Numerical solution of a dumbbell-based model for dilute polymer solutions

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joint with
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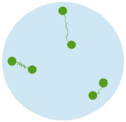
May 7, 2018

A dumbbell model

Dilute polymer solutions: a dumbbell model

- polymer molecules surrounded by Newtonian fluid
- no interactions between molecules
- polymer molecules modeled as dumbbells



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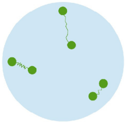

The Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = \nu\Delta_x \mathbf{u} + \mathrm{div}_x\, \mathbf{T} - \nabla_x p, \quad \mathrm{div}_x\, \mathbf{u} = 0 \qquad \text{in } (0,T)\times\Omega$$

$$\mathbf{u} = \mathbf{0} \qquad \text{on } (0,T)\times\partial\Omega$$

$$\mathbf{u}(0) = \mathbf{u}_0 \qquad \text{in } \Omega$$

$$\mathbf{T} = \gamma \int_{\mathbb{R}^d} (\mathbf{R}\otimes\mathbf{R})\psi \, d\mathbf{R} - \mathbf{I} \quad \text{(Kramer's expression)} \qquad \text{in } (0,T)\times\Omega$$

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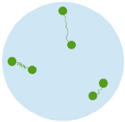

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The Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi + \mathrm{div}_R \left(\nabla_x \mathbf{u} \cdot \mathbf{R}\psi \right) = \chi \Delta_R \psi + \mathrm{div}_R \left(\mathbf{F}(\mathbf{R})\psi \right) + \epsilon \Delta_x \psi$$

Linear vs. nonlinear spring force

Hooke's law: $\mathbf{F}(\mathbf{R}) = H\mathbf{R}$, $H > 0$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi + \mathsf{div}_R \left(\nabla_x \mathbf{u} \cdot \mathbf{R}\psi\right) = \Delta_R\, \psi + \mathsf{div}_R \left(H\mathbf{R}\psi\right) + \epsilon\Delta_x\psi$$

$\approx$ kinetic Hookean model (γ, χ, ξ are constants)

- J.W. Barrett, E. Süli: *Existence of global weak solutions to the kinetic Hookean dumbbell model for incompressible dilute polymeric fluids*, Nonlinear Anal.-Real. (2017)

Linear vs. nonlinear spring force

Hooke's law: $\mathbf{F}(\mathbf{R}) = H\mathbf{R},\ H > 0$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u}\cdot\nabla_x)\psi + \mathrm{div}_R\left(\nabla_x\mathbf{u}\cdot\mathbf{R}\psi\right) = \Delta_R\,\psi + \mathrm{div}_R\left(H\mathbf{R}\psi\right) + \epsilon\Delta_x\psi$$

$\approx$ kinetic Hookean model ($\gamma,\ \chi,\ \xi$ are constants)

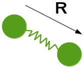
nonlinear spring law: $\mathbf{F}(\mathbf{R}) = \xi(|\mathbf{R}|^2)\mathbf{R}$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u}\cdot\nabla_x)\psi + \mathrm{div}_R\left(\nabla_x\mathbf{u}\cdot\mathbf{R}\psi\right) = \chi(|\mathbf{R}|^2)\Delta_R\,\psi + \mathrm{div}_R\left(\xi(|\mathbf{R}|^2)\mathbf{R}\psi\right) + \epsilon\Delta_x\psi$$

+ Peterlin approximation

length of the spring is replaced by the average length

$f(|\mathbf{R}|^2) \mapsto f(\langle|\mathbf{R}|^2\rangle) = f(\mathrm{tr}\,\mathbf{C})$



$$\mathrm{tr}\,\mathbf{C}(\psi) = \langle|\mathbf{R}|^2\rangle := \int_{\mathbb{R}^d} |\mathbf{R}|^2\psi(t,x,\mathbf{R})\,\mathrm{d}\mathbf{R}$$

$$\frac{\partial \psi}{\partial t} + (\mathbf{u}\cdot\nabla_x)\psi + \mathrm{div}_R\left(\nabla_x\mathbf{u}\cdot\mathbf{R}\psi\right) = \chi(\mathrm{tr}\,\mathbf{C})\Delta_R\,\psi + \mathrm{div}_R\left(\xi(\mathrm{tr}\,\mathbf{C})\mathbf{R}\psi\right) + \epsilon\Delta_x\psi$$

