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A NON-STABLE C^*-ALGEBRA WITH AN ELEMENTARY ESSENTIAL COMPOSITION SERIES

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Abstract. A C^*-algebra $\mathcal{A}$ is said to be stable if it is isomorphic to $\mathcal{A} \otimes \mathcal{K}(\ell_2)$. Hjelmborg and Rørdam have shown that countable inductive limits of separable stable C^*-algebras are stable. We show that this is no longer true in the nonseparable context even for the most natural case of an uncountable inductive limit of an increasing chain of separable stable and AF ideals: we construct a GCR, AF (in fact, scattered) subalgebra $\mathcal{A}$ of $\mathcal{B}(\ell_2)$, which is the inductive limit of length ω_1 of its separable stable ideals $\mathcal{I}_\alpha$ ($\alpha < \omega_1$) satisfying $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \cong \mathcal{K}(\ell_2)$ for each $\alpha < \omega_1$, while $\mathcal{A}$ is not stable. The sequence $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$ is the GCR composition series of $\mathcal{A}$ which in this case coincides with the Cantor-Bendixson composition series as a scattered C^*-algebra. $\mathcal{A}$ has the property that all of its proper two-sided ideals are listed as $\mathcal{I}_\alpha$s for some $\alpha < \omega_1$ and therefore the family of stable ideals of $\mathcal{A}$ has no maximal element.

By taking $\mathcal{A}' = \mathcal{A} \otimes \mathcal{K}(\ell_2)$ we obtain a stable C^*-algebra with analogous composition series $(\mathcal{J}_\alpha)_{\alpha<\omega_1}$ whose ideals $\mathcal{J}_\alpha$s are isomorphic to $\mathcal{I}_\alpha$s for each $\alpha < \omega_1$. In particular, there are nonisomorphic scattered C^*-algebras whose GCR composition series $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$ satisfy $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \cong \mathcal{K}(\ell_2)$ for all $\alpha < \omega_1$, for which the composition series differ first at $\alpha = \omega_1$.

1. INTRODUCTION

Definition 1.1. *Let $\mathcal{A}$ be a C^*-algebra and β be an ordinal. A sequence of ideals $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ of $\mathcal{A}$ is called an elementary essential composition series for $\mathcal{A}$ of length β if and only if*

(a) $\mathcal{I}_0 = \{0\}$, $\mathcal{I}_\beta = \mathcal{A}$, $\mathcal{I}_\alpha \subseteq \mathcal{I}_{\alpha'}$ for $\alpha \leq \alpha' \leq \beta$,
(b) $\mathcal{I}_\lambda = \overline{\bigcup_{\alpha<\lambda} \mathcal{I}_\alpha}$ for all limit ordinals $\lambda \leq \beta$,

For every $\alpha < \beta$

(c) $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is an essential ideal of $\mathcal{A}/\mathcal{I}_\alpha$,
(d) $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is isomorphic to $\mathcal{K}(\ell_2)$.

Definition 1.2. *A C^*-algebra $\mathcal{A}$ is called stable if it is isomorphic to $\mathcal{A} \otimes \mathcal{K}(\ell_2)$.*

The purpose of this article is to prove the following:

Theorem 1.3. *There is a nonstable C^*-subalgebra of $\mathcal{B}(\ell_2)$ which has an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$, where ω_1 is the first uncountable cardinal.*

Proof. Combine Theorems 2.5, 3.10 and 4.2. □

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Let us discuss several aspects of this construction. First recall that a C^*-algebra $\mathcal{A}$ is called approximately finite dimensional (AF) if $\mathcal{A}$ contains a directed family of finite-dimensional subalgebras whose union is dense in $\mathcal{A}$, and it is called locally-finite dimensional (LF) if every finite subset of $\mathcal{A}$ can be approximated from a finite-dimensional subalgebra of $\mathcal{A}$. See the paper [9] of Farah and Katsura for the results showing the relations between these notions in the nonseparable case. In particular, AF implies LF, and being AF is equivalent to being LF for separable C^*-algebras by a result of Bratteli [5]. More generally, for C^*-algebras of density ω_1, like our algebra from Theorem 1.3, by a result of Farah and Katsura ([9]) the two notions of AF and LF are equivalent as well. However, they also showed that the two notions are not equivalent in general.

Now let us list a few classical results concerning separable C^*-algebras which are relevant to our construction:

(1) An extension of a separable AF-algebra by a separable AF-algebra algebra is a separable AF-algebra (Brown [6], cf. [7]).
(2) Countable inductive limit of separable AF-algebras are AF-algebras (Bratteli [5], cf. [7]).
(3) An extension of a separable stable AF-algebra by a separable stable AF-algebra is stable (Blackadar [3], cf. 6.12 of [28], 7.3 of [11]).
(4) Countable inductive limits of separable stable algebras are stable (Hjelmborg and Rørdam 4.1 of [16]).

Combining these results one concludes that all ideals $\mathcal{I}_\alpha$ for $\alpha < \omega_1$ from the elementary essential composition series of our algebra $\mathcal{A}$ from Theorem 1.3 are separable stable AF-algebras. So we observe a strong failure of the permanence of stability in the context of the simplest uncountable inductive limits of separable AF-algebras:

Theorem 1.4. *There is an AF-algebra which is an uncountable inductive limit of an increasing chain of separable stable AF-ideals which is not stable.*

One should mention here other recent results concerning the failure of the permanence of stability and/or being AF for quite fundamental nonseparable C^*-algebras:

- There are 2-subhomogenous non-AF extensions of nonseparable AF-algebras by AF-algebras (Theorem 1.12 of [2]).
- There are nonstable C^*-algebras $\mathcal{A}$ satisfying the following short exact sequence:
$$0 \to \mathcal{K}(\ell_2) \xrightarrow{\iota} \mathcal{A} \to \mathcal{K}(\ell_2(2^\omega)) \to 0,$$
where $\iota[\mathcal{K}(\ell_2)]$ is an essential ideal of $\mathcal{A}$ ([12]).

A partial version of Theorem 1.4 for an inductive limit of length possibly bigger than the first uncountable ordinal ω_1 of possibly nonseparable C^*-algebras was obtained in Theorem 7.7. of [11].

Another aspect of our construction is related to the class of scattered C^*-algebras. They probably first appeared in a paper [30] of Tomiyama, but were first explicitly defined by H. Jensen in [18]. Some of many equivalent conditions defining scattered C^*-algebras are: every nonzero quotient has a minimal projection or the spectrum of every self-adjoint element is at most countable ([33]) or every subalgebra is LF [19]. For a recent survey on scattered C^*-algebras see [11]. Commutative scattered C^*-algebras are exactly of the form $C_0(X)$ where X is locally compact and scattered, i.e., its every nonempty (closed) subset has a relatively isolated point.

In fact, such spaces must be totally disconnected and so, by the Stone duality they correspond exactly to superatomic Boolean algebras ([24]). In [11] we introduced a canonical composition series $(\mathcal{I}_\alpha(\mathcal{A}))_{\alpha\leq ht(\mathcal{A})}$ for a scattered C^*-algebra $\mathcal{A}$ which corresponds to the Cantor-Bendixson derivatives in the commutative case. $\mathcal{I}_{\alpha+1}(\mathcal{A})$ is defined by requiring that $\mathcal{I}_{\alpha+1}(\mathcal{A})/\mathcal{I}_\alpha(\mathcal{A})$ is the subalgebra generated in $\mathcal{A}/\mathcal{I}_\alpha(\mathcal{A})$ by all minimal projections. In a scattered C^*-algebra the ideals $\mathcal{I}_{\alpha+1}(\mathcal{A})/\mathcal{I}_\alpha(\mathcal{A})$ for $\alpha < ht(\mathcal{A})$ are essential in $\mathcal{A}/\mathcal{I}_\alpha(\mathcal{A})$ and isomorphic to nondegenerate subalgebras of $\mathcal{K}(\ell_2(\kappa_\alpha))$ for some cardinals κ_α. Moreover the algebra $\mathcal{A}$ is scattered if and only if $\mathcal{A} = \mathcal{I}_{ht(\mathcal{A})}(\mathcal{A})$ for some ordinal $ht(\mathcal{A})$ called the height of the scattered algebra. The width of $\mathcal{A}$ is defined as $\sup\{\kappa_\alpha : \alpha < ht(\mathcal{A})\}$. Following the commutative convention, a scattered C^*-algebra of height ω_1 and width ω is called thin-tall. Then a superatomic Boolean algebra is thin-tall if its Stone space is thin-tall and a locally compact Hausdorff space X is thin-tall if $C_0(X)$ is a thin-tall C^*-algebra.

