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Chris Lambie-Hanson
Andy Zucker

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POLISH SPACE PARTITION PRINCIPLES AND THE HALPERN-LÄUCHLI THEOREM

CHRIS LAMBIE-HANSON AND ANDY ZUCKER

Abstract. The Halpern-Läuchli theorem, a combinatorial result about trees, admits an elegant proof due to Harrington using ideas from forcing. In an attempt to distill the combinatorial essence of this proof, we isolate various partition principles about products of perfect Polish spaces. These principles yield straightforward proofs of the Halpern-Läuchli theorem, and the same forcing from Harrington's proof can force their consistency. We also show that these principles are not ZFC theorems by showing that they put lower bounds on the size of the continuum.

1. Introduction

The Halpern-Läuchli theorem, first proven in [5], is a partition principle about products of finitely branching trees. While it is entirely combinatorial in nature, it has deep connections to logic, both in its original motivation — it was a key tool in Halpern and Lévy's proof [6] that, over ZF, the Boolean Prime Ideal theorem does not imply the Axiom of Choice — and in some methods of proving it. In particular, arguably the most elegant proof of the theorem, due to Harrington, uses some ideas from forcing (see [3] or [16] for a presentation of Harrington's proof). Here, we investigate some of the combinatorial ideas underlying Harrington's proof and, building on work from [18], we introduce and study a family of statements about arbitrary (i.e. not necessarily Borel) finite partitions of a product of finitely many perfect Polish spaces. The simplest of these statements are as in the following definition.

Definition 1.1. Recall that a Polish space is *perfect* if it contains no isolated points. Given $0 < d < \omega$ and a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, a *somewhere dense grid* is a subset of $\prod_{i<d} X_i$ of the form $\prod_{i<d} Y_i$, where each $Y_i \subseteq X_i$ is somewhere dense (in the ordinary topological sense).

The *Polish grid principle* in dimension d, denoted PG_d, is the statement that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$ and any coloring $\gamma \colon \prod_{i<d} X_i \to r$, there is a monochromatic somewhere dense grid. The principle PG is the statement that PG_d holds for every $d < \omega$.

More generally, for any cardinal κ, let $\mathsf{PG}_d(\kappa)$ be the statement that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces and any coloring $\gamma \colon \prod_{i<d} X_i \to \kappa$,

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there is a monochromatic somewhere dense grid, and let $\mathsf{PG}(\kappa)$ denote the statement that $\mathsf{PG}_d(\kappa)$ holds for every $d < \omega$.

We shall see in Section 3 that PG_d yields a simple, direct proof of the d-dimensional Halpern-Läuchli theorem; in fact, it will be immediate from the proof that a natural weakening of PG_d, which we will define in Section 3 and which we will denote by DDF_d, suffices for this derivation. Unlike the Halpern-Läuchli theorem, though, at least for $d > 2$ the principles PG_d and DDF_d are independent of ZFC. Much of this paper is concerned with the study of this independence; we obtain, for instance, the following complementary results.

Theorem A. *Let $2 \leq d < \omega$.*

(1) If DDF_d holds, then $2^{\aleph_0} \geq \aleph_{d-1}$.
(2) If $\mathsf{PG}_d(\aleph_0)$ holds, then $2^{\aleph_0} \geq \aleph_d$.
(3) $\mathsf{PG}_d(\aleph_0)$ holds after adding at least $(\beth_{d-1})^+$-many Cohen reals to any model of ZFC.

Clause 3 of Theorem A, together with Shoenfield absoluteness, yields a new proof of the Halpern-Läuchli theorem. This proof can be seen as a recasting of Harrington's proof that seeks to pull apart the forcing machinery and the combinatorial principles underlying the Halpern-Läuchli theorem. Indeed, our proof of Theorem A(3) is essentially an adaptation of Harrington's proof of the Halpern-Läuchli theorem.

We feel that Theorem A is also of independent interest for isolating a stratified family of natural combinatorial statements that place increasingly strong requirements on the value of the continuum. In this direction, Theorem A yields a sharp result for the principle $\mathsf{PG}_d(\aleph_0)$: for all $2 \leq d < \omega$, if $2^{\aleph_0} < \aleph_d$, then $\mathsf{PG}_d(\aleph_0)$ fails, whereas it is consistent that $2^{\aleph_0} = \aleph_d$ and $\mathsf{PG}_d(\aleph_0)$ holds (for example, after adding $\aleph_d$-many Cohen reals to a model of GCH). This can be seen as part of a line of investigation into the relationship between questions of *dimensionality* in the context of the real numbers (or uncountable Polish spaces, more generally), and the cardinals $\{\aleph_d : d < \omega\}$. Other research in this vein includes, e.g., Raghavan and Todorčević's work on Galvin's problem ([12], [13]), Komjáth's work on covering the plane by finite many clouds ([8]) and the work of several authors on additive partition relations for the real numbers ([7], [9], [17]).

The work of Raghavan and Todorčević in particular obtains results with interesting parallels to ours. They consider the problem of finding the *d-dimensional Ramsey degrees* of the topological space $\mathbb{Q}$, i.e. given a coloring of $[\mathbb{R}]^d$ into finitely many colors, find $X \subseteq \mathbb{R}$ homeomorphic to $\mathbb{Q}$ such that $[X]^d$ takes a fixed small number of colors (not depending on the starting number of colors). In [13], they show that $2^{\aleph_0} \leq \aleph_{d-2}$ implies that the d-dimensional Ramsey degree of the space $\mathbb{Q}$ is infinite, and in [12], they show that in ZFC plus some large cardinal hypotheses, the 2-dimensional Ramsey degree of the space $\mathbb{Q}$ is 2.

Note that Theorem A(1) does not yield any nontrivial information from DDF_2 or PG_2. In fact, we shall see in Section 3 that DDF_2 is a theorem of ZFC. Moreover, PG_2 is a theorem of ZFC when restricted to 2-colorings. It remains open whether the full PG_2 is a theorem of ZFC. It would be especially interesting if similar techniques as in [12] could be used to show that PG_2 is implied by some large cardinal hypotheses.

Finally, the tools developed in our proof of Clauses 1 and 2 of Theorem A yield a new proof of a recent result of Bannister, Bergfalk, Moore, and Todorčević from [1] about provable failures of the Partition Hypothesis introduced in that paper. The original proof of this fact in [1] makes heavy use of ideas from simplicial homology, whereas our proof is purely combinatorial/set theoretic.

The structure of the paper is as follows. In Section 2, we give our background notational conventions and basic definitions regarding trees and the Halpern-Läuchli theorem. In Section 3, we introduce the partition principles that form the main object of study of the paper. We show that they yield immediate derivations of relevant instances of the Halpern-Läuchli theorem. We also show that the principle DDF_2 is provable in ZFC, as is the principle PG_2 when restricted to 2-colorings. In Section 4, we prove clauses 1 and 2 of Theorem A. We also give a direct proof of the aforementioned result from [1] about provable failures of the Partition Hypothesis. In Section 5, we prove clause 3 of Theorem A. We conclude in Section 6 with a brief discussion about using the partition principles introduced here to yield variations of the Halpern-Läuchli theorem pertaining to coding trees.

Some related and partially overlapping results were recently independently obtained by Nedeljko Stefanović [14]. In particular, he proves that the principle DDF holds after adding $\beth_\omega$-many Cohen reals to any model of ZFC, thereby also yielding the proof of the Halpern-Läuchli theorem described above immediately after the statement of Theorem A. Stefanović also investigates these combinatorial principles in certain models of $\mathsf{ZF} + \neg\mathsf{AC}$, in particular proving that PG holds in Cohen's symmetric model (the same model used by Halpern and Lëvy in [6] to show that BPI does not imply AC).

2. Preliminaries

2.1. **Notation.** If a is a set of ordinals, then we will sometimes think of a as the increasing function whose domain is the order type of a (which we denote by $\mathrm{otp}(a)$). In particular, if $\eta < \mathrm{otp}(a)$, then $a(\eta)$ is the unique $\beta \in a$ such that $\mathrm{otp}(a \cap \beta) = \eta$. Similarly, if $I \subseteq \mathrm{otp}(a)$, then $a[I]$ denotes the set $\{a(\eta) \mid \eta \in I\}$. If X is any set and $n < \omega$, then $[X]^n$ denotes the set of n-element subsets of X. If X is a set of ordinals, then we will use the notation $(\alpha_0, \ldots, \alpha_{n-1}) \in [X]^n$ to denote the assertion that $\{\alpha_0, \ldots, \alpha_{n-1}\} \in [X]^n$ and $\alpha_0 < \ldots < \alpha_{n-1}$.

2.2. **Trees and the Halpern-Läuchli theorem.** A *tree* is a partially ordered set T with the property that for every $t \in T$, the set $\mathrm{pred}_T(t) := \{s \in T : s <_T t\}$ of *predecessors* of $t \in T$ is well-ordered. In particular, any subset $S \subseteq T$ equipped with the partial order induced from T is also a tree. We write $h_T(t) := \mathrm{otp}(\mathrm{pred}_T(t))$ and call this the *height* of $t \in T$. If α is an ordinal, we write $T(\alpha) := \{t \in T : h_T(t) = \alpha\}$ for level α of T. The *height* of T is the ordinal $h(T) := \sup\{h_T(t) : t \in T\}$. We write $\mathrm{succ}_T(t) := \{u \in T : t <_T u\}$, and given $\alpha > h_T(t)$, we write $\mathrm{succ}_T(t, \alpha) := \{u \in \mathrm{succ}_T(t) : h_T(u) = \alpha\}$. We write $\mathrm{IS}_T(t) := \mathrm{succ}_T(t, h_T(t) + 1)$ for the set of *immediate successors* of $t \in T$. If $S \subseteq T$, we can also write $\mathrm{succ}_T(S) := \bigcup_{t \in S} \mathrm{succ}_T(t)$ and $\mathrm{IS}_T(S) := \bigcup_{t \in S} \mathrm{IS}_T(t)$, and if $\alpha > h_T(t)$ for every $t \in S$, we can also write $\mathrm{succ}_T(S, \alpha) := \bigcup_{t \in S} \mathrm{succ}_T(t, \alpha)$. We say T is *finitely branching* if $\mathrm{IS}_T(t)$ is finite for every $t \in T$, and we say that T has *no terminal nodes* if $\mathrm{IS}_T(t) \neq \emptyset$ for every $t \in T$. We say T is *perfect* if for every $t \in T$, some $u \in \mathrm{succ}_T(t)$ has $|\mathrm{IS}_T(u)| \geq 2$. We say T is *rooted* if $|T(0)| = 1$. Unless specified otherwise, all trees in this paper will be rooted, perfect, finitely branching, height ω, and without terminal nodes.

