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Abstract

Primitive Equations(PE) are an important model which is widely used in the geophysical research and mathematical analysis. In the previous results, people derive PE from the Navier-Stokes or Euler system by asymptotic analysis or numerical approximation. Here, we give a rigorous mathematical derivation of inviscid compressible Primitive Equations from Euler system in a periodic channel, utilizing the relative entropy inequality.

Key words: Euler equations, inviscid, compressible, Primitive Equations

2010 Mathematics Subject Classifications: 35Q30, 35Q86.

1 Introduction

In this paper, we consider the following compressible Euler system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho \mathbf{u}) = 0, \\ (\rho \mathbf{u})_t + \operatorname{div}(\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = 0, \end{cases} \tag{1.1}$$

in the thin domain $(0,T) \times \Omega_\epsilon$, where $\Omega_\epsilon = \{(x,z)|x \in \mathbb{T}^2, 0 < z < \epsilon\}$, x denotes the horizontal direction and z denotes the vertical direction. Here, $\rho, \mathbf{u}, p$ represent the density, velocity and pressure, respectively. The velocity can be defined as $\mathbf{u} = (\mathbf{v}, w)$, where $\mathbf{v}(t,x,z) \in \mathbb{R}^2$ and $w(t,x,z) \in \mathbb{R}$ represent the horizonal velocity and vertical velocity respectively. Through out this paper, we use the notations $\operatorname{div}\mathbf{u} = \mathrm{div_x}\mathbf{v} + \partial_z \mathrm{w}$ and $\nabla = (\nabla_x, \partial_z)$ to denote the three-dimensional spatial divergence and gradient respectively. Here we suppose the pressure: $p(\rho) = \rho^\gamma \ (\gamma > 1)$.

Adopting the same scheme as [3], we perform the following rescaling

$$z \to \varepsilon z, w \to \epsilon w$$

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while t, x, v, ρ and p are left unchanged, then get the following rescaled Euler equations in the fixed domain $\Omega := \mathbb{T}^2 \times (0,1)$:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \rho \partial_t \mathbf{v} + \rho(\mathbf{u} \cdot \nabla)\mathbf{v} + \nabla_x p(\rho) = 0, \\ \epsilon^2 \big(\rho \partial_t w_\epsilon + \rho(\mathbf{u} \cdot \nabla) w\big) + \partial_z p(\rho) = 0, \end{cases} \tag{1.2}$$

We supplement the system with the following boundary and initial conditions:

$$(\rho_\epsilon, \mathbf{u}_\epsilon)|_{t=0} = (\rho_0, \mathbf{u}_0). \tag{1.3}$$

By setting $\varepsilon \to 0$ in the compressible Euler system (1.2), we obtain the inviscid compressible PE as the following:

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho \mathbf{v}) + \partial_z(\rho w) = 0, \\ \rho \partial_t \mathbf{v} + \rho(\mathbf{u} \cdot \nabla)\mathbf{v} + \nabla_x p(\rho) = 0, \\ \partial_z p(\rho) = 0. \end{cases} \tag{1.4}$$

In the context of geophysical flows, scientists usually use the Primitive Equations(PE) to describe and analysis the phenomena of atmosphere and ocean. One of the typical character of PE model is that there is no information for the vertical velocity in the momentum equation and the vertical velocity is determined by the horizontal velocity. Therefore, the mathematical and numerical study of the problems of PE model was unsolved until 1990s when Lions, Teman and Wang [15, 16] were first to study the PE and received fundamental results in this field. The celebrated breakthrough result was made by Cao and Titi [5], where they first proved the global well-posedness of PE in the three dimensional case. Readers can refer to [4, 6, 19] and references therein for more physical background and mathematical results.

