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**Quasilinear P.D.Es, interpolation spaces  
and Hölderian mappings**

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# QUASILINEAR P.D.ES, INTERPOLATION SPACES AND HÖLDERIAN MAPPINGS

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ABSTRACT.

As in the work of Tartar ([47]) we developed here some new results on non linear interpolation of  $\alpha$ -Hölderian mappings between normed spaces, namely, by studying the action of the mappings on  $K$ -functionals and between interpolation spaces with logarithm functors.

We apply those results to obtain regularity results on the gradient of the solution to quasilinear equations of the form

$$-\operatorname{div}(\widehat{a}(\nabla u)) + V(u) = f,$$

whenever  $V$  is a nonlinear potential,  $f$  belongs to non standard spaces as Lorentz-Zygmund spaces. We show among other that the mapping  $T : Tf = \nabla u$  is locally or globally  $\alpha$ -Hölderian under suitable values of  $\alpha$  and adequate hypothesis on  $V$  and  $\widehat{a}$ .

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## 1. Introduction - Notation - Preliminary results

### 1.1. Introduction.

The Marcinkiewicz interpolation theorems for linear operators acting on Lebesgue spaces turned out to be an essential tool for studying regularity of solutions for linear partial differential equations (P.D.Es) in  $L^p$ -spaces.

Then, Jaak Peetre ([33, 34]) introduced a method ( $K$ -method) to give a general definition of interpolation spaces between two normed spaces embedded in a same topological space. His definition allows to extend Marcinkiewicz's results of linear operators acting on abstract normed spaces. But his results allow also to go further in the study of regularity of solutions of linear equations on spaces different from  $L^p$ -spaces. The main problem to apply Peetre's definition is the identification of the interpolated spaces. For instance, we did such a study with applications to linear P.D.Es in recent papers (see [22], [2] or [24]) using new spaces as grand or small Lebesgue spaces, sometimes combining the regularity method with a duality method.

Later, in our knowledge, L. Tartar [47], under the supervision of J.L. Lions was the first to give interpolation results on nonlinear Hölderian mappings (which include Lipschitz mappings) and applied to a variety of boundary value problems as bilinear applications, semi-linear P.D.Es but also on variational inequalities .

This last paper of L. Tartar, recent results development concerning the interpolation spaces with logarithm functions (see, for instance [27], and the previous references) and the appearance of the new operators in P.D.Es as anisotropic  $\vec{p}$ -Laplacian or variable exponents  $p(\cdot)$ -Laplacian were the main motivations which lead us to reconsider the work of L.Tartar [47] and to show that we may have Hölder mappings associated to quasilinear equations in order to obtain new regularity results .

So, we extend first Tartar's results on nonlinear interpolations mappings  $T$  to couples of spaces with a logarithm function by studying the action of the mapping  $T$  on the  $K$ -functional associated to those couples . This is the purpose of the second section . Here is an example of such result:

Let  $X_1 \subset X_0, Y_1 \subset Y_0$ , be four normed spaces. Assume that  $T : X_i \rightarrow Y_i$  is globally  $\alpha$ -Hölderian for  $i = 0, 1$ . Then, for all  $a \in X_0, b \in X_1$  one has

$$K(Ta - Tb; t^\alpha) \leq 2 \text{Max}(M_0; M_1)K(a - b; t)^\alpha,$$

where  $M_i$  denotes the Hölder constants.

As a consequence, we derive the following result:

Let  $X_1 \subset X_0, Y_1 \subset Y_0$  four normed spaces. Assume that  $T : X_i \rightarrow Y_i$  is globally  $\alpha$ -Hölderian for  $i = 0, 1$ . For  $0 \leq \theta \leq 1, 1 \leq p \leq +\infty$ , if  $X_1$  is dense in  $X_0$ , then

$$T \text{ is an } \alpha\text{-Hölderian mapping form } (X_0, X_1)_{\theta,p,\lambda} \text{ into } (Y_0, Y_1)_{\theta,\frac{p}{\alpha};\lambda\alpha}.$$

The last subsection of this paragraph is devoted to some identification of interpolation spaces using couples of Lebesgue or Lorentz spaces. This allows us to recover spaces as Lorentz-Zygmund spaces or  $GT$ -gamma spaces. The list is not exhaustive but was chosen to be applied later on, in the fourth and the fifth paragraphs.

To define the adequate mappings in those last sections, we consider two types of formulations, the usual weak formulation and the entropic-renormalized formulation for the quasilinear P.D.Es of the form  $Au + V(u) = f, f$  in  $L^1(\Omega)$  where  $A$  is a Leray-Lions type operator,  $V$  a potential, and we may prove the existence and uniqueness of solution according to the space where the data  $f$  belongs. We can define a non-linear operator,  $T : L^1(\Omega) \rightarrow Y_{0i}, i = 1, \dots, n$  : to  $f \in L^1(\Omega)$  we associate the  $i$ -th component of the **gradient** of the solution in an appropriate space  $Y_{0i}$ . The main step is to prove that such a nonlinear operator is an Hölderian mapping. This is done in each application from section four to six. The fundamental lemma to obtain such a result in Marcinkiewicz spaces for  $L^1$ -data is the following result:

Let  $\nu$  be a non negative Borel measure and  $h : \Omega \rightarrow \mathbb{R}_+, g : \Omega \rightarrow \mathbb{R}_+$ , be two  $\nu$ -measurable functions. Then,  $\forall \lambda > 0, \forall k > 0$ ,

$$\nu\{h > \lambda\} \leq \frac{1}{\lambda} \int_{\{g \leq k\}} h d\nu + \nu\{g > k\}.$$

See **Lemma 3.1** for its proof.

Replacing  $L^1(\Omega)$  by other  $L^r$ -spaces we can have more regularity on the gradient of the solution.

We then apply the abstract results on interpolation mappings obtained in the second section. Let us notice that our estimates are optimal in many cases. Therefore we improved some well-known regularity results as in Lorentz spaces but also we have an easy tool to derive regularity of the gradient when the data  $f$  is in spaces as  $L^{m,r}(\text{Log } L)^\alpha$ ,  $m \geq 1$  or in small spaces  $L^{(r,\theta)}(\Omega)$  or Orlicz spaces.

For convenience, we took only models for the nonlinear operator  $A$ . More precisely, we study the regularity of the weak or entropic-renormalized solution of a p-laplacian type operators as  $-\text{div} \left( |\nabla u|^{p-2} \nabla u \right) + V(x; u) = f$ , or its anisotropic version in a bounded smooth domain  $\Omega$  of  $\mathbb{R}^n$ ,

$$-\sum_{i=1}^n \frac{\partial}{\partial x_i} \left[ \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right] + V(x; u) = f$$

$1 < p_i, p < +\infty, i = 1, \dots, n$  or the variable exponents version of  $p(\cdot)$ -Laplacian, where  $V$  is nonlinear. We only consider the Dirichlet homogeneous condition on the boundary  $u = 0$ .

An example of regularity that we can deduce from Proposition 4.4 is:

*If  $u$  is a solution of the above quasilinear equation see (33),  $2 \leq p < n$  and  $f \in L^{k,r}(\Omega)$  then the gradient of the solution  $u$  belongs to  $L^{k^*(p-1),r(p-1)}(\Omega)^n$ , with  $k \leq (p^*)'$  (conjugate of the Sobolev exponent of  $p$ ),  $k^*$  the Sobolev exponent of  $k$ . Moreover, we have*

$$\|\nabla u\|_{L^{k^*(p-1),r(p-1)}} \leq c \|f\|_{L^{k,r}}^{\frac{1}{p-1}}.$$

An example of non-standard regularity result that can be obtained from Theorem 4.3 for the solution  $u$  is:

$$\left[ \int_0^1 \left( \left( \int_t^1 |\nabla u|_*(s)^p ds \right)^{\frac{1}{p}} (1 - \log t)^{\lambda \alpha} \right)^{\frac{p}{\alpha}} \frac{dt}{t} \right]^{\frac{\alpha}{p}} \leq c \left[ \int_0^1 \left( \left( \int_t^1 f_*(s)^{(p^*)'} ds \right)^{\frac{1}{(p^*)'}} (1 - \log t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{\alpha}{p}},$$

whenever the right hand side of the inequality is finite. Here  $\alpha = \frac{1}{p-1}$ ,  $2 \leq p < n$ ,  $\lambda \in \mathbb{R}$ .

Moreover, if  $f \in L^{\frac{m'}{m'-\theta}, p_2}(\text{Log } L)^\lambda$  then  $|\nabla u| \in L^{p_\theta, p_2(p-1)}(\text{Log } L)^{\frac{\lambda}{p-1}}$ .

From Section 4 to Section 6, we give such applications of the abstract results obtained in Sections 2 and 3.

For instance, here it is the basis of the existence of an Hölderian mapping result for anisotropic equation:

Let  $u$  be the entropic-renormalized solution of equation (49). Then there exists a constant  $c > 0$  independent of  $u$  and  $f$  such that

$$(1) \text{ meas } \{|u| > k\} \leq c \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} k^{-\frac{p^*}{p'}}, \quad \forall k > 0.$$

$$(2) \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{\frac{n' p_i}{p'}}(\Omega)} \leq c \|f\|_{L^1(\Omega)}^{\frac{p'}{p_i}}, \quad i = 1, \dots, n.$$

For the sake of completeness, although the existence and uniqueness for quasilinear equations are widely done in the literature and are not the main issue of our work, we shall give some examples of proofs of uniqueness and existence. Namely, when the operator  $A$  has variable exponents, we have new results and we show in particular that:

There exists a constant  $c > 0$  depending only on  $p$ ,  $n$ ,  $\Omega$  such that

$$\text{meas } \left\{ |\nabla u|^{p(\cdot)} > \lambda \right\} \leq c \psi_1(\|f\|_1)^{\frac{1}{1+|a_1|}} \lambda^{-\frac{|a_1|}{1+|a_1|}} \quad \forall \lambda > 0.$$

Such topic is developed in the last section 6. The method is widely inspired by the previous works (see for instance [5], [37], [38], [41], [42]) and uses recent theorems as the one given in [23].

Moreover, the same method can be used to prove the existence and uniqueness of entropic-renormalized solution for general operators including the anisotropic case.

## 1.2. Notation -Preliminary results.

We shall adopt our usual notation.

For a measurable  $f \in \Omega \rightarrow \mathbb{R}$ , we set for  $t \geq 0$

$$D_f(t) = \text{meas } \left\{ x \in \Omega : |f(x)| \geq t \right\},$$

and  $f_*$  the decreasing rearrangement of  $|f|$  is defined as follows:

For  $s \in (0, |\Omega|)$ ,  $|\Omega|$  being the measure of  $\Omega$ ,

$$f_*(s) = \inf \left\{ t : D_f(t) \leq s \right\}, \text{ we also denote by } f_{**}(s) = \frac{1}{s} \int_0^s f_*(t) dt.$$

The Lorentz space  $L^{p,q}(\Omega)$ ,  $1 < p < +\infty$ ,  $1 \leq q \leq +\infty$ , is defined as the set of measurable functions  $f$  for which

$$\|f\|_{p,q} = \begin{cases} \left[ \int_0^{|\Omega|} [t^{\frac{1}{p}} f_{**}(t)]^q \frac{dt}{t} \right]^{\frac{1}{p}} & \text{if } q < +\infty, \\ \sup_{0 < t < |\Omega|} t^{\frac{1}{p}} f_{**}(t) & \text{if } q = +\infty, \end{cases} \quad \text{is finite,}$$

where  $\|v\|_q$  denotes the norm in  $L^q(\Omega)$ ,  $1 \leq q \leq +\infty$ .

If  $A_1$  and  $A_2$  are two quantities depending on some parameters, we shall write

$A_1 \lesssim A_2$  (if there exists  $c > 0$  independent of the parameters) such that  $A_1 \leq cA_2$

$A_1 \simeq A_2$  if and only if  $A_1 \lesssim A_2$  and  $A_1 \gtrsim A_2$ .

For the anisotropic problem, we will need the following Troisi's Sobolev inequalities [49, 48].

For this purpose, we shall denote by

$$\frac{1}{p} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i} \quad \text{and} \quad p^* = \frac{np}{n-p} \quad \text{if} \quad \sum_{i=1}^n \frac{1}{p_i} > 1, \quad \vec{p} = (p_1, \dots, p_n).$$

**Theorem 1.1. Poincaré-Sobolev inequality for anisotropic Sobolev spaces**

Assume that  $1 \leq p < n$ . Then, the following inequalities hold true.

(1) There exists a constant  $C = C(n, \vec{p})$  such that

$$(1) \quad \left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{n}{p^*}} \leq C \prod_{i=1}^n \left( \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx \right)^{\frac{1}{p_i}} \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

(2) For any  $\vec{\theta} = (\theta_1, \dots, \theta_n)$  such that  $\theta_i > 0$  for all  $i = 1, \dots, n$  and  $\sum_{i=1}^n \frac{1}{\theta_i} = \frac{n}{p}$ , there exists a constant  $C_{\vec{\theta}} = C(n, \vec{p}, \vec{\theta})$  such that

$$(2) \quad \left( \int_{\mathbb{R}^n} |u|^{p^*} dx \right)^{\frac{n}{p^*}} \leq C_{\vec{\theta}} \sum_{i=1}^n \left( \int_{\mathbb{R}^n} |\partial_{x_i} u|^{p_i} dx \right)^{\frac{\theta_i}{p_i}} \quad \forall u \in C_c^\infty(\mathbb{R}^n).$$

In particular, we shall use the case  $\theta_i = p_i$  for all  $i = 1, \dots, n$ .

We shall denote by  $W_0^{1, \vec{p}}(\Omega)$  the closure of  $C_c^\infty(\Omega)$  with respect to the norm:

$$\|v\|_{1, \vec{p}} = \sum_{i=0}^n \left\| \frac{\partial v}{\partial x_i} \right\|_{p_i}.$$

The following Poincaré-Sobolev inequality holds true in  $W_0^{1, \vec{p}}(\Omega)$ .

**Corollary 1.1.1. of Theorem 1.1**

(1) There exists a constant  $C = C(n, \vec{p})$  such that

$$(3) \quad \left[ \int_{\Omega} |v|^{p^*}(x) dx \right]^{\frac{1}{p^*}} \leq C \left( \sum_{i=1}^n \int_{\Omega} |\partial_i v|^{p_i} \right)^{\frac{1}{p}}$$

for all  $v \in W_0^{1, \vec{p}}(\Omega)$ , if  $\sum_{i=1}^n \frac{1}{p_i} > 1$ .



(2) If  $\sum_{i=1}^n \frac{1}{p_i} < 1$ , then

$$W_0^{1,\vec{p}}(\Omega) \subset_{\hookrightarrow} L^\infty(\Omega).$$

Moreover, there exists a constant  $C(n) > 0$  such that

$$\|v\|_\infty \leq C(n) \prod_{i=1}^{i=n} \left\| \frac{\partial v}{\partial x_i} \right\|_{p_i}^{\frac{1}{n}}.$$

(3) If  $\sum_{i=1}^n \frac{1}{p_i} = 1$ , then

$$W_0^{1,\vec{p}}(\Omega) \subset_{\hookrightarrow} L^r(\Omega)$$

for all  $r < +\infty$ .

**Remark 1.1.**

The two last statements can be found also in [46].

For the case of variable exponent spaces, for  $u : \Omega \rightarrow \mathbb{R}$  measurable, we set

$$\Phi_p(u) = \int_{\Omega} |u(x)|^{p(x)} dx$$

and we consider the norm:

$$(4) \quad \|u\|_{p(\cdot)} = \inf \left\{ \lambda > 0 : \Phi_p \left( \frac{u}{\lambda} \right) \leq 1 \right\}, \quad (\inf \emptyset = +\infty).$$

Setting

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \text{ measurable such that } \|u\|_{p(\cdot)} < +\infty\},$$

the space  $(L^{p(\cdot)}(\Omega); \|\cdot\|_{p(\cdot)})$  is a Banach function space and an equivalent norm for  $u$  is the following Amemiya norm

$$(5) \quad |u|_{p(\cdot)} = \inf_{\lambda > 0} \lambda \left( 1 + \Phi_p \left( \frac{u}{\lambda} \right) \right),$$

which is equivalent to the norm in (4) since

$$(6) \quad \|u\|_{p(\cdot)} \leq |u|_{p(\cdot)} \leq 2\|u\|_{p(\cdot)}.$$

We set

$$L_+^1(\Omega) = \{v \in L^1(\Omega) : v \geq 0\} \text{ and } L_+^{p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega) \cap L_+^1(\Omega).$$

We always assume that

$$1 < p_- = \inf\{p(x) : x \in \Omega\} \leq p_+ = \sup\{p(x) : x \in \Omega\} < \infty.$$

**Proposition 1.1.**

One has:

- $L^{p(\cdot)}(\Omega)$  is reflexive.
- For all  $u \in L^{p(\cdot)}(\Omega)$ ,

$$2\|u\|_{p(\cdot)} \leq \left( \int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p^-}} + \left( \int_{\Omega} |u(x)|^{p(x)} dx \right)^{\frac{1}{p^+}}.$$

We also have Poincaré-Sobolev type inequality for variable exponent spaces. Following [15], [12], for the next theorem, we shall consider that the function  $p(\cdot)$  being a bounded log-Hölder continuous function on a bounded open set  $\Omega$ , that is:

There exists a constant  $c_1 > 0$ :

$$\text{Log}(e + 1/|x - y|)|p(x) - p(y)| \leq c_1, \forall (x, y) \in \Omega \times \Omega.$$

Assuming also  $p_+ < n$ , one can consider the Sobolev variable exponent

$$p^*(x) = \frac{np(x)}{n - p(x)}, \quad x \in \Omega,$$

and the following Poincaré-Sobolev inequality holds true:

**Theorem 1.2.**

There exists a constant  $C = C(n, p(\cdot))$  such that

$$(7) \quad \|v\|_{p^*(\cdot)} \leq C \|\nabla v\|_{p(\cdot)} \quad \text{for all } v \in W_0^{1,p(\cdot)}(\Omega).$$

The dual of  $W_0^{1,p(\cdot)}$  is denoted by  $W^{-1,p'(\cdot)}(\Omega)$  with  $p'(x) = \frac{p(x)}{p(x) - 1}$ .

We can summarize the definitions of Lebesgue, Lorentz and Zygmund spaces as follows:

**Definition 1.1. ( Lorentz-Zygmund spaces)**

Let  $\Omega$  be a space of measure 1,  $0 < p, q \leq +\infty$ ,  $-\infty < \lambda < +\infty$ . Then the Lorentz-Zygmund space  $L^{p,q}(\log L)^\lambda$  consists of all Lebesgue measurable function  $f$  on  $\Omega$  such that :

$$\|f\|_{p,q,\lambda} = \begin{cases} \left( \int_0^1 \left[ t^{\frac{1}{p}} (1 - \text{Log } t)^\lambda f_*(t) \right]^q \frac{dt}{t} \right)^{\frac{1}{q}} & 0 < q < +\infty \\ \sup_{0 < t < 1} t^{\frac{1}{p}} (1 - \text{Log } t)^\lambda f_*(t) & q = +\infty \end{cases} \quad \text{is finite.}$$

Here  $f_*$  is the decreasing rearrangement of  $|f|$ .

We also need the next definition.

**Definition 1.2. (of  $G\Gamma(p, m; w_1, w_2)$ ) (see [22])**

Let  $w_1, w_2$  be two weights on  $(0, 1)$ ,  $m \in [1, +\infty]$ ,  $1 \leq p < +\infty$ . We assume the following conditions:

- (c1) There exists  $K_{12} > 0$  such that  $w_2(2t) \leq K_{12}w_2(t) \quad \forall t \in (1, 1/2)$ .

(c2) The function  $\int_0^t w_2(\sigma)d\sigma$  belongs to  $L^{\frac{m}{p}}(0, 1; w_1)$ .

A generalized Gamma space with double weights is the set

$$G\Gamma(p, m; w_1, w_2) = \left\{ w : \Omega \rightarrow \mathbb{R} \text{ measurable } \int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \text{ is in } L^{\frac{m}{p}}(0, 1; w_1) \right\}.$$

It is a quasi-normed space endowed with the natural quasi-norm:

$$\rho(v) = \left[ \int_0^1 w_1(t) \left( \int_0^t v_*^p(\sigma)w_2(\sigma)d\sigma \right)^{\frac{m}{p}} dt \right]^{\frac{1}{p}}.$$

If  $w_2 = 1$  we simply note  $G\Gamma(p, m; w_1, 1) = G\Gamma(p, m; w_1)$ .