Thin-tall locally compact Hausdorff spaces found an impressive amount of applications in topology usually as versions of the Ostaszewski space or the Kunen line ([23], [26], [22]). In fact the Banach spaces $C_0(X)$ where X are thin-tall play an important role in the Banach space theory as well ([13], [22], [15]). Our construction shows that, as in the commutative case, maximally noncommutative thin-tall algebras do not need not be isomorphic to each other. To state precisely a corollary of our construction we need a notion of a fully noncommutative scattered C^*-algebra:

Definition 1.5 ([11] Definition 6.1)**.** *In the above notation a scattered C^*-algebra is fully noncommutative if and only if the consecutive quotients $\mathcal{I}_{\alpha+1}(\mathcal{A})/\mathcal{I}_\alpha(\mathcal{A})$ for $\alpha < ht(\mathcal{A})$ of the Cantor-Bendixson composition series $(\mathcal{I}_\alpha(\mathcal{A}))_{\alpha\leq ht(\mathcal{A})}$ are isomorphic to the algebras of all compact operators $\mathcal{K}(\ell_2(\kappa_\alpha))$, respectively.*

Lemma 1.6. *Suppose that β is an ordinal, $\mathcal{A}$ is a C^*-algebra and $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ is a sequence of its ideals. $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ is an elementary essential composition series for $\mathcal{A}$ if and only if $\mathcal{A}$ is scattered fully noncommutative with Cantor-Bendixson composition series equal to $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ and $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \cong \mathcal{K}(\ell_2)$ for every $\alpha < \beta$.*

Proof. For the forward implication, by Definition 1.3 and Theorem 1.4 of [11] we need to note that the ideal $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ of $\mathcal{A}/\mathcal{I}_\alpha$ is equal to the ideal $\mathcal{I}^{At}(\mathcal{A}/\mathcal{I}_\alpha)$ generated by the minimal projections of $\mathcal{A}/\mathcal{I}_\alpha$. For this, by Theorem 1.2 of [11] it is enough to note that $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$ is essential in $\mathcal{A}/\mathcal{I}_\alpha$ and isomorphic to a subalgebra of $\mathcal{K}(\ell_2)$ which is the case by Definition 1.1 (c), (d).

For the backward implication we apply Theorem 1.4 of [11] to obtain (a), (b) of Definition 1.1, Proposition 4.3 and Theorem 1.4 to obtain (c) and the definition of being fully noncommutative to obtain (d). □

It turns out that being fully noncommutative is equivalent to a strong noncommutativity condition, namely that the center of the multiplier algebra of any quotient is trivial (Proposition 6.3 of [11]). Also for a fully noncommutative scattered C^*-algebra the Cantor-Bendixson composition series coincides with the GCR composition series (Proposition 6.4 of [11]) and full noncommutativity is equivalent to the stability for separable scattered C^*-algebras such that $\kappa_\alpha = \omega$ for all $\alpha < ht(\mathcal{A})$ (Lemma 7.3 of [11]). So our construction yields the following:

Theorem 1.7. *There are two scattered thin-tall fully noncommutative C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with the Cantor-Bendixson composition series $(\mathcal{I}_\alpha(\mathcal{A}))_{\alpha\leq\omega_1}$ and $(\mathcal{I}_\alpha(\mathcal{B}))_{\alpha\leq\omega_1}$ such that $\mathcal{I}_\alpha(\mathcal{A})$ is isomorphic to $\mathcal{I}_\alpha(\mathcal{B})$ for every $\alpha < \omega_1$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$, namely $\mathcal{B}$ is stable and $\mathcal{A}$ is not.*

Proof. Let $\mathcal{A}$ be the algebra satisfying Theorem 1.3 with an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$. By Lemma 1.6 the algebra $\mathcal{A}$ is scattered thin-tall fully noncommutative whose Cantor-Bendixson composition series is $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$.

Consider $\mathcal{B} = \mathcal{A}\otimes\mathcal{K}(\ell_2)$. It follows from Proposition 5.3 of [11] that $\mathcal{B}$ is scattered of the same height ω_1 whose Cantor-Bendixson composition series $(\mathcal{I}_\alpha(\mathcal{B}))_{\alpha\leq\omega_1}$ satisfies $\mathcal{I}_\alpha(\mathcal{B}) = \mathcal{I}_\alpha \otimes \mathcal{K}(\ell_2)$ for all $\alpha \leq \omega_1$. But $\mathcal{I}_\alpha \otimes \mathcal{K}(\ell_2)$ is isomorphic to $\mathcal{I}_\alpha(\mathcal{A})$ for $\alpha < \omega_1$ by the stability of $\mathcal{I}_\alpha(\mathcal{A})$ as observed after (1) - (4). Proposition 5.3. of [11] also implies that $\mathcal{I}_{\alpha+1}(\mathcal{B})/\mathcal{I}_\alpha(\mathcal{B})$ is isomorphic to $\mathcal{K}(\ell_2) \otimes \mathcal{K}(\ell_2) \cong \mathcal{K}(\ell_2)$ for every $\alpha < \omega_1$ which gives that $\mathcal{B}$ is scattered thin-tall fully noncommutative. However, Theorem 1.3 yields that $\mathcal{A}$ is not stable while $\mathcal{B}$ is trivially stable. □

In fact, our construction uses similar combinatorial ideas as the first absolute construction of two nonisomorphic thin-tall superatomic Boolean algebras from [29] due to Simon and Weese (cf. [24]). The latter corresponds to a locally compact thin-tall X which cannot be written as a disjoint union $X = X_1 \cup X_2$ where both of X_1 and X_2 are clopen and nonmetrizable. On the other hand, $X \times \mathbb{N}$ is also thin-tall locally compact but can be written as a disjoint union as above. In analogy to the above property, in our algebra $\mathcal{A}$ there are no two nonseparable subalgebras $\mathcal{A}', \mathcal{A}'' \subseteq \mathcal{A}$ such that $A'A'' = 0$ for all $A' \in \mathcal{A}'$ and $A'' \in \mathcal{A}''$ (cf. Lemma 4.1, Theorem 4.2). The combinatorial idea behind these examples is to use a Luzin almost disjoint family of subsets of $\mathbb{N}$ ([20], cf. [17]) to prevent the algebra from splitting. Recall that a family $\mathcal{F} \subseteq \wp(\mathbb{N})$ is called Luzin if it has cardinality ω_1, it is almost disjoint i.e., $A \cap B$ is finite for any two distinct $A, B \in \mathcal{F}$ and there are no separations of uncountable subfamilies i.e., given two disjoint uncountable $\mathcal{F}', \mathcal{F}'' \subseteq \mathcal{F}$ there is no $C \subseteq \mathbb{N}$ such that $A \setminus C$ and $B \cap C$ are both finite for all $A \in \mathcal{F}'$ and $B \in \mathcal{F}''$. In our case we need additional properties of a Luzin family (Theorem 2.4) which are combinatorially interesting by themselves and are published elsewhere ([14]). Such a family of subsets of $\mathbb{N}$ yields a system of noncompact operators in $\mathcal{B}(\ell_2)$ which we call a Luzin blockwise system of almost matrix units (Definitions 2.1 and 2.2). Its behaviour can be expressed in a manner similar to the commutative one described above ($=^{\mathcal{K}}$ denotes the equality modulo compact operators):

Theorem 1.8. *There is a sequence $(\mathcal{A}_\alpha)_{\alpha<\omega_1}$ of C^*-subalgebras of $\mathcal{B}(\ell_2)$ which are all isomorphic to $\mathcal{K}(\ell_2)$ and which are pairwise almost orthogonal, i.e., $AA' =^{\mathcal{K}} 0$ for all $A \in \mathcal{A}_\alpha$, $A' \in \mathcal{A}_{\alpha'}$ for any $\alpha < \alpha' < \omega_1$ with the following property:*