Linear vs. nonlinear spring force

Hooke's law: $\mathbf{F}(\mathbf{R}) = H\mathbf{R},\ H > 0$

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$\approx$ kinetic Peterlin model ($\gamma,\ \chi,\ \xi$ functions of $\mathrm{tr}\,\mathbf{C}$)

- P. Gwiazda, M. Lukáčová-Medviďová, H. Mizerová, A. Świerczewska-Gwiazda: *Existence of global weak solutions to the kinetic Peterlin model,* arXiv (2017)

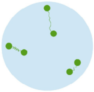
$$\boxed{\chi = \xi}$$

Multiscale model

The Navier-Stokes-Fokker-Planck system

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = \nu\Delta_x \mathbf{u} + \mathrm{div}_x\, \mathbf{T} - \nabla_x p, \quad \mathrm{div}_x\, \mathbf{u} = 0$$

$$\mathbf{T} = \gamma \mathbf{C}(\psi) - \mathbf{I}$$



Boundary and initial conditions: $\mathbf{u} = \mathbf{0}$ on $(0,T)\times\partial\Omega$, $\mathbf{u}(0) = \mathbf{u}_0$ in Ω

$$\frac{\partial \psi}{\partial t} + (\mathbf{u}\cdot\nabla_x)\psi + \mathrm{div}_R\left(\nabla_x \mathbf{u}\cdot\mathbf{R}\psi\right) = \chi\Delta_R\,\psi + \mathrm{div}_R\left(\xi\mathbf{R}\psi\right) + \epsilon\Delta_x\psi$$

Decay/boundary conditions: $\psi \to 0$ as $|\mathbf{R}| \mapsto \infty$ in $(0,T)\times\Omega$,
$\frac{\partial\psi}{\partial\mathbf{n}} = 0$ on $(0,T)\times\partial\Omega\times\mathbb{R}^d$,
and initial condition: $\psi(0) = \psi_0$ in $\Omega\times\mathbb{R}^d$

physical space: $\mathbf{x}\in\Omega\subset\mathbb{R}^d$ configuration space: $\mathbf{R}\in\mathcal{D}=\mathbb{R}^d$

Numerical approximation

Macroscopic solvent: Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = \nu\Delta_x \mathbf{u} + \mathsf{div}_x\, \mathbf{T} - \nabla_x p, \quad \mathsf{div}_x\, \mathbf{u} = 0$$

Stabilized Lagrange-Galerkin method

Conforming finite element approximation:	continuous piecewise linear finite elements
Method of characteristics:	discretization of the material derivative
Pressure-stabilization:	the Brezzi-Pitkäranta stabilization

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X^n}{\Delta t}, \mathbf{v}_h\right) = -2\nu\left(\mathrm{D}(\mathbf{u}_h^n), \mathrm{D}(\mathbf{v}_h)\right) + (\mathsf{div}\,\mathbf{v}_h, p_h^n) - (\mathsf{div}\,\mathbf{u}_h^n, q_h) + \\ - \delta_0 \sum_K h_K^2 \left(\nabla p_h^n, \nabla q_h\right)_K - (\mathsf{tr}\,\mathbf{T}_h^n, \nabla \mathbf{v}_h)$$

Numerical approximation

Molecular part: Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi - \epsilon \Delta_x \psi = -\mathsf{div}_R\,(\nabla_x \mathbf{u} \cdot \mathbf{R}\psi) + \chi \Delta_R\, \psi + \mathsf{div}_R\, \Big(\xi \mathbf{R}\psi\Big)$$

Space splitting + Hermite spectral method:

→ **configuration space ($\mathcal{D} = \mathbb{R}^2$):** $\quad \dfrac{\partial \psi}{\partial t} + \mathsf{div}_R\,(\nabla_x \mathbf{u} \cdot \mathbf{R}\psi) - \chi \Delta_R\, \psi - \mathsf{div}_R\,(\xi \mathbf{R}\psi) = 0$