We shall also need the following elementary inequalities that can be found in [30], [13].

For  $p \geq 2$ , there exists a constant  $\alpha_p > 0$  such that  $\forall \xi \in \mathbb{R}^n, \forall \xi' \in \mathbb{R}^n$

$$(8) \quad \left( |\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi - \xi' \right)_{\mathbb{R}^n} \geq \alpha_p |\xi - \xi'|^p.$$

A similar relation holds for the case  $1 < p < 2$ , namely, there exists a constant  $\alpha_p > 0$  such that  $\forall \xi \in \mathbb{R}^n, \forall \xi' \in \mathbb{R}^n$

$$(9) \quad \left( |\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi - \xi' \right)_{\mathbb{R}^n} \geq \alpha_p \frac{|\xi - \xi'|^2}{(|\xi| + |\xi'|)^{p-2}}.$$

## 2. Abstract results on nonlinear interpolation

We shall need the following results concerning real interpolation with logarithm function (see [18, 27]).

Let  $(X_0, \|\cdot\|_0), (X_1, \|\cdot\|_1)$  be two normed spaces continuously embedded in a Hausdorff topological vector space, that is,  $(X_0, X_1)$  is a compatible couple.

For  $g \in X_0 + X_1, t > 0$ , we shall denote

$$K(g, t) \doteq K(g, t; X_0, X_1) = \inf_{g=g_0+g_1} \left( \|g_0\|_0 + t\|g_1\|_1 \right)$$

For  $0 \leq \theta \leq 1, 1 \leq q \leq +\infty, \alpha \in \mathbb{R}$ , we define the interpolation space

$$(X_0, X_1)_{\theta, q; \alpha} = \left\{ g \in X_0 + X_1, \|g\|_{\theta, q; \alpha} = \|t^{-\theta-\frac{1}{q}}(1 - \text{Log } t)^\alpha K(g, t)\|_{L^q(0,1)} \text{ is finite} \right\}.$$

Next, we consider four normed spaces  $X_1 \subset X_0, Y_1 \subset Y_0$ , and  $T$  a non-linear mapping from  $X_i$  into  $Y_i, i = 0, 1$  such that:

- (1)  $\|Ta - Tb\|_{Y_0} \leq f\left(\|a\|_{X_0}; \|b\|_{X_0}\right) \|a - b\|_{X_0}^\alpha$  for all  $(a, b)$  in  $X_0$ .
- (2)  $\|Ta\|_{Y_1} \leq g\left(\|a\|_{X_0}\right) \|a\|_{X_1}^\beta, \forall a \in X_1$ .

Here  $0 < \alpha \leq 1$ ,  $\beta > 0$ ,  $g$  is a continuous increasing function, and  $f$  is continuous on  $\mathbb{R}^2$  and such that for each  $\sigma$ ,  $f(\sigma; \cdot)$  is increasing.

## 2.1. Estimating $K$ -functional related to the mapping $T$ .

### Lemma 2.1.

Under the above assumptions (1) and (2) on  $T$ , let  $G(\sigma) = \text{Max} \left( g(2\sigma); f(\sigma; 2\sigma) \right)$ ,  $\sigma \in \mathbb{R}_+$ . Then for all  $a \in X_0$ , all  $t > 0$  one has

$$K(Ta, t^\beta, Y_0, Y_1) = K(Ta, t^\beta) \leq G(\|a\|_{X_0})[K(a, t)^\beta + K(a, t)^\alpha].$$

Moreover, if  $\beta \geq \alpha$ , then

$$K(Ta, t^\beta) \leq G(\|a\|_{X_0})(1 + \|a\|_{X_0}^{\beta-\alpha})K(a, t)^\alpha.$$

### Proof :

We follow the Tartar's idea [47], (see also [32]). Let  $a \in X_0$  and  $\varepsilon > 0$ . Then, there exist functions  $a_0(\varepsilon; \cdot)$  and  $a_1(\varepsilon, \cdot)$  such that:

$a_0(\varepsilon, t) \doteq a_0(t) \in X_0$ ,  $a_1(\varepsilon, t) \doteq a_1(t) \in X_1$  with  $a = a_0(t) + a_1(t)$  such that

$$(10) \quad K(a, t) \leq \|a_0(t)\|_{X_0} + t\|a_1(t)\|_{X_1} \leq (1 + \varepsilon)K(a, t), \quad \forall t > 0.$$

We set  $Ta = b_0(t) + b_1(t)$  with  $b_1(t) = Ta_1(t)$ . Then

$$(11) \quad \begin{aligned} K(Ta, t^\beta) &\leq \|b_0(t)\|_{Y_0} + t^\beta\|b_1(t)\|_{Y_1} = \|Ta - Ta_1(t)\|_{Y_0} + t^\beta\|Ta_1(t)\|_{Y_1} \\ &\leq t^\beta g\left(\|a_1(t)\|_{X_0}\right)\|a_1(t)\|_{X_1}^\beta + f\left(\|a\|_{X_0}; \|a_1(t)\|_{X_0}\right)\|a_0(t)\|_{X_0}^\alpha. \end{aligned}$$

Since  $a \in X_0$ , then

$$(12) \quad K(a, t) \leq \|a\|_{X_0}.$$

Thus from relations (10) and (12), we have

$$(13) \quad \|a_0(t)\|_{X_0} \leq (1 + \varepsilon)\|a\|_{X_0} \quad \forall t > 0,$$

and then

$$(14) \quad \|a_1(t)\|_{X_0} \leq \|a\|_{X_0} + \|a_0(t)\|_{X_0} \leq (2 + \varepsilon)\|a\|_{X_0}.$$

Therefore, relation (11) implies that

$$(15) \quad K(Ta, t^\beta) \leq \text{Max} \left( g(\|a\|_{X_0}(2+\varepsilon)); f(\|a\|_{X_0}; (2+\varepsilon)\|a\|_{X_0}) \right) \left[ \|a_0(t)\|_{X_0}^\alpha + t^\beta\|a_1(t)\|_{X_1}^\beta \right],$$

combining this relation (15) with relation (10) and letting  $\varepsilon \rightarrow 0$ , we derive

$$(16) \quad K(Ta, t^\beta) \leq G(\|a\|_{X_0}) \left[ K(a, t)^\alpha + K(a, t)^\beta \right] \quad \forall t > 0.$$

If  $\beta \geq \alpha$ , then using relation (12), one deduces from (16) that

$$(17) \quad K(Ta, t^\beta) \leq G(\|a\|_{X_0}) (1 + \|a\|_{X_0}^{\beta-\alpha}) K(a, t)^\alpha.$$

◇

**A particular case :** as a particular case we have the following:

**Corollary 2.1.1. (of Lemma 2.1)**

Let  $X_1 \subset X_0$ ,  $Y_1 \subset Y_0$  be four normed spaces. Assume that  $T : X_1 \rightarrow Y_1$  is globally  $\alpha$ -Hölderian, i.e.  $\exists M_1 > 0$  such that

$$\|Ta - Tb\|_{Y_1} \leq M_1 \|a - b\|_{X_1}^\alpha, \quad 0 < \alpha \leq 1,$$

and  $T$  maps  $X_0$  into  $Y_0$  in the sense that  $\exists M_0 > 0$ ,  $\beta > 0$

$$\|Ta\|_{Y_0} \leq M_0 \|a\|_{X_0}^\beta.$$

Then,  $\forall a \in X_0$ ,  $\forall t > 0$ , one has

$$K(Ta, t^\beta) \leq \text{Max}(M_0; M_1) \left[ K(a, t)^\beta + K(a, t)^\alpha \right].$$

If  $\alpha \leq \beta$ , then

$$K(Ta, t^\beta) \leq (1 + \|a\|_{X_0}^{\beta-\alpha}) \text{Max}(M_0; M_1) K(a, t)^\alpha.$$

**Corollary 2.1.2. (of Lemma 2.1)**

Let  $X_1 \subset X_0$ ,  $Y_1 \subset Y_0$  be four normed spaces. Assume that  $T : X_i \rightarrow Y_i$  is globally  $\alpha$ -Hölderian for  $i = 0, 1$ . Then, for all  $a \in X_0$ ,  $b \in X_1$  one has

$$K(Ta - Tb; t^\alpha) \leq 2 \text{Max}(M_0; M_1) K(a - b; t)^\alpha,$$

where  $M_i$  denotes the Hölder constants as in Corollary 2.1.1 of Lemma 2.1. Furthermore, if  $X_1$  is dense in  $X_0$  then the above equality holds also for all  $b \in X_0$ .

**Proof :**

Let  $b \in X_1$  and define  $Sx = T(b + x) - Tb$  for  $x \in X_0$ . Then

$$\|Sx\|_{Y_0} \leq M_0 \|x\|_{X_0}^\alpha,$$

and for all  $x \in X_1$  and  $y \in X_1$  we have

$$\|Sx - Sy\|_{Y_1} \leq M_1 \|x - y\|_{X_1}^\alpha.$$

We may apply Corollary 2.1.1 of Lemma 2.1 to derive

$$K(Sx; t^\alpha) \leq 2 \text{Max}(M_0; M_1)K(x; t)^\alpha, \forall x \in X_0.$$

Choosing for  $a \in X_0$ ,  $x = a - b$  and taking into account that  $S(a - b) = Ta - Tb$ , we obtain the first result. If  $X_1$  is dense in  $X_0$ , we consider a sequence  $b_n \in X_1$  converging to  $b$  in  $X_0$ , since

$$K(b_n - b; t) \leq \|b_n - b\|_{X_0},$$

then  $K(b_n - b; t)$  converges to zero as  $n$  goes to  $\infty$ . Writing

$$K(Ta - Tb; t^\alpha) \leq K(Ta - Tb_n; t^\alpha) + K(Tb_n - Tb; t^\alpha),$$

and applying the preceding results, one has

$$K(Ta - Tb_n; t^\alpha) + K(Tb_n - Tb; t^\alpha) \leq 2 \text{Max}(M_0; M_1) \left[ K(a - b_n; t)^\alpha + K(b_n - b; t)^\alpha \right].$$

Letting  $n$  goes to  $\infty$ , we get from the two last formulae that

$$K(Ta - Tb; t^\alpha) \leq 2 \text{Max}(M_0; M_1)K(a - b; t)^\alpha,$$

for all  $(a, b) \in X_0 \times X_0$ . ◇

## 2.2. Interpolation of Hölderian mappings.

### Theorem 2.1.

Let  $X_1 \subset X_0$ ,  $Y_1 \subset Y_0$ , be four normed spaces,  $T$  the mapping satisfying (1) and (2), moreover assume that  $\alpha \leq \beta$ . Then, if  $0 \leq \theta \leq 1$ ,  $1 \leq p \leq +\infty$ , for  $a \in (X_0, X_1)_{\theta, p; \lambda}$  one has:

$$Ta \in (Y_0, Y_1)_{\theta \frac{\alpha}{\beta}, \frac{p}{\alpha}; \lambda \alpha} \quad \text{and} \quad \|Ta\|_{(Y_0, Y_1)_{\theta \frac{\alpha}{\beta}, \frac{p}{\alpha}; \lambda \alpha}} \lesssim \left[ (1 + \|a\|_{X_0}^{\beta - \alpha}) G(\|a\|_{X_0}) \right] \|a\|_{\theta, p; \lambda}^\alpha.$$

### Proof :

One has from relation (17)

$$(18) \quad K(Ta, t^\beta) \leq (G(\|a\|_{X_0})(1 + \|a\|_{X_0}^{\beta - \alpha})K(a, t)^\alpha.$$

Thus

$$(19) \quad J = \int_0^1 t^{-\theta p} (1 - \text{Log } t)^{p\lambda} K(Ta, t^\beta)^\frac{p}{\alpha} \frac{dt}{t} \leq \left[ (1 + \|a\|_{X_0}^{\beta - \alpha}) G(\|a\|_{X_0}) \right]^\frac{p}{\alpha} \|a\|_{\theta, p; \lambda}^p.$$

Set

$$J_1 = \int_0^1 \sigma^{-\theta \frac{p}{\beta}} (1 - \text{Log } (\sigma))^{p\lambda} K(Ta, \sigma)^\frac{p}{\alpha} \frac{d\sigma}{\sigma}.$$

We make the change of variables  $\sigma = t^\beta$  in the first integral  $J$  to deduce:

$$(20) \quad J = \frac{1}{\beta} \int_0^1 \sigma^{-\theta \frac{p}{\beta}} \left(1 + \frac{1}{\beta} |\text{Log}(\sigma)|\right)^{p\lambda} K(Ta, \sigma)^{\frac{p}{\alpha}} \frac{dt}{t}.$$

From which we have

$$c_1 J_1 \leq J \leq c_2 J_1$$

$$\text{with } c_1 = \begin{cases} \frac{1}{\beta} \min\left(1, \frac{1}{\beta}\right)^{p\lambda} & \text{if } \lambda \geq 0. \\ \frac{1}{\beta} \max\left(1, \frac{1}{\beta}\right)^{p\lambda} & \text{if } \lambda < 0 \end{cases} \quad \text{and } c_2 = \begin{cases} \frac{1}{\beta} \max\left(1, \frac{1}{\beta}\right)^{p\lambda} & \text{if } \lambda \geq 0, \\ \frac{1}{\beta} \min\left(1, \frac{1}{\beta}\right)^{p\lambda} & \text{if } \lambda < 0. \end{cases}$$

Then

$$J_1 \simeq J.$$

So this last relation and relations (19) to (20) lead us to

$$(21) \quad \|Ta\|_{\theta \frac{p}{\beta}, \frac{p}{\alpha}; \lambda \alpha} \lesssim (1 + \|a\|_{X_0}^{\beta - \alpha}) G(\|a\|_{X_0}) \|a\|_{\theta, p; \lambda}^\alpha.$$

◇

In view of applications in P.D.ES, we first have the following:

**Theorem 2.2.**

Let  $X_1 \subset X_0$ ,  $Y_1 \subset Y_0$  be four normed spaces. Assume that  $T : X_i \rightarrow Y_i$  is globally  $\alpha$ -Hölderian for  $i = 0, 1$ .

For  $0 \leq \theta \leq 1$ ,  $1 \leq p \leq +\infty$ , if  $X_1$  is dense in  $X_0$ , then

$$T \text{ is an } \alpha\text{-Hölderian mapping from } (X_0, X_1)_{\theta, p; \lambda} \text{ into } (Y_0, Y_1)_{\theta, \frac{p}{\alpha}; \lambda \alpha}.$$

**Proof :**

Let  $a \in (X_0, X_1)_{\theta, p; \lambda}$  and  $b \in (X_0, X_1)_{\theta, p; \lambda}$ . We have shown in the above corollary 2.1.2

$$K(Ta - Tb; t^\alpha) \leq 2 \text{Max}(M_0; M_1) K(a - b; t)^\alpha.$$

Following the same arguments as in proof of the above Theorem 2.1, we deduce from this inequality:

$$(22) \quad \|Ta - Tb\|_{\theta, \frac{p}{\alpha}; \lambda \alpha} \leq c_0 \|a - b\|_{\theta, p; \lambda}^\alpha.$$

◇

### 2.3. Identification of some interpolation spaces.

To obtain similar results as for Proposition 4.4 below with an interpolation process including a functor (as a logarithm function), we must identify the following interpolation spaces:

$$(L^1, L^m)_{\theta, p_2; \lambda}, \quad (L^1, L^{n,1})_{\theta, p_2; \lambda}, \quad (L^{s, \infty}, L^m)_{\theta, p_2; \lambda}, \quad (L^{s, \infty}, L^\infty)_{\theta, p_2; \lambda},$$

under suitable conditions on  $s$ ,  $p_2$ ,  $\theta$ . Here is the general result collecting the necessary interpolation identification that we shall need for the application. The proof can be essentially found in ([27]) (see also [1]), we give below the idea how to prove the statements.

#### Proposition 2.1.

Let  $1 \leq r < m \leq +\infty$ ,  $1 \leq q_1, q_2 \leq \infty$ ,  $1 \leq p < +\infty$ ,  $0 \leq \theta < 1$  and  $\lambda \in \mathbb{R}$ , if  $\theta = 1$  then  $\lambda < -\frac{1}{p}$ , and  $\lambda \geq -\frac{1}{p}$  if  $\theta = 0$ .

$$\|f\|_{(L^{r, q_1}, L^{m, q_2})_{\theta, p, \lambda}} \simeq \begin{cases} \left[ \int_0^1 \left( t^{\frac{1-\theta}{r} + \frac{\theta}{m}} f_*(t) (1 - \text{Log } t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}, & 0 < \theta < 1; \\ \left[ \int_0^1 \left( \left( \int_0^t s^{\frac{q_1}{r} - 1} f_*(s)^{q_1} ds \right)^{\frac{1}{q_1}} (1 - \text{Log } t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}, & \theta = 0, \quad q_1 < +\infty; \\ \left[ \int_0^1 \left( \left( \text{ess sup}_{0 < s < t} s^{\frac{1}{r}} f_*(s) \right) (1 - \text{Log } t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}, & \theta = 0, \quad q_1 = +\infty; \\ \left[ \int_0^1 \left( \left( \int_t^1 s^{\frac{q_2}{m} - 1} f_*(s)^{q_2} ds \right)^{\frac{1}{q_2}} (1 - \text{Log } t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}, & \theta = 1, \quad q_2 < +\infty; \\ \left[ \int_0^1 \left( \left( \text{ess sup}_{0 < s < t} s^{\frac{1}{m}} f_*(s) \right) (1 - \text{Log } t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}, & \theta = 1, \quad q_2 = +\infty. \end{cases}$$

#### Proof :

The above proposition can be proved directly without invoking the general framework in the cited references. Indeed, the main steps to do it are: first, use Holmsted's formula to get an equivalent expression for the K-functional between Lorentz spaces and then, we can use suitable Hardy inequalities essentially developed in [3] to get the results.  $\diamond$

We have several consequence of the above proposition. First, when we compare with the definition of Lorentz-Zygmund,  $GT$ -space, and small Lebesgue spaces, we have:

#### Proposition 2.2.

Let  $1 \leq r < m \leq \infty$ ,  $\lambda \in \mathbb{R}$ ,  $1 \leq p_2 < +\infty$ .



(1) If  $0 < \theta < 1$ , then, the interpolation space  $(L^{r;\infty}; L^m)_{\theta, p_2; \lambda}$  coincides to the Lorentz-Zygmund space  $L^{m\theta, p_2}(\text{Log } L)^\lambda$  with  $\frac{1}{m\theta} = \frac{1-\theta}{r} + \frac{\theta}{m}$ .  $(L^1, L^m)_{\theta, p_2; \lambda}$  coincides to the Lorentz-Zygmund space  $L^{\frac{m'}{1-\theta}, p_2}(\text{Log } L)^\lambda$ .

(2) If  $\theta = 0$ ,  $1 \leq q_1, q_2 \leq +\infty$ , then for  $j = 1, 2$ , we have

$$(L^{r, q_1}, L^{m, q_2})_{0, p_2; \lambda} = G\Gamma(q_j, p_2; w_1, w_{2j}),$$

with  $w_1(t) = (1 - \text{Log } t)^{\lambda p_2} t^{-1}$ ,  $w_{21}(t) = t^{\frac{q_1}{r}-1}$  if  $q_1 < \infty$ ,  $w_{22}(t) = t^{\frac{q_2}{m}-1}$  if  $q_2 < \infty$ ,  $t \in ]0, 1[$ .

The space  $(L^1, L^m)_{0, p_2; \lambda}$  is the Generalized-Gamma space  $G\Gamma(1, p_2; w_1)$  where  $w(t) = t^{-1}(1 - \text{Log } t)^{\lambda p_2}$  (see [22] or [25]).