Given any two uncountable $X, Y \subseteq \omega_1$ and any choice of $A_\alpha \in \mathcal{A}_\alpha$ for $\alpha \in X$ and $B_\beta \in \mathcal{A}_\beta$ for $\beta \in Y$ there is no projection $P \in \mathcal{B}(\ell_2)$ satisfying

$$(\perp) \qquad PA_\alpha =^{\mathcal{K}} A_\alpha \text{ for all } \alpha \in X \quad \text{and} \quad PB_\beta =^{\mathcal{K}} 0 \text{ for all } \beta \in Y.$$

Proof. We claim that the algebras $\mathcal{A}_\alpha = C^*(\mathcal{S}_\alpha)$ for a Luzin blockwise system of almost matrix units defined in Definitions 2.1 and 2.2, which exists by Proposition 2.5, satisfy the theorem. Let X, Y be as in the theorem and suppose that there is a projection $P \in \mathcal{B}(\ell_2)$ as in $(\perp)$. Let $V_\alpha, U_\beta \in \mathcal{K}(\ell_2)$ for $\alpha \in X$ and $\beta \in Y$ be such that $A_\alpha^* P - A_\alpha^* = V_\alpha$ for $\alpha \in X$ and $(1-P)B_\beta - B_\beta = U_\beta$ for $\beta \in Y$. So $(A_\alpha^* + V_\alpha)(B_\beta + U_\beta) = 0$ for $\alpha \in X, \beta \in Y$. Using the separability of $\mathbb{C}$ and of $\mathcal{K}(\ell_2)$ by thining out X and Y to uncountable subsets we may assume that there are $W_1, W_2 \in \mathcal{K}(\ell_2)$ such that $\|(A_\alpha^* - W_1)(B_\beta - W_2)\| < \|A_\alpha^*\|\|B_\beta\|/2$ for all $\alpha \in X$ and $\beta \in Y$. But this contradicts the Luzin property from Definition2.2. □

We should note that almost disjoint families and in particular Luzin families recently found several applications in constructions of interesting noncommutative objects ([32, 10, 1, 31]). Ours seems to be the first application where one considers collections of subalgebras rather than collections of elements of a C^*-algebra.

Today the diversity of thin-tall algebras in the commutative case is much better understood than at the moment of publication of [29]. In [25] Roitman showed that it is consistent that there are 2^{ω_1} (as many as possible) pairwise non-isomorphic thin-tall superatomic Boolean algebras. In [8] Dow and Simon distinguished in ZFC 2^{ω_1} nonisomorphic thin-tall superatomic Boolean algebras by analyzing the groups of automorphisms of such algebras. It would be interesting also to study these groups in the fully noncommutative case.

Another aspect of our construction is related to the structure of the family of all two-sided ideals of $\mathcal{A}$. Lemma 6.2 of [11] implies that all two-sided ideals of $\mathcal{A}$ are among the ideals $\mathcal{I}_\alpha$ for $\alpha \leq \omega_1$. In particular, they form a continuous chain where all elements are stable for $\alpha < \omega_1$ and $\mathcal{I}_{\omega_1} = \mathcal{A}$ is not stable, so we obtain:

Theorem 1.9. *There exists a C^*-algebra, where the family of all stable ideals has no maximal element. In particular, this family does not have the greatest element.*

This gives a negative answer to Question 6.5 of [28] (only) in the nonseparable case, which asks if every C^*-algebra has the greatest stable ideal. In fact, in every separable C^*-algebra there are maximal elements in the family of stable ideals (see the discussion after 6.5 of [28]). Clearly, the additional feature of this answer is the simplicity of the composition series of the algebra.

The purpose of Section 2 is to introduce the appropriate terminology and to prove the existence of a Luzin blockwise system of almost matrix units (Proposition 2.5). In Section 3 we show how to connect a Luzin blockwise system of almost matrix units with an elementary essential composition series (Theorem 3.10). The main relation between these two objects is called domination (Definition 3.2). Section 4 is devoted to the proof of the nonstablity of the algebra constructed in Section 3.

The terminology should be standard, e.g., like in [4, 7, 21]. We list here some possible exceptions. $\tilde{\mathcal{A}}$ stands for the unitization of a C^*-algebra $\mathcal{A}$. $\mathcal{K}(\ell_2)$ denotes the algebra of compact operators on the separable Hilbert space ℓ_2 and $\mathcal{B}(\ell_2)$ the algebra of all bounded operators on ℓ_2. For $A, B \in \wp(\mathbb{N})$ we use the notation $A \subseteq^{Fin} B$ to mean that $B \setminus A$ is finite, similarly $A =^{Fin} B$ if $A \subseteq^{Fin} B$ and $B \subseteq^{Fin} A$. For $A, B \in \mathcal{B}(\ell_2)$ we use the notation $A =^{\mathcal{K}} B$ to mean $A - B \in \mathcal{K}(\ell_2)$. $\cong$ stands for the isomorphism relation of C^*-algebras by which we always mean the $*$-isomorphism relation. Two projections $P, Q \in \mathcal{B}(\ell_2)$ are almost orthogonal if and only if $PQ =^{\mathcal{K}} 0$ (cf. [32]). $\delta_{x,y}$ stands for the Kronecker delta. A system of matrix units in a C^*-algebra is a family of its nonzero elements $\mathcal{T} = \{T_{j,i} : i, j \in \mathbb{N}\}$, such that for every $m, n, i, j \in \mathbb{N}$ we have

- $T^*_{j,i} = T_{i,j}$ and
- $T_{n,m}\, T_{j,i} = \delta_{m,j} T_{n,i}$.

For a set of operators $\mathcal{S} \subseteq \mathcal{B}(\ell_2)$ let $C^*(\mathcal{S})$ denote the C^*-subalgebra of $\mathcal{B}(\ell_2)$ generated by the operators in $\mathcal{S}$.

2. Blockwise systems of almost matrix units

Let $(\lambda_\alpha)_{\alpha<\omega_1}$ be the strictly increasing sequence of all countable limit ordinals (including 0). We introduce the following notations:

- $\Lambda_\alpha = [\lambda_\alpha, \lambda_\alpha + \omega) \times [\lambda_\alpha, \lambda_\alpha + \omega)$ for each $\alpha < \omega_1$,
- $\Lambda = \bigcup_{\alpha<\omega_1} \Lambda_\alpha$.

Definition 2.1. *Suppose that $\mathcal{S} = (S_{\eta,\xi} : (\xi, \eta) \in \Lambda)$ is a system of noncompact operators in $\mathcal{B}(\ell_2)$. We say that $\mathcal{S}$ is a blockwise system of almost matrix units whenever the following are satisfied:*

(1) $\mathcal{S}_\alpha = \{S_{\eta,\xi} : (\xi, \eta) \in \Lambda_\alpha\}$ is a system of matrix units in $\mathcal{B}(\ell_2)$ for every $\alpha < \omega_1$,
(2) $\{S_{\xi,\xi} : \xi \in \omega_1\}$ is a family of pairwise almost orthogonal projections.

We say that $\mathcal{S}$ is separated by a sequence of projections $(P_\alpha : \alpha < \omega_1) \subseteq \mathcal{B}(\ell_2)$ whenever the following hold:

(3) $P_{\alpha'} P_\alpha =^{\mathcal{K}} P_{\alpha'}$ for all $\alpha' \leq \alpha < \omega_1$,
(4) $P_\alpha S_{\xi,\eta} P_\alpha =^{\mathcal{K}} S_{\xi,\eta}$ for each $(\xi, \eta) \in \Lambda_{\alpha'}$ for each $\alpha' < \alpha < \omega_1$,
(5) $P^\perp_\alpha S_{\xi,\eta} P^\perp_\alpha = S_{\xi,\eta}$ for each $(\xi, \eta) \in \Lambda_\alpha$ for each $\alpha < \omega_1$.