Given a tree T, a *branch through* T is a maximal linearly ordered subset of T. We write $[T]$ for the set of branches through T. Every $x \in [T]$ satisfies $|x \cap T(m)| = 1$ for every $m < \omega$, so we write $x(m) \in T(m)$ for this unique element. We equip $[T]$ with the topology of pointwise convergence, where $x_i \to x$ iff for every $m < \omega$, we eventually have $x_i(m) = x(m)$. Because of our standing assumptions on trees, we have that $[T]$ is homeomorphic to Cantor space.

Now suppose $0 < d < \omega$ and that $\vec{T} = \langle T_0, \ldots, T_{d-1} \rangle$ is a sequence of trees with no terminal nodes. The *level product* of these trees is the set $\bigotimes \vec{T} := \bigcup_{m<\omega} \prod_{i<d} T_i(m)$. We turn $\bigotimes \vec{T}$ into a tree, where given $\vec{s} = (s_0, \ldots, s_{d-1})$ and $\vec{t} = (t_0, \ldots, t_{d-1})$ in $\bigotimes \vec{T}$, we set $\vec{s} \leq_{\bigotimes \vec{T}} \vec{t}$ iff $s_i \leq_{T_i} t_i$ for each $i < d$. To ease notation, we write $\leq_{\vec{T}}$ in place of $\leq_{\bigotimes \vec{T}}$. Note that $\left(\bigotimes \vec{T}\right)(m) = \prod_{i<d} T_i(m)$ and that as spaces, we have $[\bigotimes \vec{T}] \cong \prod_{i<d} [T_i]$. Given $\vec{x} \in \prod_{i<d} [T_i]$, if $i < d$, then we will always denote the i^{th} entry in $\vec{x}$ as $x_i \in [T_i]$. Moreover, if $m < \omega$, then we let $\vec{x}(m) := \langle x_0(m), x_1(m), \ldots, x_{d-1}(m) \rangle \in \prod_{i<d} T_i(m)$.

Definition 2.1. Let T be a tree with no terminal nodes, and let $a \subseteq \omega$ be infinite. An *a-strong subtree* of T is a subset $S \subseteq T$ built inductively as follows.

- Pick any $t_0 \in T(a(0))$ and set $S(0) = \{t_0\}$.
- Inductively assume for some $m < \omega$ that $S(m)$ has been determined and that $S(m) \subseteq T(a(m))$. Then for every $u \in \mathrm{IS}_T(S(m))$, pick some $t_u \in T(a(m+1))$, and set $S(m+1) = \{t_u : u \in \mathrm{IS}_T(S(m))\}$.

Notice that if $S \subseteq T$ is an a-strong subtree, then the set $S(m)$ from the inductive construction above is in fact level m of the tree S. Let us remark that if $\vec{T} = \langle T_0, \ldots, T_{d-1} \rangle$ is a finite sequence of trees and $S_i \subseteq T_i$ is an a-strong subtree for each $i < d$, then $\bigotimes \vec{S}$ is an a-strong subtree of $\bigotimes \vec{T}$; however, not every a-strong subtree of $\bigotimes \vec{T}$ has this particularly nice form.

We can now state the theorem of Halpern and Läuchli which is the main topic of this paper.

Theorem 2.2 (Halpern-Läuchli [5])**.** *Let $0 < d < \omega$, and let $\vec{T} = \langle T_0, \ldots, T_{d-1} \rangle$ be a sequence of trees. Let $r < \omega$, and suppose $\gamma\colon \bigotimes \vec{T} \to r$ is a coloring. Then there are an infinite $a \subseteq \omega$ and a-strong subtrees $S_i \subseteq T_i$ so that writing $\vec{S} = \langle S_0, \ldots, S_{d-1} \rangle$, we have that $\bigotimes \vec{S}$ is monochromatic for γ.*

We write HL for the statement of Theorem 2.2 and HL_d for its restriction to sequences of trees of length at most d.

3. Polish space partition principles

While HL_1 is trivial, proving HL_d by induction on d is quite difficult. Good references for proofs along these lines are [11] and [15]. However, we draw attention to a proof due to Harrington (cf. [16]) using ideas from forcing. Let us begin by giving a very brief, high-level overview of the structure of Harrington's proof. Given $\gamma\colon \bigotimes \vec{T} \to r$ as in Theorem 2.2, one considers the poset $\mathbb{P}$ for adding a large number of Cohen reals, which are viewed as members of $[\bigotimes \vec{T}]$. Upon fixing a name $\dot{U}$ for a non-principal ultrafilter on ω and various names $\dot{b}$ for $\mathbb{P}$-generic branches, certain conditions $q_{\dot b} \in \mathbb{P}$ force that for $\dot U$-many levels, the corresponding node of the branch $\dot b$ is mapped by γ to some color $i_{\dot b} < r$. By using the Erdős-Rado theorem, we can find a rich collection of such $\dot b$ so that various properties of the corresponding $q_{\dot b}$ and $i_{\dot b}$ are the same. One then uses the conditions $q_{\dot b}$ to help build the subtrees $\vec{S}$ with $\bigotimes \vec{S}$ monochromatic for γ.

It is natural to attempt to remove some of the forcing formalism from these ideas. Namely, if $U \in \beta\omega \setminus \omega$ is a non-principal ultrafilter and if $\gamma\colon \bigotimes \vec{T} \to r$ is a coloring for some $r < \omega$, we can define a coloring $\gamma_U\colon \prod_{i<d}[T_i] \to r$ via $\gamma_U(\vec{x}) = j$ iff $\{m < \omega : \gamma(\vec{x}(m)) = j\} \in U$. Of course, the coloring γ_U will typically have horrible definability properties, i.e. fail to have the Baire property. However, we can still attempt to reason about the possible Ramsey-theoretic properties of arbitrary colorings on products of Polish spaces. This line of thought naturally leads to the definition of the partition principles PG_d, which, recalling Definition 1.1, is the assertion that, for every positive $d < \omega$, any sequence $\langle X_0, \ldots, X_{d-1}\rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma : \prod_{i<d} X_i \to r$, there is a somewhere dense grid in $\prod_{i<d} X_i$ that is monochromatic for γ.

Remark 3.1. Recall that every perfect Polish space contains a dense G_δ subspace homeomorphic to Baire space ${}^\omega\omega$. Therefore to show that PG_d holds, one may assume that each X_i is the Baire space.

Proposition 3.2. PG_d *implies* HL_d*.*

Proof. Let $\gamma\colon \bigotimes \vec{T} \to r$ be a coloring. Fix a non-principal ultrafilter $U \in \beta\omega \setminus \omega$, and form the coloring $\gamma_U\colon \prod_{i<d}[T_i] \to r$ defined before Remark 3.1. Using PG_d, find somewhere dense sets $Y_i \subseteq [T_i]$ so that $\gamma_U[\prod_{i<d} Y_i] = \{j\}$ for some $j < r$. As each Y_i is somewhere dense, we can find $t_i \in T_i$ so that whenever $u \geq_{T_i} t_i$, there is $y \in Y_i$ with $u \in y$. By moving some of the t_i further up if needed to place them all on the same level, we may assume that $(t_0, \ldots, t_{d-1}) := \vec{t} \in \bigotimes \vec{T}$.

We now inductively construct an infinite $a \subseteq \omega$ and a-strong subtrees $S_i \subseteq T_i$ with $\bigotimes \vec{S}$ monochromatic for γ. To get started, for each $i < d$, pick some $y_i \in Y_i$ with $t_i \in y_i$. Setting $\vec{y} = (y_0, \ldots, y_{d-1}) \in \prod_{i<d} Y_i$, we have $\gamma_U(\vec{y}) = j$. This means that $W_0 := \{m < \omega : \gamma(\vec{y}(m)) = j\} \in U$, so in particular is infinite. Pick some $a(0) \in W_0$ above $\ell(\vec{t})$, and for each $i < d$, we set $S_i(0) = \{y_i(a(0))\}$.

Now suppose $n > 0$ and that for every $m < n$, both $a(m) < \omega$ and $S_i(m) \subseteq T_i(a(m))$ have been determined so that every $s \in S_i$ satisfies $s \geq_{T_i} t_i$. For each $u \in \mathrm{IS}_{T_i}(S_i(n-1))$, there is some $y_u \in Y_i$ with $u \in y_u$. Set $Z_i := \{y_u : u \in \mathrm{IS}_{T_i}(S_i(n-1))\}$. Because $Z_i \subseteq Y_i$, we have that $W_n := \bigcap\{\{m < \omega : \gamma(\vec{z}(m)) = j\} : \vec{z} \in \prod_{i<d} Z_i\} \in U$. Pick some $a(n) \in W_n$ with $a(n) > a(n-1)$, and for each $i < d$, set $S_i(n) = \{z(n) : z \in Z_i\}$. □

We will show in our proof of Theorem A(3) in Section 5 that one can force that PG is consistent by adding $\beth_\omega$-many Cohen reals. In particular, as HL is a $\mathbf{\Pi}^1_2$ statement, Shoenfield's absoluteness theorem then implies that HL is true in ZFC, yielding a new proof of HL. In a sense, this proof re-interprets Harrington's forcing proof of HL by actually passing to the generic extension, whereas Harrington's proof can be phrased just in terms of combinatorics on the forcing poset.