→ **physical space ($\Omega \subset \mathbb{R}^2$):** $\quad \dfrac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi - \epsilon \Delta_x \psi = 0$

$$\psi(t, \mathbf{x}, \mathbf{R}) = \sum_{z,k=0}^{N} \phi_{zk}(t, \mathbf{x}) \tilde{H}_z(r_1) \tilde{H}_k(r_2),\ \mathbf{R} = (r_1, r_2)$$

$$\tilde{H}_n(r) = \frac{\omega_\alpha^{-1}(r)}{\sqrt{2^n n!}} H_n(\alpha r), \quad \omega_\alpha(r) = e^{\alpha^2 r^2}, \quad H_n(r) = (-1)^n e^{r^2} \partial_r^n (e^{-r^2}), \quad r \in \mathbb{R}$$

$$\mathcal{D}_N = \Big\{ \mathbf{R}_{ij} = (r_{1,i}, r_{2,j}),\ i, j = 0, 1, \cdots, N;\ H_{N+1}(r_{1,i}) = H_{N+1}(r_{2,j}) = 0 \Big\}$$

Numerical approximation

Molecular part: Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + (\mathbf{u} \cdot \nabla_x)\psi - \epsilon \Delta_x \psi = -\mathsf{div}_R\,(\nabla_x \mathbf{u} \cdot \mathbf{R}\psi) + \chi \Delta_R\,\psi + \mathsf{div}_R\,\Big(\xi \mathbf{R}\psi\Big)$$

Space splitting + Hermite spectral method:

Finite difference:

$$\frac{\phi^{*}_{zk} - \phi^{n-1}_{zk}}{\Delta t} = \mathcal{L}(\phi^{*}_{zk})$$

Lagrange-Galerkin method:

$$\left(\frac{\phi^{n}_{zk} - \phi^{*}_{zk} \circ X^{n}}{\Delta t}, \varphi_h\right) + \epsilon\,(\nabla_x \phi^{n}_{zk}, \nabla_x \varphi_h) = 0$$

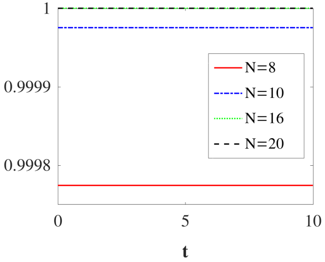
- H. Mizerová, B. She : *Multiscale simulation of dilute polymer solutions*, preprint (2017)

Conservation of discrete mass

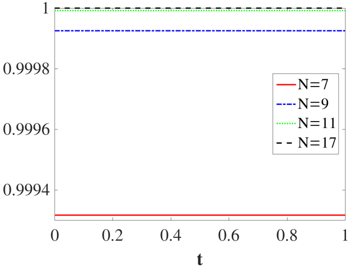
Theorem

Let $\psi_{h,N}$ be the numerical solution of the NSFP system,
and let the initial probability density satisfy $\psi(0, \mathbf{x}, \mathbf{R}) = \psi^0(\mathbf{R})$.
Then, for any n, it holds that

$$\int_{\mathcal{D}} \psi_{h,N}^{n}(\mathbf{R})\, d\mathbf{R} = \int_{\mathcal{D}} \psi_{h,N}^{0}(\mathbf{R})\, d\mathbf{R}.$$



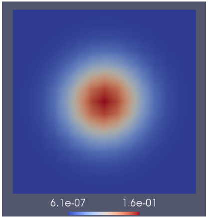
(a) FP solver: shear flow



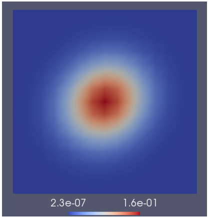
(b) NSFP solver: Poiseulle flow

Experiment 1: shear flow

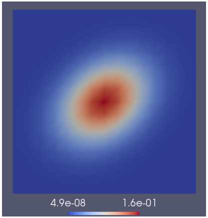
FP solver: $\mathbf{u} = (x_2, 0)^T \quad \varepsilon = \xi = \chi = 1 \quad \Delta t = 0.05 \quad N = 21$