Definition 2.2. *Suppose that $\mathcal{S} = (S_{\eta,\xi} : (\xi, \eta) \in \Lambda)$ is a blockwise system of almost matrix units. We say that $\mathcal{S}$ is Luzin if given*

(1) two uncountable subsets X, Y of ω_1,
(2) $A_\alpha \in C^*(\mathcal{S}_\alpha)$ for each $\alpha \in X$,
(3) $B_\alpha \in C^*(\mathcal{S}_\alpha)$ for each $\alpha \in Y$,
(4) $\varepsilon > 0$,
(5) $W_1, W_2 \in \mathcal{K}(\ell_2)$,

there are $\alpha \in X$ and $\beta \in Y$ such that

$$\|(A_\alpha - W_1)(B_\beta - W_2)\| \geq \|A_\alpha\|\|B_\beta\| - \varepsilon.$$

We will use the almost disjoint family as in Theorem 2.4 to show that Luzin blockwise systems of almost matrix units exist and they can be separated by families of projections. First we need the following lemma.

Lemma 2.3. *Suppose that $n \in \mathbb{N}$ and that $\mathcal{T} = \{T_{m,k} : k, m \leq n\} \subseteq \mathcal{B}(\ell_2)$ and $\mathcal{S} = \{S_{j,i} : i, j \leq n\} \subseteq \mathcal{B}(\ell_2)$ are two finite systems of matrix units. Suppose that there are pairwise orthogonal norm one vectors $(e^k_i : i, k \leq n)$ such that*

(1) $T_{m,k}(e^{k'}_i) = \delta_{k,k'} e^m_i$ for all $i, k, k', m \leq n$,
(2) $S_{j,i}(e^k_{i'}) = \delta_{i,i'} e^k_j$ for all $i, i', j, k \leq n$.

Let $A \in C^*(\mathcal{T})$ and $B \in C^*(\mathcal{S})$. Then $\|AB\| = \|A\|\|B\|$. Moreover this fact is witnessed by a norm one vector from $span(e^k_i : i, k \leq n)$.

Proof. This follows from elementary properties of tensor products, but we present a shorter complete proof producing the required vector. As $\mathcal{T} = \{T_{m,k} : k, m \leq n\}$ is a system of matrix units, the algebra $C^*(\mathcal{T})$ is isomorphic to the algebra $n \times n$ matrices and therefore is simple. Hence, for each $i \leq n$ the restriction of elements of $C^*(\mathcal{T})$ to their invariant subspace $\mathcal{H}_i = span\{e^k_i : k \leq n\}$, as a nonzero homomorphism, is an isomorphism of $C^*(\mathcal{T})$ into $\mathcal{B}(\mathcal{H}_i)$. It follows that there is $x = (x_1, \ldots, x_n) \in \ell^n_2$ such that $v_i = \sum_{k\leq n} x_k e^k_i$ is of norm one and $A(v_i) = v'_i = \sum_{k\leq n} x'_k e^k_i$ and $\|v'_i\| = \|A\|$ for each $i \leq n$. Likewise, there is $y = (y_1, \ldots, y_n) \in \ell^n_2$ such that $w_k = \sum_{j\leq n} y_j e^k_j$ is of norm one and $B(w_k) = w'_k = \sum_{k\leq n} y'_j e^k_j$ and $\|w'_k\| = \|B\|$ for each $k \leq n$.

By a direct calculation we note that vectors of ℓ_2 of the form $\sum_{j,k\leq n} \alpha_k \beta_j e^k_j$ have norms equal to the product $\|(\alpha_1, \ldots \alpha_n)\|_{\ell^n_2} \|(\beta_1, \ldots \beta_n)\|_{\ell^n_2}$. Consider norm one element $z = \sum_{j,k\leq n} x_k y_j e_j^k$. We have

$$BA(z) = B(\sum_{j\leq n} y_j A(\sum_{k\leq n} x_k e_j^k)) = B(\sum_{j\leq n} y_j (\sum_{k\leq n} x'_k e_j^k)) =$$

$$= \sum_{k\leq n} x'_k B(\sum_{j\leq n} y_j e_j^k) = \sum_{k\leq n} x'_k (\sum_{j\leq n} y'_j e_j^k) = \sum_{j,k\leq n} x'_k y'_j e_j^k.$$

So $\|BA(z)\| = \|A(v_i)\|\|B(w_k)\| = \|A\|\|B\|$ for any $i, k \leq n$. Similarly $\|AB(z)\| = \|A\|\|B\|$, which completes the proof.

□

Theorem 2.4 ([14]). *There are families $(X_\alpha : \alpha < \omega_1)$, $(Y_\alpha : \alpha < \omega_1)$ of infinite subsets of $\mathbb{N}$ and bijections $x^\alpha : \mathbb{N} \times \mathbb{N} \to X_\alpha$ for each $\alpha < \omega_1$ such that*

(1) $X_\beta \cap X_\alpha =^{Fin} \emptyset$ for all $\beta < \alpha < \omega_1$,
(2) $Y_\beta \subseteq^{Fin} Y_\alpha$ for all $\beta < \alpha < \omega_1$,
(3) $X_\beta \subseteq^{Fin} Y_\alpha$ for all $\beta < \alpha < \omega_1$,
(4) $X_\alpha \cap Y_\alpha = \emptyset$ for all $\alpha < \omega_1$,
(5) For every $\alpha < \omega_1$ and every $k \in \mathbb{N}$ for all but finitely many $\beta < \alpha$ there are $m_1 < ... < m_k$ and $n_1 < ... < n_k$ and $l_{i,j} \in \mathbb{N}$ such that

$$x^\alpha(i, n_j) = l_{i,j} = x^\beta(j, m_i)$$

for all $1 \leq i, j \leq k$.

Proposition 2.5. *There is a Luzin blockwise system of almost matrix units which is separated by a family of projections.*

Proof. Let $(X_\alpha : \alpha < \omega_1)$ be the almost disjoint family with enumerations $(x^\alpha_{n,m} : \alpha < \omega_1, n, m \in \mathbb{N})$ which is separated by a family $(Y_\alpha : \alpha < \omega_1)$ as in Theorem 2.4, where $x^\alpha_{n,m} = x^\alpha(n, m)$ for all $\alpha < \omega_1, n, m \in \mathbb{N}$. Fix an orthogonal basis $(e_n : n \in \mathbb{N})$ of ℓ_2 and define the following operators diagonal with respect to this basis:

- $P_\alpha(e_n) = \chi_{Y_\alpha}(n)e_n$,
- $S_{\lambda_\alpha+k,\lambda_\alpha+k}(e_n) = \chi_{\{x^\alpha_{k,i}:i\in\mathbb{N}\}}(n)e_n$,

for all $\alpha < \omega_1$, where χ_X denotes the characteristic function of $X \subseteq \mathbb{N}$. Moreover for every $\alpha < \omega_1$ and every $m, k \in \mathbb{N}$ define the partial isometry $S_{\lambda_\alpha+k,\lambda_\alpha+m}$ by

- $S_{\lambda_\alpha+k,\lambda_\alpha+m}(e_{x^\alpha_{m,i}}) = e_{x^\alpha_{k,i}}$ for every $i \in \mathbb{N}$,
- $S_{\lambda_\alpha+k,\lambda_\alpha+m}(e_n) = 0$ if n is not of the form $x^\alpha_{m,i}$ for some $i \in \mathbb{N}$.

It is immediate from Theorem 2.4 (1) - (4) that $\mathcal{S} = \{S_{\eta,\xi} : (\xi, \eta) \in \Lambda\}$ is a blockwise system of almost matrix units which is separated by $(P_\alpha : \alpha < \omega_1)$. We will use Theorem 2.4 (5) to conclude that it is Luzin.

So fix two uncountable subsets X, Y of ω_1 and operators $A_\alpha \in C^*(\mathcal{S}_\alpha)$ for each $\alpha \in X$, $B_\alpha \in C^*(\mathcal{S}_\alpha)$ for each $\alpha \in Y$, $\varepsilon > 0$ and two compact operators $W_1, W_2 \in \mathcal{K}(\ell_2)$. We may assume that $\varepsilon < 1$.