Upon analyzing the proof of Proposition 3.2, it becomes clear that PG is actually stronger than what we need to prove HL. We define two weakenings of PG which are still strong enough so that the proof of Proposition 3.2 goes through. The first one is the weakest such principle and is in fact true in ZFC.

Definition 3.3. If X is a topological space and $\mathcal{U}$ is a collection of open subsets of X, a *$\mathcal{U}$-set* is any $Y \subseteq X$ which meets every member of $\mathcal{U}$.

Given $0 < d < \omega$ and a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, a *finitary somewhere dense grid* is a subset $Y \subseteq \prod_{i<d} X_i$ so that for each $i < d$, there are open $U_i \subseteq X_i$ so that for every sequence $\langle \mathcal{U}_0, \ldots, \mathcal{U}_{d-1} \rangle$ with $\mathcal{U}_i$ a finite collection of non-empty open subsets of U_i, there is for each $i < d$ a $\mathcal{U}_i$-set $Y_i \subseteq X_i$ with $\prod_{i<d} Y_i \subseteq Y$.

The *finitary Polish grid principle* FPG_d states that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma \colon \prod_{i<d} X_i \to r$, there is a monochromatic finitary somewhere dense grid.

In fact, one can show in ZFC that for any $d < \omega$, any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any finitary somewhere dense grid $Y \subseteq \prod_{i<d} X_i$, and any finite coloring of Y, one of the colors contains a finitary somewhere dense grid. However, we won't say much more about FPG_d; to us at least, it seems that proofs of FPG_d either go through forcing the consistency of the much stronger PG_d or go through repeating many of the steps seen in combinatorial proofs of HL.

In between FPG_d and PG_d, we have our last principle, which was defined by the second author in [18].

Definition 3.4. Given a sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces and $I \subseteq d$, we write $\pi_I \colon \prod_{i<d} X_i \to \prod_{i \in I} X_i$ for the natural projection map. Given some $Z \subseteq \prod_{i<d} X_i$ and some $x \in \prod_{i \in I} X_i$, we write $Z_x := \pi_{d \setminus I}[\{z \in Z : \pi_I(z) = x\}]$.

We define the notion of a set $Z \subseteq \prod_{i<d} X_i$ being a *dense-by-dense filter*, DDF for short, by induction on $d > 0$.

1. $Z \subseteq X_0$ is a DDF set if it is dense.
2. Given $d > 0$, we have that $Z \subseteq \prod_{i \leq d} X_i$ is DDF if $\pi_d[Z] \subseteq \prod_{i<d} X_i$ is DDF and $\{Z_x : x \in \pi_d[Z]\}$ generates a filter of dense subsets of X_d.

We say that $Z \subseteq \prod_{i<d} X_i$ is *somewhere-DDF* if for some non-empty open sets $U_i \subseteq X_i$, Z is DDF as a subset of $\prod_{i<d} U_i$.

The principle DDF_d states that for any sequence $\langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces, any $r < \omega$, and any coloring $\gamma \colon \prod_{i<d} X_i \to r$, there is a monochromatic somewhere-DDF subset.

Since a somewhere dense grid is clearly somewhere-DDF, it is clear that PG_d implies DDF_d. The next proposition shows that a witness to DDF_d is also a witness to FPG_d.

Proposition 3.5. *Given $0 < d < \omega$ and $\langle X_0, \ldots, X_{d-1} \rangle$ a sequence of perfect Polish spaces, if $Z \subseteq \prod_{i<d} X_i$ is somewhere-DDF, then Z is also a finitary somewhere dense grid.*

Proof. We prove by induction on d that if $U_i \subseteq X_i$ are non-empty open sets with $Z \subseteq \prod_{i<d} U_i$ a DDF subset, then the U_i also witness that Z is a finitary somewhere dense grid. For $d = 1$ this is clear. Now suppose the result is known for dimension d. Let $\langle X_0, \ldots, X_d \rangle$ be perfect Polish spaces, and suppose $Z \subseteq \prod_{i \leq d} U_d$ is DDF. Fix $\langle \mathcal{U}_0, \ldots, \mathcal{U}_d \rangle$ with each $\mathcal{U}_i$ a finite collection of non-empty open subsets of U_i. As $\pi_d[Z] \subseteq \prod_{i<d} X_i$ is DDF, then by induction we can find $\mathcal{U}_i$-sets $Y_i \subseteq U_i$ with

$\prod_{i<d} Y_i \subseteq Z$. Find some finite $F_i \subseteq Y_i$ which is also a $\mathcal{U}_i$-set. As $\{Z_x : x \in \pi_d[Z]\}$ generates a filter of dense subsets of U_d, the set $Y_d = \bigcap\{Z_y : y \in \prod_{i<d} F_i\} \subseteq U_d$ is dense, so in particular is a $\mathcal{U}_d$-set. As $F_0 \times \cdots \times F_{d-1} \times Y_d \subseteq Z$, we see that Z is a finitary somewhere dense grid. $\square$

This principle serves as an interesting middle ground between FPG_d and PG_d; it is rich enough in that one can use non-combinatorial tools to investigate it, yet weak enough that one might hope to prove it in ZFC. Indeed, we have the following.

Proposition 3.6. DDF_2 *is true.*

Proof. For reasons that will be clear at the end of the proof, we prove something slightly stronger. Let X_0 be a non-meager subset of some perfect Polish space $\tilde{X}_0$, and let X_1 be a somewhere dense subset of some perfect Polish space $\tilde{X}_1$. By zooming in to the relevant open sets, we may assume that $X_0 \subseteq \tilde{X}_0$ is nowhere meager and that $X_1 \subseteq \tilde{X}_1$ is dense. Let $r < \omega$, and let $\gamma\colon X_0 \times X_1 \to r$ be a coloring. We will prove that there is a somewhere-DDF subset of $X_0 \times X_1$ that is monochromatic for γ.

The proof is by induction on r. It is trivial if $r = 1$, so assume that $r > 1$. We attempt to find a DDF subset inside $Z := \gamma^{-1}(\{r-1\})$. Let $\{U_n : n < \omega\}$ be a basis for X_0. We inductively attempt to build a decreasing collection $\{Y_n : n < \omega\}$ of dense subsets of X_1 and a dense subset $\{x_n : n < \omega\}$ of X_0 as follows. Set $Y_0 = X_1$. If $n < \omega$ and Y_n has been determined, pick any $x_n \in U_n$ such that $Z_{x_n} \cap Y_n \subseteq X_1$ is dense, and set $Y_{n+1} := Z_{x_n} \cap Y_n$. If we can do this for every $n < \omega$, then $\bigcup_{n<\omega}\{x_n\} \times Y_{n+1} \subseteq Z$ is a DDF set.

Suppose for some $k < \omega$ that we fail to construct x_k and Y_{k+1}. Let $\{V_n : n < \omega\}$ be a basis for X_1. For every $x \in U_k$, there is some $n_x < \omega$ so that $Z_x \cap Y_k \cap V_{n_x} = \emptyset$. For some $n < \omega$, the set $W := \{x \in U_k : n_x = n\}$ is non-meager. Then $W \subseteq \tilde{X}_0$ is non-meager, $Y_k \cap V_n \subseteq \tilde{X}_1$ is somewhere dense, and γ attains one fewer color on $W \times (Y_k \cap V_n)$. We can therefore apply the induction hypothesis to $\gamma \restriction W \times (Y_k \cap V_n)$ to obtain a monochromatic somewhere-DDF subset of $W \times (Y_k \cap V_n)$, and hence also of $X_0 \times X_1$. $\square$

Using similar ideas, one can also say something about the stronger PG_2 principle.

Proposition 3.7. PG_2 *restricted to* 2*-colorings is true.*

Proof. Let X_0 and X_1 be perfect Polish spaces, and let $\gamma : X_0 \times X_1 \to 2$ be a coloring. We will attempt to construct a dense grid inside $Z_1 := \gamma^{-1}(\{1\})$ and will show that if any step of the construction fails, then we can find a somewhere dense grid inside $Z_0 := \gamma^{-1}(\{0\})$. Given $j < 2$ and $x \in X_0$, let $(Z_j)_x := \{y \in X_1 : (x, y) \in Z_j\}$. Similarly, given $y \in X_1$, let $(Z_j)^y := \{x \in X_0 : (x, y) \in Z_j\}$.

For $i < 2$, let $\{U_n^i : n < \omega\}$ be a basis for X_i. We attempt to build sets $Y_i = \{x_n^i : n < \omega\}$ for $i < 2$ such that $Y_0 \times Y_1 \subseteq Z_1$ and, for all $n < \omega$ and $i < 2$, we have $x_n^i \in U_n^i$. During the construction, we will also construct $\subseteq$-decreasing sequences $\langle V_n^i : n < \omega\rangle$ of nowhere meager subsets of X_i with the property that, for all $n < \omega$, both $\{x_m^0 : m < n\} \times V_n^1$ and $V_n^0 \times \{x_m^1 : m < n\}$ are subsets of Z_1.

Begin by letting $V_0^i = X_i$ for $i < 2$. Now suppose that $n < \omega$ and we have chosen $\langle x_m^i : m < n\rangle$ and $\langle V_m^i : m \leq n\rangle$. We will choose $x_n^0 \in V_n^0 \cap U_n^0$ and a nowhere meager set $V_{n+1}^1 \subseteq V_n^i$, and then we will choose $x_n^1 \in V_{n+1}^1 \cap U_n^1$ and a nowhere meager set $V_{n+1}^0 \subseteq V_n^0$.

