INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

A short comment on two questions of Kuznetsov

Florian Oschmann

Preprint No. 51-2022

PRAHA 2022

A SHORT COMMENT ON TWO QUESTIONS OF KUZNETSOV

FLORIAN OSCHMANN

ABSTRACT. We provide a proof and a counterexample to two conjectures made by N. Kuznetsov.

1. INTRODUCTION

Mean value properties (MVP) as well as weighted MVP play a crucial role in the theory of partial differential equations. In the recent paper [Kuz22], the author investigates MVP for so-called ($\mu-$)panharmonic functions satisfying

$$\Delta u - \mu^2 u = 0 \text{ in } \Omega, \quad \mu \in \mathbb{R} \setminus \{0\}, \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a domain, and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$ is the two-dimensional Laplacian. One of their main results reads as follows:

Theorem 1 ([Kuz22, Theorem 3]). *Let $\Omega \subset \mathbb{R}^2$ be a domain. If u is panharmonic in Ω, then*

$$a(\mu r)u(x) = \fint_{D_r(x)} u(y) \log \frac{r}{|x-y|} \, \mathrm{d}y, \quad a(t) = \frac{2[I_0(t) - 1]}{t^2}, \tag{2}$$

for any $r > 0$ such that $\overline{D_r(x)} = \{y \in \mathbb{R}^2 : |x-y| \leq r\} \subset \Omega$. Here, we denoted

$$\fint_{D_r(x)} u(y) \, \mathrm{d}y = \frac{1}{\pi r^2} \int_{D_r(x)} u(y) \, \mathrm{d}y,$$

and $I_0(t)$ is the modified Bessel function of the first kind of order zero.

As $\mu \to 0$, one should expect that solutions to (1) formally converge to a harmonic function. Moreover, as $a(0) = \lim_{t \to 0} a(t) = \frac{1}{2}$, one shall also think that (2) turns into

$$\Delta u = 0 \Rightarrow \frac{1}{2} u(x) = \fint_{D_r(x)} u(y) \log \frac{r}{|x-y|} \, \mathrm{d}y.$$

Indeed, this is the content of [Kuz22, Remark 1]. Unfortunately, this claim is not proven there. The objective of the present note is to provide a short proof of this fact. To make things precise, we will show:

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be a domain, and let $\Delta u = 0$ in Ω. Then, for any $x \in \Omega$ and any $r > 0$ such that $\overline{D_r(x)} \subset \Omega$, it holds*

$$\frac{1}{2} u(x) = \fint_{D_r(x)} u(y) \log \frac{r}{|x-y|} \, \mathrm{d}y.$$

Going even further, by the properties of the Bessel function $I_0(t)$, the function $a(t)$ from (2) increases strictly monotone and satisfies $a(t) > a(0) = \frac{1}{2}$ for any $t > 0$. As a direct consequence of Theorem 1, Corollary 1 in [Kuz22] states that for $\mu > 0$ any $\mu-$panharmonic function u with $u \geq 0$ that does not identically vanish inside Ω satisfies the inequality

$$\frac{1}{2} u(x) < \fint_{D_r(x)} u(y) \log \frac{r}{|x-y|} \, \mathrm{d}y \tag{3}$$

for any admissible disc $D_r(x) \subset \Omega$. As a matter of fact, any nonnegative panharmonic function is subharmonic, that is, $-\Delta u \leq 0$ (see e.g. [Kuz22, Theorem 1 and Remark 2]). Kuznetsov therefore conjectured that inequality (3) also holds for any subharmonic function $u \geq 0$ that does not vanish identically in Ω. However, this is not true; in fact, Lemma 2 directly forces the following

Date: September 23, 2022.

Corollary 3. *Let* $\Omega = D_1(0)$ *and set* $u(x_1,x_2) = e^{x_1}\sin(x_2)+3$. *Then* u *is harmonic (in particular subharmonic) with* $u \geq 0$ *in* Ω, *but* (3) *does not hold for any* $x \in \Omega$.