By approximating W_1 and W_2 we may assume that there is $k_1 \in \mathbb{N}$ such that $\langle W_1(e_n), e_{n'}\rangle = 0 = \langle W_2(e_n), e_{n'}\rangle$ whenever $n, n' \geq k_1$. By passing to smaller uncountable subsets of X and Y respectively we may assume that there is $M > 1$ such that $\|A_\alpha\| < M$ for all $\alpha \in X$ and $\|B_\alpha\| < M$ for all $\alpha \in Y$. Passing further to uncountable subsets of X and Y we may assume that there is $k_2 \in \mathbb{N}$ such that for every $\alpha \in X$ and $\beta \in Y$ there are $k_2 \times k_2$ matrices $(a_{m,n})_{m,n \leq k_2}$, and $(b_{m,n})_{m,n \leq k_2}$ such that

$$\|A'_\alpha - A_\alpha\| < \delta, \qquad \|B'_\beta - B_\beta\| < \delta,$$

for some fixed $\delta = \delta(\varepsilon, M) > 0$, where

$$A'_\alpha = \sum_{n,m<k_2} a_{m,n} S_{\lambda_\alpha+m,\lambda_\alpha+n}$$

for each $\alpha \in X$ and

$$B'_\beta = \sum_{n,m<k_2} b_{m,n} S_{\lambda_\beta+m,\lambda_\beta+n}$$

for each $\beta \in Y$. Let $\alpha \in X$ be such that $Y \cap \alpha = Y \cap \{\beta : \beta < \alpha\}$ is infinite. By Theorem 2.4 (5) there is a finite $F \subseteq \alpha$ such that whenever $\beta \in (Y \cap \alpha) \setminus F$, then there are $m_1 < ... < m_{k_1+k_2}$ and $n_1 < ... < n_{k_1+k_2}$ and $l_{i,j} \in \mathbb{N}$ such that

$$x^\alpha_{i,n_j} = l_{i,j} = x^\beta_{j,m_i}$$

for all $1 \leq i, j \leq k_1 + k_2$. Let $G \subseteq \{1, ..., k_1 + k_2\}$ be of size k_2 such that $x^\alpha_{i,n_j} \notin \{1, ..., k_1\}$ for $i \in G$ and any $1 \leq j \leq k_1 + k_2$. Now note that A'_α and B'_β satisfy the hypothesis of Lemma 2.3 on the finite dimensional spaces spanned by $\{e_{x^\alpha_{i,n_j}} : i, j \in G\}$. By the choice of G the operators W_1, W_2 are null on this subspace. So Lemma 2.3 implies that $\|(A'_\alpha - W_1)(B'_\beta - W_2)\| = \|A'_\alpha\|\|B'_\beta\|$. This means that if δ is sufficiently small, then

$$\|(A_\alpha - W_1)(B_\beta - W_2)\| \geq \|A_\alpha\|\|B_\beta\| - \varepsilon.$$

□

3. Dominating a blockwise system of almost matrix units by a representing sequence

The following are two main definitions of this section:

Definition 3.1. *Let $\mathcal{A}$ be a C^*-algebra with an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha \leq \beta}$ for some ordinal β, and let $\pi_\alpha : \mathcal{A} \to \mathcal{A}/\mathcal{I}_\alpha$ for $\alpha < \beta$ be the quotient homomorphisms. A system $\mathcal{T} = (T_{\alpha+1,m,n} : n, m \in \mathbb{N}, \alpha < \beta)$ of elements of $\mathcal{A}$ is called a representing sequence for $\mathcal{A}$ if for each $\alpha < \beta$ the following hold:*

(1) $(T_{\alpha+1,m,n} : n, m \in \mathbb{N})$ is a system of matrix units in $\mathcal{A}$.

(2) $(\pi_\alpha(T_{\alpha+1,m,n}) : n, m \in \mathbb{N})$ is a system of matrix units in $\mathcal{A}/\mathcal{I}_\alpha$ which generates $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$.

We also say that $(T_{\alpha+1,m,n} : m, n \in \mathbb{N})$ represents the $(\alpha + 1)$-th level of $\mathcal{A}$.

Definition 3.2. *Suppose that $\mathcal{S} = \{S_{\eta,\xi} : (\xi, \eta) \in \Lambda\}$ is a blockwise system of almost matrix units which is separated by a family of projections $\mathcal{P} = (P_\alpha : \alpha < \omega_1)$. Let $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ be a C^*-algebra with an elementary essential composition series with a representing sequence $\mathcal{T} = (T_{\alpha+1,m,n} : n, m \in \mathbb{N}, \alpha < \omega_1)$. We say that $\mathcal{T}$ dominates $\mathcal{S}$ if for every $0 < \alpha < \omega_1$, $m, n \in \mathbb{N}$, we have*

$$(*) \qquad T_{\alpha+1,m,n} = P_\alpha T_{\alpha+1,m,n} P_\alpha + S_{\lambda_\alpha+m,\lambda_\alpha+n}.$$

The reason that the case $\alpha = 0$ is excluded in the definition above, is that in Theorem 3.10 we would like $(T_{1,m,n} : n, m \in \mathbb{N})$ to be the standard system of matrix units for $\mathcal{K}(\ell_2)$, and $S_{\lambda_0+n,\lambda_0+m}$ the way constructed in Proposition 2.5

are noncompact operators, hence do not satisfy the condition of Definition 3.2 for $\alpha = 0$.

We will show in Theorem 3.10 that given any blockwise system of almost matrix units $\mathcal{S}$ we can find a C^*-algebra $\mathcal{A}$ with an essential elementary composition series with a representing sequence $\mathcal{T}$ which dominates $\mathcal{S}$. For this we need some lemmas.

Definition 3.3. *Let $\mathcal{A}$ be a C^*-subalgebra of $\mathcal{B}(\ell_2)$ with an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ for some ordinal β, and let $\pi_\alpha : \mathcal{A} \to \mathcal{A}/\mathcal{I}_\alpha$ for $\alpha < \beta$ be the quotient homomorphisms. For each element $A \in \mathcal{A}$ we define its height $ht(A)$ by*

$$ht(A) = \min\{\alpha \leq \beta : A \in \mathcal{I}_\alpha\}.$$

Lemma 3.4. *Suppose that $\mathcal{S}, \mathcal{P}, \mathcal{A}$ and $\mathcal{T}$ are as in Definition 3.2 and $0 \leq \alpha < \omega_1$. If $\mathcal{T}$ dominates $\mathcal{S}$, then for every $A \in \mathcal{A}$ of height $\leq \alpha+1$ we have $A =^{\mathcal{K}} P_\alpha A P_\alpha + C$, where $C \in C^*(\mathcal{S}_\alpha)$. If $\alpha > 0$ and $ht(A) = \alpha + 1$, then $C \neq 0$.*

Proof. The elements $A \in \mathcal{B}(\ell_2)$ of the form $A =^{\mathcal{K}} P_\alpha A P_\alpha + C$ where $\alpha < \omega_1$ and $C \in C^*(\mathcal{S}_\alpha)$ form a subalgebra of $\mathcal{B}(\ell_2)$ because $P_\alpha^\perp C P_\alpha^\perp = C$ for each $C \in C^*(\mathcal{S}_\alpha)$ by Definition 2.1 (5).

By Definition 2.1 items (3), (4) we have that $P_{\alpha'} P_\alpha =^{\mathcal{K}} P_{\alpha'}$ and moreover $P_\alpha S_{\lambda_{\alpha'}+m,\lambda_{\alpha'}+n} P_\alpha =^{\mathcal{K}} S_{\lambda_{\alpha'}+m,\lambda_{\alpha'}+n}$ for each $\alpha' < \alpha < \omega_1$ and each $n, m \in \mathbb{N}$. So by $(*)$ we have $P_\alpha T_{\alpha'+1,m,n} P_\alpha =^{\mathcal{K}} T_{\alpha'+1,m,n}$ for each $\alpha' < \alpha$ and each $n, m \in \mathbb{N}$. By $(*)$ $T_{\alpha+1,m,n} = P_\alpha T_{\alpha+1,m,n} P_\alpha + C$ for $C \in C^*(\mathcal{S}_\alpha)$. The first part of the lemma follows from the fact that $T_{\alpha'+1,m,n}$s for $\alpha' \leq \alpha$ and each $n, m \in \mathbb{N}$ generate $\mathcal{I}_{\alpha+1}$.