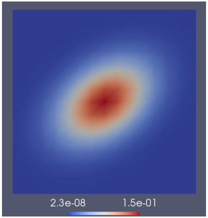
(a) t=0



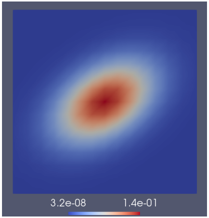
(b) t=0.1



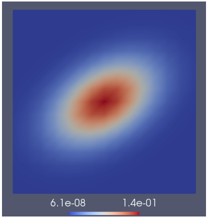
(c) t=0.5



(d) t=1



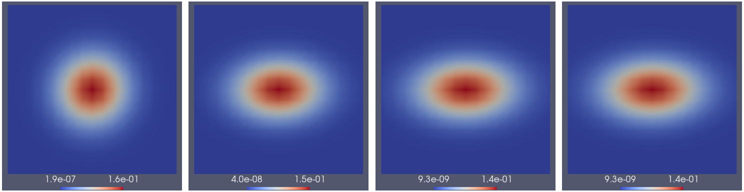
(e) t=2



(f) t=10

Experiment 2: extensional flow

FP solver: $\nabla_x \mathbf{u} = \mathrm{diag}\{\kappa, -\kappa\}$ $\quad \kappa = 0.5 \quad \xi = \chi = 1 \quad \varepsilon = 0 \quad \Delta t = 0.05 \quad N = 40$



(a) t=0 (b) t=1 (c) t=10 (d) steady state

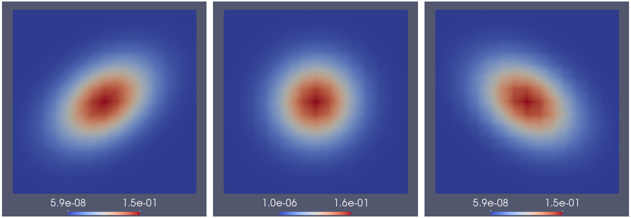
exact steady-state solution: $\psi_{\text{ref}}(\mathbf{R}) = cM e^{\mathbf{R}^T(\nabla_x \mathbf{u})\mathbf{R}}$

numerical error: $e_\psi = \psi_{\text{ref}} - \psi_{h,N}$

N	5	8	10	16	20	30	40
$\|e_\psi\|_{L^2(\mathcal{D})}$	3.4e-2	2.1e-2	1.3e-2	3.3e-3	1.3e-3	1.5e-4	1.8e-5
$\|e_\psi\|_{L^\infty(\mathcal{D})}$	1.9e-2	7.6e-3	4.8e-3	1.2e-3	5.0e-4	5.7e-5	8.0e-6

Experiment 3: Poiseulle flow

NSFP solver: $\Omega_h = [0,1]^2 \quad \mathbf{u}^0 = \left(x_2(1-x_2), 0\right)^T$
$\nu = 0.5 \quad \varepsilon = 0 \quad \chi = \xi = \gamma = 1 \quad \Delta t = h \quad T = 1$

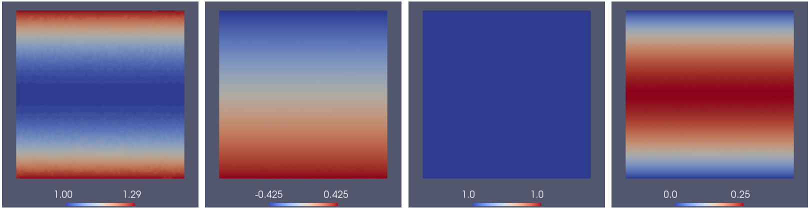


(a) $\mathbf{x} = (0.75, 0)$ (b) $\mathbf{x} = (0.75, 0.5)$ (c) $\mathbf{x} = (0.75, 1)$

Experiment 3: Poiseulle flow

NSFP solver: $\Omega_h = [0,1]^2 \quad \mathbf{u}^0 = \Big(x_2(1-x_2), 0\Big)^T$

$\nu = 0.5 \quad \varepsilon = 0 \quad \chi = \xi = \gamma = 1 \quad \Delta t = h \quad T = 1$