If we are able to find $x \in V^0_n \cap U^0_n$ such that $V^1_n \cap (Z_1)_x$ is nowhere meager, then let x^0_n be such an x and let $V^1_{n+1} := V^1_n \cap (Z_1)_{x_n}$. Suppose momentarily that we were unable to find such an x. Then, for every $x \in V^0_n \cap U^0_n$, there is $k_x < \omega$ such that $V^1_n \cap (Z_1)_x \cap U^1_{k_x}$ is meager. Then there is a fixed $k < \omega$ and a non-meager set $W_0 \subseteq V^0_n \cap U^0_n$ such that $k_x = k$ for all $x \in W_0$. Let W'_0 be a countable somewhere dense subset of W_0. Then $V^1_n \cap U^1_k \cap \bigcup\{(Z_1)_x : x \in W'_0\}$ is meager, so $W_1 := V^1_n \cap U^1_k \cap \bigcap\{(Z_0)_x : x \in W'_0\}$ is nonmeager. In particular, W_1 is somewhere dense, so $W'_0 \times W_1$ is a somewhere dense grid contained in Z_0.

We can therefore assume that we were able to construct x^0_n and V^1_{n+1} and continue to the second half of step n of the construction, where a symmetric argument shows that we can either find

(1) $y \in V^1_{n+1}$ such that $V^0_n \cap (Z_1)^y$ is nowhere meager; or
(2) a somewhere dense grid contained in Z_0.

We can therefore again assume we are in case (1), let x^1_n be such a y, and let $V^0_{n+1} := V^0_n \cap (Z_1)^{x^1_n}$.

At the end of the construction, we have produced a dense grid $Y_0 \times Y_1$. To see that it is a subset of Z_1, fix $m, n \leq \omega$. If $m \leq n$, then we ensured that $x^1_n \in V^1_{n+1} \subseteq V^1_{m+1}$, and hence $(x^0_m, x^1_n) \in Z_1$. Similarly, if $m > n$, then we ensured that $x^0_m \in V^0_m \subseteq V^0_{n+1}$, so again $(x^0_m, x^1_n) \in Z_1$. $\square$

It remains open whether the full PG_2 is true in ZFC. For $d > 2$, we will show in Theorem 4.6 that DDF_d, even restricted to 2-colorings, implies that $\mathfrak{c} \geq \aleph_{d-1}$.

4. Consistent failure of DDF_d

Definition 4.1. Let S be an infinite set, and suppose that $\mathcal{H}$ is a collection of subsets of S. We say that $\mathcal{H}$ is *weakly partition regular* if for every finite partition $S = \bigcup_{j<k} P_j$ of S, there is $j < k$ such that $P_j \in \mathcal{H}$

Given an infinite cardinal κ, we say that $\mathcal{H}$ is *weakly κ-partition regular* if for every partition $S = \bigcup_{\eta<\kappa} P_\eta$ of S into κ-many parts, there is $\eta < \kappa$ such that $P_\eta \in \mathcal{H}$.

A further weakening of this notion will be useful for us. Given an infinite regular cardinal κ, we say that $\mathcal{H}$ is *weakly κ-partition subregular* if for every partition $S = \bigcup_{\eta<\kappa} P_\eta$ of S into κ-many parts, there is $\xi < \kappa$ such that $\bigcup_{\eta<\xi} P_\eta \in \mathcal{H}$.

Remark 4.2. It will often be more convenient to phrase these partition regularity properties in terms of colorings instead of partitions. For instance, a collection $\mathcal{H}$ of subsets of a set S is weakly κ-partition subregular if, for every set X of cardinality κ and every coloring $c : S \to X$, there is $Y \in \mathcal{H}$ such that $|c[Y]| < \kappa$. Throughout, we shall interchangeably use the partition and coloring formulations without explicit comment.

Given $0 < d < \omega$ and $\vec{X} = \langle X_0, ..., X_{d-1}\rangle$ a sequence of perfect Polish spaces, let $\mathsf{DDF}(\vec{X})$ denote the set of subsets of $\prod_{i<d} X_i$ which contain a somewhere-DDF subset. So the principle DDF_d says that for every such $\vec{X}$, the collection $\mathsf{DDF}(\vec{X})$ is weakly partition regular. Now consider another perfect Polish space X_d. We want to consider how partition properties of $\mathsf{DDF}(\vec{X})$ affect those of $\mathsf{DDF}(\vec{X}^\frown X_d)$.

Proposition 4.3. *Fix $d \geq 2$ and let $\vec{X}$ and X_d be as above. If $\mathsf{DDF}(\vec{X})$ is not weakly ω-partition subregular, then there is a 2-coloring of $\prod_{i \leq d} X_i$ witnessing that $\mathsf{DDF}(\vec{X}^\frown X_d)$ is not weakly partition regular.*

Proof. Assume that $\mathsf{DDF}(\vec{X})$ is not weakly ω-partition subregular, and fix a partition $\prod_{i<d} X_i = \bigcup_{j<\omega} P_j$ such that, for all $k < \omega$, $\bigcup_{j<k} P_j$ does not contain a somewhere-DDF set.

Let $\{S_n : n < \omega\}$ be a sequence of open subsets of X_d with the following property:

- For every non-empty open $U \subseteq X_d$, there is $N < \omega$ such that for all $n \geq N$, we have $S_n \cap U \neq \emptyset$ and $\mathrm{Int}(X_d \setminus S_n) \cap U \neq \emptyset$.

We now describe a partition $\prod_{i \leq d} X_i = P_0^* \cup P_1^*$ that will witness that $\mathsf{DDF}(\vec{X})$ is not weakly partition regular. Given $\vec{x} = \langle x_0, \ldots, x_d \rangle \in \prod_{i \leq d} X_i$, first let $j(\vec{x})$ be the unique natural number j such that $\langle x_0, \ldots, x_{d-1} \rangle \in P_j$. Now put $\vec{x}$ into P_0^* if $x_d \in S_{j(\vec{x})}$ and into P_1^* otherwise.

We claim that this partition is as desired. Suppose for the sake of contradiction that there are non-empty open $U_i \subseteq X_i$ such that P_0^* (wlog) contains a set Y which is DDF in $\prod_{i \leq d} U_i$. However, let $N < \omega$ be such that for all $n \geq N$, we have that $S_n \cap U_d$ and $\mathrm{Int}(X_d \setminus S_n) \cap U_d$ are non-empty. Then for every $\vec{x} = (x_0, \ldots, x_d) \in Y$, we must have that $(x_0, \ldots, x_{d-1}) \in \bigcup_{j<N} P_j$. This contradicts the assumption that $\bigcup_{j<N} P_j$ does not contain a somewhere-DDF set. $\square$

We now show that if the continuum is too small, then for a given $d < \omega$ and any $\vec{X}$ as above, we have that $\mathsf{DDF}(\vec{X})$ is not weakly ω-partition subregular. To that end, we now define a sequence of colorings $\langle c_n : 1 \leq n < \omega \rangle$, where $c_n : [\omega_n]^{n+1} \to \omega$ for all $1 \leq n < \omega$. The definition is by recursion on n. First, let $c_1 : [\omega_1]^2 \to \omega$ be any function such that, for all $\beta < \omega_1$, the fiber $c_1(\cdot, \beta) : \beta \to \omega$ is injective. Now suppose that $1 \leq n < \omega$ and we have defined c_n. For each $\beta < \omega_{n+1}$, let $e_\beta : \beta \to \omega_n$ be an injective function. Then, for each $\{\alpha_0, \ldots, \alpha_n, \beta\} \in [\omega_{n+1}]^{n+2}$ with $\alpha_0 < \ldots < \alpha_n < \beta$, set

$$c_{n+1}(\alpha_0, \ldots, \alpha_n, \beta) = c_n(e_\beta(\alpha_0), \ldots, e_\beta(\alpha_n)).$$

(Note that the $(n+1)$-tuple $(e_\beta(\alpha_0), \ldots, e_\beta(\alpha_n))$ may not be increasing, but it is certainly injective.)

Now, given $1 \leq n < \omega$ and $a \in [\omega_n]^{n+1}$, we specify a distinguished element $a(*) \in a$. We do this by recursion on n. First, if $n = 1$, then we simply let $a(*) = \min(a)$. Next, if $n > 1$, then let $\beta = \max(a)$, let $a_1 := \{e_\beta(\alpha) : \alpha \in a \setminus \{\beta\}\}$, and let $a(*) = e_\beta^{-1}(a_1(*))$.

Lemma 4.4. *Suppose that $1 \leq n < \omega$, $a, b \in [\omega_n]^{n+1}$, $a(*) \neq b(*)$, and $a \setminus \{a(*)\} = b \setminus \{b(*)\}$. Then $c_n(a) \neq c_n(b)$.*

Proof. We proceed by induction on n. If $n = 1$, then there is $\beta > \max\{a(*), b(*)\}$ such that $a = \{a(*), \beta\}$ and $b = \{b(*), \beta\}$. Then $c_1(a) \neq c_1(b)$ follows from the fact that $c_1(\cdot, \beta)$ is injective.

Next, suppose that $n > 1$. Let $\beta = \max(a) = \max(b)$, let $a_1 = \{e_\beta(\alpha) : \alpha \in a \setminus \{\beta\}\}$, and let $b_1 = \{e_\beta(\alpha) : \alpha \in b \setminus \{\beta\}\}$. Then

(1) $a(*) = e_\beta^{-1}(a_1(*))$ and $b(*) = e_\beta^{-1}(b_1(*))$;
(2) $a_1 \setminus \{a_1(*)\} = b_1 \setminus \{b_1(*)\}$;
(3) $c_n(a) = c_{n-1}(a_1)$ and $c_n(b) = c_{n-1}(b_1)$.

Items (1) and (2), combined with the induction hypothesis, imply that $c_{n-1}(a_1) \neq c_{n-1}(b_1)$, and then item (3) implies that $c_n(a) \neq c_n(b)$. □

Lemma 4.5. *Fix $1 \leq n < \omega$. There is a coloring $c\colon (\omega_n)^{n+1} \to (n+2) \times \omega$ and a sequence of natural numbers $\langle m_k : k < \omega \rangle$ such that, for every $k < \omega$ and every sequence $\langle A_i : i \leq n \rangle$ of elements of $[\omega_n]^{m_k}$, we have*

$$\left| c \left[\prod_{i \leq n} A_i \right] \right| > k.$$

Proof. We first define $c\colon (\omega_n)^{n+1} \to (n+2) \times \omega$. Suppose that $\vec{\alpha} = \langle \alpha_i : i \leq n \rangle \in (\omega_n)^{n+1}$. If there are $i < j \leq n$ such that $\alpha_i = \alpha_j$, then let $c(\vec{\alpha}) := (n+1, 0)$. Otherwise, set $a_{\vec{\alpha}} := \{\alpha_i : i \leq n\}$, and note that $a_{\vec{\alpha}} \in [\omega_n]^{n+1}$. Let $i_{\vec{\alpha}}$ be the unique $i \leq n$ such that $a_{\vec{\alpha}}(*) = \alpha_i$, and let $c(\vec{\alpha}) = (i_{\vec{\alpha}}, c_n(a_{\vec{\alpha}}))$.