By Definition 3.1 (2) $ht(A) = \alpha + 1$ implies that there is $B \in C^*(\mathcal{T}_\alpha) \setminus \{0\}$ such that $A - B \in \mathcal{I}_\alpha$, where $\mathcal{T}_\alpha = \{T_{\alpha+1,m,n} : n, m \in \mathbb{N}\}$. As noted in the first part of the proof $P_\alpha D P_\alpha =^{\mathcal{K}} D$ for each $D \in \mathcal{I}_\alpha$. So for the second part of the lemma it is enought to prove that $P_\alpha^\perp B P_\alpha^\perp \notin \mathcal{K}(\ell_2)$.

By $(*)$ and Definition 2.1 (5) the range of $P_\alpha^\perp$ is invariant for $C^*(\mathcal{T}_\alpha)$, and restricting elements of $C^*(\mathcal{T}_\alpha)$ to it is an isomorhpism. By $(*)$ and the hypothesis that $\alpha > 0$ the range of this isomorphism is $C^*(\mathcal{S}_\alpha)$ which satisfies $C^*(\mathcal{S}_\alpha) \cap \mathcal{K}(\ell_2) = \{0\}$. It follows that $P_\alpha^\perp B P_\alpha^\perp \notin \mathcal{K}(\ell_2)$ as required. □

Lemma 3.5. *Suppose $\mathcal{A}, \mathcal{A}' \subseteq \mathcal{B}(\ell_2)$ are C^*-subalgebras of $\mathcal{B}(\ell_2)$ which both contain $\mathcal{K}(\ell_2)$. Then every isomorhism $\Phi : \mathcal{A} \to \mathcal{A}'$ is given by $\Phi(A) = U^* A U$ for all $A \in \mathcal{A}$ for some fixed unitary $U \in \mathcal{B}(\ell_2)$.*

Proof. Note that minimal projections of $\mathcal{A}$ and of $\mathcal{A}'$ are one dimensional, and Φ preserves minimal projections, which means that $\Phi[\mathcal{K}(\ell_2)] = \mathcal{K}(\ell_2)$. Every automorphism of $\mathcal{K}(\ell_2)$ is induced by conjugating with a unitary in $\mathcal{B}(\ell_2)$ (Theorem 2.4.8 of [21]). So for some unitary U we have $\Phi(T) = U^* T U$, for every T in $\mathcal{K}(\ell_2)$. Fix an orthogonal basis $\{e_n : n \in \mathbb{N}\}$ for ℓ_2 and let P_n for $n \in \mathbb{N}$ denotes the projection on the one dimensional subspace spanned by e_n. Let $Q_n = U P_n U^*$ for every $n \in \mathbb{N}$. For every $T \in \mathcal{A}$ and m, n we have

$$\begin{aligned}(\Phi(T)e_n, e_m) &= (P_m \Phi(T) P_n e_n, e_m) = (\Phi(Q_m T Q_n) e_n, e_m)\\ &= (U^* Q_m T Q_n U e_n, e_m) = (P_m U^* T U P_n e_n, e_m)\\ &= (U^* T U e_n, e_m).\end{aligned}$$

Therefore $\Phi(T) = U^* T U$, for every $T \in \mathcal{A}$. □

Lemma 3.6. *Suppose that β is an ordinal and $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ is an elementary essential composition series for a stable C^*-algebra $\mathcal{A}$. Then there is a stable C^*-algebra $\mathcal{B}$ with an essential elementary composition series $(\mathcal{J}_\alpha)_{\alpha\leq\beta+1}$ satisfying $\mathcal{J}_\alpha = \mathcal{I}_\alpha$ for $\alpha \leq \beta$ and $\mathcal{J}_{\beta+1} = \mathcal{B}$ with a sequence $\{T_{\beta+1,m,n} : n, m \in \mathbb{N}\}$ representing the $\beta + 1$-th level of $\mathcal{B}$.*

Proof. By Lemma 1.6 $\mathcal{A}$ is scattered C^*-algebra with the Cantor-Bendixson composition series $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$. So by Lemma 7.5 of [11] and again Lemma 1.6 the algebra $\mathcal{B} = \tilde{\mathcal{A}} \otimes \mathcal{K}(\ell_2)$, where $\tilde{\mathcal{A}}$ is the unitization of $\mathcal{A}$, is the required algebra with the identification of $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{K}(\ell_2)$ which is justified by the stability of $\mathcal{A}$. $T_{\beta+1,m,n} = 1 \otimes e_{m,n}$, where $e_{m,n}$s are the standard matrix units in $\mathcal{K}(\ell_2)$. □

Lemma 3.7. *Suppose that $\beta < \omega_1$ and $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$ is an elementary essential composition series for a stable C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\ell_2)$ such that $\mathcal{I}_1 = \mathcal{K}(\ell_2)$. Then there is a stable C^*-subalgebra $\mathcal{B}$ of $\mathcal{B}(\ell_2)$ with an elementary essential composition series $(\mathcal{J}_\alpha)_{\alpha\leq\beta+1}$ satisfying $\mathcal{J}_\alpha = \mathcal{I}_\alpha$ for $\alpha \leq \beta$ and $\mathcal{J}_{\beta+1} = \mathcal{B}$ with a sequence $\{T_{\beta+1,m,n} : n, m \in \mathbb{N}\}$ representing the $\beta + 1$-th level of $\mathcal{B}$*

Proof. We need to find a C^*-subalgebra $\mathcal{B}$ of $\mathcal{B}(\ell_2)$ such that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B}$ satisfies Lemma 3.6. First consider a $\mathcal{B}'$ which satisfies Lemma 3.6, not necessarily a subalgebra of $\mathcal{B}(\ell_2)$. The second step is to obtain an algebra $\mathcal{B}''$ isomorphic to $\mathcal{B}'$ and satisfying $\mathcal{K}(\ell_2) \subseteq \mathcal{B}'' \subseteq \mathcal{B}(\ell_2)$. To get it note that $\mathcal{K}(\ell_2)$ must be an essential ideal of $\mathcal{B}'$ by Definition 1.1, so, since $\mathcal{B}(\ell_2)$ is the multiplier algebra of $\mathcal{K}(\ell_2)$, there is an embedding $\Phi : \mathcal{B}' \to \mathcal{B}(\ell_2)$ with image $\phi[\mathcal{B}'] = \mathcal{B}''$ such that $\Phi[\mathcal{K}(\ell_2)] = \mathcal{K}(\ell_2)$. Let $\mathcal{A}'' = \Phi[\mathcal{A}]$. By Lemma 3.5 there is a unitary $U \in \mathcal{B}(\ell_2)$ such that the conjugation by U is an isomorphism from $\mathcal{A}''$ onto $\mathcal{A}$. Since the conjugation by U is an automorphism of the entire $\mathcal{B}(\ell_2)$ we conclude that $\mathcal{B} = \{U^*BU : B \in \mathcal{B}''\}$ works. □

Lemma 3.8. *Let β be an ordinal. Suppose that a C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ has an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha\leq\beta}$, with $\mathcal{I}_1 = \mathcal{K}(\ell_2)$ and that there is an infinite rank projection $P \in \mathcal{B}(\ell_2)$ such that $A - PAP \in \mathcal{K}(\ell_2)$ for each $A \in \mathcal{A}$. Then $P\mathcal{A}P$ is a C^*-subalgebra of $\mathcal{B}(\ell_2)$ with an elementary essential composition series $(P\mathcal{I}_\alpha P)_{\alpha\leq\beta}$, and $\mathcal{A}$ is generated by $\mathcal{K}(\ell_2)$ and $P\mathcal{A}P$.*

Proof. Since $A - PAP \in \mathcal{K}(\ell_2)$ for each $A \in \mathcal{A}$ and $\mathcal{K}(\ell_2) \subseteq \mathcal{A}$, we have that $P\mathcal{A}P \subseteq \mathcal{A}$. $P\mathcal{K}(\ell_2)P$ is an essential ideal of $P\mathcal{B}(\ell_2)P$ and so an essential ideal of $P\mathcal{A}P$ which is isomorphic to $\mathcal{K}(\ell_2)$ since P has infinite rank.