(3) In particular, we have for  $\lambda > -\frac{1}{p_2}$ , the link with small Lebesgue spaces as:

(a) If  $1 < q_1 < +\infty$ ,  $L^{(q_1, \alpha_1)}(\Omega) = (L^{q_1}, L^{m, q_2})_{0, p_1; \lambda} \quad \forall q_2 \in [1, +\infty], \quad \forall m \in ]q_1, +\infty], \quad \alpha_1 = q_1'(\lambda p_2 + 1), \quad \frac{1}{q_1} + \frac{1}{q_1'} = 1.$

(b) If  $1 < q_2 < +\infty$ ,  $L^{(q_2, \alpha_2)}(\Omega) = (L^{r, q_1}, L^{q_2})_{0, p_2; \lambda}$  with  $\alpha_2 = q_2'(\lambda p_2 + 1)$ ,  $1 = \frac{1}{q_2} + \frac{1}{q_2'}, \quad \forall r \in [1, q_2], \quad \forall q_1 \in [1, +\infty].$

For the case  $0 < \theta < 1$ , we may apply the following duality result (see [22, 32, 9, 45]).

### Proposition 2.3.

Let  $X_1 \subset X_0$  two Banach function spaces. Then the associate space of  $(X_0, X_1)_{\theta, p_2; \lambda}$  with  $0 < \theta < 1$ ,  $1 \leq p_2 < +\infty$ ,  $\lambda \in \mathbb{R}$ , is the space

$$(X_1', X_0')_{1-\theta, p_2'; -\lambda} \text{ with } \frac{1}{p_2} + \frac{1}{p_2'} = 1, \text{ where } X_i' \text{ is the associate space of } X_i, \quad i = 0, 1.$$

This Proposition 2.3 leads us to:

### Corollary 2.3.1. of Proposition 2.3

Let  $1 < m < +\infty$ ,  $1 < p_2' \leq +\infty$ ,  $\lambda \in \mathbb{R}$ ,  $0 < \theta < 1$ ,  $m' = \frac{m}{m-1}$ . Then

$$(L^m, L^\infty)_{\theta, p_2'; \lambda} \text{ is the associate space of } (L^1, L^{m'})_{1-\theta, p_2'; -\lambda},$$

that is the Lorentz-Zygmund space (up to equivalence of norms)

$$(L^m, L^\infty)_{\theta, p_2'; \lambda} = L^{\frac{m}{1-\theta}, p_2'}(\text{Log } L)^\lambda.$$

Moreover, we have

$$(L^m, L^\infty)_{\theta, p_2'; \lambda} = (L^{m, \infty}, L^\infty)_{\theta, p_2'; \lambda}.$$

Another theorem that we shall need, concerns a reiteration of Lorentz-Zygmund spaces, following the Lions-Peetre's lemma (see [4, 27]).

**Proposition 2.4.** (see [27])

Let  $1 \leq p_0, q_0, p_1, q_1 \leq +\infty, p_1, 0 < \theta < 1, r_i \in \mathbb{R}$ . Then

$$\left( L^{p_0, q_0} \left( \text{Log } L \right)^{r_0}, L^{p_1, q_1} \left( \text{Log } L \right)^{r_1} \right)_{\theta, q; r} = L^{p_\theta, q} \left( \text{Log } L \right)^{r_\theta}$$

$$\text{with } \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \text{ and } r_\theta = (1-\theta) \frac{r_0 q}{q_0} + \frac{\theta r_1 q}{q_1} + \frac{r}{q}.$$

Before starting the application of those interpolation formulae, we shall introduce a very useful lemma inspired by the work of Benilan and al. (see [5] Lemma 4.2). But we state it in a general framework in view of applications for a large number of estimates that we shall use in the next section.

### 3. A fundamental lemma for estimates in Marcinkiewicz space

**Lemma 3.1. Fundamental lemma of Benilan's type**

Let  $\nu$  be a non negative Borel measure and  $h : \Omega \rightarrow \mathbb{R}_+, g : \Omega \rightarrow \mathbb{R}_+$ , two  $\nu$ -measurable functions. Then,  $\forall \lambda > 0, \forall k > 0$ , we have

$$\nu \{ h > \lambda \} \leq \frac{1}{\lambda} \int_{\{g \leq k\}} h d\nu + \nu \{ g > k \}.$$

**Proof :**

Since  $t \rightarrow \nu \{ h > t \}$  is non decreasing,  $\forall \lambda > 0$ , then

$$\begin{aligned} \nu \{ h > \lambda \} &\leq \frac{1}{\lambda} \int_0^\lambda \nu \{ h > t \} dt \\ &= \frac{1}{\lambda} \int_0^\lambda \left( \nu \{ h > t \} - \nu \{ h > t, g > k \} \right) dt + \frac{1}{\lambda} \int_0^\lambda \nu \{ f > t, g > k \} dt. \end{aligned}$$

We have  $\nu \{ h > t, g > k \} \leq \nu \{ g > k \}$  and

$$\nu \{ h > t \} - \nu \{ h > t : g > k \} = \nu \{ h > t : g \leq k \}.$$

Therefore, we obtain

$$\nu \{ h > \lambda \} \leq \frac{1}{\lambda} \int_0^{+\infty} \nu \{ h > t : g \leq k \} dt + \nu \{ g > k \}.$$

By the Cavalieri's principle, one has

$$\int_0^{+\infty} \nu \{ h > t : g \leq k \} dt = \int_{\{g \leq k\}} h d\nu.$$

With those two last inequalities, we get the result.  $\diamond$

Besides the applications that we shall give in the next section, we recall some estimates that we have already used in a previous work ([14]).

Let us recall that if  $\omega$  is an integrable weight function on  $\Omega$ , the weighted Marcinkiewicz space is defined by

$$L^{q,\infty}(\Omega, \omega) = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable s.t. } \sup_{\lambda > 0} \lambda^q \int_{|v| > \lambda} \omega(x) dx < +\infty \right\}, 0 < q < +\infty.$$

If  $\omega = 1$ ,  $L^{q,\infty}(\Omega, 1)$  is the same as the Lorentz space  $L^{q,\infty}(\Omega)$  defined in the first section.

### Theorem 3.1. A generalized Benilan type result

Let  $\omega$  be an integrable weight function on  $\Omega$ ,  $1 \leq p < +\infty$ , and let  $u \in W_{loc}^{1,1}(\Omega)$  be such that, for a constant  $M_f > 0$ , we have

$$\int_{\Omega} |\nabla T_k(u)|^p \omega(x) dx + \int_{\Omega} |T_k(u)|^p \omega(x) dx \leq M_f k, \quad \forall k > 0,$$

with  $T_k(t) = \min(|t|; k) \text{sign}(t)$ ,  $t \in \mathbb{R}$ .

Assume furthermore that we have a continuous Sobolev embedding,

$$W^1 L^p(\Omega, \omega) \subsetneq L^{p^*}(\Omega, \omega) \quad \text{for some } p^* > p.$$

Then, one has:

$$\int_{|\nabla u| > \lambda} \omega dx \leq c M_f^{\frac{p^*}{p^* + p'}} \lambda^{-\frac{pp^*}{p^* + p'}} \quad \forall \lambda > 0,$$

where  $c > 0$  is a constant depending only on  $p$ ,  $\Omega$ ,  $p^*$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ . Hence  $|\nabla u| \in L^{q,\infty}(\Omega, \omega)$ , with  $q = \frac{pp^*}{p^* + p'}$ . If  $q > 1$ , then  $u \in W^1 L^r(\Omega, \omega)$ ,  $1 \leq r < q$ .

**Proof :**

For a measurable set  $E \subset \Omega$ , we set  $\nu(E) = \int_E \omega(x) dx$  and we apply the above fundamental Lemma 3.1 to derive that for all  $\lambda > 0$ ,  $\forall k > 0$ ,

$$(23) \quad \nu(\{|\nabla u|^p > \lambda\}) \leq \frac{1}{\lambda} \int_{\Omega} |\nabla T_k(u)|^p \omega dx + \nu(\{|u| > k\}).$$

By the first assumption of the theorem, we get, for all  $k > 0$  and  $\lambda > 0$ , that

$$(24) \quad \nu(\{|\nabla u|^{p^*} > \lambda\}) \leq \frac{k}{\lambda} M_f + \nu(\{|u| > k\}).$$

Following Benilan and al. [5], we have  $\{|u| > \varepsilon\} = \{|T_k(u)| > \varepsilon\}$  for  $\varepsilon < k$ . Therefore

$$(25) \quad \nu(\{|u| > \varepsilon\}) \leq \frac{1}{\varepsilon^{p^*}} \int_{\Omega} |T_k(u)|^{p^*} \omega dx.$$

Using Sobolev inequality, we have

$$(26) \quad \nu(\{|u| > \varepsilon\}) \leq \frac{1}{\varepsilon^*} c_s \left[ \int_{\Omega} |\nabla T_k(u)|^p \omega dx + \int_{\Omega} |T_k(u)|^p dx \right]^{\frac{p^*}{p}} \leq \frac{1}{\varepsilon^*} c_s k^{\frac{k^*}{k}}.$$

As  $\varepsilon \rightarrow k$ , we have, for all  $k > 0$ ,

$$(27) \quad \nu(\{|u| > k\}) = \int_{|u|>k} \omega(x) dx \leq c_s k^{-\frac{p^*}{p'}},$$

where  $c_s$  is the Sobolev constant.

Combining relations (24) and (27) one has

$$(28) \quad \nu(\{|\nabla u|^p > \lambda\}) \leq \inf_{k>0} \left\{ \frac{M_f}{\lambda} k + c_s k^{-\frac{p^*}{p'}} \right\}.$$

Computing the infimum, we have  $\forall \lambda > 0$

$$(29) \quad \nu(\{|\nabla u|^p > \lambda\}) \lesssim M_f^{\frac{p^*}{p^*+p'}} \lambda^{-\frac{p^*}{p^*+p'}}.$$

This implies the result ◇

Here are some weighted spaces in which we have a Sobolev embedding (see [28, 44]).

### Proposition 3.1.

Assume that  $\Omega$  is a bounded open Lipschitz set of  $\mathbb{R}^n$ . Let  $\alpha \geq 0$ , and let  $\omega$  be one of the following weights

- $\omega(x) = \text{dist}(x; \partial\Omega)^\alpha = \delta(x)^\alpha$ ,
- $\omega(x) = \text{dist}(x; x_0)^\alpha$ ,  $x_0 \in \partial\Omega$ .

For  $1 \leq p < n + \alpha$ , we have  $p^* = \frac{(n + \alpha)p}{n + \alpha - p}$  and

$$\left[ \int_{\Omega} |v|^{p^*} \omega dx \right]^{\frac{1}{p^*}} \leq c \left[ \left( \int_{\Omega} |v|^p \omega dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |\nabla v|^p \omega dx \right)^{\frac{1}{p}} \right].$$

As a consequence of the above Theorem and Proposition 3.1, here it is a proposition that we have already stated and used in [14] (see Proposition 13).

### Proposition 3.2.

Let  $v \in L^1(\Omega, \delta^\alpha)$ , and  $\alpha \in [0, 1]$ . Assume that there exists a constant  $c_0 > 0$  such that for all  $k > 0$

$$T_k(v) := \min(|v|; k) \text{ sign}(v) \in W^1 L^2(\Omega, \delta^\alpha),$$

and

$$(30) \quad \int_{\Omega} |\nabla T_k(v)|^2 \delta^\alpha dx + \int_{\Omega} |T_k(v)|^2 \delta^\alpha dx \leq c_0 k.$$

Then, there exists a constant  $c$ , depending continuously on  $c_0 > 0$ , such that for all  $\lambda > 0$

$$\int_{\{x:|\nabla v|(x)>\lambda\}} \delta^\alpha(x)dx \leq \frac{c}{\lambda^{1+\frac{1}{n+\alpha-1}}}.$$

In particular, if  $(v_j)_j$  is a sequence converging weakly in  $L^1(\Omega)$  to a function  $v$ , satisfying the inequality (30) and such that

$$\int_{\Omega} |\nabla T_k(v_j)|^2 \delta^\alpha dx \leq c_0 k \quad \forall j, \forall k,$$

then  $(v_j)_j$  converges to  $v$  weakly in  $W^{1,q}(\Omega')$  for all  $q \in \left[1, \frac{n+\alpha}{n+\alpha-1}\right]$  and all  $\Omega' \subset\subset \Omega$ , and there exists a subsequence  $(v_j)_j$  such that  $v_j(x) \rightarrow v(x)$  a.e. in  $\Omega$ .

#### 4. Application to the regularity of the solution of a $p$ -Laplacian

Let  $\Omega$  be a bounded set of  $\mathbb{R}^n$ . Let us consider  $f \in L^1(\Omega) \cap W^{-1,p'}(\Omega)$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ ,  $1 < p < +\infty$ ,  $V$  be a Caratheodory function from  $\Omega \times \mathbb{R}$  into  $\mathbb{R}$  such that

**H1:** for all  $\sigma \in \mathbb{R}$ ,  $x \in \Omega \rightarrow V(x; \sigma)$  is in  $L^\infty(\Omega)$ .

**H2:** for a.e.  $x \in \Omega, \sigma \in \mathbb{R} \rightarrow V(x; \sigma)$  is continuous and non decreasing with  $V(x; 0) = 0$ .

Using the Leray-Lions' method for monotone operators (see Lions's book [29]) or the usual fixed point theorem of Leray-Schauder's type (see Gilbarg -Trudinger [26]) we have:

##### Proposition 4.1.

Let  $f$  be in  $L^1(\Omega) \cap W^{-1,p'}(\Omega)$ ,  $\frac{1}{p} + \frac{1}{p'} = 1$ . Then there exists a unique element  $u \in W_0^{1,p}(\Omega)$  such that

$$(31) \quad \int_{\Omega} \varphi(x)V(x; u)dx + \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} f \varphi dx \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega).$$

We call such solution a weak solution of the Dirichlet equation  $-\Delta_p u + V(x; u) = f$ .

##### Remark 4.1. (on the above existence and uniqueness)

If  $p > n$ , then  $L^1(\Omega) \subset W^{-1,p'}(\Omega)$ .

If  $p \leq n$ , then the dual space  $L^{(p^*)}'(\Omega) \subset W^{-1,p'}(\Omega)$  whenever  $p^* = \frac{np}{n-p}$  if  $p < n$ , and  $p^*$  is any finite number if  $p = n$ .

Therefore, the above result can be applied for these cases. In the paragraph concerning the equation with variable exponents, we give the idea on how to prove the above proposition.

We can define a nonlinear mapping:

$$\begin{aligned} T : L^1(\Omega) \cap W^{-1,p'}(\Omega) &\longrightarrow L^p(\Omega)^n \\ f &\longmapsto Tf = \nabla u. \end{aligned}$$

We shall need sometimes the following additional growth assumption for  $V$ .

**H3:** There exist  $m_1 \in [p-1, \bar{m}_1[$ ,  $\bar{m}_1 = \begin{cases} (p-1) \left(1 + \frac{1}{n-p}\right) & \text{if } p < n \\ < +\infty & \text{if } p \geq n, \end{cases}$

and a constant  $c > 0$  such that

$$|V(x, \sigma)| \leq c|\sigma|^{m_1} \quad \forall \sigma \in \mathbb{R}, \text{ a.e. } x \in \Omega.$$

We want to extend above mapping over all  $L^1(\Omega)$ . When  $p = n$  and  $f \in L^1(\Omega)$  the Iwaniec-Sbordone's method ensures the existence and uniqueness of a weak solution that is under the above formulation (31) or even in the sense of distribution, see for instance ([19], [31], [20]). So the above mapping is well defined on  $L^1(\Omega)$ .

When  $p < n$  and the data  $f$  is only in  $L^1(\Omega)$ , the formulation by equation (31) cannot ensure the uniqueness of the solution. Here it is an equivalent formulation which summarizes various definitions introduced by different authors (see for instance [16, 5, 7, 8, 41, 40, 10]).

We consider again the usual truncation

$$T_k : \mathbb{R} \rightarrow \mathbb{R} \text{ defined by } T_k(\sigma) = \{|k + \sigma| - |k - \sigma|\}/2,$$

and we define as in [38, 41] (see also [5]), the following  $T$ -space or  $T$ -set:

$$\mathbb{S}_0^{1,p} = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable such that } \tan^{-1}(v) \in W_0^{1,1}(\Omega), \right. \\ \left. \text{and } \forall k > 0, T_k(v) \in W_0^{1,p}(\Omega), \sup_{k>0} k^{-\frac{1}{p}} \|\nabla T_k(v)\|_{L^p(\Omega)} = \kappa < +\infty \right\}.$$

**Definition 4.1 (of an entropic-renormalized solution).**

We will say that a function  $u$  defined on  $\Omega$  is an entropic-renormalized solution associated to the Dirichlet problem

$$(32) \quad -\Delta_p u + V(x; u) = f \in L^1(\Omega) \quad u = 0 \text{ on } \partial\Omega$$

if

$$(1) \quad u \in \mathbb{S}_0^{1,p}(\Omega), \quad V(\cdot, u) \in L^1(\Omega).$$

$$(2) \quad \forall \eta \in W^{1,r}(\Omega), \quad r > n, \quad \forall \varphi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega) \text{ and all } B \in W^{1,\infty}(\mathbb{R}) \text{ with } B(0) = 0, \\ B'(\sigma) = 0 \text{ for all } \sigma \text{ such } |\sigma| \geq \sigma_0 > 0, \text{ one has:}$$

$$(33) \quad \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla (\eta B(u - \varphi)) dx + \int_{\Omega} V(x; u) \eta B(u - \varphi) dx = \int_{\Omega} f \eta B(u - \varphi) dx.$$

If  $f \in L^{p'}(\Omega)$  the above formulation (33) is equivalent to the formulation (31) (i.e. a weak solution is an entropic-renormalized solution); see [38] or [31] for the case  $p = n$ .

It has been proved in the above references (see [5, 42, 40, 7, 10]) that we have existence and uniqueness of an entropic-renormalized solution.

**Theorem 4.1.**

Let  $f \in L^1(\Omega)$  and assume (H1) and (H2). Then there exists a unique entropic-renormalized solution of equation (32). Moreover, if the sequence  $(f_j)_j$  converges to  $f$  in  $L^1(\Omega)$ , then the sequence  $(\nabla u_j(x))_j$  converges to  $\nabla u(x)$  almost everywhere in  $\Omega$  for a subsequence still denoted by  $(u_j)_j$ .

When  $p > 2 - \frac{1}{n}$ , the solution  $u \in W_0^{1,1}(\Omega)$ .

**Comments on the proofs of Theorem 4.1 and Theorem 4.2**

- For any  $v \in \mathbb{S}_0^{1,p}(\Omega)$ , the gradient of  $v$  exists a.e. in the sense that if we denote by  $\{\vec{e}_1, \dots, \vec{e}_n\}$  the canonical basis of  $\mathbb{R}^n$ , then the following limit exists almost everywhere in  $\Omega$

$$\lim_{t \rightarrow 0} \frac{v(x + t\vec{e}_i) - v(x)}{t} \doteq \frac{\partial v}{\partial x_i}(x)$$

and

$$DB(v)(x) = B'(v(x))Dv(x), \text{ whenever } B \in W^{1,\infty}(\mathbb{R}).$$

This result is only given in [40] (see also[41]).

- Let  $f_1 \in L^1(\Omega)$ ,  $f_2 \in L^\infty(\Omega)$ ,  $u_1$  be the entropic-renormalized solution associated to  $f_1$  and  $u_2$  be the weak solution of equation (31) associated to  $f_2$ . Then, choosing  $\eta = 1$ ,  $B = \tan^{-1}(T_k)$ , for  $k > 0$ ,  $\varphi = u_2$  is in  $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ . One has, dropping the non negative term,

$$(34) \quad \int_{\Omega} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2 \right) \cdot \nabla B(u_1 - u_2) dx \leq \frac{\pi}{2} \int_{\Omega} |f_1 - f_2| dx.$$

The relation (34) implies the uniqueness of the entropic-renormalized solution for all  $p \in ]1, n[$ .

