Density of the wild data for the barotropic Euler system

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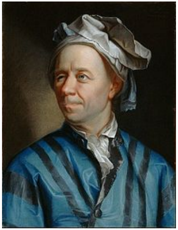
Euler system of gas dynamics



Leonhard Paul Euler
1707–1783

Equation of continuity – Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation – Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0, \ \ p(\varrho) \approx \varrho^\gamma$$

Impermeable boundary

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ \ \Omega \subset R^d, \ \ d = 2, 3$$

Initial state (data)

$$\varrho(0, \cdot) = \varrho_0, \ \ (\varrho \mathbf{u})(0, \cdot) = \varrho_0 \mathbf{u}_0$$

Admissibility

Energy

$$E(\varrho, \mathbf{u}) = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)$$

Pressure potential

$$P'(\varrho)\varrho - P(\varrho) = p(\varrho),\ P(\varrho) = \frac{a}{\gamma - 1}\varrho^\gamma$$

Dissipative (weak) solutions

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E(\varrho, \mathbf{u})\ \mathrm{d}x \leq 0,\ \int_\Omega E(\varrho, \mathbf{u})(\tau, \cdot)\ \mathrm{d}x \leq \int_\Omega E(\varrho_0, \mathbf{u}_0)\ \mathrm{d}x$$

Admissible (weak) solutions

$$\partial_t E(\varrho, \mathbf{u}) + \mathrm{div}_x\Big(E(\varrho, \mathbf{u})\mathbf{u} + p(\varrho)\mathbf{u}\Big) \leq 0,\ E(\varrho, \mathbf{u})(\tau, \cdot) \nearrow E(\varrho_0, \mathbf{u}_0),\ \tau \to 0$$

Euler system, good and bad news

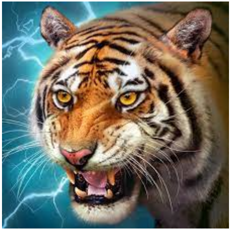
- smooth data $\Rightarrow$ local–in–time smooth solutions
- finite time blow up for a "generic" class of initial data
- infinitely many *weak solutions* for any finite energy initial data (not dissipative in general) [Chiodaroli, De Lellis, Székelyhidi]
- infinitely many dissipative/admissible weak solution for special initial data (L^∞, C^ν) [Chiodaroli, De Lellis, Giri, Klingenberg, Kreml, Kwon, Markfelder]
- infinitely many admissible weak solutions for special smooth initial data [Chiodaroli, Kreml, Macha, Schwarzacher]

Wild data

Initial state

$$\varrho(0,\cdot) = \varrho_0,\ (\varrho\mathbf{u})(0,\cdot) = \varrho_0\mathbf{u}_0$$

The initial data are *wild* if there exists $T > 0$ such that the Euler system admits infinitely many (weak) *admissible* solutions on any time interval $[0,\tau]$, $0 < \tau < T$



Theorem (E. Chiodaroli, EF 2022) The set of wild data is dense in $L^p \times L^p$, $1 \leq p < \infty$

E. Chiodaroli (Pisa)

Related results for the incompressible Euler system by Székelyhidi–Wiedemann, Daneri–Székelyhidy

Related results for the barotropic Euler system by Ming, Vasseur, and You

$$\int_\Omega E(\varrho,\mathbf{u})(\tau)\ \mathrm{d}x \leq \int_\Omega E(\varrho_0,\mathbf{u}_0)\ \mathrm{d}x,\ \tau \geq 0$$

Density of wild data – exact statement

Periodic boundary conditions (for simplicity)

$$\Omega = \mathbb{T}^d,\ \mathbb{T}^d = \left([-1,1]|_{\{-1;1\}} \right)^d,\ d = 2,3$$

Theorem (Density of wild data)

Suppose $p \in C^\infty(a,b)$, $p' > 0$ in (a,b), for some $0 \le a < b \le \infty$. Then for any

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \le \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1,$$

any $\varepsilon > 0$, and any $1 \le p < \infty$, there exist wild data $\varrho_{0,\varepsilon}$, $\mathbf{u}_{0,\varepsilon}$ such that

$$\| \varrho_{0,\varepsilon} - \varrho_0 \|_{L^p(\mathbb{T}^d)} < \varepsilon,\ \| \mathbf{u}_{0,\varepsilon} - \mathbf{u}_0 \|_{L^p(\mathbb{T}^d;R^d)} < \varepsilon.$$

Local existence for smooth data

Theorem (Local existence for smooth data)

Suppose $p \in C^\infty(a,b)$, $p' > 0$ *in* (a,b), *for some* $0 \leq a < b \leq \infty$. *Then for any initial data*

$$\varrho_0 \in W^{k,2}(\mathbb{T}^d),\ a < \inf_{\mathbb{T}^d} \varrho_0 \leq \sup_{\mathbb{T}^d} \varrho_0 < b, \mathbf{u}_0 \in W^{k,2}(\mathbb{T}^d; R^d),\ k > \frac{d}{2} + 1$$

there exists $T_{\max} > 0$ *such that the compressible Euler system admits a classical solution* $(\varrho, \mathbf{u})$ *unique in the class*

$$\varrho \in C([0,T]; W^{k,2}(\mathbb{T}^d)),\ a < \varrho < b,\ \mathbf{u} \in C([0,T]; W^{k,2}(\mathbb{T}^d; R^d))$$

for any $0 < T < T_{\max}$.