(a) C_{11} (b) C_{12} (c) C_{22} (d) u_1

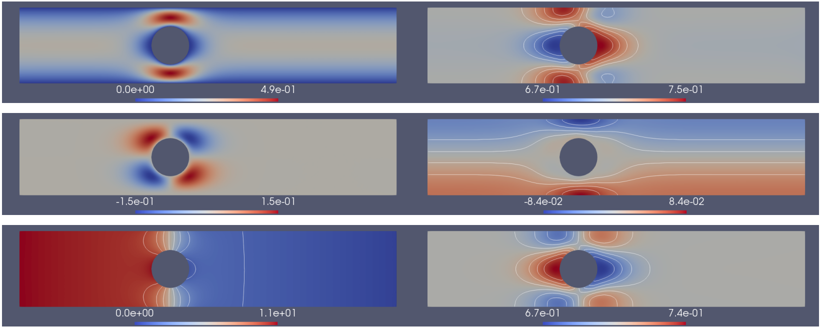
exact solution: $C_{11} = 1 + \frac{1}{2}\left|\frac{\partial u_1}{\partial x_2}\right|^2 \Big(1 - (2t+1)\mathrm{e}^{-2t}\Big), \; C_{12} = \frac{1}{2}\frac{\partial u_1}{\partial x_2}(1 - \mathrm{e}^{-2t}), \; C_{22} = 1$

$1/h$	N	$\|e_{\mathbf{u}}\|_{L^2(\Omega)}$	$\|e_{\mathbf{u}}\|_{H^1(\Omega)}$	$\|e_{C_{11}}\|_{L^2(\Omega)}$	$\|e_{C_{12}}\|_{L^2(\Omega)}$	$\|e_{C_{22}}\|_{L^2(\Omega)}$
16	8	2.15e-3	1.11e-2	3.17e-2	6.41e-2	2.82e-2
32	12	5.17e-4	4.33e-3	5.30e-3	1.45e-2	2.64e-3
64	16	1.30e-4	2.24e-3	2.58e-3	7.85e-3	1.53e-3

Experiment 4: flow past cylinder

NSFP solver: $\gamma = 1 \quad \chi = \operatorname{tr}\mathbf{C} \quad \xi = (\operatorname{tr}\mathbf{C})^2 \quad \varepsilon = 1 \quad T = 4 \quad \Delta t = 0.01 \quad \nu = 0.59$

inlet velocity $\mathbf{u} = \left(\frac{1}{4}x_2(1 - x_2), 0\right)^T$



solution of u_1, u_2, p, (left) C_{11}, C_{12}, C_{22} (right)

The Peterlin macroscopic model

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{u} = \nu\Delta_x\mathbf{u} + \mathrm{div}_x\,\mathbf{T} - \nabla_x p, \quad \mathrm{div}_x\,\mathbf{u} = 0$$

$$\mathbf{T} = \gamma(\mathrm{tr}\,\mathbf{C})\mathbf{C}$$

$$\frac{\partial \mathbf{C}}{\partial t} + (\mathbf{u}\cdot\nabla_x)\mathbf{C} - (\nabla_x\mathbf{u})\mathbf{C} - \mathbf{C}(\nabla_x\mathbf{u})^T = \chi(\mathrm{tr}\,\mathbf{C})\mathbf{I} - \xi(\mathrm{tr}\,\mathbf{C})\mathbf{C} + \varepsilon\Delta_x\mathbf{C}$$

$$\text{boundary conditions:} \quad \mathbf{u} = \mathbf{0}, \quad \varepsilon\frac{\partial \mathbf{C}}{\partial \mathbf{n}} = 0$$

$$\text{initial conditions:} \quad \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{C}(0) = \mathbf{C}_0$$

- M. Lukáčová-Medvid'ová, H. Mizerová, Š. Nečasová: *Global existence and uniqueness result for the diffusive Peterlin viscoelastic model*, Nonlinear Anal.-Theor. 120 (2015)

$$\boxed{\gamma = \chi = \mathrm{tr}\,\mathbf{C}, \ \ \xi = (\mathrm{tr}\,\mathbf{C})^2}$$

- M. Lukáčová-Medvidďová, H. Mizerová, Š. Nečasová, M. Renardy: *Global existence result for the generalized Peterlin viscoelastic model*, SIAM J. Math. Anal. 49-4 (2017)

Stabilized Lagrange-Galerkin method:

I. Nonlinear scheme

Conforming finite element approximation:	continuous piecewise linear finite elements
Method of characteristics:	discretization of the material derivative
Pressure-stabilization:	the Brezzi-Pitkäranta stabilization
Fully implicit:	time discretization