We now define $\langle m_k : k < \omega \rangle$. First, let $m_0 = 1$. If $k > 0$, then first let $m_k^* < \omega$ be large enough so that

$$m_k^* \to (n+2)_k^{n+1},$$

i.e., for every coloring $r\colon [m_k^*]^{n+1} \to k$, there is $H \in [m_k^*]^{n+2}$ such that $r \restriction [H]^{n+1}$ is constant. Then let $m_k = (n+1) \cdot m_k^*$.

We claim that $\langle m_k : k < \omega \rangle$ is as desired. If $k = 0$, then this is clear, so fix $k > 0$ and a sequence $\langle A_i : i \leq n \rangle$ of elements of $[\omega_n]^{m_k}$, and suppose for sake of contradiction that $\left| c \left[\prod_{i \leq n} A_i \right] \right| \leq k$. First, since $m_k = (n+1) \cdot m_k^*$, we can find a sequence of pairwise disjoint sets $\langle A_i^* : i \leq n \rangle$ such that, for all $i \leq n$, $A_i^* \in [A_i]^{m_k^*}$. In particular, for all $\vec{\alpha} \in \prod_{i \leq n} A_i^*$, $\vec{\alpha}$ is injective and hence, if $c(\vec{\alpha}) = (i, \ell)$, then $i \leq n$ and $\ell = c_n(a_{\vec{\alpha}})$.

For each $i \leq n$, enumerate A_i^* in increasing order as $\langle \alpha_j^i \mid j < m_k^* \rangle$, and define a coloring r of $[m_k^*]^{n+1}$ as follows. Given $u \in [m_k^*]^{n+1}$, let $\vec{\alpha}_u^* := \langle \alpha_{u(i)}^i \mid i \leq n \rangle$, and let $r(u) = c(\vec{\alpha}_u^*)$. Since $\left| c \left[\prod_{i \leq n} A_i \right] \right| \leq k$, the coloring r takes at most k-many colors. Therefore, by our choice of m_k^*, we can find $H \in [m_k^*]^{n+2}$ such that $r \restriction [H]^{n+1}$ is constant, say with value (i^*, ℓ^*). By the last sentence of the previous paragraph, we know that $i^* \leq n$.

Enumerate H in increasing order as $\langle \ell_0, \ldots, \ell_{i^*-1}, \ell_{i^*,0}, \ell_{i^*,1}, \ell_{i^*+1}, \ldots, \ell_n \rangle$, i.e.:

- the first i^*-many elements of H are $\langle \ell_0, \ldots, \ell_{i^*-1} \rangle$;
- the next 2 elements of H are $\langle \ell_{i^*,0}, \ell_{i^*,1} \rangle$;
- the final $(n - i^*)$-many elements of H are $\langle \ell_{i^*+1}, \ldots, \ell_n \rangle$.

Let $u_0 = H \setminus \{\ell_{i^*,1}\}$ and $u_1 = H \setminus \{\ell_{i^*,0}\}$, let $\vec{\alpha}^0 = \vec{\alpha}_{u_0}^*$ and $\vec{\alpha}^1 = \vec{\alpha}_{u_1}^*$, and let $a^0 = a_{\vec{\alpha}^0}$ and $a^1 = a_{\vec{\alpha}^1}$. Since $r(u_0) = r(u_1) = (i^*, \ell^*)$, we have $c(\vec{\alpha}^0) = c(\vec{\alpha}^1) = (i^*, \ell^*)$, and hence

- $a_0(*) = \alpha_{\ell_0^*}^{i^*}$ and $a_1(*) = \alpha_{\ell_1^*}^{i^*}$;
- $c_n(a_0) = c_n(a_1) = \ell^*$.

However, we also know that $a_0(*) \neq a_1(*)$ and $a_0 \setminus \{a_0(*)\} = a_1 \setminus \{a_1(*)\}$ and hence, by Lemma 4.4, $c_n(a_0) \neq c_n(a_1)$. This is a contradiction. □

Theorem 4.6. *Suppose that $2 \leq d < \omega$ and $2^{\aleph_0} \leq \aleph_{d-1}$. Then, for every d-sequence $\vec{X}$ of perfect Polish spaces, $\mathsf{DDF}(\vec{X})$ is not weakly ω-partition subregular.*

We are not making any real attempt to optimize the value of m_k here; we just need to know that a natural number m_k with the desired properties exists.

Proof. Proceed by induction on d. Fix a sequence $\vec{X} = \langle X_0, \ldots, X_{d-1} \rangle$ of perfect Polish spaces. We can assume that we in fact have $2^{\aleph_0} = \aleph_{d-1}$, since if $2^{\aleph_0} = \aleph_m < \aleph_{d-1}$, then the induction hypothesis will imply that $\mathsf{DDF}(\vec{X}\restriction(m+1))$ is not weakly ω-partition subregular, which immediately implies that $\mathsf{DDF}(\vec{X})$ is not weakly ω-partition subregular either. For all $i < d$, X_i is a perfect Polish space, so we can injectively enumerate X_i as $\langle x_{i,\alpha} : \alpha < \omega_{d-1} \rangle$.

By Lemma 4.5 with $n = d-1$, we can find a coloring $c\colon \prod_{i<d} X_i \to (d+1) \times \omega$ and a sequence of natural numbers $\langle m_k : k < \omega \rangle$ such that, for every $k < \omega$ and every sequence $\langle A_i : i < d \rangle$ such that A_i is a subset of X_i of size m_k for each $i < d$, we have $|c[\prod_{i<d} A_i]| > k$. By Proposition 3.5, every $Y \in \mathsf{DDF}(\vec{X})$ contains arbitrarily large finite products, which by the discussion above implies that $c[Y]$ is infinite. □

Corollary 4.7. *Suppose that $2 \leq d < \omega$ and $2^{\aleph_0} \leq \aleph_{d-1}$. Then, for every $(d+1)$-sequence $\vec{X}$ of perfect Polish spaces, there is a 2-coloring of $\prod_{i<d} X_i$ witnessing that $\mathsf{DDF}(\vec{X})$ is not weakly partition regular.*

Proof. This is immediate from Proposition 4.3 and Theorem 4.6. □

We have now established clauses (1) and (2) of Theorem A.

Corollary 4.8. *Suppose that $2 \leq d < \omega$.*

(1) *If DDF_d holds, then $2^{\aleph_0} \geq \aleph_{d-1}$.*
(2) *If $\mathsf{PG}_d(\aleph_0)$ holds, then $2^{\aleph_0} \geq \aleph_d$.*

Proof. Note that DDF_d is equivalent to the assertion that, for every d-sequence $\vec{X}$ of perfect Polish spaces, $\mathsf{DDF}(\vec{X})$ is weakly partition regular, and $\mathsf{PG}_d(\aleph_0)$ implies that $\mathsf{DDF}(\vec{X})$ is weakly ω-partition subregular. Clauses (1) and (2) then immediately follow from Corollary 4.7 and Theorem 4.6, respectively. □

4.1. **On the Partition Hypothesis.** In [1], Bannister, Bergfalk, Moore, and Todorčević introduce a partition hypothesis denoted $\mathrm{PH}_n(\Lambda)$, where $n < \omega$ is a dimensional parameter and Λ is an arbitrary directed quasi-order. They prove there that, for all $n < \omega$, $\mathrm{PH}_n(\omega_n)$ fails, where ω_n has the usual ordinal ordering. The proof of this fact presented in [1] makes heavy use of ideas coming from simplicial homology. Here, we show how the results from this section yield a direct, purely combinatorial proof. We first recall the following definitions from [1].

Definition 4.9. Suppose that $1 \leq n < \omega$ and Λ is a directed quasi-order.

(1) If $\vec{x}, \vec{y} \in \Lambda^{\leq n}$, then we write $\vec{x} \trianglelefteq \vec{y}$ to indicate that $\vec{x}$ is a subsequence of $\vec{y}$ (not necessarily an initial segment). $\vec{x} \triangleleft \vec{y}$ indicates that $\vec{x}$ is a *proper* subsequence of $\vec{y}$.
(2) A function $F : \Lambda^{\leq n} \to \Lambda$ is *n-cofinal* if
    (a) $x \leq F(\langle x \rangle)$ for all $x \in \Lambda$;
    (b) $F(\vec{x}) \leq F(\vec{y})$ for all $\vec{x} \trianglelefteq \vec{y}$ in $\Lambda^{\leq n}$.
(3) Let $\Lambda^{[\![n]\!]} \subseteq \prod_{i<n} \Lambda^{i+1}$ consist of all $\sigma \in \prod_{i<n} \Lambda^{i+1}$ that are $\trianglelefteq$-increasing. If $F : \Lambda^{\leq n} \to \Lambda$ is n-cofinal, define $F^* : \Lambda^{[\![n]\!]} \to \Lambda^n$ by letting $F^*(\sigma) = F \circ \sigma = \langle F(\sigma(i)) \mid i < n \rangle$ for all $\sigma \in \Lambda^{[\![n]\!]}$.

Definition 4.10. Suppose that $n < \omega$ and Λ is a directed quasi-order. The Partition Hypothesis $\mathrm{PH}_n(\Lambda)$ is the following assertion: for all $c : \Lambda^{n+1} \to \omega$, there is an $(n+1)$-cofinal function $F : \Lambda^{\leq n+1} \to \Lambda$ such that $c \circ F^* : \Lambda^{[\![n+1]\!]} \to \omega$ is constant.

