Self-adjointness of Sturm–Liouville operators with strongly singular coefficients

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The main aim of the talk is

a presentation of constructive sufficient conditions for self-adjointness of semibounded in Hilbert space $L^2(\mathbb{R})$ Sturm–Liouville operators with strongly singular coefficients.

Content

1. Schrödinger operators.
2. Sturm–Liouville operators.
3. Sturm–Liouville operators with strongly singular coefficients.
4. References.

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Let us consider a differential operator in Hilbert space $L^2(\mathbb{R})$

$$L_{00} \colon y(\cdot) \to -y''(x) + q(x)y(x),$$

where real-valued function $q \in L^2_{loc}(\mathbb{R})$, with a domain

$$Dom\,(L_{00}) = C^{\infty}_{comp}(\mathbb{R}).$$

It is densely defined and symmetric in $L^2(\mathbb{R})$.

The operator is important in quantum mechanics and mathematical physics problems.

At the same time, only those operators L_{00} that are essentially self-adjoint have a physical meaning. For this purpose, the function q must satisfy certain conditions. The simplest of them is given by

Theorem (H. Weyl, 1910)

If

$$q(x) \geq const, \quad x \in \mathbb{R},$$

then the operator L_{00} is semibounded below and essentially self-adjoint.

Carleman showed that this Theorem can be generalized to multidimensional Schrödinger operators

$$L_{00} = -\triangle + q(x), \quad x \in \mathbb{R}^n,$$

in Hilbert space $L^2(\mathbb{R}^n)$.

Povzner (1953) and later independently Wienholtz (1958) showed that if the function q is real-valued and *continuous*, and the operator L_{00} is semibounded, then it is essentially self-adjoint in space $L^2(\mathbb{R}^n)$.

Later, the conditions for the regularity of the potential q were significantly weakened.

Schrödinger operators with a *singular* potential q naturally arise in mathematical models of physical processes in strongly inhomogeneous media. In these models the potential is not a locally summable function, it is a Radon measure or a distribution.

In this connection, a question arises.

How to define the Schrödinger operator in this case?

Since the middle of 20th century, the study of this topic has been devoted to a large number of papers, mainly of a physical character, where various approaches are used.

In the one-dimensional case, the most natural and general approach is the interpretation of an *ordinary* differential expression with singular coefficients as quasi-differential in the Shin-Zettl sense.

Applied to the one-dimensional Schrödinger operator, it consists of the following.

Let a *formal* differential expression be given

$$ly = -y''(x) + Q'(x)y(x), \quad x \in \mathbb{R}, \tag{1}$$

where the function Q is real-valued and belongs to the class $L^2_{loc}(\mathbb{R})$, and the derivative is in the sense of the theory of distributions.

In particular, if the function Q has a locally bounded variation, then Q' is a Radon measure on $\mathbb{R}$.

We determine the quasi-derivatives of the function y by putting

$$D^{[0]}y := y,$$
$$D^{[1]}y := y' - Qy,$$
$$D^{[2]}y := -\left(D^{[1]}y\right)' - QD^{[1]}y - Q^2 y =: -ly.$$

If the function Q is smooth, then this definition is equivalent to the standard one. In general case

$$Dom(L) = \{y \in L^2(\mathbb{R}) \cap AC_{loc}(\mathbb{R}) : D^{[2]}y \in L^2(\mathbb{R})\}.$$

We define the operator L_{00} as a restriction L to a set of functions with a compact support.

We can show that

- The set $Dom(L_{00})$ is dense in Hilbert space $L^2(\mathbb{R})$.
- The operator L_{00} is symmetric in $L^2(\mathbb{R})$.

We denote by L_0 the closure of the operator L_{00} in space $L^2(\mathbb{R})$.

Theorem 1.

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- Necessary and sufficient conditions for the lower boundedness of the operator L_0 were obtained in paper [1].
- Constructive sufficient conditions for the lower boundedness of the operator L_0 are obtained in the paper [2].
- Theorem 1 allows a generalization to matrix-valued potentials [3].

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Let's consider the operators that are generated by the differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru'),$$

with real-valued coefficients $\{p, q, r\}$ given on $\mathbb{R}$.

If these coefficients are sufficiently regular, then the mapping

$$L_{00} : u \to l[u], \quad u \in C^{\infty}_{comp}(\mathbb{R})$$

defines a densely defined preminimal symmetric operator L_{00} in the Hilbert space $L^2(\mathbb{R})$.

Naturally, the question arises about the self-adjointness conditions of the closure of this operator L_0.

Hartman (1948) and Rellich (1951) showed that if the operator L_{00} is bounded below and the conditions are satisfied

$$r \equiv 0, \quad 0 < p \in C^2(\mathbb{R}),$$

q is piecewise continuous on $\mathbb{R}$ and function p satisfies the condition

$$\int_0^\infty p^{-1/2}(t)\mathrm{d}t = \int_{-\infty}^0 p^{-1/2}(t)\mathrm{d}t = \infty, \tag{2}$$

then the minimal operator L_0 is self-adjoint.