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X^n}{\Delta t}, \mathbf{v}_h\right) = -2\nu\left(\mathrm{D}(\mathbf{u}_h^n), \mathrm{D}(\mathbf{v}_h)\right) + (\operatorname{div}\mathbf{v}_h, p_h^n) - (\operatorname{div}\mathbf{u}_h^n, q_h) + - \delta_0 \sum_K h_K^2 \left(\nabla p_h^n, \nabla q_h\right)_K - (\operatorname{tr}\mathbf{C}_h^n\,\mathbf{C}_h^n, \nabla \mathbf{v}_h)$$

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X^n}{\Delta t}, \mathbf{D}_h\right) = 2\left((\nabla \mathbf{u}_h^n)\mathbf{C}_h^n, \mathbf{D}_h\right) + \left(\operatorname{div}\mathbf{u}_h^n (\mathbf{C}_h^n)^{\#}, \mathbf{D}_h\right) + + (\operatorname{tr}\mathbf{C}_h^n\,\mathbf{I}, \mathbf{D}_h) - \left((\operatorname{tr}\mathbf{C}_h^n)^2\mathbf{C}_h^n, \mathbf{D}_h\right) - \varepsilon\left(\nabla \mathbf{C}_h^n, \nabla \mathbf{w}_h\right)$$

▶ M. Lukáčová-Medvid'ová, H. Mizerová, H. Notsu, M. Tabata: *Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange-Galerkin method, Part I: A nonlinear scheme*, ESAIM: M2AN 51 (2017)

Stabilized Lagrange-Galerkin method:

II. Linear scheme

Conforming finite element approximation:	continuous piecewise linear finite elements
Method of characteristics:	discretization of the material derivative
Pressure-stabilization:	the Brezzi-Pitkäranta stabilization
Semi-implicit:	time discretization

$$\left(\frac{\mathbf{u}_h^n - \mathbf{u}_h^{n-1}\circ X^n}{\Delta t}, \mathbf{v}_h\right) = -2\nu\left(\mathrm{D}(\mathbf{u}_h^n), \mathrm{D}(\mathbf{v}_h)\right) + \left(\operatorname{div}\mathbf{v}_h, p_h^n\right) - \left(\operatorname{div}\mathbf{u}_h^n, q_h\right) +$$

$$-\delta_0 \sum_K h_K^2 \left(\nabla p_h^n, \nabla q_h\right)_K - \left(\operatorname{tr}\mathbf{C}_h^n\, \mathbf{C}_h^{n-1}, \nabla\mathbf{v}_h\right)$$

$$\left(\frac{\mathbf{C}_h^n - \mathbf{C}_h^{n-1}\circ X^n}{\Delta t}, \mathbf{D}_h\right) = 2\left((\nabla\mathbf{u}_h^n)\mathbf{C}_h^{n-1}, \mathbf{D}_h\right) +$$

$$+\left(\operatorname{tr}\mathbf{C}_h^{n-1}\,\mathbf{I}, \mathbf{D}_h\right) - \left((\operatorname{tr}\mathbf{C}_h^{n-1})^2\mathbf{C}_h^n, \mathbf{D}_h\right) - \varepsilon\left(\nabla\mathbf{C}_h^n, \nabla\mathbf{D}_h\right)$$

▶ M. Lukáčová-Medvid'ová, H. Mizerová, H. Notsu, M. Tabata: *Numerical analysis of the Oseen-type Peterlin viscoelastic model by the stabilized Lagrange-Galerkin method, Part II: A linear scheme*, ESAIM: M2AN 51 (2017)

Error estimates

Scheme	nonlinear	linear
ε	≥ 0	> 0
d	2	2 and 3

Theorem (nonlinear scheme)

For any $(h, \Delta t)$ *s. t.* $h \in (0, h_0]$, $\Delta t \in (0, \Delta t_0]$, *it holds that*

$$\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)},\ \|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)},\ |p_h - p|_{\ell^2(|\cdot|_h)},$$
$$\|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(L^2)},\ |\mathbf{C}_h - \mathbf{C}|_{\ell^2(H^1)},\ \left\|\mathrm{tr}\,(\mathbf{C}_h - \mathbf{C})(\mathbf{C}_h - \mathbf{C})\right\|_{\ell^2(L^2)} \leq c_\dagger (h + \Delta t).$$

Theorem (linear scheme)