We are now ready to give a direct proof of the aforementioned result from [1].

Theorem 4.11. *For all $n < \omega$, $\mathrm{PH}_n(\omega_n)$ fails.*

Proof. As noted in [1], the identity function $c : \omega \to \omega$ readily witnesses the failure of $\mathrm{PH}_0(\omega)$. Therefore, fix $1 \leq n < \omega$, and let $c\colon (\omega_n)^{n+1} \to (n+2) \times \omega$ be the function defined in the proof of Lemma 4.5.

We claim that c witnesses the failure of $\mathrm{PH}_n(\omega_n)$. Towards a contradiction, suppose that $F\colon (\omega_n)^{n+1} \to \omega_n$ is an $(n+1)$-cofinal function such that $c \circ F^*$ is constant, taking value $(i^*, k^*) \in (n+2) \times \omega$. As shown in [1, Lemma 7.8], we may assume that F is *strictly increasing*, i.e., $F(\vec{x}) \lneq F(\vec{y})$ for all $\vec{x} \lhd \vec{y}$ in $(\omega_n)^{\leq n+1}$. In particular, we can assume that $F^*(\sigma) = F \circ \sigma$ is *injective* for all $\sigma \in \omega_n^{[\![n+1]\!]}$, and hence, recalling the definition of c, we know that $i^* \neq n+1$. In addition, since $F \circ \sigma$ is strictly increasing for all $\sigma \in \omega_n^{[\![n+1]\!]}$, the definition of c implies that we in fact have $i^* < n$.

Let $\alpha^* := F(\langle 0, 1, \ldots, i^* \rangle) + 1$. Now define $\sigma_0, \sigma_1 \in \omega_n^{[\![n+1]\!]}$ as follows:

- for all $i < i^*$, $\sigma_0(i) = \sigma_1(i) := \langle 0, 1, \ldots, i \rangle$;
- $\sigma_0(i^*) := \langle 0, 1, \ldots, i^* \rangle$;
- $\sigma_1(i^*) := \langle 0, 1, \ldots, i^* - 1, \alpha^* \rangle$;
- for all $\ell < n - i^*$, $\sigma_0(i^* + \ell + 1) = \sigma_1(i^* + \ell + 1) := \langle 0, 1, \ldots, i^*, \alpha^*, \alpha^* + 1, \ldots, \alpha^* + \ell \rangle$.

Let $a_0 := F^*(\sigma_0)$ and $a_1 := F^*(\sigma_1)$. Since σ_0 and σ_1 only differ in their i^*-th entry and $F^*(\sigma_0)$ and $F^*(\sigma_1)$ are strictly increasing, we have $a_0(i) = a_1(i)$ for all $i \in (n+1) \setminus \{i^*\}$. By the definition of c and the fact that $c \circ F^*$ is constant with value (i^*, k^*), it follows that $a_0(*) = a_1(*) = i^*$ (recall the notation from the paragraph preceding Lemma 4.4). Moreover, since $\alpha^* > F(\sigma_0(i^*))$ and F is $(n+1)$-cofinal, we know that

$$a_0(i^*) = F(\sigma_0(i^*)) < \alpha^* \leq F(\sigma_1(i^*)) = a_1(i^*).$$

Then Lemma 4.4 implies that $c_n(a_0) \neq c_n(a_1)$. However, again by the fact that $c \circ F^*$ is constant with value (i^*, k^*), it must be the case that $c_n(a_0) = c_n(a_1) = k^*$, which is a contradiction. $\square$

5. Forcing PG_d

We now finish the proof of Theorem A by proving part (3). We show that adding $\beth_{d-1}^+$-many Cohen reals to any model of ZFC yields a model of $\mathsf{PG}_d(\aleph_0)$. In particular, by starting with a model of GCH, Theorem 4.6 is consistently sharp. We will need the notion of *uniform n-dimensional Δ-system* isolated in [10]. We begin by recalling the relevant definitions.

Definition 5.1. Suppose that a and b are sets of ordinals.

(1) We say that a and b are *aligned* if $\mathrm{otp}(a) = \mathrm{otp}(b)$ and $\mathrm{otp}(a \cap \gamma) = \mathrm{otp}(b \cap \gamma)$ for all $\gamma \in a \cap b$. In other words, if γ is a common element of two aligned sets a and b, then it occupies the same relative position in both a and b.

(2) If a and b are aligned then we let $\mathbf{r}(a, b) := \{i < \mathrm{otp}(a) : a(i) = b(i)\}$. Notice that, in this case, $a \cap b = a[\mathbf{r}(a, b)] = b[\mathbf{r}(a, b)]$.

Definition 5.2. Suppose that H is a set of ordinals, n is a positive integer, and u_b is a set of ordinals for all $b \in [H]^n$. We call $\langle u_b : b \in [H]^n \rangle$ a *uniform n-dimensional Δ-system* if there are an ordinal ρ and, for each $\mathbf{m} \subseteq n$, a set $\mathbf{r}_{\mathbf{m}} \subseteq \rho$ satisfying the following statements.

(1) $\mathrm{otp}(u_b) = \rho$ for all $b \in [H]^n$.
(2) For all $a, b \in [H]^n$ and $\mathbf{m} \subseteq n$, if a and b are aligned with $\mathbf{r}(a,b) = \mathbf{m}$, then u_a and u_b are aligned with $\mathbf{r}(u_a, u_b) = \mathbf{r_m}$.
(3) For all $\mathbf{m}_0, \mathbf{m}_1 \subseteq n$, we have $\mathbf{r}_{\mathbf{m}_0 \cap \mathbf{m}_1} = \mathbf{r}_{\mathbf{m}_0} \cap \mathbf{r}_{\mathbf{m}_1}$.

The following is a corollary of the main lemma of [10].

Corollary 5.3. [10, Corollary 3.16] *Suppose that $1 \leq n < \omega$, and let $\mu = \beth_{n-1}^+$. If $\langle u_b : b \in [\mu]^n \rangle$ is a sequence of finite sets of ordinals and $g : [\mu]^n \to \omega$ is a function, then there is $H \in [\mu]^{\aleph_1}$ such that $\langle u_b : b \in [H]^n \rangle$ is a uniform n-dimensional Δ-system and $g \restriction [H]^n$ is constant.*

We now turn to the proof of Theorem A(3). Recall that every perfect Polish space contains a dense G_δ subspace homeomorphic to Baire space ${}^\omega\omega$. Therefore to show that $\mathsf{PG}_d(\aleph_0)$ holds, one may assume that each X_i is the Baire space. It will be helpful to write $X_i = [T_i]$, where each T_i is a copy of the tree ${}^{<\omega}\omega$. Let $\theta := \beth_{d-1}^+$, and let $\mathbb{P} = \mathrm{Add}(\omega, \beth_{d-1}^+)$ be the forcing to add $\beth_{d-1}^+$-many Cohen reals. We think of the conditions in $\mathbb{P}$ as being all finite partial functions $p : \theta \to \prod_{i<d} T_i$, and $q \leq_{\mathbb{P}} p$ if and only if $\mathrm{dom}(q) \supseteq \mathrm{dom}(p)$ and, for all $\alpha \in \mathrm{dom}(p)$ and $i < d$, we have $p(\alpha)(i) \leq_{T_i} q(\alpha)(i)$. For each $\alpha < \theta$ and $i \leq n$, let $\dot{x}_i^\alpha$ be the canonical $\mathbb{P}$-name for $\{p(\alpha)(i) \mid p \in \dot{G}\}$, where $\dot{G}$ is the canonical $\mathbb{P}$-name for the generic filter. By standard arguments, $\dot{x}_i^\alpha$ is forced to be an element of $[T_i]$ (as defined in $V^{\mathbb{P}}$).

For each $p \in \mathbb{P}$, we define a "collapsed" version of p, denoted $\bar{p}$, as follows. Let $\ell := |\mathrm{dom}(p)|$, and enumerate $\mathrm{dom}(p)$ in increasing order as $\langle \alpha_k : k < \ell \rangle$. Then define $\bar{p} : \ell \to \prod_{i<d} T_i$ by letting $\bar{p}(k) = p(\alpha_k)$ for all $k < \ell$. Note that each collapsed condition is a function from some natural number to a countable set, so there are only countably many such collapsed conditions.

Let $\dot{c}_0$ be a $\mathbb{P}$-name for a function from $\prod_{i<d}[T_i] \to \omega$. We will really only be interested in the values of $\dot{c}_0$ on $(n+1)$-tuples of the generic branches $\dot{x}_i^\alpha$, so let $\dot{c}$ be a $\mathbb{P}$-name for a function from $[\theta]^d$ to ω defined in the following way: for all $(\alpha_0, \ldots, \alpha_{d-1}) \in [\theta]^d$, let $\dot{c}(\alpha_0, \ldots, \alpha_{d-1}) = \dot{c}_0(\dot{x}_0^{\alpha_0}, \dot{x}_1^{\alpha_1}, \ldots, \dot{x}_{d-1}^{\alpha_{d-1}})$ (recall our convention that the notation $(\alpha_0, \ldots, \alpha_{d-1}) \in [\theta]^d$ implies that $\alpha_0 < \alpha_1 < \ldots < \alpha_{d-1}$).

Fix an arbitrary $p \in \mathbb{P}$. We will find $q \leq_{\mathbb{P}} p$ and $j < \omega$ such that q forces the existence of a sequence $\langle Y_i \mid i < d \rangle$ such that each Y_i is a somewhere dense subset of $[T_i]$ and $c \restriction \prod_{i<d} Y_i$ is constant, taking value j.

For each $a \in [\theta]^d$, find a condition $q_a \leq p$ and a $j_a < \omega$ such that $q_a \Vdash_{\mathbb{P}}$ "$\dot{c}(a) = j_a$". Let $u_a := \mathrm{dom}(q_a)$. Without loss of generality, assume that $a \subseteq u_a$ for every $a \in [\theta]^d$.

