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Marek Kobera

Qualitative properties of radiation magnetohydrodynamics

Mathematical Institute of Charles University

Supervisor of the doctoral thesis: RNDr. Šárka Nečasová, DSc.

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Title: Qualitative properties of radiation magnetohydrodynamics

Author: Marek Kobera

Institute: Mathematical Institute of Charles University

Supervisor: RNDr. Šárka Nečasová, DSc., Institute of the Czech Academy of Sciences

Abstract: We consider a simplified model based on the Navier-Stokes-Fourier system coupled to a transport equation and the Maxwell system, proposed to describe radiative flows in stars. We establish global-in-time existence for the associated initial-boundary value problem in the framework of weak solutions.

Next, we study a hydrodynamical model describing the motion of internal stellar layers based on compressible Navier-Stokes-Fourier-Poisson system. We suppose that the medium is electrically charged, we include energy exchanges through radiative transfer and we assume that the system is steadily rotating.

We analyze the singular limit of this system when the Mach number, the Alfvén number, the Péclet number and the Froude number go to zero in a certain way and prove convergence to a 3D incompressible MHD system with a stationary linear transport equation for transport of radiation intensity. Finally, we show that the energy equation reduces to a steady equation for the temperature corrector.

Keywords: compressible non-ideal resistive radiative magnetohydrodynamics, existence of a global-in-time weak solution, singular limit for small Mach, Péclet, Froude, Alfvén numbers low stratification, tachoclines and upper radiative zones in giant stars

I would like to dedicate this thesis to my beloved wife Mgr. Veronika Koberová who supported me during the period of writing this thesis. This thesis had not been completed without the precious help of my advisor, RNDr. Šárka Nečasová, who also suggested this topic. Finally, I would like to express my genuine gratefulness and warmest regard to my whole family for supporting me during my studies and to teachers of Faculty of Mathematics and Physics for their teaching efforts and patience with me.

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Preface

The fluid mechanics is a vast discipline with numerous applications in physical sciences and engineering. In both fields to arrive at particular solutions one usually needs to solve certain differential equations numerically. As one faces the problem of various time and length scales connected with values of physical parameters appearing in the equations the complexity of the full numerical resolution of the problem at all scales can be overwhelming and also superfluous. Therefore practitioners often take recourse to some popular simplified models.

The mathematical fluid mechanics brings to this situation mathematical rigour and order. Some models which have been derived by formal arguments like asymptotic power series analysis can be rederived rigorously. Moreover the proofs can shed light on the sources of instabilities that can be observed in numerical approximations and serve as inspirations for the particular choices of the most suitable numerical schemes.

Magnetohydrodynamics is the simplest of possible descriptions of plasma. Plasma is the most prevalent state of the matter in the visible part of the Universe. Matter emiting electromagnetic radiation makes approximately 4.9 % of its energy-matter content. It was said that thereof about 99.999 % is in a plasmatic state (Puerta, Martín , 1998, page 57). However, not all matter that is partially ionized is called plasma by definition. Plasma has to satisfy the following requirements (Karlický , 2014): *the number of particles within the Debye sphere* has to be greater than one (i. e. plasma is dense enough to ensure that the movement of its particles is "collective"), *quasineutrality* (that is the Debye shielding length λ_D is small in comparison with the typical referential length of the system and therefore rather bulk than interfacial effects dominate) and *electron plasma frequency* is much larger than electron-neutral collision frequency, meaning that the behaviour of plasma is dominated by electromagnetic forces and not by ordinary gas-like collisions.

The magnetohydrodynamical (MHD) approximation consists in neglecting any solenoidal part of the electric field (as a consequence of quasineutrality) and Maxwell's displacement current. This is the last term that completed the full Maxwell system, particularly Ampère's law, and was introduced by James Clerk Maxwell himself. As it contains a prefactor that equals in vacuum c^{-2}, where c is the speed of light (in vacuum), it seems plausible to assume that the time change of the electric induction field $\vec{D}$ is not of that magnitude.