It is conventional using the Boussinesq and hydrostatic approximation to derive PE model from incompressible Navier-Stokes equations. Therefore, how to derive it rigorously is an interesting question in the mathematical community. During the last decades, deriving incompressible PE model has progressed by concentrated the mathematical arguments and a vast amount of published literature exists. More precisely, Azérad and Guillén [1] proved the incompressible Navier-Stokes equations converge to PE in the sense of weak solutions. Li and Titi [14] proved the weak solutions of incompressible Navier-Stokes equations converge to strong solutions of PE. Donatelli and Juhasz [7] proved the convergence in downwind-matching coordinates. On the other hand, Grenier [12] used the energy estimates and Brenier [3] used the relative entropy inequality to prove the smooth solutions of incompressible Euler system converge to smooth solutions of inviscid PE. However, as is well known, the atmosphere and ocean, consists mostly of air and water, should be considered as compressible fluid. Therefore, people consider that whether we could deduce the PE model in compressible case. Recently, Ersoy et al. [8] used the asymptotic analysis combined with dimensionless to obtain the compressible PE model where the viscosity coefficients are depending on the density. Necasova and her authors [11] deduce the compressible PE from anisotropic Navier-Stokes equations with constant viscosity coefficient. To the the authors' best knowledge, there is no result concerning on the inviscid compressible PE equations. Inspired by Brenier's work [3], we give a rigourous mathematical justification to deduce the inviscid compressible PE model.

The goal of this work is to investigate the limit process $\epsilon \to 0$ in the system of (1.2) converges in a certain sense to the inviscid compressible Primitive Equations (1.4). The paper is organized as follows. In Section 2, we introduce the relative entropy inequality, give some useful lemmas and state the main theorem. Section 3 is devoted to the proof of the convergence.

2 Main result

2.1 Relative entropy inequality

Motivated by [3, 9], for any smooth solution $(\rho, \mathbf{u})$, where $\mathbf{u} = (\mathbf{v}, w)$, to the compressible Euler system (1.2), we introduce the relative entropy functional

$$\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U}) = \int_\Omega [\frac{1}{2}\rho|\mathbf{v} - \mathbf{V}|^2 + \frac{\epsilon^2}{2}\rho|w - W|^2 + P(\rho) - P'(r)(\rho - r) - P(r)]dxdz \tag{2.1}$$

where $r > 0$, $\mathbf{U} = (\mathbf{V}, W)$ are smooth "test" functions, r, $\mathbf{U}$ compactly supported in Ω. Here we have used $rP'(r) - P(r) = p(r)$. For simplicity, we use the notation $\int_\Omega f$ instead of $\int_\Omega f dxdz$.

Lemma 2.1. *Let* $(\rho, \mathbf{u})$ *be a smooth solution to the compressible Euler system* (1.2)*, and let* $(r, \mathbf{U})$ *be smooth solutions to the inviscid compressible PE system* (1.4)*. Then we have*

$$\begin{aligned}
&\frac{d}{dt}\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U}) \\
&= -\int_\Omega \rho w(\mathbf{v} - \mathbf{V})(w - W)\partial_z \mathbf{V} - \int_\Omega (\mathbf{v} - \mathbf{V})(\nabla_x p(\rho) - \frac{\rho}{r}\nabla_x p(r)) \\
&- \int_\Omega \rho(\mathbf{v} - \mathbf{V})^2 \nabla_x \mathbf{V} + \varepsilon^2 \int_\Omega \rho(w - W)(\partial_t w - \partial_t W) \\
&+ \varepsilon^2 \int_\Omega \rho \mathbf{v}|w - W|\nabla_x(w - W) + \varepsilon^2 \rho w|w - W|\partial_z(w - W) \\
&- \int_\Omega [P'(\rho) - P'(r)](div_x(\rho \mathbf{v}) + \partial_z(\rho w)) + \int_\Omega P''(r)(\rho - r)(div_x(r\mathbf{V}) + \partial_z(rW))
\end{aligned} \tag{2.2}$$

Proof. We calculate the terms in (2.1) one by one. Firstly, we compute the first items on the right side of (2.2) as:

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\rho|\mathbf{v} - \mathbf{V}|^2 dxdz = \int_\Omega \frac{1}{2}\rho_t|\mathbf{v} - \mathbf{V}|^2 + \int_\Omega \rho(\mathbf{v} - \mathbf{V})(\partial_t \mathbf{v} - \partial_t \mathbf{V}).$$

From the continuity equation $(1.2)_1$, it is easy to find

$$\begin{aligned}
\int_\Omega \frac{1}{2}\rho_t|\mathbf{v} - \mathbf{V}|^2 &= -\frac{1}{2}\int_\Omega \mathrm{div}_x(\rho \mathbf{v})|\mathbf{v} - \mathbf{V}|^2 dxdz - \frac{1}{2}\int_\Omega \partial_z(\rho w)|\mathbf{v} - \mathbf{V}|^2 \\
&= \int_\Omega \rho \mathbf{v}(\mathbf{v} - \mathbf{V})(\nabla_x \mathbf{v} - \nabla_x \mathbf{V}) + \int_\Omega \rho w(\mathbf{v} - \mathbf{V})(\partial_z \mathbf{v} - \partial_z \mathbf{V}).
\end{aligned} \tag{2.3}$$

And from the momentum equation $(1.2)_2$ and $(1.4)_2$, we have

$$\begin{aligned}\int_\Omega \rho(\mathbf{v}-\mathbf{V})(\partial_t\mathbf{v}-\partial_t\mathbf{V}) &= -\int_\Omega (\mathbf{v}-\mathbf{V})(\rho\mathbf{v}\nabla_x\mathbf{v}+\rho w\partial_z\mathbf{v}+\nabla_x p(\rho)-\rho\mathbf{V}\nabla_x\mathbf{V}-\rho W\partial_z\mathbf{V}-\frac{\rho}{r}\nabla_x p(r))\\ &= -\int_\Omega (\mathbf{v}-\mathbf{V})[\rho\mathbf{v}\nabla_x(\mathbf{v}-\mathbf{V})+\rho(\mathbf{v}-\mathbf{V})\nabla_x\mathbf{V}\\ &\qquad +\rho w\partial_z(\mathbf{v}-\mathbf{V})+\rho(w-W)\partial_z\mathbf{V}+\nabla_x p(\rho)-\frac{\rho}{r}\nabla_x p(r)].\end{aligned} \tag{2.4}$$

Combining (2.3) and (2.4), we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega [\frac{1}{2}\rho|\mathbf{v}-\mathbf{V}|^2 &= -\int_\Omega \rho w(\mathbf{v}-\mathbf{V})(w-W)\partial_z\mathbf{V}-\int_\Omega(\mathbf{v}-\mathbf{V})(\nabla_x p(\rho)-\frac{\rho}{r}\nabla_x p(r))\\ &\quad -\int_\Omega \rho(\mathbf{v}-\mathbf{V})^2\nabla_x\mathbf{V}.\end{aligned} \tag{2.5}$$

Similarly, by virtue of continuity equation, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \frac{\varepsilon^2}{2}\rho|w-W|^2\\ &=\frac{\varepsilon^2}{2}\int_\Omega \rho_t|w-W|^2+\varepsilon^2\int_\Omega \rho(w-W)(\partial_t w-\partial_t W)\\ &=-\frac{\varepsilon^2}{2}\int_\Omega \mathrm{div}_x(\rho\mathbf{v})|w-W|^2-\frac{\varepsilon^2}{2}\int_\Omega \partial_z(\rho w)|w-W|^2+\varepsilon^2\int_\Omega \rho(w-W)(\partial_t w-\partial_t W)\\ &=\varepsilon^2\int_\Omega [\rho\mathbf{v}|w-W|\nabla_x(w-W)+\varepsilon^2\rho w|w-W|\partial_z(w-W)]+\varepsilon^2\int_\Omega \rho(w-W)(\partial_t w-\partial_t W).\end{aligned} \tag{2.6}$$