By Corollary 5.3, we can find $H \in [\theta]^{\aleph_1}$, a "collapsed" condition $\bar{q}^*$, natural numbers j^* and ρ, and a set $\mathbf{r}^* \in [\rho]^d$ such that

- for all $a \in [H]^d$, we have $\bar{q}_a = \bar{q}^*$ and $j_a = j^*$
- $\langle u_a : a \in [H]^d \rangle$ forms a uniform d-dimensional Δ-system; and
- for all $a \in [H]^d$, we have $|u_a| = \rho$ and $a = u_a[\mathbf{r}^*]$.

By taking an initial segment if necessary, assume that $\mathrm{otp}(H) = \omega_1$. Let $\langle \mathbf{r_m} : \mathbf{m} \subseteq d \rangle$ witness the fact that $\langle u_a : a \in [H]^d \rangle$ is a uniform d-dimensional Δ-system. For each $m < d$ and each $a \in [H]^m$, define u_a and q_a by choosing any $b \in [H]^d$ for which $b[m] = a$ and setting $u_a := u_b[\mathbf{r}_m]$ and $q_a := q_b \restriction u_a$. By our uniformization of H (cf. [2, Lemma 2.3]), these definitions are independent of our choice of b.

Let $q := q_\emptyset$. Since $q_a \leq_{\mathbb{P}} p$ for every $a \in [H]^d$, it follows that $\mathrm{dom}(p) \subseteq u_\emptyset$ and hence $q_\emptyset \leq p$. Also, for each $i < d$, let $s_i := \bar{q}(\mathbf{r}^*(i))(i)$. In other words, $s_i \in T_i$ is such that, for all $a = (\alpha_0, \ldots, \alpha_{d-1})$ in $[H]^d$, we have $q_a(\alpha_i)(i) = s_i$. We claim that q is as desired; in particular, q forces the existence of a sequence of sets $\langle Y_i : i < d\rangle$ such that

- for all $i < d$, Y_i is s_i-dense in $[T_i]$, i.e., for all $t \geq_{T_i} s_i$, there is $y \in Y_i$ such that $t \in y$;
- $\dot{c}_0 \restriction \prod_{i<d} Y_i$ is constant, taking value j^*.

Claim 5.4. *Suppose that $m < d$, $a \in [H]^m$, and $\gamma \in H \setminus (\max(a) + 1)$. Then the set $D_{a,\gamma} := \{q_{a^\frown\langle\beta\rangle} : \beta \in H \setminus \gamma\}$ is pre-dense below q_a in $\mathbb{P}$.*

Proof. Let $r \leq_{\mathbb{P}} q_a$ be arbitrary. The set $\{u_{a^\frown\langle\beta\rangle} : \beta \in H\setminus\gamma\}$ forms an uncountable Δ-system with root u_a. We can therefore find $\beta \in H\setminus\gamma$ for which $\mathrm{dom}(r)\cap(u_{a^\frown\langle\beta\rangle} \setminus u_a) = \emptyset$. We also know that $q_{a^\frown\langle\beta\rangle} \restriction u_a = q_a$, and $r \leq_{\mathbb{P}} q_a$. It follows that r and $q_{a^\frown\langle\beta\rangle}$ are compatible in $\mathbb{P}$, so $D_{a,\gamma}$ is indeed pre-dense below q_a. □

Now let G be $\mathbb{P}$-generic over V with $q \in G$, and let c be the realization of $\dot{c}$ in $V[G]$. By recursively applying Claim 5.4 d-many times, we can find a set $\delta \in [H]^d$, enumerated in increasing order as $\langle\delta_0, \ldots, \delta_{d-1}\rangle$ such that

- for all $i < d - 1$, $H \cap (\delta_i, \delta_{i+1})$ is infinite;
- $H \cap \delta_0$ is infinite;
- $q_\delta \in G$.

Let H_0 denote the set of the first ω-many elements of H, and for $i < d-1$, let H_{i+1} denote the set of the first ω-many elements of $H \cap (\delta_i, \delta_{i+1})$. Note that each H_i is an element of V. Now, working in $V[G]$, we will recursively construct a matrix of ordinals $\langle\alpha_{i,k} : i < d,\ k < \omega\rangle$ such that, setting $A_i := \{\alpha_{i,k} : k < \omega\rangle$ and $Y_i := \{x_i^\alpha : \alpha \in A_i\}$ for all $i < d$, we have the following:

- for all $i < d$, $A_i \subseteq H_i \cup \{\delta_i\}$;
- for all $i < d$, Y_i is s_i-dense in $[T_i]$;
- for all $a \in \prod_{i<d} A_i$, we have $q_a \in G$, and hence $c \restriction \prod_{i<d} Y_i$ is constant, taking value j^*.

The construction of the matrix of ordinals is by recursion on $k < \omega$ and, for fixed k, by recursion on $i < d$; in other words, the construction is by recursion on the anti-lexicographic ordering of $d \times \omega$.

For each pair $(i, k) \in d \times \omega$ and each $j < d$, let $A_j \restriction (i, k)$ be the portion of A_j constructed before stage (i, k) of the process, i.e., $A_j \restriction (i, k) = \{\alpha_{j,\ell} : \ell \leq k\}$ if $j < i$ and $A_j \restriction (i, k) = \{\alpha_{j,\ell} : \ell < k\}$ if $j \geq i$. Our recursion hypothesis will be the assumption that $q_a \in G$ for all $a \in \prod_{j\leq k} A_j \restriction (i, k)$ by the time we have reached stage (i, k) of the construction. Enumerate ${}^{<\omega}\omega$ as $\langle t_k \mid k < \omega\rangle$, with $t_0 = \emptyset$. We will also maintain the requirement that, for all $(i, k) \in d \times \omega$, $x_i^{\alpha_{i,k}}$ extends $s_i{}^\frown t_k$; this is what will ensure that Y_i is s_i-dense in $[T_i]$.

Begin by setting $\alpha_{i,0} := \delta_i$ for all $i \leq n$. The fact that $q_\delta \in G$ ensures that this satisfies the recursion hypotheses. Now suppose that $(i, k) \in d \times \omega$, with $i \geq 1$, and we have reached stage (i, k) of the construction. Let

$$r_{i,k} = \bigcup\{q_a : a \in \prod_{j<d} A_j \restriction (i, k)\}.$$

By our recursion hypothesis, $r_{i,k}$ is a condition in $\mathbb{P}$ and is in fact in G. Let $B_0 := \prod_{j<i} A_j \restriction (i,k)$ and $B_1 := \prod_{i<j<d} A_j \restriction (i,k)$. Note that both B_0 and B_1 are in V, as they are finite sets of finite sequences of ordinals. For each $\alpha \in H_i$, let

$$q^*_\alpha := \bigcup \{q_{b_0 \frown \langle \alpha \rangle \frown b_1} : b_0 \in B_0,\ b_1 \in B_1\}.$$

Notice that, for all $b_0, b'_0 \in B_0$ and $b_1, b'_1 \in B_1$, $b_0 \frown \langle \alpha \rangle \frown b_1$ and $b'_0 \frown \langle \alpha \rangle \frown b'_1$ are aligned; it follows that q^*_α is a condition in $\mathbb{P}$. Moreover, for all $(b_0, b_1) \in B_0 \times B_1$, we have $q_{b_0 \frown \langle \alpha \rangle \frown b_1}(\alpha)(i) = s_i$, and hence $q^*_\alpha(\alpha)(i) = s_i$. We can therefore extend q^*_α to a condition q^{**}_α with the same domain by letting $q^{**}_\alpha(\alpha)(i) = s_i \frown t_k$ and $q^{**}_\alpha(\eta)(j) = q^*_\alpha(\eta)(j)$ for all $(\eta, j) \in (\mathrm{dom}(q^*_\alpha) \times d) \setminus \{(\alpha, i)\}$.

Claim 5.5. *The set $E := \{q^{**}_\alpha : \alpha \in H_i \setminus \{\alpha_{i,\ell} \mid \ell < k\}\}$ is predense in $\mathbb{P}$ below $r_{i,k}$.*

Proof. Fix an arbitrary condition $r \leq_{\mathbb{P}} r_{i,k}$; we will find a condition in E that is compatible with r. Let $\mathbf{m} := d \setminus \{i\}$, and let $H^* := H_i \setminus \{\alpha_{i,\ell} \mid \ell < k\}$. For each $(b_0, b_1) \in B_0 \times B_1$, the set $\{u_{b_0 \frown \langle \alpha \rangle \frown b_1} : \alpha \in H^*\}$ forms a Δ-system whose root is equal to $v_{b_0,b_1} := u_{b_0 \frown \langle \alpha \rangle \frown b_1}[\mathbf{r_m}]$ for some (and therefore all) $\alpha \in H^*$. Since there are only finitely many such pairs (b_0, b_1) and since H^* is infinite, we can therefore fix $\alpha \in H^*$ such that, for all $(b_0, b_1) \in B_0 \times B_1$, we have

$$(u_{b_0 \frown \langle \alpha \rangle \frown b_1} \setminus v_{b_0,b_1}) \cap \mathrm{dom}(r) = \emptyset.$$

In particular, we have $\alpha \notin \mathrm{dom}(r)$.