When  $p \geq 2$ , we can have more inequalities for  $u_1 - u_2$ . Indeed, we can use the strong coercivity of the  $p$ -Laplacian, that is inequality (8) (or see below(44)), and we let  $k \rightarrow +\infty$  to obtain:

$$\int_{\Omega} \frac{|\nabla(u_1 - u_2)|^p}{1 + |u_1 - u_2|^2} dx \leq \frac{\pi}{2} \int_{\Omega} |f_1 - f_2| dx.$$

From this relation, we have for all  $1 \leq q < \frac{n}{n-1}(p-1)$

$$\int_{\Omega} |\nabla(u_1 - u_2)|^q \leq c \|f_1 - f_2\|_{L^1}^{\frac{q}{p}} \left[ 1 + \left( \int_{\Omega} |\nabla(u_1 - u_2)|^q dx \right)^{m_2} \right]$$

with  $m_2 = q^* \left( \frac{1}{q} - \frac{1}{p} \right)$  and  $\frac{1}{q^*} = \frac{1}{q} - \frac{1}{n}$  if  $p < n$ , so that  $m_2 < 1$ ; and  $q^*$  is any number so that  $m_2 < 1$  if  $p \geq n$ .

Hence, using Young inequality,

$$(35) \quad \left( \int_{\Omega} |\nabla(u_1 - u_2)|^q \right)^{1/q} \leq c \left[ \|f_1 - f_2\|_{L^1}^{\beta_1} + \|f_1 - f_2\|_{L^1}^{\beta_2} \right]$$

where  $\beta_i$ ,  $i = 1, 2$ ,  $c$  depend only  $p$ ,  $n$ ,  $\Omega$ ,  $\beta_i > 0$ .

This is the method used in [37, 39]. The above inequality gives a stability and uniqueness result.

The technique developed by Benilan and al. [5] gives a more precise result than the above relation (35). Indeed the same arguments as for having (34) with  $B = T_k$ , leads to:

$$(36) \quad \int_{\Omega} |\nabla T_k(u_1 - u_2)|^p dx \leq k \int_{\Omega} |f_1 - f_2| dx, \quad \forall k \geq 0.$$

If  $f_1 \neq f_2$ , we set  $w = \frac{u_1 - u_2}{\|f_1 - f_2\|_{L^1}^{\frac{1}{p-1}}}$ .  $\lambda = k \|f_1 - f_2\|_{L^1}^{-\frac{1}{p-1}}$  and we deduce that

$$(37) \quad \int_{\Omega} |\nabla T_{\lambda}(w)|^p dx \leq \lambda.$$

From this inequality, the Benilan's technique (see [5] or the above Theorem 3.1) implies that

$$(38) \quad \|\nabla w\|_{L^{n'(p-1), \infty}} \leq c(p, \Omega).$$

This implies the second statement of the Theorem. Another proof of this regularity result (38) is in [43].  $\diamond$

### Proposition 4.2.

Let  $u$  be the solution of equation (32) with  $f$  being in  $L^p(\Omega)$ . Then

- (1) If  $f \in L^{\frac{n}{p}; \frac{1}{p-1}}(\Omega)$ ,  $p \leq n$  (Lorentz space), then  $u \in L^{\infty}(\Omega)$ . Moreover,
 
$$\begin{cases} \|u\|_{L^{\infty}} \leq c \|f\|_{L^{\frac{n}{p}; \frac{1}{p-1}}(\Omega)}^{\frac{1}{p-1}}, & \text{if } p \leq n, \\ \|u\|_{\infty} \leq C \|f\|_1^{\frac{1}{p-1}}, & \text{if } p > n. \end{cases}$$
- (2) If we assume (H3) and  $f \in L^{n,1}(\Omega)$ , then

$$\nabla u \in L^{\infty} \text{ and } \|\nabla u\|_{L^{\infty}} \leq c \left( 1 + \|f\|_{L^1}^{\frac{m_1+1-p}{(p-1)^2}} \right) \|f\|_{L^{n,1}(\Omega)}^{\frac{1}{p-1}}.$$

All the constants denoted by  $c$  depend only on  $p$ ,  $\Omega$  and  $V$ .



This proposition gathers well known results (see for instance [11] for statement (1), and [36, 35] for statement (2)).

The growth of the gradient in Proposition 4.2 comes from the following Lemmas.

**Lemma 4.1.**

Assume (H3). Let  $m_3 = \frac{n}{n-p}(p-1)$  if  $1 < p < n$ , and  $m_3$  any finite number in  $[nm_1, +\infty[$ , and let  $u \in L^\infty(\Omega)$ . Then

$$\|V(\cdot; u(\cdot))\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|u\|_\infty \cdot \|u\|_{L^{m_3,1}}^{\frac{m_1+1-p}{p-1}}.$$

**Proof :**

One has from (H3)

$$\begin{aligned} \|V(\cdot; u(\cdot))\|_{L^{n,1}}^{\frac{1}{p-1}} &\leq c \left[ \int_0^{|\Omega|} t^{\frac{1}{n}} |u|_*^{m_1}(t) \frac{dt}{t} \right]^{\frac{1}{p-1}} \\ &\lesssim \|u\|_\infty \left[ \int_0^{|\Omega|} t^{\frac{1}{n}} |u|_*^{m_1+1-p}(t) \frac{dt}{t} \right]^{\frac{1}{p-1}} \\ (39) \quad &\lesssim \|u\|_\infty \|u\|_{L^{n(m_1+1-p), m_1+1-p}}^{\frac{m_1+1-p}{p-1}}. \end{aligned}$$

If  $1 < p < n$ , one has  $n(m_1 + 1 - p) < \frac{n}{n-p}(p-1) = m_3$  since  $m_1 < (p-1) \left[ 1 + \frac{1}{n-p} \right]$ ,  $n(m_1 + 1 - p) < m_3$  if  $p \geq n$ . Therefore, the last inequality implies the result.  $\diamond$

**Lemma 4.2.**

Assume (H3) and let  $m_3$  be as in Lemma 4.1. If  $f \in L^{n,1}(\Omega)$  and  $u$  is a weak solution of equation (31), then

$$\|V(\cdot; u(\cdot))\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|f\|_{L^{n,1}}^{\frac{1}{p-1}} \cdot \|f\|_{L^1}^{\frac{m_1+1-p}{p-1}}.$$

**Proof :**

Following Proposition 4.2,  $u$  is bounded and  $\|u\|_\infty \lesssim \|f\|_{L^{n,1}}^{\frac{1}{p-1}}$ . Since we have

$$(40) \quad \|u\|_{L^{n'(p-1)*},1} \leq c \|\nabla u\|_{L^{n'(p-1),\infty}},$$

by Sobolev-Poincaré's inequality (see [35] ) and

$$(41) \quad \|\nabla u\|_{L^{n'(p-1),\infty}} \lesssim \|f\|_{L^1}^{\frac{1}{p-1}},$$

by Theorem 4.1.

From (40) and (41), we have

$$(42) \quad \|u\|_{L^{n'(p-1)*},1} \lesssim \|f\|_{L^1}^{\frac{1}{p-1}}.$$

Since  $n'(p-1)^* = m_3$ , from Lemma 4.1 we derive the result.  $\diamond$

The statement 2.) of Proposition 4.2 is then a consequence of the above Lemmas since  $-\Delta_p u = f - V(\cdot; u) \in L^{n,1}$  if  $u$  is a weak solution of (31) for  $f \in L^{n,1}(\Omega)$ . The Cianchi-Mazja's result implies

$$(43) \quad \|\nabla u\|_{L^\infty} \lesssim \|f - V(\cdot; u)\|_{L^{n,1}}^{\frac{1}{p-1}} \lesssim \|f\|_{L^{n,1}} + \|V(\cdot; u)\|_{L^{n,1}}^{\frac{1}{p-1}}.$$

This inequality (43) and Lemma 4.2 implies the estimate 2.) in Proposition 4.2.

#### 4.1. The Hölderian mappings for the case $p \geq 2$ .

We start with the Hölder property in the case  $2 \leq p < n$ .

##### Theorem 4.2.

If  $f_i \in L^1(\Omega)$ ,  $i = 1, 2$  and if  $u_i$  are the corresponding entropic-renormalized solution, one has for  $p \geq 2$ .

- (1)  $\int_{\Omega} \left| \nabla T_k(u_1 - u_2) \right|^p dx \leq k \int_{\Omega} |f_1 - f_2| dx.$
- (2)  $u_i \in W_0^{1,1}(\Omega)$ , and moreover

$$\|\nabla(u_1 - u_2)\|_{L^{n'(p-1),\infty}(\Omega)} \leq c \|f_1 - f_2\|_{L^1(\Omega)}^{\frac{1}{p-1}},$$

where  $c$  is a constant depending only on the data  $p$ ,  $\Omega$  and  $V$ ,  $n' = \frac{n}{n-1}$ .

##### Proof :

We use the stability result. Indeed, let  $f_{1j} = T_j(f_1)$  (resp.  $f_{2j} = T_j(f_2)$ ). Then,  $u_{ij}$   $i = 1, 2$  associated to  $f_{ij}$  are solutions of (31).

We note first that if  $u_1$  is an entropic-renormalized solution associated to  $f_1$  and  $f_{1j} = T_j(f_1) \in L^\infty(\Omega)$ , then the weak solution  $u_{1j}$  of equation (31) satisfies

$$\|\nabla u_{1j} - \nabla u_1\|_{L^{n'(p-1),\infty}(\Omega)} \xrightarrow{j \rightarrow +\infty} 0.$$

Therefore using relation (8) and  $\varphi = T_k(u_{1j} - u_{2j})$  as a test function in relation (31), we derive the statement (1) of the theorem using the convergences for each  $\nabla u_i$ . While for statement (2), we may apply Theorem 3.1 with  $u = u_1 - u_2 \in W_{loc}^{1,1}(\Omega)$  and  $\omega = 1$ .  $\diamond$

As consequence of this theorem, we have the following Corollary which proves Theorem 4.2.

##### Corollary 4.2.1 (of Theorem 4.1).

Under the assumptions (H1) and (H2), we extend the mapping  $T$  :

$$\begin{array}{ccc} L^1(\Omega) & \longrightarrow & \left[ L^{n'(p-1),\infty}(\Omega) \right]^n \\ f & \longmapsto & Tf \end{array}$$

with  $Tf = \nabla u$ , where  $u$  is the unique entropic-renormalized solution of the Dirichlet equation (32). Then, for  $p \geq 2$ , there exists a constant  $c(p, \Omega) > 0$  independent of  $V$  such that

$$\|Tf_1 - Tf_2\|_{L^{n'(p-1),\infty}} \leq c(p, \Omega) \|f_1 - f_2\|_{L^1(\Omega)}^{\frac{1}{p-1}}.$$

**Proof :**

We derive this result from the statement (2) of Theorem 4.2. This stability implies the desired result.  $\diamond$

**Lemma 4.3.**

Assume (H1) and (H2). If  $p \geq 2$ , then the preceding mapping  $T$  is  $\frac{1}{p-1} = \alpha$ -Hölderian from  $L^{(p^*)}'$  into  $L^p(\Omega)^n$  with  $\frac{1}{(p^*)}' = \frac{1}{p'} + \frac{1}{n}$ ,  $p^* = \frac{np}{n-p}$ ,  $1 < p < n$ ,  $p'$  is the conjugate of  $p$ .

**Proof :**

For  $p \geq 2$ , we recall that there exists a constant  $\alpha_p > 0$  such that  $\forall \xi \in \mathbb{R}^n$ ,  $\forall \xi' \in \mathbb{R}^n$

$$(44) \quad \left( |\xi|^{p-2}\xi - |\xi'|^{p-2}\xi', \xi - \xi' \right)_{\mathbb{R}^n} \geq \alpha_p |\xi - \xi'|^p.$$

Therefore, for two data  $f_1$  and  $f_2$  in  $L^{p'}(\Omega)$ , dropping non negative term; we have

$$\begin{aligned} c_p \int_{\Omega} |Tf_1 - Tf_2|^p dx &\leq \int_{\Omega} \left( |\nabla u_1|^{p-2} \nabla u_1 - |\nabla u_2|^{p-2} \nabla u_2, \nabla(u_1 - u_2) \right) dx \\ &\leq \int_{\Omega} (f_1 - f_2)(u_1 - u_2) dx \quad (\text{by Poincaré-Sobolev inequality}) \\ &\leq c_{1p} \|f_1 - f_2\|_{L^{(p^*)}'} \|Tf_1 - Tf_2\|_{L^p}, \end{aligned}$$

so that

$$\|Tf_1 - Tf_2\|_{L^p(\Omega)} \leq c_{2p} \|f_1 - f_2\|_{L^{(p^*)}'(\Omega)}^{\frac{1}{p-1}}.$$

This implies the result using a density argument.  $\diamond$

To apply the abstract results given in the second section for interpolation spaces, we need to use some well-known results concerning some identification. The first one can be deduced from the famous reiteration process of Lions-Peetre or from Proposition 2.1.

**Proposition 4.3.**

For all  $r \in [1, +\infty]$ ,  $1 \leq m \leq +\infty$ ,  $1 < q \leq +\infty$ ,  $m < k < q$ , we have

$$\left( L^m(\Omega), L^q(\Omega) \right)_{\theta, r; 0} = L^{k, r}(\Omega) \quad \text{with} \quad \frac{1}{k} = \frac{1-\theta}{m} + \frac{\theta}{q}.$$

Notice that the interpolation space  $(X_0, X_1)_{\theta, r; 0}$  is the same as Peetre interpolation space  $(X_0, X_1)_{\theta, r}$  since  $X_1 \subset X_0$ .

**Proposition 4.4.**

Assume (H1) and (H2). Let  $p^* = \frac{np}{n-p}$  with  $2 \leq p < n$ ,  $(p^*)'$  its conjugate,  $1 \leq k \leq (p^*)' = \frac{np}{np-n+p}$ ,  $p^*$  is any finite number if  $p \geq n$ .  $r \in [1, +\infty]$ . Then

$$\|Tf_1 - Tf_2\|_{L^{k^*(p-1), r(p-1)}} \leq c \|f_1 - f_2\|_{L^{k,r}}^{\frac{1}{p-1}},$$

for  $f_1, f_2$  in  $L^{k,r}(\Omega)$ , with  $k^* = \frac{kn}{n-k}$  if  $k < n$  and any finite number if  $k \geq n$ .

In particular if  $f \in L^{k,r}(\Omega)$  then the gradient of the solution  $u$  of (33) belongs to  $L^{k^*(p-1), r(p-1)}(\Omega)^n$ .

**Proof :**

The mapping  $T$  is  $\alpha = \frac{1}{p-1}$ -Hölderian from  $L^1(\Omega)$  into  $L^{n'(p-1), \infty}(\Omega)^n$  and from  $L^{q'}(\Omega)$  into  $L^p(\Omega)^n$  with  $q' = (p^*)'$ . Moreover, we have  $L^{k,r}(\Omega) = (L^1, L^{q'})_{\theta, r}$  with  $\theta = p^* \left(1 - \frac{1}{k}\right)$  and  $L^{k^*(p-1), r(p-1)} = \left(L^{n'(p-1), \infty}, L^p\right)_{\theta, r(p-1)}$ .

From the abstract theorem, Theorem 2.2, we have for  $f_1, f_2$  in  $L^{k,r}(\Omega)$ :

$$\|Tf_1 - Tf_2\|_{\theta, r(p-1); 0} \lesssim \|f_1 - f_2\|_{\theta, r}^{\frac{1}{p-1}}.$$

Noticing that bounded functions are dense in  $L^{k,r}(\Omega)$ , this gives the result.  $\diamond$

Proposition 4.4 improves previous known results in fact, one can consider the case  $k = r$ , the usual estimate is only obtained in  $\left[L^{k^*(p-1)}(\Omega)\right]^n$  (see for instance [11]).

Let us apply now those identification of the interpolation spaces to obtain precise regularity of the gradient of an entropic-renormalized solution.

**Theorem 4.3.**

Assume (H1) and (H2) and let  $m = \frac{np}{(n+1)p-n}$  if  $2 \leq p < n$  and  $m \in [1, +\infty[$  if  $p \geq n$ ,  $1 \leq p_2 < +\infty$ ,  $\lambda \in \mathbb{R}$ ,  $0 < \theta < 1$ . Then, the mapping  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $L^{\frac{m'}{m'-\theta}, p_2}(\text{Log } L)^\lambda$  into  $L^{p_\theta, p_2(p-1)}(\text{Log } L)^{\frac{\lambda}{p-1}}$  with

$$\frac{1}{p_\theta} = \frac{(1-\theta)(n-1)}{n(p-1)} + \frac{\theta}{p} \quad \text{and} \quad m' = \frac{m}{m-1}.$$

If  $2 \leq p < n$  and  $\theta = 0$ , the mapping  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $G\Gamma(1, p; t^{-1}(1 - \text{Log } t)^{\lambda p})$  into  $(L^{n'(p-1), \infty}(\Omega), L^p(\Omega))_{0, p(p-1); \lambda/(p-1)}$ . This latter space has norm equivalent to

(\*)

$$\|f\|_{(L^{n'(p-1), \infty}(\Omega), L^p(\Omega))_{0, p(p-1); \lambda/(p-1)}} \approx \left[ \int_0^1 \left( \sup_{0 < s < t^\sigma} s^{\frac{p'}{p m'}} f_*(s) (1 - \text{Log } t)^{\frac{\lambda}{p-1}} \right)^{p(p-1)} \frac{dt}{t} \right]^{\frac{1}{p(p-1)}},$$

where  $1/\sigma = p'/(pn') - 1/p$ .

In the case  $\theta = 1$ , the mapping  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $(L^1(\Omega), L^{(p^*)}'(\Omega))_{1,p;\lambda}$  into  $(L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{1,p/\alpha;\lambda\alpha}$ . The first space has (quasi)norm

$$\left[ \int_0^1 \left( \left( \int_t^1 f_*(s)^{(p^*)}' ds \right)^{\frac{1}{(p^*)'}} (1 - \log t)^\lambda \right)^p \frac{dt}{t} \right]^{\frac{1}{p}}$$

while the second has (quasi)norm

$$\left[ \int_0^1 \left( \left( \int_t^1 f_*(s)^p ds \right)^{\frac{1}{p}} (1 - \log t)^{\lambda\alpha} \right)^{\frac{p}{\alpha}} \frac{dt}{t} \right]^{\frac{\alpha}{p}}.$$

Here  $\alpha = \frac{1}{p-1}$ .

**Proof :**

Since  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $L^1$  into  $L^{n'(p-1),\infty}(\Omega)^n$  and  $L^m$  into  $L^p(\Omega)^n$ , and since smooth functions are dense in Lorentz-Zygmung space  $L^{p_1,p_2}(\text{Log } L)^\gamma$ ,  $1 \leq p_1, p_2 < +\infty$  and  $(L^1, L^m)_{\theta,p_2;\lambda} = L^{\frac{m'}{m'-\theta},p_2}(\text{Log } L)^\lambda$  according to Proposition 2.2, we deduce from Theorem 2.2 that  $T$  maps  $(L^1, L^m)_{\theta,p_2;\lambda}$  into  $(L^{n'(p-1)}, L^p)_{\theta,\frac{p_2}{\alpha};\lambda\alpha}$  with  $\alpha = \frac{1}{p-1}$ .

The identification of the last space given by Corollary 2.3.1 of Proposition 2.3 proves the results.

Let  $2 \leq p < n$  and  $\theta = 0$ . First of all, as before, the mapping  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $L^1(\Omega)$  into  $L^{n'(p-1),\infty}(\Omega)^n$ . On the other hand, in this case, by Lemma 4.3 we know that the mapping  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $L^{(p^*)}'(\Omega)$  into  $L^p(\Omega)^n$ , where as usual  $(p^*)' = np/(np - n + p)$ . Noticing that by Proposition 2.2,  $(L^1, L^{(p^*)}'(\Omega))_{0,p;\lambda} = G\Gamma(1, p; t^{-1}(1 - \text{Log } t)^{\lambda p})$  and that  $L^{(p^*)}'(\Omega)$  is dense therein, we can therefore apply Theorem 2.2 and get that  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $(L^1(\Omega), L^{(p^*)}'(\Omega))_{0,p;\lambda}$  into  $(L^{n'(p-1),\infty}(\Omega), L^p(\Omega))_{0,p/\alpha;\lambda\alpha}$ . Then the assertion follows, because the domain space has been identified in Remark 2.1 while the target space has been identified in Proposition 2.1. The same argument holds for  $\theta = 1$ .  $\diamond$

To obtain boundedness of the solution in a more general situation, we need to assume (H3).