Finally,

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (P(\rho)-P'(r)(\rho-r)-P(r))\\ &=\int_\Omega (P('\rho)\partial_t\rho-P''(r)\partial_t r(\rho-r)-P'(r)\partial_t\rho)\\ &=\int_\Omega [(P'(\rho)-P'(r))\partial_t\rho-P''(r)\partial_t r(\rho-r)\\ &=-\int_\Omega [P'(\rho)-P'(r)](\mathrm{div}_x(\rho\mathbf{v})+\partial_z(\rho w))+\int_\Omega P''(r)(\rho-r)(\mathrm{div}_x(r\mathbf{V})+\partial_z(rW)).\end{aligned} \tag{2.7}$$

Getting (2.5)-(2.7) together, we complete the proof of Lemma 2.1.

□

Based on the relative entropy inequality, we can obtain the following lemma from [9]:

Lemma 2.2. *Let $0<a<b<\infty$. Then there exists $c=c(a,b)>0$ such that for all $\rho\in[0,\infty)$ and $r\in[a,b]$ there holds*

$$P(\rho)-P'(r)(\rho-r)-P(r)\geq\begin{cases} C|\rho-r|^2, & \text{when } \frac{r}{2}<\rho<r,\\ C(1+\rho^\gamma), & \text{otherwise},\end{cases}$$

where $C=C(a,b)$.

2.2 Main result

Now, we are ready to state our main result.

Theorem 2.1. *Let $(\rho, \mathbf{u})$ and $(r, \mathbf{U})$ be smooth solutions with $\rho > 0$ and $r > 0$, respectively, the compressible Euler equations* (1.2) *and the inviscid compressible PE system* (1.4)*, and T_{max} be the life time of smooth solutions. Suppose that they emerges at the same initial data, that is $\rho|_{t=0} = r|_{t=0}$, $\mathbf{u}|_{t=0}\mathbf{U}|_{t=0}$, then for all $t \in [0, T_{max}]$, we have*

$$\sup_{t\in(0,T_{max})} \mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U}) \leq C_T \varepsilon^2,$$

where C_T is independent of ε.

Remark 2.1. *The compressible Euler system* (1.2) *can be written as a positive, symmetric, hyperbolic system and Kato [13] proved the system possess a unique, local C^1 solution with $\rho > 0$, provided the initial data are sufficiently regular. We suggest readers can refer to [20] for more results and background about compressible Euler system.*

Remark 2.2. *Brenier [2], Masmoudi and Wong [18] obtained the local existence of smooth solution for inviscid incompressible PE system. Liu and Titi [17] proved the local existence of strong solutions to compressible PE model. While, the corresponding result for inviscid compressible PE model is still not known, which is left for future study.*

3 Proof of Theorem 2.1

The Theorem 2.1 can proved directly by Lemma 2.1. We just need to control the right hand side of (2.2). Firstly, we rewrite and control the proceeding first terms at right side of (2.2):

$$\begin{aligned}
\int_\Omega (\mathbf{v}-\mathbf{V})(\nabla_x p(\rho) - \frac{\rho}{r}\nabla_x p(r)) &= -\int_\Omega (\mathbf{v}-\mathbf{V})\frac{(r-\rho)\nabla_x p(\rho) + \rho(\nabla_x p(\rho) - \nabla_x(r))}{r} \\
&= \int_\Omega (\mathbf{v}-\mathbf{V})(\rho - r)\frac{\nabla_x p(r)}{r} - \int_\Omega (\mathbf{v}-\mathbf{V})\frac{\rho}{r}\big(p'(\rho)\nabla_x\rho - p'(r)\nabla_x r\big) \\
&= I_1 + I_2.
\end{aligned} \tag{3.1}$$