We claim that q^{**}_α and r are compatible. Since $\alpha \notin \mathrm{dom}(r)$, it suffices to show that $q_{b_0 \frown \langle \alpha \rangle \frown b_1}$ and r are compatible for all $(b_0, b_1) \in B_0 \times B_1$. Thus, fix such a (b_0, b_1). Note that $b_0 \frown \langle \alpha \rangle \frown b_1$ and $b_0 \frown \langle \delta_i \rangle \frown b_1$ are aligned and that $q_{b_0 \frown \langle \alpha \rangle \frown b_1} \restriction v_{b_0,b_1} = q_{b_0 \frown \langle \delta_k \rangle \frown b_1} \restriction v_{b_0,b_1}$. Since $r_{i,k} \leq_{\mathbb{P}} q_{b_0 \frown \langle \delta_k \rangle \frown b_1}$ and $r \leq_{\mathbb{P}} r_{i,k}$, we know that r is compatible with $q_{b_0 \frown \langle \alpha \rangle \frown b_1} \restriction v_{b_0,b_1}$, and since $u_{b_0 \frown \langle \alpha \rangle \frown b_1} \setminus v_{b_0,b_1}$ is disjoint from $\mathrm{dom}(r)$, it follows that r is compatible with $q_{b_0 \frown \langle \alpha \rangle \frown b_1}$ and therefore with q^{**}_α. $\square$

By the claim and the fact that $r_{i,k} \in G$, we can fix an $\alpha_{i,k} \in H_i \setminus \{\alpha_{i,\ell} : \ell < k\}$ such that $q^{**}_{\alpha_{i,k}} \in G$ and proceed to the next stage of the recursive construction. At the end of the construction, our recursion hypothesis ensures that, for all $a \in \prod_{i<d} A_i$, we have $q_a \in G$ and hence $c \restriction \prod_{i<d} A_i$ is constant, taking value j^*. It follows that $c_0 \restriction \prod_{i<d} Y_i$ is constant, also taking value j^*. Finally, our construction ensures that for every $i < d$ and every $t \in {}^{<\omega}\omega$, there is $x \in Y_i$ extending $s_i \frown t$, and hence Y_i is s_i-dense in $[T_i]$, as desired. This completes the proof of Theorem A(3).

Corollary 5.6. *For every $1 \leq d < \omega$, it is consistent that $2^{\aleph_0} = \aleph_d$ and, for every d-sequence $\vec{X}$ of perfect Polish spaces, $\mathsf{DDF}(\vec{X})$ is weakly ω-partition regular.*

6. PG_d and coding trees

We conclude with a brief discussion of using PG to prove more detailed versions of the Halpern-Läuchli theorem pertaining to *coding trees*. In recent years, coding trees in various forms have been developed in [4] and [19] to code countable structures in a finite binary language, such as the Rado graph or Henson's triangle-free graph. Halpern-Läuchli theorems for the *strict similarity types* of [4] or the *aged embeddings* of [19], proven using a Harrington-style forcing argument, form the pigeon-hole principle used to show that certain Fraïssé classes have finite big Ramsey degrees. Indeed, this is a major motivation for trying to develop new proofs of the Halpern-Läuchli theorem, as these new proofs might generalize to previously unknown settings.

With this in mind, we show how PG can be used to prove a version of HL for coding trees of a simple form. Fix $0 < d, k < \omega$, and for each $i < d$, let T_i be a copy of the tree ${}^{<\omega}k$. We assume that $T_i \cap T_j = \emptyset$ for $i \neq j < d$. We now enrich each T_i to a structure $\mathbf{T}_i$ by declaring that for each $m < \omega$, at most one node of $\bigcup_{i<d} T_i(m)$ is a *coding node*. Write $\mathbf{T} = \langle \mathbf{T}_0, ..., \mathbf{T}_{d-1} \rangle$ for this sequence of structures. If $\bigcup_{i<d} T_i(m)$ contains a coding node, we write $c_{\mathbf{T}}(m)$ for this node. We assume that for every $i < d$ and every $t \in T_i$, there is some $n < \omega$ so that $c_{\mathbf{T}}(n) \in T_i$ and $t \sqsubseteq c_{\mathbf{T}}(n)$.

An *embedding* of $\mathbf{T}$ into itself is an injection $f \colon \bigcup_{i<d} T_i \to \bigcup_{i<d} T_i$ satisfying the following properties:

(1) $f[T_i] \subseteq T_i$ for each $i < d$,
(2) f preserves tree order, meets, relative levels, and lexicographic order. Write $\tilde{f} \colon \omega \to \omega$ for the induced function on levels.
(3) If $c_{\mathbf{T}}(m)$ exists, then so does $c_{\mathbf{T}}(\tilde{f}(m))$, and we have $f(c_{\mathbf{T}}(m)) = c_{\mathbf{T}}(\tilde{f}(m))$.

Write $\mathrm{Emb}(\mathbf{T}, \mathbf{T})$ for the set of embeddings of $\mathbf{T}$ into itself.

Note that if we remove the extra coding node structure and item (3) from the above, then items (1) and (2) describe strong subtrees of the form appearing in the ordinary Halpern-Läuchli theorem. To state the version for these coding trees, first observe that the level product $\bigotimes \mathbf{T}_i$ now contains $d+1$ different types of elements; given $(t_0, ..., t_{d-1}) \in \bigotimes \mathbf{T}_i$, either none of the t_i is a coding node, or exactly one of the t_i is a coding node. We refer to the former case as *type* -1 and the latter as *type* i for a given $i < d$. Given $\mathsf{p} \in \{-1, ..., d-1\}$, let $\mathbf{T}(\mathsf{p}) \subseteq \bigotimes \mathbf{T}_i$ denote those tuples of type p.

Theorem 6.1. *Given $\mathsf{p} \in \{-1, 0, ..., d-1\}$, $r < \omega$, and a coloring $\gamma \colon \mathbf{T}(\mathsf{p}) \to r$, there is $f \in \mathrm{Emb}(\mathbf{T}, \mathbf{T})$ such that $f[\mathbf{T}(\mathsf{p})] \subseteq \mathbf{T}(\mathsf{p})$ is monochromatic for γ.*

We opt to give a relatively straightforward proof using PG_d. Then by Shoenfield absoluteness, this yields a ZFC proof.

Proof. For each $i < d$, let $X_i \subseteq [T_i]$ consist of those branches which contain infinitely many coding nodes. Then X_i is a dense G_δ subspace of $[T_i]$, so is itself a perfect Polish space. In the case where $\mathsf{p} = -1$, the proof is almost identical to that of Proposition 3.2, the key difference being that one works inside the space $\prod_{i<d} X_i$ so that when building f, we can ensure there are coding nodes where we need them.

So now assume that without loss of generality $\mathsf{p} = 0$. For each $y \in X_0$, let $U \colon X_0 \to \beta\omega$ be a function such that for every $y \in X_0$, we have that $C(y) := \{n < \omega : c_{\mathbf{T}}(n) \in y\} \in U(y)$. Form a coloring $\tilde{\gamma} \colon \prod_{i<d} X_i$ by setting $\tilde{\gamma}(y_0, ..., y_{d-1}) = j < r$ iff $\{n \in C(y_0) : \gamma(y_0(n), ..., y_{d-1}(n)) = j\} \in U(y_0)$. Using PG_d, find somewhere dense sets $Y_i \subseteq X_i$ for each $i < d$ such that $\prod_{i<d} Y_i$ is monochromatic, say with color $j < r$. Say that $(s_0, ..., s_{d-1}) \in \bigotimes \mathbf{T}_i$ is such that each Y_i is dense above s_i. We now proceed to define $f \in \mathrm{Emb}(\mathbf{T}, \mathbf{T})$. Suppose $m < \omega$ and that f has been defined on $\bigcup_{\ell<m} \mathbf{T}_i(\ell)$. If $m = 0$, then letting $\emptyset_i \in \mathbf{T}_i(0)$ denote the root, define $f'(\emptyset_i) = s_i$. If $m > 0$, we define the map $f' \colon \bigcup_{i<d} \mathbf{T}_i(m)$, where given $t \in \bigcup_{i<d} \mathbf{T}_i(m)$ with $t = s^\frown b$ for some $s \in \bigcup_{i<d} \mathbf{T}_i(m-1)$ and some $b < k$, then we set $f'(t) = f(s)^\frown b$. If $\mathbf{T}_0(m)$ does not contain a coding node, then we can define f on $\bigcup_{i<d} \mathbf{T}_i(m)$ as follows. If $c_{\mathbf{T}}(m)$ exists, pick any $n < \omega$ such that $c_{\mathbf{T}}(n) \sqsupseteq f'(c_{\mathbf{T}}(m))$, set $f(c_{\mathbf{T}}(m)) = c_{\mathbf{T}}(n)$, and for every other $t \in \bigcup_{i<d} \mathbf{T}_i(m)$, let $f(t) \in \mathrm{succ}(f'(t), n)$ be any node. If $c_{\mathbf{T}}(m)$ does not exist, then pick any large enough $n < \omega$ and simply let $f(t) \in \mathrm{succ}(f'(t), n)$. If $c_{\mathbf{T}}(m) \in \mathbf{T}_0(m)$, first for each $i < d$ and each $t \in \mathbf{T}_i(m)$, fix a branch $y_t \in Y_i$ with $f'(t) \in y_0$. Writing $y = y_{c_{\mathbf{T}}(m)}$, then for any choice of $t_i \in \mathbf{T}_i(m)$ for $0 < i < d$, we have $\{n \in C(y) : \gamma(y(n), y_{t_i}(n), ..., y_{t_{d-1}}(n)) = j\} \in U(y)$. Hence we can find $n \in C(y)$ so that for every $0 < i < d$ and every $t_i \in \mathbf{T}_i(m)$, we have that $\gamma(y(n), y_{t_1}(n), ..., y_{t_{d-1}}(n)) = j$. We then set $f(t) = y_t(n)$ for every $t \in \bigcup_{i<d} \mathbf{T}_i(m)$. □

One major difficulty in generalizing the above argument to the more general coding trees and aged embeddings of [19] is that in general, the forcing one needs to use is not Cohen forcing. This suggests that rather than the principle PG, one would ask for a combinatorial principle corresponding to each specific type of forcing used. Nonetheless, the following seems like a worthwhile question to ask.

Question 6.2. *Is there a family of consistent combinatorial principles about partitions of structures on products of Polish spaces which implies all of the variants of the Halpern-Läuchli theorem appearing in* [4] *and* [19]*?*

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Institute of Mathematics of the Czech Academy of Sciences
Žitná 25, Prague 1, 115 67, Czech Republic
Email address: lambiehanson@math.cas.cz
URL: http://math.cas.cz/lambiehanson

Department of Pure Mathematics
University of Waterloo
200 University Ave. W.
Waterloo, ON, Canada N2L 3G1
Email address: a3zucker@uwaterloo.ca