Any plasma will generally emit by its movement electromagnetic radiation in forms of photon quanta. In dense (dusty) plasmas this effect may be small and the radiation may be absorbed by nearby objects. This is not the case we treat here. We mainly think of astrophysical plasmas that are in *local thermal equilibrium* (LTE). Even in this case we can mentally divide photon quanta forming a photon gas into two categories. The first one are photons that are in equilibrium with the studied plasma (typically electrons, ions and neutrals). This part behaves according to the black body radiation, Stefan-Boltzmann law, with a given temperature ϑ. However, due to mechanisms of synchrotron radiation (so called Bremsstrahlung) and inverse Compton scattering we encounter photons in the second category — radiation that is not in equilibrium with the plasmatic matter.

The photon quanta that are in equilibrium augment the pressure, energy and entropy functions for gas, and its thermal conductivitity (a radiation contribution) as well. The latter that are out of equilibrium we describe by means of a transport equation. This equation of transfer or *radiative transport equation* (RTE) was derived by Chandrasekhar (Chandrasekhar , 1950, page 9) and in his notation it reads

$$\frac{dI_\nu}{ds} = -\kappa_\nu \rho I_\nu + j_\nu \rho, \tag{1}$$

where I_ν is the (specific) intensity (of radiation), s is the arclength of the radiation pencil (thickness in a direction of propagation), ρ is the density of the material, κ_ν is the mass absorption coefficient for radiation at a frequency ν and finally j_ν is the emission coefficient. In the LTE we apply Kirchhoff's law

$$j_\nu = \kappa_\nu \mathfrak{B}(\nu, \vartheta) \tag{2}$$

where $\mathfrak{B}(\nu,\vartheta) = 2h\vartheta^3 c^{-2}/(\exp(\frac{h\nu}{k_B\vartheta}) - 1)$ is the Planck function of black body radiation.

This thesis is organized as follows. In the first part we introduce some notations and some of the issues of incompressible and compressible Navier-Stokes and Navier-Stokes-Fourier systems. This is a review of many results obtained by various researchers in the past.

The second part then investigates the issue of existence of global-in-time weak solutions of the problem of viscous resistive radiative magnetohydrodynamics. The balance of momentum and the balance of energy contain radiative contributions and the entropy inequality contains additional terms that are indefinite. The proof of existence relies on the Feireisl - Novotný theory for compressible Navier-Stokes-Fourier equations that was recalled in the first part and on velocity averaging lemmas due to Golse et al. (Golse, Lions, Perthame, Sentis , 1988). This part extends the result of Ducomet and Feireisl (Ducomet, Feireisl , 2006) to the case of a nonequilibrial part of the photon gas.

The third part briefly introduces a kind reader to the problem of singular limits in fluid dynamics generally, and to the problem of incompressible limits for a small Mach number specially. We present the basics of the Lighthill's acoustic analogy and properties of the wave equation for the acoustic potential. In the case of a bounded domain Ω the theory developed in the paper of Layton and Novotný (Layton, Novotný , 2010) is concisely reminded.