Let $\pi_1 : \mathcal{A} \to \mathcal{A}/\mathcal{K}(\ell_2)$ and $\sigma_1 : P\mathcal{A}P \to P\mathcal{A}P/P\mathcal{K}(\ell_2)P$ be the quotient maps. $\Psi : \mathcal{A}/\mathcal{K}(\ell_2) \to P\mathcal{A}P/P\mathcal{K}(\ell_2)P$ given by $\Psi(\pi_1(A)) = \sigma_1(PAP)$ for $A \in \mathcal{A}$ is a well defined isomorphism since e.g., $PABP - PAPBP \in \mathcal{K}(\ell_2)$ for any $A, B \in \mathcal{A}$. It is clear that $\mathcal{A}/\mathcal{I}_1$ has an elementary essential composition series $(\mathcal{I}_\alpha/\mathcal{I}_1)_{\alpha\leq\beta}$ and so $P\mathcal{A}P/P\mathcal{K}(\ell_2)P$ has $(\Psi[\mathcal{I}_\alpha/\mathcal{I}_1])_{\alpha\leq\beta}$ as such a series. It follows that $P\mathcal{A}P$ has an essential composition series $(\mathcal{J}_\alpha)_{\alpha\leq\beta}$ where $\mathcal{J}_1 = P\mathcal{K}(\ell_2)P$ and

$$\mathcal{J}_\alpha = \sigma_1^{-1}[\Psi[\mathcal{I}_\alpha + \mathcal{K}(\ell_2)]] = \sigma^{-1}[P\mathcal{I}_\alpha P + P\mathcal{K}(\ell_2)P] = P\mathcal{I}_\alpha P$$

for $1 \leq \alpha \leq \beta$ as required. The fact that $\mathcal{A}$ is generated by $\mathcal{K}(\ell_2)$ and $P\mathcal{A}P$ follows directly from the assumptions. □

Lemma 3.9. *Let $\beta > 0$ be an ordinal. Suppose that $P \in \mathcal{B}(\ell_2)$ is an infinite rank projection and a C^*-algebra $\mathcal{A}$ satisfying $P\mathcal{K}(\ell_2)P \subseteq \mathcal{A} \subseteq P\mathcal{B}(\ell_2)P$ has an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha\leq\beta+1}$ with $\mathcal{I}_1 = P\mathcal{K}(\ell_2)P$ and with*

$\mathcal{T} = \{T_{\beta+1,n,m} : m,n \in \mathbb{N}\}$ representing the $(\beta+1)$-th level of $\mathcal{A}$. Let $\mathcal{S} = \{S_{n,m} : m,n \in \mathbb{N}\} \subseteq P^\perp \mathcal{B}(\ell_2) P^\perp$ be a system of noncompact matrix units and define

$$R_{m,n} = T_{\beta+1,m,n} + S_{m,n}$$

for each $m,n \in \mathbb{N}$.

Then the C^*-algebra $\mathcal{B} \subseteq \mathcal{B}(\ell_2)$ generated by $\mathcal{I}_\beta$, $\mathcal{K}(\ell_2)$ and $\{R_{m,n} : n,m \in \mathbb{N}\}$ has an elementary essential composition series $(\mathcal{J}_\alpha)_{\alpha \leq \beta+1}$ such that $\mathcal{J}_1 = \mathcal{K}(\ell_2)$, $\mathcal{J}_\alpha = \mathcal{I}_\alpha + \mathcal{K}(\ell_2)$ for $1 \leq \alpha < \beta+1$ and $\mathcal{R} = \{R_{m,n} : m,n \in \mathbb{N}\}$ represents the $(\beta+1)$-th level of $\mathcal{B}$.

Proof. Let $\Phi : C^*(\mathcal{T}) \to C^*(\mathcal{S})$ be the isomorphism such that $\Phi(T_{\beta+1,m,n}) = S_{m,n}$. As $\mathcal{T}$ represent $\beta+1$-th level of $\mathcal{A}$, each element of $\mathcal{B}$ is of the form $A+\Phi(A)+B+K$, where $A \in C^*(\mathcal{T})$, $B \in \mathcal{I}_\beta$ and $K \in \mathcal{K}(\ell_2)$. It follows that $\Psi : \mathcal{B}/\mathcal{K}(\ell_2) \to \mathcal{A}/P\mathcal{K}(\ell_2)P$ given by

$$\Psi(A + \Phi(A) + B + \mathcal{K}(\ell_2)) = A + B + P\mathcal{K}(\ell_2)P$$

is a well defined isomorphism such that $\Psi(R_{m,n} + \mathcal{K}(\ell_2)) = T_{\beta+1,m,n} + P\mathcal{K}(\ell_2)P$ for each $n,m \in \mathbb{N}$ and $\Psi[\mathcal{I}_\alpha + \mathcal{K}(\ell_2)] = \mathcal{I}_\alpha + P\mathcal{K}(\ell_2)P$ for $\alpha \leq \beta$. So the properties of $\mathcal{B}$ follow from the properties of $\mathcal{A}$.

□

Theorem 3.10. *Suppose that $\mathcal{S} = \{S_{\eta,\xi} : (\xi,\eta) \in \Lambda\}$ is a blockwise system of almost matrix units which is separated by a system of projections $(P_\alpha : \alpha < \omega_1)$. Then there is C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(\ell_2)$ with an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha \leq \omega_1}$ with a representing sequence $\mathcal{T} = (T_{\alpha+1,m,n} : n,m \in \mathbb{N}, \alpha < \omega_1)$ such that $\mathcal{T}$ dominates $\mathcal{S}$.*

Proof. By recursion on $\beta < \omega_1$ we build operators $(T_{\alpha+1,m,n} : n,m \in \mathbb{N}, \alpha < \beta)$ and stable C^*-algebras $\mathcal{I}_\beta \subseteq \mathcal{B}(\ell_2)$ with an elementary essential composition series $(\mathcal{I}_\alpha)_{\alpha \leq \beta}$, such that $(T_{\alpha+1,m,n} : n,m \in \mathbb{N}, \alpha < \beta)$ forms a representing sequence for $\mathcal{I}_\beta$ satisfying $(*)$ from Definition 3.2. Then $\mathcal{A} = \bigcup_{\alpha<\omega_1} \mathcal{I}_\alpha$ is the required algebra.

Let $(T_{1,m,n} : n,m \in \mathbb{N})$ be the standard system of matrix units of $\mathcal{K}(\ell_2)$ and let $\mathcal{I}_1 = \mathcal{K}(\ell_2)$. If β is a countable limit ordinal, then $\mathcal{I}_\beta$ is the closure of the union of $\mathcal{I}_\alpha$s for $\alpha < \beta$.

Suppose now that $(T_{\alpha+1,m,n} : n,m \in \mathbb{N}, \alpha < \beta)$ and $\mathcal{I}_\beta$ are constructed for $\beta < \omega_1$ and the condition of Definition 3.2 is satisfied, i.e., for every $0 < \alpha < \beta$ we have

$$(*) \qquad T_{\alpha+1,m,n} = P_\alpha T_{\alpha+1,m,n} P_\alpha + S_{\lambda_\alpha+m,\lambda_\alpha+n}$$

for each $n,m \in \mathbb{N}$. We construct $\mathcal{I}_{\beta+1}$ and $(T_{\beta+1,m,n} : n,m \in \mathbb{N})$ as follows. Definition 2.1 (3)-(5) and the above inductive requirement $(*)$ for every $\alpha < \beta$, imply that $A - P_\beta A P_\beta \in \mathcal{K}(\ell_2)$ for every $A \in \mathcal{I}_\beta$. It follows from Lemma 3.8 that $P_\beta \mathcal{I}_\beta P_\beta$ is a C^*-algebra with an essential elementary composition series $(P_\beta \mathcal{I}_\alpha P_\beta)_{\alpha \leq \beta}$ and $\mathcal{I}_\beta$ is generated by $P_\beta \mathcal{I}_\beta P_\beta$ and $\mathcal{K}(\ell_2)$. Since $\mathcal{K}(\ell_2) = \mathcal{I}_1 \subseteq \mathcal{I}_\beta$, for each $\alpha \leq \beta$ we have that

$$(**) \qquad \mathcal{I}_\alpha = P_\beta \mathcal{I}_\alpha P_\beta + \mathcal{K}(\ell_2).$$