We divide I_1 into two parts as

$$I_1 = \int_{\frac{r}{2}\leq\rho\leq 2r} (\mathbf{v}-\mathbf{V})(r-\rho)\frac{\nabla_x p(\rho)}{r} + \int_{\rho\leq\frac{r}{2},\rho\geq 2r} (\mathbf{v}-\mathbf{V})(r-\rho)\frac{\nabla_x p(\rho)}{r}.$$

We know that $(\rho, \mathbf{u})$ and $(r, \mathbf{U})$ are smooth solution with $\rho > 0$ and $r > 0$. By virtue of Lemma 2.2 and Cauchy inequality, we get

$$\int_{\frac{r}{2}\leq\rho\leq 2r} (\mathbf{v}-\mathbf{V})(r-\rho)\cdot\frac{\nabla_x p(\rho)}{r} = \int_{\frac{r}{2}\leq\rho\leq 2r} \sqrt{\rho}(\mathbf{v}-\mathbf{V})\cdot(r-\rho)\frac{\nabla_x p(\rho)}{\sqrt{\rho}r}$$

$$\leq \int_\Omega \rho|\mathbf{v}-\mathbf{V}|^2 + C\int_{\frac{r}{2}\leq\rho\leq 2r}(\rho-r)^2 \leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U}), \quad (3.2)$$

and

$$\int_{\rho\leq\frac{r}{2},\rho\geq 2r}(\mathbf{v}-\mathbf{V})(r-\rho)\frac{\nabla_x p(\rho)}{r} = \int_{\rho\leq\frac{r}{2},\rho\geq 2r}\sqrt{\rho}(\mathbf{v}-\mathbf{V})(r-\rho)\frac{\nabla_x p(\rho)}{\rho r}$$
$$\leq \int_\Omega \rho|\mathbf{v}-\mathbf{V}|^2 + C\int_{\rho\leq\frac{r}{2},\rho\geq 2r}1 \leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U}), \quad (3.3)$$

where we have used $\int_{\frac{r}{2}\leq\rho\leq 2r}(\rho-r)^2\leq\mathcal{E}$ and $\int_{\rho\leq\frac{r}{2},\rho\geq 2r}1\leq\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})$.

For the term I_2,

$$-I_2 = \int_\Omega(\mathbf{v}-\mathbf{V})\frac{\rho}{r}(p'(\rho)-p'(r))\nabla_x\rho + \int_\Omega(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p'(r)\nabla_x(\rho-r) = I_{21}+I_{22}.$$

Using the mean value theorem, we see $I_{21}=\int_\Omega(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p''(\xi_1)(\rho-r)\nabla_x\rho$ where $\xi_1=\theta\rho+(1-\theta)r, \theta\in(0,1)$.

$$I_{22} = \int_{\frac{r}{2}\leq\rho\leq 2r}(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p''(\xi_1)(\rho-r)\nabla_x\rho + \int_{\rho\leq\frac{r}{2},\rho\geq 2r}(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p''(\xi_1)(\rho-r)\nabla_x\rho$$

By virtue of Cauchy inequality and Holder inequality, we get

$$\int_{\frac{r}{2}\leq\rho\leq 2r}(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p''(\xi_1)(\rho-r)\nabla_x\rho$$
$$\leq C\int_{\frac{r}{2}\leq\rho\leq 2r}(\rho-r)^2 + C\int_{\frac{r}{2}\leq\rho\leq 2r}\rho|\mathbf{v}-\mathbf{V}|^2$$
$$\leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})$$

and

$$\int_{\rho\leq\frac{r}{2},\rho\geq 2r}(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p''(\xi_1)(\rho-r)\nabla_x\rho$$
$$\leq \int_{\rho\leq\frac{r}{2},\rho\geq 2r}\rho|\mathbf{v}-\mathbf{V}|^2 + C\int_{\rho\leq\frac{r}{2},\rho\geq 2r}\rho$$
$$\leq C\int_\Omega\rho|\mathbf{v}-\mathbf{V}|^2 + C[\int_{\rho\leq\frac{r}{2},\rho\geq 2r}\rho^\gamma + \int_{\rho\leq\frac{r}{2},\rho\geq 2r}1]$$
$$\leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})$$