The final part of the thesis represents another original contribution. We investigate a singular limit for a simplified system in comparison to the second part. Here the simplification consists in negligence of the momentum transfer from the plasma to the nonequilibrial part of the photon gas and vice versa. The omission of this term has been studied in literature and is considered to be well-grounded. An advantage of that simplified system lies in the preservation of the original Planck function $\mathfrak{B}(\nu,\vartheta)$. We add non-inertial terms due to rotation: the centrifugal and the Coriolis force. The existence theory developed in the second part is applied here. The particular limit regime that can be used for example in outer radiation zones of giant stars has got three "orders" of smallness. If ε denotes a quantity that is small, we choose the Froude number of the order $\sqrt{\varepsilon}$, the Mach and Alfvén numbers of the order ε and the Péclet number of the order ε^2. Moreover the infrarelativistic number is of order ε^{-1}. In the terms of physics that means that the material we study is under given conditions nearly incompressible (slightly compressible), lowly stratified by an external force (gravity), subject to a strong magnetic field and well in the realm of classical, not relativistic physics. Moreover, the heat transfer is mainly due to radiation and conduction, not convection. The programme of our investigation vaguely follows the strategy delineated in the third part. We prove that a weak solution of the system of compressible radiative magnetohydrodynamics in this limit converges to a weak solution of a radiative-MHD problem that consists of an incompressible Navier-Stokes system with an effective pressure and an extra right hand side, a system for the magnetic field, a stationary transport equation for the deviation of the radiative intensity from the Planck function $\mathfrak{B}$ and a decoupled steady heat equation for the "third order" correction to the equilibrial temperature $\overline{\vartheta}$. This equation has a diffusion term, an advection term, a heat source corresponding to heating by absorption of the deviation of the radiative intensity, all with constant coefficients, and a corrector uniform in space that preserves the zero mean property of the temperature correction. This limit transition is studied under thermal and mechanical isolation of the domain Ω. For the electric and magnetic field we choose the boundary conditions for a contact with a perfect conductor and for the radiative intensity the boundary condition expressing no reflection at the boundary $\partial\Omega$. The initial conditions are well-prepared, although an adaptation to the ill-prepared case would not be difficult.

The contents of the second part has been submitted as the paper (Ducomet, Kobera, Nečasová , 2014) and the contents of the fourth part as the paper (Donatelli, Ducomet, Kobera, Nečasová , 2016).

1. Introduction

1.1 Notation

- $\Omega \subset \mathbb{R}^N$ domain — connected open set in an Euclidean space. We use $N = 3$.
- $\mathbb{I}_Q$ characteristic function of a set Q
- $B(a, r)$ open ball with its center at $a \in \mathbb{R}^N$ and its radius $r > 0$
- $\mathbb{S}, \mathbb{T}, \ldots$ tensors
- $\mathbb{I}$ identity (operator, tensor)
- $a \otimes b$ tensor product
- $a \times b$ antisymmetric part of $a \otimes b$ — i. e. vector product
- $|Q|$ Lebesgue measure of a set Q
- $\nabla_x \vec{a}$ matrix with entries $\{\partial_{x_j} a_i\}_{i,j=1}^N$
- $\mathrm{div}_x \mathbb{B}$ column vector with entries $\sum_{j=1}^N \partial_{x_j} B_{i,j}$, $i = 1, \ldots, N$
- $\mathrm{curl}_x \vec{c}$ vorticity $\nabla_x \vec{c} - (\nabla_x \vec{c})^T$ antisymmetric matrix, in 3D represented by a vector
- $C(\overline{Q}, X)$ for $Q \subset \mathbb{R}^N$ and X Banach space is the (Bochner) Banach space of continuous functions endowed with the sup norm
- $BC(Q)$ for $Q \subset \mathbb{R}^N$ is the space of bounded and continuous functions endowed with the sup norm
- $C_w(\overline{Q}, X)$ for $Q \subset \mathbb{R}^N$ and X Banach space is the Banach space of bounded weakly continuous functions
- $C^k(\overline{Q}, X)$ for $Q \subset \mathbb{R}^N$, $k \in \mathbb{N}$ and X Banach space is the (Bochner) Banach space of restrictions of $k-$times differential functions endowed with the norm $||v||_{C^k(\overline{Q},X)} := \max_{|\vec{\alpha}| \le k} \sup_{x \in \overline{Q}} \|\partial^{\vec{\alpha}} v(x)\|_X$, where $\vec{\alpha}$ is a multiindex
- $C^{k,\nu}(\overline{Q}, X)$ for $Q \subset \mathbb{R}^N$, $k \in \mathbb{N}$, $\nu \in (0, 1]$ and X Banach space is the (Bochner) Banach space endowed with the norm $||v||_{C^{k,\nu}(\overline{Q},X)} := ||v||_{C^k(\overline{Q},X)}$ $+ \max_{|\vec{\alpha}|=k} \sup_{(x,y) \in Q^2 : x \neq y} \frac{\|\partial^{\vec{\alpha}} v(x)\|_X}{|x-y|^\nu}$
- $C_c^k\left(Q, \mathbb{R}^M\right)$ for $Q \subset \mathbb{R}^N$ and $k, M \in \mathbb{N}$ the space of the $k-$times differential vector fields with a compact support in Q
- $\mathcal{D}\left(Q, \mathbb{R}^M\right) = C_c^\infty\left(Q, \mathbb{R}^M\right)$ for $Q \subset \mathbb{R}^N$ and $k, M \in \mathbb{N}$ the space of the $k-$times differential vector fields with a compact support in Q endowed with the local uniform convergence topology (of all derivatives)