Convex integration ansatz

regular initial data $(\varrho_0, \mathbf{u}_0) \Rightarrow$ smooth solution $(\widetilde{\varrho}, \widetilde{\mathbf{u}})$
in $[0, T] \times \mathbb{T}^d$, $T < T_{\max}$, $\widetilde{\mathbf{m}} = \widetilde{\varrho}\widetilde{\mathbf{u}}$

$$\partial_t \widetilde{\varrho} + \mathrm{div}_x \widetilde{\mathbf{m}} = 0$$

$$\partial_t \widetilde{\mathbf{m}} + \mathrm{div}_x \left(\frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} + p(\widetilde{\varrho})\mathbb{I} \right) = 0$$

Ansatz:

$$\varrho = \widetilde{\varrho},\ \mathbf{m} = \varrho \mathbf{u} = \widetilde{\mathbf{m}} + \mathbf{v},$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \widetilde{\mathbf{m}}) \otimes (\mathbf{v} + \widetilde{\mathbf{m}})}{\widetilde{\varrho}} - \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right) = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

Application of convex integration

Data:

$$\mathbb{H} = \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} - \frac{1}{d}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\mathbb{I} \in C^1([0,T] \times \mathbb{T}^d; R^{d\times d}_{0,\mathrm{sym}})$$

$$e = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + \boxed{\Lambda},\ \Lambda = \Lambda(t) \in C([0,T] \times \mathbb{T}^d)$$

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\underbrace{\frac{(\mathbf{v}+\widetilde{\mathbf{m}}) \otimes (\mathbf{v}+\widetilde{\mathbf{m}})}{\widetilde{\varrho}} - \frac{1}{d}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}\mathbb{I}}_{\text{traceless tensor}} - \mathbb{H} \right) = 0$$

$$\underbrace{\frac{1}{2}\frac{|\mathbf{v}+\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}}_{\text{kinetic energy}} = e$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0$$

Solutions with initial energy jump

Theorem (Existence with a small initial energy jump)

Let $\Lambda \in C[0,T]$, $\inf_{t \in [0,T]} \Lambda(t) > 0$ *be given. Let* $\mathbf{v}_0 = 0$. *Then the problem admits infinitely many weak solutions* $\mathbf{v}$ *in* $L^\infty((0,T) \times \mathbb{T}^d; R^d)$.

See **Theorem 13.2.1** in

E. Feireisl.
Weak solutions to problems involving inviscid fluids.
In *Mathematical Fluid Dynamics, Present and Future*, volume 183 of *Springer Proceedings in Mathematics and Statistics*, pages 377–399. Springer, New York, 2016

Solutions without initial energy jump

Theorem (Existence without initial energy jump)

Let $\Lambda \in C[0,T]$, $\inf_{t\in[0,T]} \Lambda(t) > 0$ *be given.*
Then there exists a sequence $\tau_n \to 0$ *and* $\mathbf{v}_{0,n}$,

$$\mathbf{v}_{0,n} \to 0 \text{ weakly-(*) in } L^\infty(\mathbb{T}^d; R^d)$$

such that the problem admits infinitely many weak solutions in $(\tau_n, T) \times \mathbb{T}^d$ *satisfying*

$$\mathbf{v}(\tau_n, \cdot) = \mathbf{v}_{0,n},\ \mathbf{v}(T,\cdot) = 0,\ \boxed{\frac{1}{2}\frac{|\mathbf{v} + \widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}}(\tau_n,\cdot) = e(\tau_n)}.$$

See **Theorem 13.6.1** in

E. Feireisl.
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Adjusting the energy profile

$$e = \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + \boxed{\Lambda},\ \Lambda = \Lambda(t)$$

Desired properties:

- $$\limsup_{n \to 0} \|\mathbf{v}_{0,n}\|_{L^2(\mathbb{T}^d;R^d)} < \varepsilon;$$
- the energy inequality holds for $\mathbf{u} = \mathbf{v} + \widetilde{\mathbf{m}}$, $\varrho = \widetilde{\varrho}$, at least on a short time interval.

$$\Lambda(0) \text{ small enough},\ \Lambda' + \Lambda \mathrm{div}_x \widetilde{\mathbf{u}} + \nabla_x \left[\frac{1}{\widetilde{\varrho}} \left(\frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) + \Lambda + p(\widetilde{\varrho}) \right) \right] \cdot \mathbf{v} \leq 0$$

$$\Lambda(t) = \varepsilon \exp\left(-\frac{t}{\varepsilon^2} \right) \ \Rightarrow\ \Lambda' \leq 0 \ \Rightarrow\ \|\mathbf{v}\|_{L^\infty} \text{ controlled by } \Lambda(0)$$