As observed after (1) -(4) of the Introduction the separable ideals of an elementary essential composition series are stable and so is $P_\beta \mathcal{I}_\beta P_\beta$. Now working with $P_\beta \mathcal{I}_\beta P_\beta$ inside $P_\beta \mathcal{B}(\ell_2) P_\beta$ apply Lemma 3.7 to obtain a stable C^*-algebra $\mathcal{I}'_{\beta+1} \subseteq P_\beta \mathcal{B}(\ell_2) P_\beta$ with an elementary essential composition series $(\mathcal{J}_\alpha)_{\alpha \leq \beta+1}$ satisfying $\mathcal{J}_\alpha = P_\beta \mathcal{I}_\alpha P_\beta$

for $\alpha \leq \beta$ and with a sequence $\{T'_{m,n} : n, m \in \mathbb{N}\}$ representing the $\beta+1$-th level of $\mathcal{J}_{\beta+1}$. Now for each $n, m \in \mathbb{N}$ define

$$T_{\beta+1,m,n} = T'_{m,n} + S_{\lambda_\beta+m,\lambda_\beta+n},$$

so $(*)$ of Definition 3.2 is satisfied. By Definition 2.1 (5) we have $P_\beta^\perp S_{\lambda_\beta+m,\lambda_\beta+n} P_\beta^\perp = S_{\lambda_\beta+m,\lambda_\beta+n}$, so we are in the position of applying Lemma 3.9 to obtain the required $\mathcal{I}_{\beta+1}$. Note that by $(**)$ and Lemma 3.9 the essential elementary composition series of $\mathcal{I}_{\beta+1}$ agrees with that of $\mathcal{I}_\beta$, in particular, its β-th element is $\mathcal{I}_\beta$ and its representing sequence is $(T_{\alpha+1,m,n} : n, m \in \mathbb{N}, \alpha < \beta+1)$. This finishes the construction and the proof. □

4. The nonstability

In the previous section we showed that for any blockwise system of almost matrix units $\mathcal{S}$ of size ω_1 which is separated by a system of projections, we can find a C^*-algebra $\mathcal{A}$ with an essential elementary composition series of length ω_1 with a representing sequence which dominates $\mathcal{S}$. In this section we will show that if $\mathcal{S}$ above is a Luzin blockwise system of almost matrix units, then such C^*-algebra $\mathcal{A}$ is not stable.

Lemma 4.1. *Suppose that $\mathcal{A}$ is a stable C^*-algebra with an essential elementary composition series $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$. Then there is a sequence of projections $(R_\alpha, Q_\alpha : \alpha < \omega_1) \subseteq \mathcal{A}$ such that $ht(R_\alpha) = ht(Q_\alpha) = \alpha+1$ for every $\alpha < \omega_1$ and we have*

$$R_{\alpha_1} Q_{\alpha_2} = 0$$

for every $\alpha_1, \alpha_2 < \omega_1$.

Proof. Since $\mathcal{A}$ is stable we have $\mathcal{A} \cong \mathcal{A} \otimes \mathcal{K}(\ell_2)$. As $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha \cong \mathcal{K}(\ell_2)$, there are projections in $\mathcal{I}_{\alpha+1}/\mathcal{I}_\alpha$, so let $R''_\alpha \in \mathcal{I}_{\alpha+1}$ be a lifting of such a projection. We can choose this lifting to be a projection since each $\mathcal{I}_\alpha(\mathcal{A})$ is a separable AF-ideal (see Lemma III. 6. 1. of [7]). Let R, Q be two orthogonal projections in $\mathcal{K}(\ell_2)$. Put $R'_\alpha = R''_\alpha \otimes R$ and $Q'_\alpha = R''_\alpha \otimes Q$ for every $\alpha < \omega_1$. Clearly $R'_{\alpha_1} Q'_{\alpha_2} = 0$ for every $\alpha_1, \alpha_2 < \omega_1$. Also $ht(R'_\alpha) = ht(Q'_\alpha) = \alpha+1$, by Proposition 5.3 of [11] and Lemma 1.6. So R_α and Q_α obtained from R'_α and Q'_α via the isomorphism between $\mathcal{A}$ and $\mathcal{A} \otimes \mathcal{K}(\ell_2)$, satisfy the lemma. □

Theorem 4.2. *Suppose that $\mathcal{A}$ is C^*-algebra with an essential elementary composition series $(\mathcal{I}_\alpha)_{\alpha\leq\omega_1}$ with a representing sequence that dominates a Luzin blockwise system of almost matrix units. Then $\mathcal{A}$ is not stable.*

Proof. Let $\mathcal{T}_\mathcal{A} = (T_{\alpha+1,m,n} : n, m \in \mathbb{N}, \alpha < \omega_1) \subseteq B(\ell_2)$ be a representing sequence of $\mathcal{A}$ (Definition 3.1) which dominates (Definition 3.2) a Luzin blockwise system of almost matrix units $\mathcal{S} = \{S_{\eta,\xi} : (\xi, \eta) \in \Lambda\}$ which is separated by a family of projection $(P_\alpha : \alpha < \omega_1)$ (Definition 2.1). We will derive a contradiction from the hypothesis that $\mathcal{A}$ is stable.

By Lemma 4.1 there is a sequence $(R_\alpha, Q_\alpha : \alpha < \omega_1) \subseteq \mathcal{A}$ of projections such that $ht(R_\alpha) = ht(Q_\alpha) = \alpha+1$ for every $\alpha < \omega_1$ and $R_\alpha Q_\beta = 0$ for every $\alpha < \beta < \omega_1$.

By Lemma 3.4 we have

$$R_\alpha = A_\alpha + P_\alpha R_\alpha P_\alpha + F_\alpha, \quad Q_\alpha = B_\alpha + P_\alpha Q_\alpha P_\alpha + G_\alpha,$$

where F_α, G_α are compact operators and $A_\alpha, B_\alpha \in C^*(S_\alpha)$. Note that $\|A_\alpha\|, \|B_\alpha\| > 0$ for all $0 < \alpha < \omega_1$ as $ht(R_\alpha) = ht(Q_\alpha) = \alpha+1$.

So by passing to an uncountable subset $X \subseteq \omega_1$, we may assume that $M > \|A_\alpha\|, \|B_\alpha\| > 2\sqrt{\varepsilon}$ and $M > \|F_\alpha\|, \|G_\alpha\|$ for each $\alpha \in X$ and some $M > 1$ and $0 < \varepsilon < 1$. Passing further down to an uncountable subset of X and using the separability of $\mathcal{K}(\ell_2)$ we may assume that there are compact operators F, G such that $\|P_\alpha^\perp F_\alpha - F\|, \|P_\alpha^\perp G_\alpha - G\| < \varepsilon/2M$ and $\|F\|, \|G\| < M$ for every $\alpha \in X$. For every $\alpha, \beta \in X$ we have

$$\begin{aligned}
0 = \|R_\alpha Q_\beta\| &\geq \|P_\alpha^\perp R_\alpha Q_\beta P_\beta^\perp\| \\
&= \|P_\alpha^\perp (A_\alpha + F_\alpha)(B_\beta + G_\beta) P_\beta^\perp\| = \|(A_\alpha + P_\alpha^\perp F_\alpha)(B_\beta + G_\beta P_\beta^\perp)\| \\
&\geq \|(A_\alpha + F)(B_\beta + G)\| - \|(P_\alpha^\perp F_\alpha - F)(B_\beta + G)\| \\
&- \|(A_\alpha + F)(G_\beta P_\beta^\perp - G)\| - \|(P_\alpha^\perp F_\alpha - F)(G_\beta P_\beta^\perp - G)\| \\
&\geq \|(A_\alpha + F)(B_\beta + G)\| - 2\varepsilon - \frac{\varepsilon^2}{M^2} \\
&> \|(A_\alpha + F)(B_\beta + G)\| - 3\varepsilon.
\end{aligned}$$

Therefore $\|(A_\alpha + F)(B_\beta + G)\| < 3\varepsilon$ for every $\alpha, \beta \in X$. However, by the Luzin property (Definition 2.2) we can find $\alpha, \beta \in X$ such that

$$\|(A_\alpha + F)(B_\beta + G)\| \geq \|A_\alpha\|\|B_\beta\| - \varepsilon > (2\sqrt{\varepsilon})^2 - \varepsilon = 3\varepsilon$$

which gives the required contradiction. Therefore $\mathcal{A}$ is not stable. □

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