As Stetkaer–Hansen (1966) showed, the conditions for the regularity of the coefficients can be weakened by replacing them with the following ones

$$r \equiv 0, \quad 0 < p \quad \text{is locally Lipshitz}, \quad q \in L^2_{loc}(\mathbb{R}).$$

Clark and Gesztesy (2003) obtained another sufficient condition for the self-adjointness of the operator L_0.
It has the form

$$\|p\|_{L^\infty(-\rho,-\rho/2)}, \quad \|p\|_{L^\infty(\rho/2,\,\rho)} = O(\rho^2), \quad \rho \to \infty. \tag{3}$$

At the same time, it is assumed that

$$r \equiv 0, \quad 0 < p \in W^1_{2,loc}(\mathbb{R}).$$

The examples show that the conditions (2) and (3) are independent.

As Wienholtz (1958) showed, similar results are valid for multidimensional elliptic operators of the second order in space $L^2(\mathbb{R}^n)$.

The more general result on the self-adjointness of such operators with smooth complex-valued coefficients was obtained by Berezansky (1968).

He showed that such operator L_{00} given on $C^{\infty}_{comp}(\mathbb{R})$ is essentially self-adjoint in space $L^2(\mathbb{R}^n)$ under the condition of globally finite rate of propagation.
That is, if every solution of a hyperbolic differential equation

$$u_{tt} + Lu = 0,$$

which has a compact support for $t = 0$ has compact support for any $t > 0$.
These results were further developed.

We introduce and investigate symmetric operators L_0 associated in the complex Hilbert space $L^2(\mathbb{R})$ with a formal differential expression

$$l[u] := -(pu')' + qu + i((ru)' + ru')$$

under minimal conditions on the regularity of the coefficients.

They are assumed to satisfy conditions

$$q = Q' + s, \quad \frac{1}{|\rho|^{1/2}}, \quad \frac{Q}{|\rho|^{1/2}}, \quad \frac{r}{|\rho|^{1/2}} \quad \in L^2_{loc}(\mathbb{R}), \tag{4}$$

$$s \in L^1_{loc}(\mathbb{R}), \quad 1/p \neq 0, \quad \text{a. e.},$$

where the derivative of function Q is understood in the sense of a theory of distributions, and all functions {p, Q, r, s} are real-valued.

In particular, the coefficients q and r' can be Radon measures on $\mathbb{R}$, while function p can be discontinuous.

These operators are defined using Shin–Zettl quasi-derivatives.

$$u^{[0]} := u, \quad u^{[1]} := pu' - (Q+ir)u, \tag{5}$$

$$u^{[2]} := \left(u^{[1]}\right)' + \frac{Q-it}{p}u^{[1]} + \left(\frac{Q^2+r^2}{p} - s\right)u = -l[u].$$

$$Dom(l) := \{u \colon \mathbb{R} \to \mathbb{C} | \quad u, u^{[1]} \in AC_{loc}(\mathbb{R})\}.$$

This definition is motivated by the fact that

$$\langle -u^{[2]}, u\rangle \equiv \langle -(pu')' + qu + i\left((ru)' + ru'\right), u\rangle \quad u \in C^{\infty}_{comp}(\mathbb{R})$$

in the sense of a theory of distributions.

We define for the quasi-differential expression l the operators L and L_{00} as:

$$Lu := l[u], \quad Dom(L) := \{u \in L^2(\mathbb{R}) | \quad u, u^{[1]} \in AC_{loc}(\mathbb{R}), \quad l[u] \in L^2(\mathbb{R})\},$$

$$L_{00}u := Lu, \quad Dom(L_{00}) := \{u \in Dom(L) | \quad supp\, u \subset\subset \mathbb{R}\}.$$

Their definitions coincide with the classical ones if the coefficients l are sufficiently smooth.

One can prove that the operator L_{00} is densely defined in $L^2(\mathbb{R})$ and is symmetric.

Theorem 2.

Let the coefficients of the formal differential expression l satisfy the assumptions (4) and also

i) $p \in W^1_{2,loc}(\mathbb{R}), \quad p > 0,$

ii) $\int_{-\infty}^{0} p^{-1/2}(t)\mathrm{d}t = \int_0^\infty p^{-1/2}(t)\mathrm{d}t = \infty.$

Then, if operator L_{00} is bounder below, then it is essentially self-adjoint and

$$L_{00}^* = L^*.$$

In the next theorem, additional conditions on coefficient p are imposed not on the entire axis, but only on a sequence of finite interval. However, outside of these intervals the function p *can vanish and be discontinuous.*

Theorem 3.

Suppose the assumptions (4) satisfy and the operator L_{00} is bounded below. Suppose the sequence of intervals $\triangle_n := [a_n, b_n]$ exist such that

$$-\infty < a_n < b_n < \infty, \quad b_n \to -\infty, \quad n \to -\infty, \quad a_n \to \infty, \quad n \to \infty,$$

where the coefficients p satisfy the additional conditions

i) $p_n := p|\triangle_n \in W_2^1(\triangle_n), \quad p_n > 0,$

ii) $\exists c > 0 \colon p_n(x) \leq c|\triangle_n|^2, \quad n \in \mathbb{Z},$

where $|\triangle_n|$ is the length of interval $\triangle_n$.

Then semibounded operator L_{00} is essentially self-adjoint and

$$L_{00}^* = L^*.$$

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Thank you for your attention!