On the other hand, it is similar to I_{21} to get $\int_\Omega(\mathbf{v}-\mathbf{V})\frac{\rho}{r}p'(r)\nabla_x(\rho-r)\leq\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U})$. Moreover, due to the smooth solutions, it is easy to get the following

$$|\int_\Omega\rho(\mathbf{v}-\mathbf{V})^2\nabla_x\mathbf{V}|\leq C\mathcal{E}(\rho,\mathbf{u}|r,\mathbf{U}).$$

Next, we turn to estimate the term about P. By the same token, we use Lemma to get

$$\int_\Omega[P'(\rho)-P'(r)](\mathrm{div}_x(\rho\mathbf{v})+\partial_z(\rho w))$$

$$
\begin{aligned}
&= \int_{\Omega} P''(\xi_2)(\rho - r)(\mathrm{div}_x(\rho\mathbf{v}) + \partial_z(\rho w)),\ \xi_2 = \theta\rho + (1-\theta)r,\ \theta \in (0,1) \\
&= \int_{\frac{r}{2}\le\rho\le 2r} P''(\xi_2)(\rho - r)(\mathrm{div}_x(\rho\mathbf{v}) + \partial_z(\rho w)) + \int_{\rho\le\frac{r}{2},\rho\ge 2r} P''(\xi_2)(\rho - r)(\mathrm{div}_x(\rho\mathbf{v}) + \partial_z(\rho w)) \\
&\le \int_{\frac{r}{2}\le\rho\le 2r} |P''(\xi_2)|(\rho - r)^2|\mathrm{div}_x(\rho\mathbf{v}) + \partial_z(\rho w)| + C\int_{\rho\le\frac{r}{2},\rho\ge 2r} 1 \\
&\le C\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\Omega} P''(r)(\rho - r)(\mathrm{div}_x(r\mathbf{V}) + \partial_z(rW)) \\
&\quad= \int_{\frac{r}{2}\le\rho\le 2r} P''(r)(\rho - r)(\mathrm{div}_x(r\mathbf{V}) + \partial_z(rW)) + \int_{\rho\le\frac{r}{2},\rho\ge 2r} P''(r)(\rho - r)(\mathrm{div}_x(r\mathbf{V}) + \partial_z(rW)) \\
&\quad\le \int_{\frac{r}{2}\le\rho\le 2r} |P''(r)|(\rho - r)^2|\mathrm{div}_x(r\mathbf{V}) + \partial_z(rW)| + C\int_{\rho\le\frac{r}{2},\rho\ge 2r} 1 \\
&\quad\le C\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})
\end{aligned}
$$

Finally, we will control the term about vertical velocity.

$$
\begin{aligned}
\int_{\Omega} \rho(w - W)(\mathbf{v} - \mathbf{V})\partial_z\mathbf{V} &\le \int_{\Omega} \sqrt{\rho}(\mathbf{v} - \mathbf{V}) \cdot \sqrt{\rho}(w - W)\partial_z\mathbf{V} \\
&\le C\int_{\Omega} \rho|\mathbf{v} - \mathbf{V}|^2 + C\int_{\Omega} \rho \\
&\le C\mathcal{E} + C\int_{\frac{r}{2}\le\rho\le 2r} \rho + C\int_{\rho\le\frac{r}{2},\rho\ge 2r} \rho \\
&\le C\mathcal{E} + +C\int_{\frac{r}{2}\le\rho\le 2r} (\rho - r)^2 + C[\int_{\rho\le\frac{r}{2},\rho\ge 2r} \rho^\gamma + \int_{\rho\le\frac{r}{2},\rho\ge 2r} 1] \\
&\le C\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})
\end{aligned}
$$

