeed, it can be seen by making the following associations: $h_i \sim g_i$, $i = 1, \dots, ln$, $\omega(\mathbf{h}_{l,n}) \sim F(\mathbf{x}_l)$, $h^*(h_1) \dots h^*(h_{l,n})(\mathbf{z}_{ln}) \sim \omega_G(\mathbf{z}_{l,n})$.

7. EXAMPLE: CODIMENSION ONE INVARIANTS

In this section, using the double complex construction of Sections 3–5, we provide a generalization of the Godbillon–Vey invariant [18] for codimension one foliations. We call a map $\bar{F} \in C_k^l$ closed if $D_k^l \bar{F} = 0$. For $k \geq 1$. It is exact if there exists $\bar{F} \in C_{k-1}^{l+1}$ such that $\bar{F} = D_k^l \bar{F}$. Taking into account the correspondence with the Čech-de Rham complex due to [10], we reformulate the derivation of Godbillon–Vey invariant in restricted meromorphic functions terms. For $\bar{F} \in C_k^l$ we call the cohomology class of mappings $[\bar{F}]$ the set of all closed forms that differs from \bar{F} by an exact mapping, i.e., for $\bar{F}_1 \in C_{k+1}^{l-1}$, $[\bar{F}] = \bar{F} + D_{k+1}^{l-1} \bar{F}_1$, assuming that both parts of the last formula belongs to the same space C_k^l .

For a three-dimensional smooth complex manifold, consider a codimension one foliation \mathcal{F} . Following the construction of Definition (3.1), we take k -tuples of one-dimensional transversal sections. For $u_j \in G$, $w_j \in U_j$, for each section we attach the form $\nu_G(x_j)$, $x_j = (u_j, w_j)$. We then work with mappings $\varphi \in C_k^{3l}$. As in the setup of differential forms, a mapping $\varphi \in C_k^{3l}$ defines a codimension one foliation. As we see from (6.4) and definition of the action of the derivative, it satisfies properties similar to differential forms. The integrability condition for mappings that belong to C_k^l has the form: $\bar{F}_0 * D_k^l \bar{F}_0 = 0$. It results with the Frobenius theorem, i.e., that there exist $\bar{F}_2 \in C_{k'}^{l'}$, such that

$$D_k^l \bar{F}_0 = \bar{F}_0 * \bar{F}_2, \quad (7.1)$$

which uniquely determines a foliation with parameters of ν_G -forms satisfying Definition 1 conditions. Since both sides of (7.1) are in the same double complex space let us apply limitations to possible combinations of (l, k) and (l', k') . In the following proposition we show that the integrability condition delivers a generalization of Godbillon–Vey invariant.

Proposition 4. *The cohomology classes $[(D_k^l \Delta) * \Delta]$, for $\Delta = \bar{F}, \bar{F}_1, \bar{F}'$, and combinations $(l, k) = (1, 2), (0, 3), (1, t)$ correspondingly, with non-vanishing $((D_k^l \Delta) * \Delta)$, are independent on the choice of Δ .*

Remark 2. The case $t = 1$ corresponds to the classical Godbillon–Vey invariant.

Proof. Consider two maps $\bar{F}(g_1) \in C_2^1$ and $\bar{F}_1 \in C_3^0$. The ortogonality condition has the form

$$\bar{F} * D_3^0 \bar{F}_1 = 0. \quad (7.2)$$

Then there exists $\bar{F}'(g_2) \in C_m^n$ such that

$$D_3^0 \chi = \bar{F} * \bar{F}', \quad (7.3)$$

and $1 = 1 + n - r$, $2 = 2 + m - t$, i.e., $n = r$, which results in $r = 1$; $m = t$, $0 \leq t \leq 2$, i.e., $\bar{F}' \in C_t^1$. Here r and t are numbers of common ν_G -forms. Consider now (7.2). Taking into account definition of the derivative action we obtain

$$D_{4-t'}^{2-r'}(\bar{F}' * D_3^0 \bar{F}_1) = (D_2^1 \bar{F}) * D_3^0 \bar{F}_1 + \bar{F} * D_2^1 D_3^0 \bar{F}_1 = (D_2^1 \bar{F}) * D_3^0 \bar{F}_1 = (D_2^1 \bar{F}) * \bar{F} * \bar{F}'.$$

Thus $0 = D_{3-t'}^{3-r'} D_{4-t'}^{2-r'}(\bar{F} * D_3^0 \bar{F}_1) = D_{3-t'}^{3-r'}((D_2^1 \bar{F}) * \bar{F} * \bar{F}')$, and $((D_2^1 \bar{F}) * \bar{F} * \bar{F}')$ is closed. From (7.2) it follows that $0 = D_2^1 \bar{F} * D_3^0 \bar{F}_1 - \bar{F} * D_2^1 D_3^0 \bar{F}_1 = (\bar{F} * D_3^0 \bar{F}_1)$. Thus $D_2^1 \bar{F} * D_3^0 \bar{F}_1 = D_2^1 \bar{F} * \bar{F} * \bar{F}' = 0$. Consider (7.3). Acting by D_2^1 and substituting back we obtain $0 = D_2^1 D_3^0 \bar{F}_1 = D_2^1(\bar{F} * \bar{F}') = D_2^1(\bar{F}) * \bar{F}' - \bar{F} * D_t^1 \bar{F}'$. thus $D_2^1(\bar{F}) * \bar{F}' = \bar{F} * D_t^1 \bar{F}'$. The last equality trivializes on applying D_{t+1}^3 to both sides.

Let us show now the non-vanishing property of $((D_2^1 \bar{F}) * \bar{F})$. Indeed, suppose $(D_2^1 \bar{F}) * \bar{F} = 0$. Then there exists $\gamma \in C_m^n$, such that $D_2^1 \bar{F} = \bar{F}_3 * \bar{F}$. Both sides of the last equality should belong to the same double complex space but one can see that it is not possible. Thus, $(D_2^1 \bar{F}) * \bar{F}$ is non-vanishing. One proves in the same way that $(D_3^0 \bar{F}_1) * \bar{F}_1$ and $(D_t^1 \bar{F}') * \bar{F}'$ do not vanish too. Now let us show that $[(D_2^1 \bar{F}) * \bar{F}]$ is invariant, i.e., it does not depend on the choice of $\bar{F} \in C_2^1$. Substitute \bar{F} by $(\bar{F} + \bar{F}_4) \in C_2^1$. We have

$$\begin{aligned} (D_2^1(\bar{F} + \bar{F}_4)) * (\bar{F} + \bar{F}_4) &= (D_2^1 \bar{F}) * \bar{F} + ((D_2^1 \bar{F}) * \bar{F}_4 - \bar{F} * D_2^1 \bar{F}_4) \\ &\quad + (\bar{F} * D_2^1 \bar{F}_4 + D_2^1 \bar{F}_4 * \bar{F}) + (D_2^1 \bar{F}_4) * \bar{F}_4. \end{aligned} \quad (7.4)$$

Since $(\bar{F} * D_2^1 \bar{F}_4 + (D_2^1 \bar{F}_4) * \bar{F}) = \bar{F} D_2^1 \bar{F}_4 - (D_2^1 \bar{F}_4) \bar{F} + (D_2^1 \bar{F}_4) \bar{F} - \bar{F} D_2^1 \bar{F}_4 = 0$, then (7.4) represents the same cohomology class $[(D_2^1 \bar{F}) * \bar{F}]$. The same holds for $[(D_3^0 \bar{F}_1) * \bar{F}_1]$, and $[(D_t^1 \bar{F}') * \bar{F}']$. \square

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