Proof. A short calculation shows that $u \geq 0$ in Ω. Moreover, obviously, $\Delta u = 0$, so the assumptions of Lemma 2 are satisfied and we conclude easily. □

Remark 4. *Since* $e^{x_1}\sin(x_2)$ *is analytic, we obviously can exchange* $\Omega = D_1(0)$ *in the previous Corollary by any domain* $\Omega \subset \{(x_1,x_2) \in \mathbb{R}^2 : x_1 < c\}$ *for some* $c \in \mathbb{R}$ *and recall the proof for* $\tilde{u}(x_1,x_2) = e^{x_1}\sin(x_2) - \min_{(x_1,x_2)\in\Omega} e^{x_1}\sin(x_2)$.

The fact that inequality (3) holds for $\mu-$panharmonic functions with $\mu > 0$ is due to $a(t) > a(0) = \frac{1}{2}$ for any $t > 0$ and the MVP (2). Of course, one might instead ask whether inequality (3) holds for any function $u \geq 0$ with $u \not\equiv 0$ which is *strictly* subharmonic, i.e., $-\Delta u \lneq 0$ in Ω.

2. Proof of Lemma 2

Before proving Lemma 2, we recall the well-known fact that harmonic functions satisfy both the mean value property and the spherical mean value property, i.e.,

$$u(x) = \fint_{D_r(x)} u(y)\ \mathrm{d}y = \frac{1}{\pi r^2}\int_{D_r(x)} u(y)\ \mathrm{d}y \tag{4}$$

$$= \fint_{\partial D_r(x)} u(y)\ \mathrm{d}\sigma(y) = \frac{1}{2\pi r}\int_{\partial D_r(x)} u(y)\ \mathrm{d}\sigma(y), \tag{5}$$

for any admissible $r > 0$. Note further that by rescaling, we have from (5)

$$u(x) = \fint_{\partial D_r(x)} u(y)\ \mathrm{d}\sigma(y) = \fint_{\partial D_1(0)} u(x+ry)\ \mathrm{d}\sigma(y) = \frac{1}{2\pi}\int_0^{2\pi} u(x+re^{i\varphi})\ \mathrm{d}\varphi, \tag{6}$$

where we identified the plane $\mathbb{R}^2$ with the complex numbers $\mathbb{C} = \mathbb{R} + i\,\mathbb{R} \simeq \mathbb{R}^2$, $i^2 = -1$.

We are now in the position to prove Lemma 2.

Proof of Lemma 2. First, note that

$$\fint_{D_r(x)} u(y)\log\frac{r}{|x-y|}\ \mathrm{d}y = \fint_{D_r(x)} u(y)\log r\ \mathrm{d}y - \fint_{D_r(x)} u(y)\log|x-y|\ \mathrm{d}y$$
$$= u(x)\log r - \fint_{D_r(x)} u(y)\log|x-y|\ \mathrm{d}y$$

since u is harmonic and so satisfies (4). In turn, it is enough to show

$$\fint_{D_r(x)} u(y)\log|x-y|\ \mathrm{d}y = u(x)\left[\log r - \frac{1}{2}\right].$$

For the sequel, we set without loss of generality $x = 0$ (otherwise do a transformation $z = x - y$ in the integrals and repeat the computations for $v_x(z) = u(x-z)$). Using that u also satisfies (6), we deduce

$$\int_{D_r(0)} u(y)\log|y|\ \mathrm{d}y = \int_0^r\int_0^{2\pi} u(se^{i\varphi})\log s\cdot s\ \mathrm{d}\varphi\ \mathrm{d}s = \int_0^r s\log s\int_0^{2\pi} u(se^{i\varphi})\ \mathrm{d}\varphi\ \mathrm{d}s$$
$$\overset{(6)}{=} 2\pi u(0)\int_0^r s\log s\ \mathrm{d}s = 2\pi u(0)\left[\frac{r^2}{2}\log r - \frac{r^2}{4}\right] = \pi r^2 u(0)\left[\log r - \frac{1}{2}\right].$$

Dividing by πr^2, this finishes the proof. □

3. Acknowledgements

The author was supported by the Czech Science Foundation (GAČR) project GA22-01591S.

References

[Kuz22] Nikolay Kuznetsov, *Characterizations of discs via weighted means*, arXiv preprint arXiv:2209.10281v1 (2022).

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic.
Email address: `oschmann@math.cas.cz`