We have the following:

**Theorem 4.4.**

Assume (H1), (H2) and (H3). Let  $0 \leq \theta < 1$ ,  $1 < p_2 < +\infty$ ,  $\lambda \in \mathbb{R}$ ,  $f \in L^{\frac{n'}{n'-\theta},p_2}(\text{Log } L)^\lambda$   $n' = \frac{n}{n-1}$ ,  $0 < \theta < 1$  and  $f \in G\Gamma(1, p_2; w_2)$  with  $w_2(t) = t^{-1}(1 - \text{Log } t)^{\lambda p_2}$  if  $\theta = 0$ .

Then the entropic-renormalized solution  $u$  of the Dirichlet equation (33) has its gradient in  $L^{\frac{n(p-1)}{(1-\theta)(n-1)},p_2(p-1)}(\text{Log } L)^{\frac{\lambda}{p-1}}$ .

**Proof :**

Since  $T$  is  $\frac{1}{p-1}$ -Hölderian from  $L^1$  into  $L^{n'(p-1,\infty)}(\Omega)^n$  and  $T$  since from Proposition 4.2 is bounded from  $L^{n,1}(\Omega)$  into  $L^\infty(\Omega)$ , then following Theorem 2.1,  $T$  bounded from  $(L^1, L^n)_{\theta,p_2;\lambda}$  into  $(L^{n'(p-1)}, L^\infty)_{\theta,p_2(p-1);\frac{\lambda}{p-1}}$ . With the identification of those interpolation spaces we obtain the result.  $\diamond$

#### 4.2. Few results on the case $1 < p < 2$ .

Some of the above results remain true in the case  $1 < p < 2$ . The fundamental changes concern the Hölder properties than can exist but are not sharp as for the case  $p \geq 2$ , and the Hölder constant appearing depend on the data. Here is an example.

**Theorem 4.5. (local Lipschitz contraction when  $1 < p < 2$ )**

Let  $1 < p < 2$ ,  $p^* = \frac{np}{n-p}$ ,  $n \geq 2$ ,  $(p^*)' = \frac{np}{np+p-n}$  its conjugate and  $f_1$  (resp  $f_2$ ) in  $L^{(p^*)'}(\Omega)$ . Then, for the weak solution  $u$  (resp.  $v$ ) of (31), say  $-\Delta_p u + V(x; u) = f_1$ , whenever  $V$  satisfies (H1) and (H2), one has:

- (1)  $\|\nabla u\|_{L^p} \leq c \|f_1\|_{L^{(p^*)}'},$  (idem for  $v$ ).
- (2)  $\|\nabla(u-v)\|_{L^p} \leq c \left( \|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} \right)^{2-p} \|f_1 - f_2\|_{L^{(p^*)}'}$ .

Here the constant  $c$  depends only on  $p$  and  $\Omega$ .

**Proof :**

Since we have stability result, we may assume that  $f_1$  and  $f_2$  are bounded. Arguing as before, one has, using Poincaré Sobolev inequality, that

$$\int_{\Omega} |\nabla u|^p dx + \int_{\Omega} uV(x; u) dx \leq \|f_1\|_{L^{(p^*)}'} \leq c \|\nabla u\|_{L^p} \cdot \|f_1\|_{L^{(p^*)}'}$$

Dropping the non negative term  $\int_{\Omega} uV(x; u) dx \geq 0$ , we obtain (1).

As to the second statement, we use the following inequality (see [30] or [13]) concerning the  $p$ -Laplacian, namely, setting  $\hat{a}(\nabla u) = |\nabla u|^{p-2} \nabla u$  we have

$$(45) \quad \left( \hat{a}(\nabla u) - \hat{a}(\nabla v) \right) \cdot \nabla(u-v) \geq \alpha \frac{|\nabla(u-v)|^2}{\left( |\nabla u| + |\nabla v| \right)^{2-p}} \text{ a.e. in } \Omega.$$

Therefore, making the differences between the two equations and dropping non negative terms containing  $V$ , we have from relation (45)

$$(46) \quad \int_{\Omega} \frac{|\nabla(u-v)|^2}{\left( |\nabla u| + |\nabla v| \right)^{2-p}} dx \leq c \|f - g\|_{L^{(p^*)}'} \|\nabla(u-v)\|_{L^p}.$$

We have used the Poincaré-Sobolev inequality.

Now we estimate  $\int_{\Omega} |\nabla(u - v)|^p dx$ .

Adding the term  $\left(|\nabla u| + |\nabla v|\right)^{(p-2)\frac{p}{2}}$ , the Hölder inequality yields

$$\int_{\Omega} |\nabla(u - v)|^p dx \leq \left( \int_{\Omega} |\nabla(u - v)|^2 (|\nabla u| + |\nabla v|)^{p-2} dx \right)^{\frac{p}{2}} \left[ \int_{\Omega} (|\nabla u| + |\nabla v|)^p dx \right]^{1-\frac{p}{2}},$$

and with the help of relation (46), we have

$$\|\nabla(u - v)\|_{L^p} \lesssim \|f_1 - f_2\|_{L^{(p^*)'}} \left( \|\nabla u\|_{L^p} + \|\nabla v\|_{L^p} \right)^{2-p}.$$

◇

### Corollary 4.5.1. of Theorem 4.5

*Under the same assumptions as in Theorem 4.5, there exists a constant  $c$  depending only on  $p$  and  $\Omega$  such that*

$$\|\nabla(u - v)\|_{L^p} \leq c \left( \|f_1\|_{L^{(p^*)'}}^{\frac{1}{p-1}} + \|f_2\|_{L^{(p^*)'}}^{\frac{1}{p-1}} \right) \|f_1 - f_2\|_{L^{(p^*)'}}.$$

*In particular, the mapping  $T$  is locally Lipschitz from  $L^{(p^*)'}(\Omega)$  into  $L^p(\Omega)^n$  with  $\frac{1}{(p^*)'} = 1 - \frac{1}{p} + \frac{1}{n}$ .*

Therefore, we can have a weaker version of Proposition 4.4 when  $1 < p < 2$ .

### Proposition 4.5.

*Assume (H1), (H2) and (H3). Let  $1 < p < 2$ ,  $(p^*)' < k < n$ ,  $r \in [1, +\infty]$ .*

*Then, the non linear mapping  $T$  is bounded from  $L^{k,r}(\Omega)$  into  $L^{k_1,r}(\Omega)$ ,  $k_1 = \frac{1}{1 - \theta(p-1)}$ ,*

*with  $\theta = p^*(1 - \frac{1}{k})$*

### Proof :

Following Corollary 4.5.1 of Theorem 4.5 and Proposition 4.2 , the hypothesis of Theorem 2.1 are valid for  $T$  with  $X_0 = L^{(p^*)'}(\Omega)$ ,  $X_1 = L^{n,1}(\Omega)$ ,  $\lambda = 0$ ,  $Y_0 = L^p(\Omega)^n$ ,  $Y_1 = L^\infty(\Omega)^n$ ,  $\alpha = 1$  and  $\beta = \frac{1}{p-1}$ . According to Theorem 2.1,  $T$  is then a locally bounded mapping from

$(X_0, X_1)_{\theta,r}$  into  $(Y_0, Y_1)_{\theta(p-1),r}$  with  $\theta \in [0, 1]$  such that  $\frac{1}{k} = \frac{1-\theta}{(p^*)'} + \frac{\theta}{n}$ .

Therefore  $(X_0, X_1)_{\theta,r} = L^{k,r}(\Omega)$  and  $(Y_0, Y_1)_{\theta(p-1),r} = L^{k_1,r}(\Omega)$  with  $k_1 = \frac{1}{1 - \theta(p-1)}$ . ◇

**Remark 4.2.**

- One can make precise the bound for  $T$  locally, according to Theorem 2.1, Corollary 4.5.1 of Theorem 4.5, and Proposition 4.2.

- If  $p > n$ , then the mapping  $T$  is Lipschitz from  $L^1(\Omega)$  into  $[L^p(\Omega)]^n$ : this is a consequence of the Poincaré Sobolev inequality that have recalled in Proposition 4.2. Therefore, in view of the Cianchi-Mazja's regularity result, the application  $T$  is bounded from  $(L^1, L^{n,1})_{\theta,q,\lambda}$  into  $[(L^p, L^\infty)_{\theta,q,\lambda}]^n$ .

-The list of applications of the above applications is not exhaustive, the reader might derive more results combining those abstract theorems and propositions.

## 5. Application of the interpolation for the regularity of the solution of the anisotropic equation

### 5.1. Preliminary results on anisotropic equations.

We want to provide similar results as before for the solution of

$$(47) \quad \begin{cases} -\Delta_{\vec{p}}u + V(x; u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here  $\Delta_{\vec{p}}u = -\sum_{i=1}^n \frac{\partial}{\partial x_i} \left( \left| \frac{\partial u}{\partial x_i} \right|^{p_i-2} \frac{\partial u}{\partial x_i} \right)$ ,  $\vec{p} = (p_1, \dots, p_n)$ ,  $1 < p_i < +\infty$ ,  $\vec{p}' = (p'_1, \dots, p'_n)$ , where  $p'_i$  is the conjugate of  $p_i$ .

The main differences reside in the exponent appearing in different directions of the space  $R^n$ .

Moreover, the estimates concern directly the derivatives in each direction of the  $\mathbb{R}^n$ -space.

Let us recall from the introduction, the real number  $p$  is defined as  $\frac{1}{p} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$ . When

$\sum_{i=1}^n \frac{1}{p_i} > 1$  (say  $p < n$ ), we set  $p^* = \frac{np}{n-p}$ . We will focus first on the case  $p < n$  for having the Hölderian property of the mapping  $T$ . We set

$$W_0^{1,\vec{p}}(\Omega) = \left\{ \varphi \in W_0^{1,1}(\Omega) \text{ such that } \partial_i \varphi \in L^{p_i}(\Omega), i = 1, \dots, n \right\}$$

$$\mathbb{S}_0^{1,\vec{p}} = \left\{ v : \Omega \rightarrow \mathbb{R} \text{ measurable s.t. } \tan^{-1}(v) \in W_0^{1,1}(\Omega) \right.$$

$$\left. \text{and } T_k(v) \in W_0^{1,\vec{p}}(\Omega) \text{ with } \sup_{k>0} \left[ \max_{1 \leq i \leq n} k^{\frac{1}{p_i}} \|\partial_i T_k(v)\|_{L^{p_i}} < +\infty \right] \right\}.$$

The definition of an entropic-renormalized solution is similar to Definition 4.1; we replace the operator and the spaces by the above ones:



**Definition 5.1. entropic renormalized solution for anisotropic equation**

For  $\xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n$  we consider the vector field  $\widehat{a}_{\vec{p}}(\xi) = (|\xi|^{p_1-2}\xi_1, \dots, |\xi_n|^{p_n-2}\xi_n)$

We will say that a function  $u$  defined on  $\Omega$  is an entropic-renormalized solution associated to the Dirichlet problem

$$(48) \quad -\Delta_{\vec{p}}u + V(x; u) = f \in L^1(\Omega) \quad u = 0 \text{ on } \partial\Omega$$

if

- (1)  $u \in \mathbb{S}_0^{1, \vec{p}}(\Omega)$ ,  $V(\cdot, u) \in L^1(\Omega)$ .
- (2)  $\forall \eta \in W^{1, \infty}(\Omega)$ ,  $\forall \varphi \in W_0^{1, \vec{p}}(\Omega) \cap L^\infty(\Omega)$  and all  $B \in W^{1, \infty}(\mathbb{R})$  with  $B(0) = 0$   $B'(\sigma) = 0$  for all  $\sigma$  such that  $|\sigma| \geq \sigma_0 > 0$ , one has:

$$(49) \quad \int_{\Omega} \widehat{a}_{\vec{p}}(\nabla u) \cdot \nabla (\eta B(u - \varphi)) dx + \int_{\Omega} V(x; u) \eta B(u - \varphi) dx = \int_{\Omega} f \eta B(u - \varphi) dx.$$

Concerning the existence and uniqueness, let  $p^*$  be the number defined for the validity of the Poincaré Sobolev inequality:  $\exists c > 0$  such that

$$\forall v \in W_0^{1, \vec{p}}(\Omega) \quad \left( \int_{\Omega} |v|^{p^*}(x) dx \right)^{\frac{1}{p^*}} \leq c \left( \sum_{i=1}^n \int_{\Omega} |\partial_i v|^{p_i} dx \right)^{\frac{1}{p}}$$

Considering the main operator

$$Au = -\operatorname{div}(\widehat{a}_{\vec{p}}(\nabla u)) + V(\cdot, u)$$

which is strongly monotonic from  $W_0^{1, \vec{p}}(\Omega)$  into its dual  $W^{-1, \vec{p}'}(\Omega)$ , for  $f \in L^1(\Omega) \cap W^{-1, \vec{p}'}(\Omega)$ , the usual well-known Leray-Lions method or the Leray-Schauder fixed point can be used for having the existence and uniqueness. Moreover, if  $f \in L^\infty(\Omega)$ , the maximum principle holds true, using for instance the rearrangement technic (see for instance [13, 21, 35]) and noticing that the operator  $\widehat{a}_{\vec{p}}$  satisfies the following coercivity condition : there exists  $c_1 > 0$  such that for all  $\xi \in \mathbb{R}^n$ ,

$$\widehat{a}_{\vec{p}}(\xi) \cdot \xi \geq |\xi|^{p_-} - c_1 \text{ with } p_- = \min\{p_i, i \in \{1, \dots, n\}\}.$$

Once the  $L^\infty$ -estimate is obtained, one may apply standard technics (approximation method and compactness results) (see [29, 13, 36]) to obtain the following proposition:

**Proposition 5.1.**

Let  $f \in L^1(\Omega) \cap W^{-1, \vec{p}'}(\Omega)$ . Then we have a unique weak solution  $u \in W_0^{1, \vec{p}}(\Omega)$  satisfying:

$$(50) \quad \int_{\Omega} \widehat{a}_{\vec{p}}(\nabla u) \cdot \nabla \varphi dx + \int_{\Omega} \varphi V(x; u) dx = \int_{\Omega} f \varphi dx, \quad \forall \varphi \in W_0^{1, \vec{p}}(\Omega) \cap L^\infty(\Omega).$$

Moreover, one has the following energy estimates, for  $f \in L^{(p^*)'}(\Omega)$ ,  $p < n$ ,

$$(51) \quad \sum_{i=1}^n \int_{\Omega} |\partial_i u|^{p_i} dx + \int_{\Omega} u V(x; u) dx \leq c \|f\|_{L^{(p^*)}'}^{p'}.$$

The constant  $c$  depends only on  $\Omega$  and  $p$ .

If  $f \in L^\infty(\Omega)$ , then  $u \in L^\infty(\Omega)$ , and there are constants  $c_i$  independent of  $V$  and  $f$  so that:

$$(52) \quad \|u\|_\infty \leq c_1 + c_2 \|f\|_\infty^{\frac{p'_-}{p_-}}$$

with  $p_- = \min \{p_i, i = 1, \dots, n\}$  and  $p'_-$  its conjugate.

**Remark 5.1.**

- a.) The fact that the constants  $c_1$  and  $c_2$  in relation (52) do not depend on  $V$  is due to the hypothesis on  $V$  which implies that  $\sigma V(x; \sigma) \geq 0$ ,  $\forall \sigma \in \mathbb{R}$ .
- b.) Compactness results concerning anisotropic equation in general form can be found in [17] (see also [37, 39]).
- c.) When  $f \in L^{(p^*)'}(\Omega)$ , the weak formulation is equivalent to the entropic-renormalized formulation. The proof is the same as in [38].
- d.) The entropic-renormalized solution is specially made for  $f \in L^1(\Omega)$ . But the proof of the uniqueness for the solution of (49) (see Definition 5.1) is the same as Benilan and al. [5] or Rakotoson [38], since the operator

$$Au = -\operatorname{div}(\widehat{a}_{\vec{p}}(\nabla u)) + V(\cdot; u)$$

is monotonic. It can be shown that, if  $u_1$  and  $u_2$  are two solutions in a  $T$ -space  $\mathbb{S}_0^{1, \vec{p}}(\Omega)$ , then necessarily, one has for all  $k > 0$

$$\int_{|u_1 - u_2| \leq k} [\widehat{a}_{\vec{p}}(\nabla u_1) - \widehat{a}_{\vec{p}}(\nabla u_2)] \cdot \nabla(u_1 - u_2) dx \leq 0.$$

As to the existence, it follows using standard approximation technique by replacing  $f \in L^1(\Omega)$  by the sequence  $f_j \in L^\infty(\Omega)$  such that  $\|f - f_j\|_{L^1} \xrightarrow{j \rightarrow \infty} 0$ ,  $\|f_j\|_1 \leq \|f\|_1$ . Then, one can obtain uniform estimates for the unique weak solution  $u_j \in W_0^{1, \vec{p}}(\Omega) \cap L^\infty(\Omega)$

$$(53) \quad -\Delta_{\vec{p}} u_j + V(x; u_j) = f_j.$$

The proof of the following theorem follows the same arguments as in [5] and [37, 38, 39, 41, 40].

**Theorem 5.1.**

Assume that (H1) and (H2). Then there is a unique entropic-renormalized solution  $u$  of (49) given in Definition 5.1.

Moreover, for a subsequence denoted by  $u_j$ ,  $Du_j(x) \rightarrow Du(x)$  a.e. in  $\Omega$ .

**Remark 5.2.**

In the next paragraph, we will give new and precise spaces where the gradient should be, under various conditions. In the case

$$\min_i p_i = p_- \geq \max \left\{ \frac{p'}{n'}, 1 \right\}, \quad n' = \frac{n}{n-1}, \quad p' = \frac{p}{p-1} \text{ conjugate of } p,$$

we have  $u \in W_0^{1,1}(\Omega)$ .

## 5.2. The definition of the mappings $\tilde{T}_i$ from $L^1(\Omega)$ into $L^{\frac{n'p_i}{p'}, \infty}(\Omega)$ .

### Theorem 5.2.

Let  $u$  be the entropic-renormalized solution of equation (47). Then, there exists a constant  $c > 0$  independent of  $u$  and  $f$  such that :

$$(1) \text{ meas } \{|u| > k\} \leq c \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} k^{-\frac{p^*}{p'}}, \quad \forall k > 0.$$

$$(2) \left\| \frac{\partial u}{\partial x_i} \right\|_{L^{\frac{n'p_i}{p'}, \infty}(\Omega)} \leq c \|f\|_{L^1(\Omega)}^{\frac{p'}{p_i}}, \quad i = 1, \dots, n.$$

### Proof :

For the statement (1), we follow the arguments of Benilan et al [5].

If  $k > 0$ , then the equation satisfied by  $u$  implies

$$(54) \quad \sum_{i=1}^n \int_{\Omega} |\partial_i T_k(u)|^{p_i} dx \leq k \|f\|_{L^1(\Omega)}.$$

If  $\varepsilon < k$ , then we have

$$\text{meas } \{|u| > \varepsilon\} = \text{meas } \{|T_k(u)| > \varepsilon\} \leq \frac{1}{\varepsilon^{p^*}} \int_{\Omega} |T_k(u)|^{p^*} dx.$$

Using Poincaré-Sobolev inequality, we have

$$(55) \quad \text{meas } \{|T_k(u)| > \varepsilon\} \leq \frac{c_s}{\varepsilon^{p^*}} \left( \sum_{i=1}^n \int_{\Omega} |\partial_i T_k(u)|^{p_i} dx \right)^{\frac{p^*}{p}}$$

$$(56) \quad \leq \frac{c_5 \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} k^{\frac{p^*}{p}}}{\varepsilon^{p^*}}.$$

As  $\varepsilon \rightarrow k$  in relation (56), we get (1).