Then due to the solution is smooth solution, so that

$$
\varepsilon^2\int_{\Omega} \rho\mathbf{v}|w - W|\nabla_x(w - W) + \varepsilon^2\rho w|w - W|\partial_z(w - W) + \varepsilon^2\int_{\Omega} \rho(w - W)(\partial_t w - \partial_t W) \le O(\varepsilon^2)
$$

Therefore, combining all the above estimates together, we have

$$
\frac{d}{dt}\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})(\tau) \le C\int_0^\tau h(t)\mathcal{E}(\rho, \mathbf{u}|r, \mathbf{U})(t)dt + o(\epsilon^2). \tag{3.4}
$$

Then applying the Gronwall's inequality, we finish the proof of Theorem 2.1.

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References

[1] P. Azérad and F. Guillén, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, *SIAM J. Math. Anal.*, 33 (2001), 847-859.

[2] Y. Brenier, Homogeneous hydrostatic flows with convex velocity profiles, *Nonlinearity*, 12 (1999), 495-512.

[3] Y. Brenier, Remarks on the derivation of the hydrostatic Euler equations, *Bull. Sci. Math.*, 127 (2003), 585-595.

[4] D. Bresch, F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido, On the uniqueness of weak solutions of the two-dimensional primitive equations, *Differential Integral Equations*, 16 (2003), 77-94.

[5] C. S. Cao and E. S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, *Ann. of Math.*, 166 (2007), 245-267.

[6] C. S. Cao, J. K. Li and E. S. Titi, Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion, *Comm. Pure Appl. Math.*, 69 (2016), 1492-1531.

[7] D. Donatelli and N. Juhasz, The primitive equations of the polluted atmosphere as a weak and strong limit of the 3D Navier-Stokes equations in downwind-matching coordinates, *Discrete Contin. Dyn. Syst.*, 42 (2022), 2859-2892.

[8] M. Ersoy, T. Ngom and M. Sy, Compressible primitive equations: formal derivation and stability of weak solutions, *Nonlinearity*, 24 (2011), 79-96.

[9] E. Feireisl, J. B. Jin and A. Novotný, Relative entropies, suitable weak solutions, and weak-strong uniqueness for the compressible Navier-Stokes system, *J. Math. Fluid Mech.*, 14 (2012), 717-730.

[10] E. Feireisl and A. Novotný, Singular limits in thermodynamics of viscous fluids, Advances in Mathematical Fluid Mechanics, Birkhäuser, Basel, 2009.

[11] H. J. Gao, Š. Nečasová and T. Tang, On the hydrostatic approximation of compressible anisotropic Navier-Stokes equations Justification, J. Math. Fluid Mech., 24 (2022), Paper No. 86.

[12] E. Grenier, On the derivation of homogeneous hydrostatic equations, *M2AN Math. Model. Numer. Anal.*, 33 (1999), 965-970.

[13] T. Kato, The Cauchy problem for quasi-linear symmetric hyperbolic systems, *Arch. Ration. Mech. Anal.*, 58 (1975), 181-205.

[14] J. K. Li, and E. S. Titi, The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation, *J. Math. Pures Appl.*, 124 (2019), 30-58.

[15] J. L. Lions, R. Temam and S. H. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5 (1992), 1007-1053.

[16] J. L. Lions, R. Temam and S. H. Wang, New formulations of the primitive equations of atmosphere and applications, *Nonlinearity*, 5 (1992), 237-288.

[17] X. Liu and E. S. Titi, Local well-posedness of strong solutions to the three-dimensional compressible Primitive Equations, *Arch. Ration. Mech. Anal.*, 241 (2021), 729-764.

[18] N. Masmoudi and T. K. Wong, On the H^s theory of hydrostatic Euler equations, *Arch. Ration. Mech. Anal.*, 204 (2012), 231-271.

[19] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, 2004.

[20] D. Serre, Solutions classiques globales des equations d'Euler pour un fluide parfait compressible, Ann. Inst. Fourier Grenoble, 47 (1997), 139-159.