For any $(h, \Delta t)$ *s. t.* $h \in (0, h_0]$, $\Delta t \leq \dfrac{c_0}{(1 + |\log h|)^{1/2}}\ (d = 2)$ *or* $\Delta t \leq c_0 h^{1/2}\ (d = 3)$ *it holds that*

$$\left.\begin{array}{l}\|\mathbf{u}_h - \mathbf{u}\|_{\ell^\infty(L^2)},\ \|\mathbf{u}_h - \mathbf{u}\|_{\ell^2(H^1)},\ |p_h - p|_{\ell^2(|\cdot|_h)}, \\ \|\mathbf{C}_h - \mathbf{C}\|_{\ell^\infty(H^1)},\ \left\|\bar{D}_{\Delta t}\mathbf{C}_h - \frac{\partial \mathbf{C}}{\partial t}\right\|_{\ell^2(L^2)}\end{array}\right. \leq c(h + \Delta t).$$

Error estimates

Scheme	nonlinear	linear
ε	≥ 0	> 0
d	2	2 and 3

	nonlinear, $d = 2$	
Existence	$\varnothing$	
	$\varepsilon > 0$	$\varepsilon = 0$
Uniqueness	$O\Big((1 + \lvert\log h\rvert)^{-2}\Big)$	$O(h)$
Optimal error estimates	$\varnothing$	

	linear, $\varepsilon > 0$	
Existence	$\varnothing$	
Uniqueness	$\varnothing$	
	$d = 2$	$d = 3$
Optimal error estimates	$O\Big((1 + \lvert\log h\rvert)^{-1/2}\Big)$	$O\Big(\sqrt{h}\,\Big)$

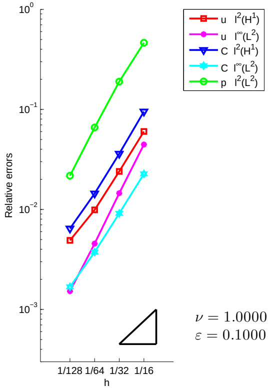
Experimental order of convergence

Semi-implicit linear scheme

- computational domain $\Omega = (0,1)^2$
- final time $T = 0.5$
- mesh size $h = 1/16, 1/32, 1/64, 1/128$
- time step $\Delta t = h/2$
- pressure-stabilization constant $\delta_0 = 1$

h	e_u $l^2(H^1)$	EOC	e_u $l^\infty(L^2)$	EOC
1/16	6.01e-02	–	4.46e-02	–
1/32	2.40e-02	1.33	1.45e-02	1.62
1/64	9.90e-03	1.27	4.56e-01	1.67
1/128	4.90e-03	1.02	1.52e-02	1.58
h	e_c $l^2(H^1)$	EOC	e_c $l^\infty(L^2)$	EOC
1/16	9.51e-02	–	2.27e-02	–
1/32	3.60e-02	1.40	9.13e-03	1.31
1/64	1.44e-02	1.32	3.75e-03	1.28
1/128	6.44e-03	1.16	1.68e-03	1.15
h	e_p $l^2(L^2)$	EOC		
1/16	4.64e-01	–		
1/32	1.90e-01	1.29		
1/64	6.59e-02	1.52		
1/128	2.17e-02	1.60		

- $\nu =$ fluid viscosity
- $\varepsilon =$ elastic stress diffusivity



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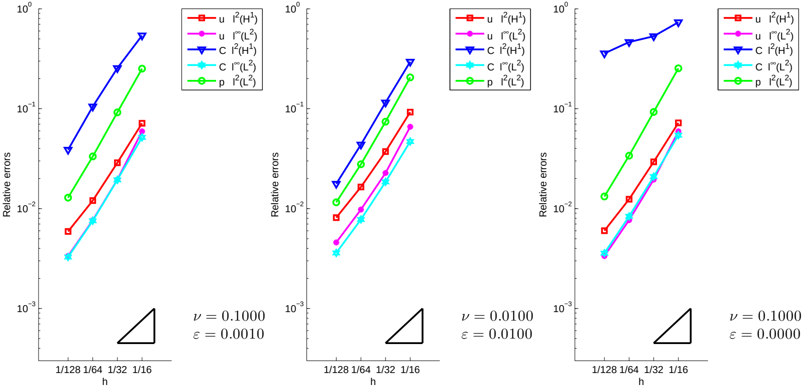
- ν = fluid viscosity
- ε = elastic stress diffusivity

$\boldsymbol{\varepsilon = 0}$



Thank you for your attention!