While for the second statement (2), we apply the fundamental lemma of Benilan's type (see Lemma 3.1, in the third paragraph) choosing  $h = \left| \frac{\partial u}{\partial x_i} \right|^{p_i}$  and  $g = |u|$ , to deduce that for  $\lambda > 0$  and for all  $k > 0$ :

$$(57) \quad \text{meas } \left\{ \left| \frac{\partial u}{\partial x_i} \right|^{p_i} > \lambda \right\} \leq \frac{1}{\lambda} \int_{|u| \leq k} \left| \frac{\partial u}{\partial x_i} \right|^{p_i} dx + \text{meas } \{|u| > k\}$$

$$(58) \quad \leq \frac{k}{\lambda} \|f\|_{L^1(\Omega)} + c_s \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} k^{-\frac{p^*}{p'}}.$$

This implies

$$\text{meas } \left\{ \left| \frac{\partial u}{\partial x_i} \right|^{p_i} > \lambda \right\} \lesssim \min_{k>0} \left\{ \frac{1}{\lambda} \|f\|_{L^1(\Omega)} k + \|f\|_{L^1(\Omega)}^{\frac{p^*}{p}} k^{-\frac{p^*}{p'}} \right\}.$$

Computing the infimum, one has

$$\text{meas} \left\{ \left| \frac{\partial u}{\partial x_i} \right|^{p_i} > \lambda \right\} \lesssim \|f\|_{L^1}^{a+1} \lambda^{-\frac{n'}{p'}} \text{ with } a = n' \left( \frac{1}{p} - \frac{1}{p^*} \right).$$

This last inequality implies the result. ◇

In order to derive an Hölderian mapping, we recall the following elementary inequalities (see [13, 30]): if  $t_1 \in \mathbb{R}$ ,  $t_2 \in \mathbb{R}$ ,  $1 < q < +\infty$  then

$$(59) \quad \left( |t_1|^{q-2}t_1 - |t_2|^{q-2}t_2 \right) (t_1 - t_2) \geq \alpha |t_1 - t_2|^q, \text{ if } q \geq 2,$$

and

$$(60) \quad \left( |t_1|^{q-2}t_1 - |t_2|^{q-2}t_2 \right) (t_1 - t_2) \geq \alpha \frac{|t_1 - t_2|}{(|t_1| + |t_2|)^{2-q}}, \text{ if } q < 2, \text{ if } |t_1| + |t_2| \neq 0.$$

Here  $\alpha > 0$  is a universal constant depending only on  $q$ . We will deal with different situations, let us start with the case  $p_i \geq 2$  for all  $i$ .

### Theorem 5.3.

Assume (H1), (H2) and  $p_- \geq \max \left\{ \frac{p'}{n}; 2 \right\}$ . Let  $i \in \{1, \dots, n\}$ . Then, the mapping

$$\begin{aligned} \tilde{T}_i : L^1(\Omega) &\longrightarrow L^{\frac{n'p_i}{p'}, \infty}(\Omega) \\ f &\longmapsto \tilde{T}_i f = \frac{\partial u}{\partial x_i} \end{aligned}$$

where  $u$  is the unique entropic-renormalized solution is:

- (1)  $\frac{p'}{p_i}$ -Hölderian if  $p' < p_i$ ,
- (2) globally Lipchitz if  $p' = p_i$ ,
- (3) locally Lipchitz if  $p' > p_i$ .
- (4) More precisely, we have a constant  $M_1 > 0$  such that for all  $f_1$  and  $f_2$  in  $L^1(\Omega)$

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{L^{\frac{n'p_i}{p'}, \infty}} \leq M_1 \|f_1 - f_2\|_{L^1}^{\frac{p'}{p_i}}, \quad i \in \{1, \dots, n\}.$$

### Proof :

Let  $f_1$  and  $f_2$  be in  $L^1(\Omega)$ . Due to the stability property, we may assume that  $f_1$  and  $f_2$  are in  $L^\infty(\Omega)$ . Let  $u_1$  (resp.  $u_2$ ) be the weak solution of (47) associated to  $f_1$  (resp.  $f_2$ ). Then, for all  $k > 0$ , using relation (48) one has:

$$(61) \quad \alpha \sum_{m=1}^n \int_{\Omega} |\partial_m T_k(u_1 - u_2)|^{p_m} dx \leq \|f_1 - f_2\|_{L^1}$$

Arguing as in Theorem 5.2, one deduces that  $\forall k > 0$

$$(62) \quad \text{meas} \{|u_1 - u_2| > k\} \leq c_\alpha \|f_1 - f_2\|_{L^1}^{\frac{p^*}{p}} k^{-\frac{p^*}{p'}}.$$

From relation (62), by the same argument as before, which uses the fundamental lemma of Benilan's type (see Lemma 3.1, in the second paragraph) with adequate choices of  $h$  and  $g$ , we deduce we deduce

$$\|\partial_i(u_1 - u_2)\|_{L^{\frac{n'p_i}{p'}, \infty}} \leq c \|f_1 - f_2\|_{L^1}^{\frac{p'}{p_i}}.$$

This gives the result.  $\diamond$

We have another Hölderian mapping when the data is in  $L^{(p^*)'}(\Omega)$

**Theorem 5.4.**

Assume (H1) and (H2). Let  $\frac{1}{p} = \frac{1}{n} \sum_{i=1}^n \frac{1}{p_i}$  with  $\sum_{i=1}^n \frac{1}{p_i} > 1$ , and let  $f_1$  and  $f_2$  be two functions  $L^{(p^*)'}(\Omega)$  with  $p^* = \frac{np}{n-p}$ . Furthermore, we assume that  $p_- \geq 2$ . Then, for two weak solutions  $u_1$  and  $u_2$  associated to  $f_1$  and  $f_2$ , one has

$$(1) \sum_{i=1}^n \int_{\Omega} |\partial_i(u_1 - u_2)|^{p_i} dx \leq c \|f_1 - f_2\|_{L^{(p^*)}'},$$

$$(2) \|\partial_i(u_1 - u_2)\|_{L^{p_i}} \leq c \|f_1 - f_2\|_{L^{(p^*)}'}, \text{ for } i = 1, \dots, n.$$

**Proof :**

The proof is straightforward using  $u_1 - u_2$  as a test function in the weak formulation for equation (48).  $\diamond$

Now, we apply the abstract results concerning interpolations, at first for usual Lorentz spaces as we did before.

**Theorem 5.5.**

Assume (H1), (H2), and  $p_- \geq \max\{2; p'\}$ ,  $1 \leq k \leq (p^*)'$ ,  $r \in [1, +\infty]$ . Then for each  $i \in \{1, \dots, n\}$ , the application  $\tilde{T}_i$  is an Hölderian mapping from  $L^{k,r}(\Omega)$  into  $L^{\frac{k^*p_i}{p'}, \frac{rp_i}{p'}}(\Omega)$ , with  $k^* = \frac{kn}{n-k}$ . More precisely, for all  $f_1, f_2$  in  $L^{k,r}(\Omega)$ ,  $\tilde{T}_i f_j = \frac{\partial u_j}{\partial x_i}$ ,  $i = 1, \dots, n$ ,  $j = 1, 2$ , we have

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{L^{\frac{k^*p_i}{p'}, \frac{rp_i}{p'}}} \leq M_2 \|f_1 - f_2\|_{L^{k,r}}^{\frac{p'}{p_i}}.$$

**Proof :**

We argue as in Proposition 4.4, following Theorem 2.2. We have

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{(L^{\frac{n'p_i}{p'}, \infty}, L^{p_i})_{\theta, \frac{rp_i}{p'}}} \lesssim \|f_1 - f_2\|_{(L^1, L^{(p^*)}')_{\theta, r}}^{\frac{p'}{p_i}}$$

whenever  $\theta = p^* \left(1 - \frac{1}{k}\right)$ , and the identification process (Proposition 4.3) shows that

$$\left(L^{\frac{n'p_i}{p'}, \infty}, L^{p_i}\right)_{\theta, \frac{rp_i}{p'}} = L^{\frac{k^*p_i}{p'}, \frac{rp_i}{p'}}.$$

This gives the results.  $\diamond$

We may also use the interpolation with a function  $(1 - \text{Log} t)^\lambda$ . Here is an example.

**Theorem 5.6.**

Assume (H1), (H2), and  $p' \leq p_i$  for each  $i \in \{1, \dots, n\}$ ,  $m = (p^*)'$ ,  $\lambda \in \mathbb{R}$   $1 \leq q_1 < +\infty$ ,

$$\tilde{T}_i \text{ is } \alpha_i = \frac{p'}{p_i}\text{-Hölderian mapping from } L^{\frac{p^*}{p^*-1}, q_1}(\text{Log } L)^\lambda \text{ into } L^{\frac{p\theta_i q_1}{\alpha_i}}(\text{Log } L)^{\lambda \alpha_i}$$

$$\text{with } \frac{1}{p\theta_i} = \frac{1-\theta}{r_i} + \frac{\theta}{p_i}, \quad r_i = \frac{n'p_i}{p'}, \quad \alpha_i = \frac{p'}{p_i}.$$

**Proof :**

We apply the abstract result stated in Theorem 2.2, with

$$X_0 = L^1, \quad X_1 = L^{(p^*)'}, \quad Y_0 = L^{r_i, \infty}, \quad Y_1 = L^{p_i}, \quad r_i = \frac{n'p_i}{p'},$$

the Hölder exponent being  $\alpha_i = \frac{p'}{p_i}$ .

Since  $\tilde{T}_i$  is  $\alpha_i$ -Hölderian mapping from  $L^1$  into  $L^{r_i, \infty}$  and from  $L^{(p^*)'}$  into  $L^{p_i}$  we deduce that

$$\tilde{T}_i : \left( L^1, L^{(p^*)'} \right)_{\theta, q_1; \lambda} \longrightarrow \left( L^{r_i, \infty}, L^{p_i} \right)_{\theta, \frac{q}{\alpha_i}; \lambda \alpha_i}$$

is  $\alpha_i$ -Hölderian mapping and the identification space gives the right result.  $\diamond$

We can have similar results for variable exponent but computations are more complicated and are not optimal. So we restrict ourselves to some estimates.

**5.3. The Local Hölderian mappings for the case**  $\sum_{i=1}^n \frac{1}{p_i} < 1$ . The purpose of this para-

graph is to show the followings result, which deals with the case  $\sum_{j=1}^n \frac{1}{p_j} = \frac{n}{p} < 1$

**Theorem 5.7.**

Assume (H1) and (H2). Let  $f \in L^1(\Omega)$ ,  $p > n$ . Then the unique solution  $u \in W_0^{1, \vec{p}}(\Omega)$  of the equation (47) satisfies:

**i.):**  $\|u\|_\infty \leq c \|f\|_1^{\frac{1}{p-1}}$ .

**ii.):**  $\sum_{i=1}^n \|\partial_i u\|_{p_i}^{p_i} \leq c \|f\|_1^{p'}$ ,  $\frac{1}{p'} + \frac{1}{p} = 1$ .

**iii.):** In particular if  $u_1$  (resp.  $u_2$ ) is the solution associated to  $f_1$  (resp.  $f_2$ ), we have for  $i \in \{1, \dots, n\}$

(1) If  $p_i \geq 2$ , then:

$$\|\partial_i(u_1 - u_2)\|_{p_i}^{p_i} \lesssim \|f_1 - f_2\|_1 \|u_1 - u_2\|_\infty.$$

(2) If  $p_i < 2$ , then :

$$\int_{\Omega} \frac{|\partial_i(u_1 - u_2)|^2}{(|\partial_i u_1| + |\partial_i u_2|)^{2-p_i}} dx \leq \|f_1 - f_2\|_1 \|u_1 - u_2\|_{\infty}.$$

**Proof :**

Note that when  $p > n$ ,  $L^1(\Omega)$  is a subspace of the dual of  $W_0^{1,\vec{p}}(\Omega)$ , therefore the existence and uniqueness follows from standard theorem concerning monotone operator (see Lions [29]) or using fixed point theorem. So we have for the solution  $u \in W_0^{1,\vec{p}}(\Omega)$ , noticing that  $uV(x, u) \geq 0$ , that

$$(63) \quad \sum_{i=1}^n \|\partial_i u\|_{p_i}^{p_i} \leq \int_{\Omega} f u dx \leq \|f\|_1 \|u\|_{\infty}.$$

This implies

$$(64) \quad \left[ \prod_{i=1}^n \|\partial_i u\|_{p_i} \right]^{\frac{1}{n}} \leq (\|f\|_1 \|u\|_{\infty})^{\frac{1}{p}}.$$

Using the Poincaré-Sobolev inequality given in Corollary 1.1.1 of Theorem 1.1, we derive

$$(65) \quad \|u\|_{\infty} \lesssim \|f\|_1^{\frac{1}{p-1}}.$$

Combining relations (63) and (65), we get the statement i.).

Let  $u_1$  (resp.  $u_2$ ) be the solution associated to  $f_1$  (resp.  $f_2$ ).

Since  $(V(x; u_1) - V(x; u_2))(u_1 - u_2) \geq 0$ , the equation (47) implies using elementary inequalities (see relations (59) and (60)), that

$$\sum_{\{i:p_i \geq 2\}} \|\partial_i(u_1 - u_2)\|_{p_i}^{p_i} + \sum_{\{i:p_i < 2\}} \int_{\Omega} \frac{|\partial_i(u_1 - u_2)|^2}{(|\partial_i u_1| + |\partial_i u_2|)^{2-p_i}} dx \leq \|f_1 - f_2\|_1 \|u_1 - u_2\|_{\infty},$$

from which we derive the result.  $\diamond$

### Corollary 5.7.1. of Theorem 5.7

Let  $p > n$ ,  $i \in \{1, \dots, n\}$ . Then the mapping  $\tilde{T}_i : L^1(\Omega) \longrightarrow L^{p_i}(\Omega)$   
 $f \longmapsto \frac{\partial u}{\partial x_i}$ , where  $u$  is the unique solution of (47), verifies

1<sup>st</sup> case:

If  $p_i \geq 2$ , then  $\tilde{T}$  is locally  $\frac{1}{p_i}$ -Hölderian mapping and for  $f_1 \in L^1(\Omega)$ ,  $f_2 \in L^2(\Omega)$

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{p_i} \lesssim \left[ \|f_1\|_1^{\frac{1}{p-1}} + \|f_2\|_1^{\frac{1}{p-1}} \right]^{\frac{1}{p_i}} \|f_1 - f_2\|_1^{\frac{1}{p_i}}.$$



2<sup>nd</sup> case:

If  $1 < p_i < 2$ , then  $\tilde{T}_i$  is locally  $\frac{1}{2}$ -Hölderian mapping and

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{p_i} \lesssim G_0\left(\|f_1\|; \|f_2\|\right) \|f_1 - f_2\|^{\frac{1}{2}}$$

with  $G_0(t; \sigma) = \left(t^{p'} + \sigma^{p'}\right)^{\frac{1}{p_i} - \frac{1}{2}} \left(t^{\frac{1}{p-1}} + \sigma^{\frac{1}{p-1}}\right)^{\frac{1}{2}}$  for  $(t, \sigma) \in [0, +\infty[ \times [0, +\infty[$ .

**Proof :**

If  $i$  is such that  $p_i \geq 2$ , then following Theorem 5.7,

$$\|\partial_i(u_1 - u_2)\|_{p_i} \lesssim \|f_1 - f_2\|_1 \left[\|u_1\|_1 + \|u_2\|_\infty\right] \leq \left[\|f_1\|_1^{\frac{1}{p-1}} + \|f_2\|_1^{\frac{1}{p-1}}\right] \|f_1 - f_2\|_1.$$

This gives the first statement.

Let  $i$  be such that  $1 < p_i < 2$ . From Hölder's inequality, using iii.)2), we have

$$(66) \quad \|\partial_i(u_1 - u_2)\|_{p_i}^{p_i} \leq \left[\|f_1 - f_2\|_1 \|u_1 - u_2\|_\infty\right]^{\frac{p_i}{2}} \left[\|\partial_i u_1\|_{p_i}^{p_i} + \|\partial_i u_2\|_{p_i}^{p_i}\right]^{1 - \frac{p_i}{2}}.$$

Using i.) and ii.) of Theorem 5.7,

$$\|\partial_i(u_1 - u_2)\|_{p_i} \lesssim \left[\|f_1\|_1^{\frac{1}{p-1}} + \|f_2\|_1^{\frac{1}{p-1}}\right]^{\frac{1}{2}} \left[\|f_1\|_1^{p'} + \|f_2\|_1^{p'}\right]^{\frac{1}{p_i} - \frac{1}{2}} \|f_1 - f_2\|_1^{\frac{1}{2}}.$$

This gives the results. ◇

As we observed, if  $p_i \geq 2 \forall i$ , we may have a global-Hölderian or Lipschitzian mapping.

**Corollary 5.7.2. (of Theorem 5.7)**

If  $p_- = \min_i p_i \geq 2$ , then for all  $i \in \{1, \dots, n\}$

$$\|\tilde{T}_i f_1 - \tilde{T}_i f_2\|_{p_i} \lesssim \|f_1 - f_2\|_1^{\frac{p'}{p_i}} \quad \forall f_1 \text{ and } \forall f_2 \text{ in } L^1(\Omega).$$

## 6. Few estimates for the solution of $-\Delta_{p(\cdot)} u + V(x; u) = f \in L^1(\Omega)$

### 6.1. Existence and uniqueness for $-\Delta_{p(\cdot)} u + V(x; u) = f \in L^\infty(\Omega)$ .

For the  $p(\cdot)$ -Laplacian associated to variable exponent, we shall consider the same framework that we introduced in the first paragraph, in particular  $p : \Omega \rightarrow ]0, +\infty[$ , will be a bounded log-Hölder continuous function

$$1 < p_- = \min_{x \in \Omega} p(x) \leq p_+ = \max_{x \in \Omega} p(x) < n, \quad p^*(x) = \frac{np(x)}{n - p(x)},$$

whose conjugate is denoted by  $[p^*(\cdot)]' = (p^*)'(\cdot)$ . Moreover, we set

$$p'(x) = \frac{p(x)}{p(x) - 1}, \quad x \in \bar{\Omega}, \quad p_-^* = \min_{x \in \Omega} p^*(x), \quad p_+^* = \max_{x \in \Omega} p^*(x), \quad \text{idem for } p' \text{ conjugate of } p.$$

For convenience, we shall add the following assumption for  $V$ :

$$(H4) : \exists \varepsilon > 0, f_0 \in \mathbb{R}_+, \text{ such that } \text{sign}(t)V(x;t) \geq |t|^\varepsilon - f_0, \text{ for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}.$$

Such assumption is true if for instance  $V(x;t) = |t|^{p(x)-2}t$  with  $\varepsilon = p_- - 1$ . We need (H4) only to ensure boundedness of solution when the right hand side is bounded. We first have:

**Proposition 6.1.**

Assume (H1), (H2), and (H4), and let  $f$  be in  $L^\infty(\Omega)$ . Then we have a unique element  $u \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  such that:

$$(67) \quad \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} \varphi V(x;u) \, dx = \int_{\Omega} f \varphi \, dx \quad \forall \varphi \in W_0^{1,p(\cdot)}(\Omega).$$

Moreover, we have

$$(68) \quad \int_{\Omega} |\nabla u|^{p(x)} \, dx + \int_{\Omega} u V(x;u) \, dx \leq C \left[ \|f\|_{(p^*(\cdot))'}^{p'_-} + \|f\|_{(p^*(\cdot))'}^{p'_+} \right]$$

$$(69) \quad \|u\|_{\infty} \leq M + 1, \text{ with } \left( f_0 + \|f\|_{\infty} \right)^{\frac{1}{\varepsilon}} \doteq M.$$

**Idea of the proof**

Let  $k = M + 1$ , and define the operator  $A$  from  $W = W_0^{1,p(\cdot)}(\Omega)$  into  $W' = W^{-1,p'(\cdot)}(\Omega)$  by

$$Av = -\Delta_{p(\cdot)} v + V(\cdot; T_k(v)), \quad v \in W.$$

Due to the assumption (H1) and (H2) on  $W$ , one can check that  $A$  is hemi-continuous, monotonic and coercitive (see Lions's book for the definition [29]).

Therefore,  $\forall f \in W'$ , we have an element  $u \in W : Au = f$ . Since the  $p(\cdot)$ -Laplacian is strictly monotonic and  $L^\infty(\Omega) \subset W'$ , we deduce that  $u$  is unique and solves

$$(70) \quad \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx + \int_{\Omega} \varphi V(x; T_k(u)) \, dx = \int_{\Omega} f \varphi \, dx, \quad \text{for all } \varphi \in W_0^{1,p(\cdot)}(\Omega).$$

Let us show the  $L^\infty$ -estimates. For this purpose, we consider

$$\varphi = \left( |T_k(u)| - M \right)_+ \text{sign}(u) \in W_0^{1,p(\cdot)}(\Omega).$$

Then, dropping the first term, we have:

$$(71) \quad \int_{\Omega} \left( |T_k(u)| - M \right)_+ \text{sign}(u) \cdot V(x; T_k(u)) \, dx \leq \|f\|_{\infty} \int_{\Omega} \left( |T_k(u)| - M \right)_+ \, dx.$$

Taking into account the hypothesis (H4), we derive from relation (71) that

$$(72) \quad \int_{\Omega} \left( |T_k(u)| - M \right)_+ \left[ |T_k(u)|^\varepsilon - \left( f_0 + \|f\|_{\infty} \right) \right] \, dx \leq 0.$$

The set  $\{|T_k(u)| > M\}$  is equal to  $\{|T_k(u)|^\varepsilon > (f_0 + \|f\|_\infty)\}$ . So we deduce from (72) that  $\{|T_k(u)| > M\}$  is of measure zero, i.e.  $|T_k(u)| \leq M$  a.e. in  $\Omega$ . But  $k > M$  and this implies that  $|u(x)| \leq k$  almost everywhere in  $\Omega$ . This relation and equation (70) imply that  $u$  is a solution of (67). The uniqueness follows from the fact that

$$\left[\widehat{a}_{p(\cdot)}(\xi) - \widehat{a}_{p(\cdot)}(\xi')\right][\xi - \xi'] > 0 \quad \text{if } \xi \neq \xi', \quad \widehat{a}_{p(\cdot)}(\xi) = |\xi|^{p(x)-2}\xi. \quad \diamond$$

### Remarks

We may have  $\|u\|_\infty \leq M = (f_0 + \|f\|_\infty)^{\frac{1}{\varepsilon}}$  if  $f_0 > 0$  or  $\|f\|_\infty > 0$ , using the same argument but choosing  $k = M$ ,  $\varphi = \left[|T_k(u)| - M + \eta\right]_+ \text{sign}(u)$  with  $\eta$  small enough so that  $M > \eta$ .

The energy inequality is obtained by choosing  $\varphi = u$  and applying Poincaré-Sobolev inequality to derive

$$\int_{\Omega} fu \leq c \|f\|_{[p^*(\cdot)]'} \|\nabla u\|_{p(\cdot)}.$$

Knowing that

$$2\|\nabla u\|_{p(\cdot)} \leq \left(\int_{\Omega} |\nabla u(x)|^{p(x)} dx\right)^{\frac{1}{p^-}} + \left(\int_{\Omega} |\nabla u(x)|^{p(x)} dx\right)^{\frac{1}{p^+}},$$

we derive

$$\int_{\Omega} |\nabla u|^{p(x)} dx + \int_{\Omega} uV(x; u) dx \leq c \left[\|f\|_{[p^*(\cdot)]'}^{p'_-} + \|f\|_{[p^*(\cdot)]'}^{p'_+}\right].$$

$\diamond$

The Proposition 6.1 is the basis of the existence results when we change the definition of weak solution in (67) by entropic solution or renormalized solution, or simply taking the data  $f$  in the space  $L^1(\Omega) \cap W^{-1,p'(\cdot)}(\Omega)$ . Here is an example of such a result:

### Corollary 6.1.1. of Proposition 6.1

For  $f \in L^{(p^*(\cdot))'}$ , there exists a unique weak solution  $u$  of (67) with the test functions  $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ , which means that

$$(73) \quad \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx + \int_{\Omega} \varphi V(x; u) dx = \int_{\Omega} f \varphi dx.$$

### Sketch of the proof

Let  $f_j = T_j(f) \in L^\infty(\Omega)$ . Then  $\forall \lambda > 0$

$$\int_{\Omega} \left| \frac{f_j(x)}{\lambda} \right|^{(p^*(x))'} dx \leq \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{(p^*(x))'} dx.$$

Therefore,

$$\|f_j\|_{[p^*(\cdot)]'} \leq \|f\|_{[p^*(\cdot)]'}$$

Following Proposition 4.2, we have  $u_j \in W_0^{1,p(\cdot)}(\Omega)$  such that (67) and (68) hold. We derive

$$(74) \quad \int_{\Omega} |\nabla u_j|^{p(x)} dx + \int_{\Omega} u_j V(x; u_j) dx \leq c \left[ \|f\|_{[p^*(\cdot)]'}^{p^-} + \|f\|_{[p^*(\cdot)]'}^{p^+} \right].$$

Since  $W_0^{1,p(\cdot)}(\Omega)$  is a reflexive space, we have  $u \in W_0^{1,p(\cdot)}(\Omega)$  and a subsequence still denoted by  $u_j$  such that the sequence  $(u_j)_j$  converges weakly to a function  $u$  in  $W_0^{1,p(\cdot)}(\Omega)$ , almost everywhere in  $\Omega$  and strongly (by compactness) in  $L^{p^-}(\Omega)$ .

Moreover, the fact that  $0 \leq \int_{\Omega} u_j V(x; u_j) dx \leq C_f < +\infty$  implies that

$$(75) \quad \sup_j \int_{\Omega} |V(x; u_j)| dx \leq C'_f < +\infty.$$

Hence we have  $\int_{\Omega} |V(x; u)| dx \leq C'_f$  using Fatou's lemma. Moreover, choosing

$\varphi = \left( |u_j| - t \right)_+ \text{sign}(u_j)$ ,  $t > 0$ , we derive from (67)

$$(76) \quad \int_{|u_j| > t} |V(x; u_j)| dx \leq \int_{|u_j| > t} |f_j| dx.$$

Therefore we get

$$(77) \quad \lim_{j \rightarrow +\infty} \int_{\Omega} |V(x; u_j) - V(x; u)| dx = 0.$$

◇

For the strong convergence of the gradient, we recall the following lemma, which is based on the monotonicity of the mapping  $\widehat{a}_{p(\cdot)}(\xi) = |\xi|^{p(x)-2}\xi$  in our case (see [37, 39]).

### Lemma 6.1.

Let  $(u_j)_j$  be a sequence of  $W_0^{1,p(\cdot)}(\Omega)$  having the following properties :

- (1) There exists  $q(\cdot)$ ,  $1 < q_- \leq q(\cdot) \leq p(\cdot)$ , such that  $(u_j)_j$  remains in a bounded set of  $W_0^{1,q(\cdot)}(\Omega)$  and  $(u_j)_j$  converges weakly and a.e. to a function  $u$ .
- (2)  $z_j^k = T_k(u_j)$  remains in a bounded set of  $W_0^{1,p(\cdot)}(\Omega)$  for all  $k > 0$ .
- (3)  $\forall k > 0$ , we have a real function  $c_k$ , such that

$$\forall 0 < \varepsilon < \varepsilon_0, \quad \limsup_j \int_{|u_j - T_k(u)| < \varepsilon} \widehat{a}_{p(\cdot)}(\nabla u_j) \cdot \nabla (u_j - T_k(u)) dx \leq c_k(\varepsilon) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0} c_k(\varepsilon) = 0.$$

Then, for a subsequence still denoted by  $u_j$ :

- (a)  $\nabla u_j(x) \xrightarrow{j \rightarrow +\infty} \nabla u(x)$  a.e in  $\Omega$ .

(b) If furthermore the conjugate  $s$  of  $s'(\cdot) = \frac{q(\cdot)}{p(\cdot) - 1}$  satisfies  $\lim_{m \rightarrow \infty} \left[ \int_{\Omega} s^m(x) dx \right]^{\frac{1}{m}} = 0$ , then

$$\lim_j \int_{\Omega} \left| |\nabla u_j|^{q(x)-2} \nabla u_j dx - \int_{\Omega} |\nabla u|^{q(x)-2} \nabla u \right| dx = 0.$$

(c) In particular, for all  $\varphi \in W_0^{1,q'(\cdot)}(\Omega)$

$$\lim_j \int_{\Omega} |\nabla u_j|^{q(x)-2} \nabla u_j \cdot \nabla \varphi dx = \int_{\Omega} |\nabla u|^{q(x)-2} \nabla u \cdot \nabla \varphi dx.$$

**Proof :**

The proof of the first statement is similar to Lemma 2 of [39] (see also [37]) or Lemma A.5 ([40]) for a more general case, so we drop it. But for the second statement, we need to use Theorem 2.1 of [23] and the Vitali's convergence lemma.

Indeed, let us set

$$h_j = \left| |\nabla u_j|^{q(x)-2} \nabla u_j dx - \int_{\Omega} |\nabla u|^{q(x)-2} \nabla u \right|.$$

Then, the preceding statement shows that  $h_j(x) \rightarrow 0$  almost everywhere in  $\Omega$ . Besides applying Hölder's inequality, we have the following uniform integrability, for all measurable set  $E \subset \Omega$ :

$$\int_E |h_j(x)| dx \leq c \|\chi_E\|_{s(\cdot)},$$

for some constant  $c$  independent of  $j$  and  $E$ . Following Theorem 2.1 of [23], the condition on  $s$  implies that  $\|\chi_E\|_{s(\cdot)}$  tends to zero as  $\text{meas}(E)$  tends to zero. Thus the conditions of Vitali's convergence lemma are fulfilled, so that  $\lim_j \int_{\Omega} |h_j(x)| dx = 0$ .  $\diamond$

Since  $T_k(u) \in W_0^{1,p(\cdot)}(\Omega)$ , for all  $k > 0$ , then for any  $\varepsilon > 0$  the function  $T_\varepsilon(u_j - T_k(u))$  is a suitable test function in relation (67). We then derive the third statement of Lemma 6.1. Therefore we have the necessary convergences for the gradient to pass to the limit in the following equation,

$$(78) \quad \int_{\Omega} |\nabla u_j|^{p(x)-2} \nabla u_j \nabla \varphi dx + \int_{\Omega} \varphi V(x; u_j) dx = \int_{\Omega} f_j \varphi \quad \varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega).$$

so that  $u$  solves the equation (73).

The uniqueness is a consequence of strong monotonicity of the  $\widehat{a}_{p(\cdot)}$ .  $\diamond$

**Corollary 6.1.2. (of Proposition 6.1, Local Hölderian mapping)**

Assume that  $p_- \geq 2$ .

The mapping  $T^* : L^{(p^*(\cdot))'}(\Omega) \longrightarrow \left[ L^{p(\cdot)}(\Omega) \right]^n$  is  $\alpha_1 = \frac{p'_-}{p_+}$ -local Hölderian mapping.  
 $f \longmapsto T^*f = \nabla u$

Here  $u$  is the solution of (73) associated to  $f$ .

More precisely, we have:  $\forall f_1, f_2$  in  $L^{[p^*(\cdot)]'}(\Omega)$

$$\|T^*f_1 - T^*f_2\|_{p(\cdot)} \lesssim \Phi_2(f_1; f_2) \|f_1 - f_2\|_{[p^*(\cdot)]'}^{\alpha_1}$$

where  $\Phi(f_1; f_2) = \|f_1 - f_2\|_{L^{[p^*(\cdot)]}'}^{1-\alpha_1} + \|f_1 - f_2\|_{L^{[p^*(\cdot)]}'}^{\alpha_2 - \alpha_1} + 1$  with  $\alpha_2 = \frac{p'_+}{p_-}$ .

**Proof :**

If  $u_1$  (resp  $u_2$ ) is the solution of (67) with  $f = f_1$  (resp  $f_2 \in L^{[p^*(\cdot)]'}(\Omega)$ ), then

$$\int_{\Omega} |\nabla u_1 - \nabla u_2|^{p(x)} dx \leq c \left[ \|f_1 - f_2\|_{[p^*(\cdot)]'}^{p_-} + \|f_1 - f_2\|_{[p^*(\cdot)]'}^{p_+} \right].$$

Since we have

$$2\|\nabla(u_1 - u_2)\|_{p(\cdot)} \leq \left[ \int_{\Omega} |\nabla(u_1 - u_2)|^{p(x)} dx \right]^{\frac{1}{p_-}} + \left[ \int_{\Omega} |\nabla(u_1 - u_2)|^{p(x)} dx \right]^{\frac{1}{p_+}},$$

we get the result, noticing that  $1 \geq \alpha_2 = \frac{p'_+}{p_-} \geq \frac{p'_-}{p_+} = \alpha_1$ . ◇

## 6.2. A priori estimates for variable exponents with data in $L^1(\Omega)$ .

We only give priori estimates starting with the equation (67).

### Proposition 6.2.

For a solution  $u$  of (67), one has:

- (1)  $\int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \leq k \|f\|_1, \quad \forall k > 0.$
- (2)  $\|\nabla T_k(u)\|_{p(\cdot)} \leq \text{Max} \left\{ (k \|f\|_1)^{\frac{1}{p_+}}; (k \|f\|_1)^{\frac{1}{p_-}} \right\}.$
- (3)  $\|T_k(u)\|_{p^*(\cdot)} \lesssim \text{Max} \left\{ (k \|f\|_1)^{\frac{1}{p_+}}; (k \|f\|_1)^{\frac{1}{p_-}} \right\}.$

**Proof :**

Taking as a test function  $\varphi = T_k(u)$ , we get (1). In order to get (2), we use the estimate

$$2\|\nabla T_k(u)\|_{p(\cdot)} \leq \left( \int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \right)^{\frac{1}{p_+}} + \left( \int_{\Omega} |\nabla T_k(u)|^{p(x)} dx \right)^{\frac{1}{p_-}}$$

and statement (1).

Finally, the last statement is a consequence of the Poincaré-Sobolev inequality. ◇

Next we want to study the decay of  $\text{meas} \left\{ |\nabla u|^{p(\cdot)} > \lambda \right\}$ , for  $\lambda > 0$ .

To make our computation easier, we will take the new variable  $k = \bar{k} \|f\|_1, \bar{k} > 0$ . We have:

**Proposition 6.3.**

For all  $\lambda > 0$ , all  $\bar{k} > 0$ , we have

$$\text{meas} \left\{ |\nabla u|^{p(\cdot)} > \lambda \right\} \leq \frac{k}{\lambda} + \text{meas} \left\{ |u| > \bar{k} \right\}.$$

**Proof :**

We use first the fundamental lemma of Benilan's type, see Lemma 3.1 with  $h = |\nabla u|^{p(\cdot)}$ ,  $g = |u|$  and then we apply the statement (1) of the preceding Proposition 6.2 to conclude.  $\diamond$

Next, we need to estimate the decay of  $\text{meas} \left\{ |u| > \bar{k} \right\}$ . One has:

**Proposition 6.4.**

Let  $a_1 = \frac{p_+^*}{p_-} - p_-^*$ ,  $\psi_1(t) = \text{Max} \left\{ t^{p_+^*}; t^{p_-^*} \right\}$  for  $t > 0$ . Assume that  $a_1 < 0$ , that is

$$\frac{n - p_-}{n - p_+} < p_- \left( \frac{p_-}{p_+} \right).$$

Then,

$$\text{meas} \left\{ |u| > \bar{k} \right\} \lesssim \psi_1 \left( \|f\|_1 \right) k^{-|a_1|} \quad \text{with } k = \bar{k} \|f\|_1.$$

**Proof :**

We know that for  $\varepsilon < \bar{k}$ , one has  $\left\{ |u| > \varepsilon \right\} = \left\{ |T_{\bar{k}}(u)| > \varepsilon \right\}$ .

Then the same argument as before leads to

$$(79) \quad \text{meas} \left\{ |u| > \bar{k} \right\} \leq \max \left\{ \frac{\|f\|_1^{p_+^*}}{k^{p_+^*}}; \frac{\|f\|_1^{p_-^*}}{k^{p_-^*}} \right\} \int_{\Omega} |T_{\bar{k}}(u)|^{p^*(x)} dx,$$

from which we get, using statement (3),

$$\text{meas} \left\{ |u| > \bar{k} \right\} \lesssim \psi_1 \left( \|f\|_1 \right) \text{Max} \left\{ k^{-p_+^*}; k^{-p_-^*} \right\} \max \left\{ M_1(k)^{p_+^*}; M_1(k)^{p_-^*} \right\}$$

where  $M_1(k) = \max \left\{ k^{\frac{1}{p_+}}; k^{\frac{1}{p_-}} \right\}$ .

If  $k \geq 1$ , then the above estimate is reduced to

$$\text{meas} \left\{ |u| > \bar{k} \right\} \leq \psi \left( \|f\|_1 \right) k^{-|a_1|}, \quad a_1 = \frac{p_+^*}{p_-^*} - p_-^*.$$

If  $k \leq 1$ , then it is reduced to

$$\text{meas} \left\{ |u| > \bar{k} \right\} \leq \psi_1 \left( \|f\|_1 \right) k^{a_2}, \quad \text{with } a_2 = \frac{p_-^*}{p_+} - p_+^*.$$

But we have

$$a_1 - a_2 = \frac{n^2(p_+ - p_-)}{(n - p_+)(n - p_-)} \left[ \frac{1}{p_+} + \frac{1}{p_-} + \frac{n - 1}{n} \right] > 0 : a_1 \geq a_2,$$

and therefore for  $k \leq 1$ ,  $k^{a_2} \leq k^{-|a_1|}$ .

So for all  $k > 0$ , one has

$$\text{meas} \left\{ |u| > \bar{k} \right\} \lesssim \psi_1(\|f\|) k^{-|a_1|}.$$

◇

**Theorem 6.1. (main estimate for the  $L^1$ -data)**

Under the same assumptions as for Proposition 6.4, there exists a constant  $c > 0$  depending only on  $p$ ,  $n$ ,  $\Omega$  such that

$$\text{meas} \left\{ |\nabla u|^{p(\cdot)} > \lambda \right\} \leq c \psi_1(\|f\|_1)^{\frac{1}{1+|a_1|}} \lambda^{-\frac{|a_1|}{1+|a_1|}} \quad \forall \lambda > 0.$$

**Proof :**

From Proposition 6.3 and Proposition 6.4, we have, for all  $k > 0$ ,

$$\text{meas} \left\{ |\nabla u|^{p(\cdot)} > \lambda \right\} \leq \frac{k}{\lambda} + c_1 \psi_1(\|f\|_1) k^{-|a_1|}$$

where  $c_1$  depends only the Sobolev constant that is on  $\Omega$ ,  $n$ ,  $p$ . Taking the infimum of the right hand side, we derive the result. ◇

**Corollary 6.1.1. (of Theorem 6.1)**

Assume that  $\frac{|a_1|}{1+|a_1|} p_- > 1$ . Then for all  $q \in \left[ \frac{p_+}{p_-}, \frac{|a_1|}{1+|a_1|} p_+ \right]$  we have

$$\int_{\Omega} |\nabla u|^{\frac{q}{p_+} p(x)} dx \leq c \psi_1(\|f\|_1)^{\frac{q}{p_+|a_1|}}$$

where  $c$  depends only on  $\Omega$ ,  $p$ ,  $n$ .

**Proof :**

From Theorem 6.1, we deduce

$$\int_{\Omega} |\nabla u|^{\frac{q}{p_+} p(x)} dx \leq c \psi_1(\|f\|_1)^{\frac{q}{p_+|a_1|}} \int_0^{|\Omega|} t^{-\frac{1+|a_1|}{|a_1|} \frac{q}{p_+}} dt < +\infty.$$

◇

**Remark**

We recover all the condition that we obtained in the preceding paragraph when  $p(x) = p$  is constant. In particular, the condition  $\frac{|a_1|}{1+|a_1|} p_- > 1$  is equivalent to  $p > 2 - \frac{1}{n}$  since we have

$$p \frac{|a_1|}{1+|a_1|} = \frac{n}{n-1} (p-1).$$



### 6.3. Appendix : An existence and uniqueness result of an entropic-renormalized solution for variable exponents.

Although it is not the purpose of our paper, we will show now how to prove the existence of an entropic-renormalized solution. The principle is the same as we did in our previous papers, by for convenience, here we give the main steps.

#### Theorem 6.2.

Let  $q$  be as in Corollary 6.1.1 of Theorem 6.1. Assume (H1), (H2), and (H4), that  $q > \frac{p_+}{p_-}(p_+ - 1)$ , and let  $f \in L^1(\Omega)$ .

Then there exists a unique solution  $u \in W_0^{1,q(\cdot)}(\Omega)$  with  $q(x) = \frac{q}{p_+}p(x)$  such that  $\forall \eta \in W^{1,\infty}(\Omega)$ ,  $\forall B \in W^{1,\infty}(\mathbb{R})$  with  $B(0) = 0$ ,  $B'(\sigma) = 0$  for  $|\sigma| \geq \sigma_0 > 0$ ,  $\forall \varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$

$$\int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla (\eta B(u - \varphi)) dx + \int_{\Omega} \eta B(u - \varphi) \cdot V(x; u) dx = \int_{\Omega} \eta B(u - \varphi) f dx.$$

#### Proof :

We only give the main steps for the existence. Consider  $f_j = T_j(f) \in L^\infty(\Omega)$ . Following Proposition 6.1, we have a unique function  $u_j \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  satisfying relation (67). Moreover the above Corollary 6.1.1 of Theorem 6.1 shows that  $u_j$  remains in a bounded set of  $W_0^{1,q(\cdot)}(\Omega)$ , and we have

$$(80) \quad \sup_j \int_{\Omega} \|\nabla u_j\|^{q(x)} dx \leq c \psi_1 (\|f\|_1)^{\frac{q}{p_+|a_1|}}.$$

Taking as a test function  $T_k(u_j) = \varphi$  in relation (67), we deduce

$$(81) \quad \int_{\Omega} |\nabla T_k(u_j)|^{p(x)} dx \leq k \|f\|_1.$$

Since  $1 < q(\cdot) < p_+ < +\infty$ , the space  $W_0^{1,q(\cdot)}(\Omega)$  is reflexive, and we may subtract a sequence still denoted  $u_j$ , and have an element  $u \in W_0^{1,q(\cdot)}(\Omega)$  such that

- $u_j$  converges weakly to  $u$  in  $W_0^{1,q(\cdot)}(\Omega)$ .
- $u_j(x) \xrightarrow{j \rightarrow +\infty} u(x)$  a.e in  $\Omega$ .
- $T_k(u_j)$  converges weakly to  $T_k(u)$  in  $W_0^{1,p}(\Omega)$  for all  $k > 0$ .

Taking as a test function  $\varphi = \left(|u_j| - t\right)_+ \text{sign}(u_j)$ ,  $t > 0$ , and dropping non negative term, we have

$$(82) \quad \int_{|u_j| > t} |V(x; u_j)| dx \leq \int_{|u_j| > t} |f| dx.$$

This relation with the pointwise convergence and assumptions (H1) and (H2), implies

$$(83) \quad \lim_j \int_{\Omega} |V(x; u_j) - V(x; u)| dx = 0.$$

Next, we choose as a test function  $\varphi = T_{\varepsilon}(u_j - T_k(u))$  with  $\varepsilon > 0$ , so we have

$$(84) \quad \int_{|u_j - T_k(u)| < \varepsilon} \widehat{a}_{p(\cdot)}(\nabla u_j) \cdot \nabla (u_j - T_k(u)) dx \leq \varepsilon \left[ \|f\|_1 + \int_{\Omega} |V(x; u_j)| dx \right].$$

Therefore, we have

$$(85) \quad \limsup_j \int_{|u_j - T_k(u)| < \varepsilon} \widehat{a}_{p(\cdot)}(\nabla u_j) \cdot \nabla (u_j - T_k(u)) dx \leq \varepsilon \left[ \|f\|_1 + \int_{\Omega} |V(x; u)| dx \right].$$

We may invoke Lemma 6.1 to derive for a sequence still denoted  $(u_j)$

$$\bullet \nabla u_j(x) \xrightarrow{j \rightarrow +\infty} \nabla u(x) \text{ a.e. in } \Omega.$$

The condition that  $q > \frac{p_+}{p_-}(p_+ - 1)$  implies, for all  $x$ ,

$$q(x) = \frac{q}{p_+} p(x) \geq \frac{q}{p_+} p_- > p_+ - 1 \geq p(x) - 1 > p_- - 1 > 0. \text{ Therefore}$$

$$(86) \quad \lim_j \int_{\Omega} \left| |\nabla u_j(x)|^{p(x)-2} \nabla u_j(x) - |\nabla u(x)|^{p(x)-2} \nabla u(x) \right| dx \equiv 0.$$

Indeed, let us set  $g_j(x) = \left| |\nabla u_j|^{p(\cdot)-2} \nabla u_j - |\nabla u|^{p(\cdot)-2} \nabla u \right|(x)$ .

Since  $r(x) \doteq \frac{q(x)}{p(x) - 1} \geq \frac{qp_-}{p_+(p_+ - 1)} > 1$ ,  $r \in C(\overline{\Omega})$ , we may apply Theorem 2.1 ([23]) to derive that for all measurable set  $E \subset \Omega$ ,

$$(87) \quad \lim_{|E| \rightarrow 0} \|\chi_E\|_{r'(\cdot)} = 0 \quad \text{with } r'(x) = \frac{r(x)}{r(x) - 1}, \chi_E \text{ is the characteristic function of } E.$$

But the boundedness of the sequence  $(u_j)_j$  in  $W_0^{1,q(\cdot)}(\Omega)$  and Hölder inequality imply, for all measurable set  $E$ , that

$$(88) \quad \sup_j \int_E |g_j(x)| dx \leq c \|\chi_E\|_{r'(\cdot)}.$$

Thus, we may apply Vitali's convergence theorem to derive

$$\lim_{j \rightarrow +\infty} \int_{\Omega} |g_j(x)| dx = 0,$$

since  $g_j(x) \xrightarrow{j \rightarrow +\infty} 0$  a.e., so that we have the uniform integrability given by (88).

The convergences given by relation (83) and relation (86) are enough to prove the existence of a weak solution when  $f \in L^1(\Omega)$ .

To obtain an entropic-renormalized solution, we need further estimates:

### Lemma 6.2. (Gradient behavior)

One has for all  $m \geq 0$ , all  $j \geq 0$ :

$$\begin{aligned}
 (1) \quad & \int_{\{x:m \leq |u_j| \leq m+1\}} |\nabla u_j(x)|^{p(x)} dx \leq \int_{\Omega} |f| \left| T_{m+1}(u_j) - T_m(u_j) \right| dx. \\
 (2) \quad & \int_{\{x:m \leq |u_j| \leq m+1\}} |\nabla u(x)|^{p(x)} dx \leq \limsup_j \int_{\{x:m \leq |u_j| \leq m+1\}} |\nabla u_j(x)|^{p(x)} dx \\
 & \leq \int_{\Omega} |f(x)| \left| T_{m+1}(u) - T_m(u) \right| dx \xrightarrow{m \rightarrow +\infty} 0.
 \end{aligned}$$

**Proof :**

We can take as test function  $\psi_{mj} = T_{m+1}(u_j) - T_m(u_j)$ . Since

$$\int_{\Omega} \psi_{mj} V(x; u_j) dx \geq 0,$$

and

$$\int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u_j) \cdot \nabla \psi_{mj} dx = \int_{m \leq |u_j| \leq m+1} |\nabla u_j(x)|^{p(\cdot)} dx,$$

we get statement (1). On the other hand, statement (2) follows from (1) using Fatou's lemma and pointwise convergences of the gradient for the lower bound and the pointwise convergence of  $u_j$  for the upper bound, combined with the Lebesgue dominated convergence.  $\diamond$

For convenience for the next results, for  $v \in L^1(\Omega)$ , we shall denote  $v^m = T_m(v)$  and we define  $h_m \in W^{1,\infty}(\mathbb{R})$  :

$$h_m(\sigma) = \begin{cases} 1 & \text{if } |\sigma| \leq m, \\ 0 & \text{if } |\sigma| \geq m+1, \\ m+1 - |\sigma| & \text{otherwise.} \end{cases}$$

**Lemma 6.3.**

Let  $\eta \in W^{1,r}(\Omega)$ ,  $r > n$ ,  $b \in W^{1,\infty}(\mathbb{R})$  with  $B(0) = 0$ ,  $B'(\sigma) = 0$  for  $|\sigma| \geq \sigma_0 > 0$ ,  $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$  and set  $\varphi_{mj} = \eta B(T_{m+1}(u) - \varphi) h_m(u_j)$ . Then

$$(1) \quad \varphi_{mj} \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega), \quad \forall m \geq 0, \quad \forall j \geq 0.$$

$$\begin{aligned}
 (2) \quad & \left| \int_{\Omega} |\nabla u_j^{m+1}|^{p(x)-2} \nabla u_j^{m+1} \nabla \left( \eta B(u^{m+1} - \varphi) \right) h_m(u_j) + \int_{\Omega} \varphi_{mj} \left[ V(x; u_j) - f_j \right] dx \right| \\
 & \leq \|\eta\|_{\infty} \|B\|_{\infty} \int_{m \leq |u_j| \leq m+1} |\nabla u_j|^{p(x)} dx.
 \end{aligned}$$

**Proof :**

Since  $h_m(u_j) \in W^{1,p(\cdot)}(\Omega)$ , we can take  $\varphi_{mj}$  as a test function, and

$$|\nabla h_m(u_j)| \leq \begin{cases} |\nabla u_j| & \text{if } m \leq |u_j| \leq m+1, \\ 0 & \text{elsewhere,} \end{cases}$$

so that we derive statement (2).

Let us note that  $\eta B(u^{m+1} - \varphi)$  is in  $W_0^{1,p(\cdot)}(\Omega)$ ,  $\widehat{a}_p(0) = 0$ . ◇

**Lemma 6.4.**

For fixed  $m$ ,

$$\widehat{a}_{p(\cdot)}(\nabla u_j^{m+1}) \text{ converges weakly to } \widehat{a}_{p(\cdot)}(\nabla u^{m+1}) \text{ in } L^{p'(\cdot)}(\Omega)^n, \quad p'(x) = \frac{p(x)}{p(x) - 1}.$$

**Proof :**

The pointwise convergence of the gradient implies

$$\widehat{a}_{p(\cdot)}(\nabla u_j^{m+1}) \rightarrow \widehat{a}_p(\nabla u^{m+1}) \text{ a.e in } \Omega.$$

Furthermore, we know that

$$\|\widehat{a}_{p(\cdot)}(\nabla u_j^{m+1})\|_{L^{p'(\cdot)}} \leq c_m < +\infty.$$

By the reflexivity of  $L^{p'(\cdot)}(\Omega)^n$ , we derive the result. ◇

**Corollary 6.2.1. (of Lemma 6.2, 6.3, 6.4)**

The function  $u$  satisfies, for all  $m \geq 0$

$$\begin{aligned} & \left| \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u^{m+1}) \cdot \nabla(\eta B(u^{m+1} - \varphi)) h_m(u) + \int_{\Omega} h_m(u) \eta B(u^{m+1} - \varphi) [V(x; u) - f] dx \right| \\ & \leq \|\eta\|_{\infty} \|B\|_{\infty} \int_{\Omega} |f(x)| |T_{m+1}(u) - T_m(u)| dx. \end{aligned}$$

**Proof :**

Since  $\nabla(\eta B(u^{m+1} - \varphi)) h_m(u_j)$  converges strongly to  $\nabla(\eta B(u^{m+1} - \varphi)) h_m(u)$  in  $L^{p(\cdot)}(\Omega)^n$ , combining with the weak convergence of Lemma 6.4, we obtain

$$(89) \quad \lim_{j \rightarrow +\infty} \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u_j^{m+1}) \nabla(\eta B(u^{m+1} - \varphi)) h_m(u_j) dx = \int_{\Omega} \widehat{a}_p(\nabla u^{m+1}) \cdot \nabla(\eta B(u^{m+1} - \varphi)) h_m(u) dx.$$

Since  $V(\cdot; u_j)$  (resp.  $f_j$ ) converge strongly to  $V(\cdot; u)$  (resp  $f$ ) in  $L^1(\Omega)$ , we have

$$(90) \quad \lim_{j \rightarrow +\infty} \int_{\Omega} \varphi_{mj} [V(x; u_j) - f_j] dx = \int_{\Omega} h_m(u) \eta B(u^{m+1} - \varphi) [V(x; u) - f] dx.$$

Combining with Lemma 6.2, the two last relations and Lemma 6.3 give the result. ◇

We then have:

**Lemma 6.5.**

$$(1) \quad \lim_{m \rightarrow +\infty} \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u^{m+1}) \nabla \left( \eta B(u^{m+1} - \varphi) \right) h_m(u) dx = \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla \left( \eta B(u - \varphi) \right) dx.$$

$$(2) \quad \lim_{m \rightarrow +\infty} \int_{\Omega} h_m(u) \eta B(u^{m+1} - \varphi) \left[ V(x; u) - f \right] dx = \int_{\Omega} \eta B(u - \varphi) \left[ V(x; u) - f \right] dx.$$

**Proof :**

As we have already observed before,

$$(91) \quad \widehat{a}_{p(\cdot)}(\nabla u^{m+1}) \cdot \nabla \left( \eta B(u^{m+1} - \varphi) \right) h_m(u) = \widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla \left( \eta B(u - \varphi) \right) h_m(u),$$

this is due to the definition of  $h_m$ ,  $\widehat{a}_{p(\cdot)}(0) = 0$ .

Moreover, when expanding the gradient, we have:

$$\widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla \left( \eta B(u - \varphi) \right) = \widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla \eta B(u - \varphi) + \widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla u B'(u - \varphi) \eta.$$

Since  $B'(u - \varphi) = 0$  if  $|u - \varphi| > \sigma_0$ , then, setting  $k_0 = \|f\|_{\infty} + \sigma_0$ , we have:

$$\widehat{a}_{p(\cdot)}(\nabla u) \cdot \nabla u B'(u - \varphi) \eta = \widehat{a}_{p(\cdot)}(\nabla u^{k_0}) \cdot \nabla u^{k_0} B'(u - \varphi) \eta.$$

Hence we deduce, from the preceding decomposition, the following estimate:

$$\left| \widehat{a}_{p(\cdot)}(\nabla u) \nabla \left( \eta B(u - \varphi) \right) \right| \leq c \left[ |\nabla u(x)|^{p(x)-1} + |\nabla u^{k_0}|^{p(x)} \right] \doteq R(x).$$

Here  $c > 0$  is independent of  $u, \varphi$ .

One has  $R \in L^1(\Omega)$ . Therefore by the Lebesgue dominated theorem, we have:

$$(92) \quad \lim_{m \rightarrow +\infty} \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u) \cdot \left( \nabla \eta B(u - \varphi) \right) h_m(u) = \int_{\Omega} \widehat{a}_{p(\cdot)}(\nabla u) \nabla \left( \eta B(u - \varphi) \right).$$

Both relations (91) and (92) infer the first statement (1) of Lemma 6.5 while the second one comes from the Lebesgue dominated theorem.  $\diamond$

### End of the proof of the main theorem

Letting  $m \rightarrow +\infty$  in Corollary 6.2.1 of Lemmas 6.2 to 6.4 with the help of Lemma 6.5, we get that  $u$  is an entropic -renormalized solution.

For the uniqueness, we may use the method of Benilan et al [5] since any entropic-renormalized solution is also an entropic solution in their sense. Note that here, in our case, the solution is always in  $W_0^{1,1}(\Omega)$ . The second method consists in noticing that since  $f_1$  (resp  $f_2$ ) are two elements of  $L^1(\Omega)$  and  $u_1$  (resp  $u_2$ ), we have:

**Lemma 6.6.**

$$(93) \quad \int_{\Omega} \frac{\Delta[u_1 - u_2]}{1 + |u_1 - u_2|} dx \leq \frac{\pi}{2} \int_{\Omega} |f_1 - f_2| dx$$

whenever  $\Delta[u_1; u_2] = \left[ \widehat{a}_{p(\cdot)}(\nabla u_1) - \widehat{a}_{p(\cdot)}(\nabla u_2) \right] \cdot \nabla u(u_1 - u_2) \geq 0$ .

The proof of this lemma needs the following result, which can be carried out even for more general situation:

**Lemma 6.7.**

Let  $w$  be in  $W_{loc}^{1,1}(\Omega)$  such that for all  $k > 0$ ,  $T_k(w) = w^k \in W_0^{1,p(\cdot)}(\Omega)$  and let  $B \in W^{1,\infty}(\mathbb{R})$  with  $B(0) = 0$ ,  $B'(\sigma) = 0$  for all  $\sigma$  such  $|\sigma| \geq \sigma_0 > 0$ ,  $\varphi \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ . Then  $B(w - \varphi)$  is in  $W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ .

**Proof of Lemma 6.7**

If we choose  $k = \sigma_0 + \|\varphi\|_\infty$ , then  $B(w^k - \varphi)$  is in  $W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ . Moreover, almost everywhere in  $\Omega$ ,

$$\nabla B(w^k - \varphi) = B'(w - \varphi) \nabla(w - \varphi) = \nabla B(w - \varphi).$$

Since  $\nabla B(w - \varphi) \in W_{loc}^{1,1}(\Omega)$ , then the above equality holds in the sense of distribution and implies the result.  $\diamond$

**Proof of Lemma 6.6**

The main theorem shows that if  $f_{1j} = T(f_1)$ , then necessarily any weak solution  $(v_{1j})_j$  associated to  $f_{1j} = T_j(f_1) \in L^\infty(\Omega)$  remains in a bounded set of  $W_0^{1,q(\cdot)}(\Omega)$ , and there exists a subsequence  $(v_{1\sigma(j)})_j$  associated to  $f_{1j} \in L^\infty(\Omega)$  and a function  $v \in W_0^{1,q(\cdot)}(\Omega)$  which satisfy  $\nabla v_{1\sigma(j)} \rightarrow \nabla v$  and  $v_{1\sigma(j)} \rightarrow v$  a.e in  $\Omega$  and  $v_{1j}$ .

Let us show that we have necessarily  $\nabla u_1 \equiv \nabla v$ .

Indeed, for  $k > 0$ ,  $B = \tan^{-1}(T_k)$  is in  $W^{1,\infty}(\mathbb{R})$ ,  $B'(0) = 0$  if  $|\sigma| > k$ . Then, according to Lemma 6.7,  $\varphi = v_{1\sigma(j)}$  and  $B(u - \varphi)$  are suitable test functions for both equations (weak formulation and entropic-renormalized formulation), hence we then have after letting  $k \rightarrow +\infty$ :

$$\int_{\Omega} \left[ \widehat{a}_{p(\cdot)}(\nabla u_1) - \widehat{a}_{p(\cdot)}(v_{1\sigma(j)}) \right] \cdot \frac{\nabla(u - v_{1\sigma(j)}) dx}{1 + |u_1 - v_{1\sigma(j)}|^2} \leq \frac{\pi}{2} \int_{\Omega} |f_1 - T_{\sigma(j)} f_1|.$$

Letting  $j \rightarrow +\infty$

$$\int_{\Omega} \frac{\Delta(u_1; v)}{1 + |u_1 - v|^2} dx = 0$$

from which  $\Delta(u_1; v) = 0$  a.e., so that  $\nabla u_1 = \nabla v$ .

This result shows that all the sequence  $(v_j)_j$  must satisfy  $\lim_{j \rightarrow +\infty} \int_{\Omega} |\nabla u_1 - \nabla v_j| dx = 0$ .

This remark shows us if  $f_1$  and  $f_2$  are in  $L^1(\Omega)$ , then we have a subsequence  $T_{\sigma(j)}(f_1)$ ,  $T_{\sigma(j)}(f_2)$  whose weak solutions  $(v_{1\sigma(j)})_j$ ,  $(v_{2\sigma(j)})_j$  satisfy

$$\lim_{j \rightarrow +\infty} \nabla v_{i\sigma(j)}(x) = \nabla v_i(x) \quad \text{a.e in } \Omega.$$

As before, we easily have

$$\int_{\Omega} \frac{\Delta[v_{1\sigma(j)}; v_{2\sigma(j)}]}{1 + |v_{1\sigma(j)} - v_{2\sigma(j)}|^2} dx \leq \frac{\pi}{2} \int_{\Omega} |T_{\sigma(j)}f_1 - T_{\sigma(j)}f_2|.$$

Letting  $j \rightarrow +\infty$ , we get

$$\int_{\Omega} \frac{\Delta[u_1; u_2]}{1 + |u_1 - u_2|^2} \leq \frac{\pi}{2} \int |f_1 - f_2| dx,$$

from which we get the uniqueness. ◇

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