- $W^{m,p}(Q)$ for $m \in \mathbb{Z}$, $p \in [1,\infty]$ and $Q \subset \mathbb{R}^N$ Sobolev space, for $m \notin \mathbb{Z}$ Aronszajn-Slobodetskii space
- $\dot{L}^p(Q) = \left\{ v \in L^p(Q) : \int_Q v(x)\,\mathrm{d}x = 0 \right\}$
- $\dot{W}^{1,p}(Q) = \dot{L}^p(Q) \cap W^{1,p}(Q)$
- $|w|^+ := \max\{w, 0\}$
- $|w|^- := \min\{w, 0\}$
- $\mathcal{R} = \triangle^{-1}\nabla_x \otimes \nabla_x$ double Riesz transform
- $\mathcal{M}(X)$ the set of signed Borel measures on X
- $\mathcal{M}^+(X)$ the cone of non-negative elements of $\mathcal{M}(X)$
- $BV(Q)$ the function space of functions on Q with bounded variation
- $\mathcal{H}(\Omega) = \{\vec{U} \in L^2(\Omega;\mathbb{R}^3),\ \mathrm{div}_x\vec{U} = 0 \text{ in } \Omega,\ \vec{U}\Big|_{\partial\Omega} = 0\}$
- $\mathcal{U}(\Omega) = \mathcal{H}(\Omega) \cap W_0^{1,2}(\Omega;\mathbb{R}^3))$
- $\mathcal{V}(\Omega) = \left\{\vec{b} \in L^2(\Omega;\mathbb{R}^3),\ \mathrm{div}_x\vec{b} = 0,\ \vec{b}\cdot\vec{n}\Big|_{\partial\Omega} = 0\right\}$
- $\mathcal{W}(\Omega) = \mathcal{V}(\Omega) \cap W_0^{1,2}(\Omega;\mathbb{R}^3)$

1.2 Introduction to models in fluid mechanics

A fluid is a substance that cannot sustain stress in its equilibrium — it starts to flow[1]. Examples of fluids are liquids, gases and plasmas. Fluid dynamics is applied in engineering, meteorology, astrophysics. We distinguish various levels of description of the flow of fluids.

The first level is the level of *molecular dynamics*. At this level we assume we can follow individual molecules/particles or at least some of their characteristics if we take the quantum theory into account. We can relatively easily formulate a dynamical system for $n-$body dynamics of molecules. In the classical physics it is based on Newton's equation, Lagrange or Hamilton's formalism. In the quantum non-relativistic case we use the Schrödinger equation, the Liouville-von Neumann equation and Fock space. In the quantum relativistic case there are ways how to generalize one particle Dirac, Klein-Gordon, Proca or generally Bargmann-Wigner equation to the many body case. In the general relativistic case we encounter Einstein's equations. The Newtonian dynamic is fully time reversible. The natural disadvantage of these models is that the typical n is huge (cf. the Avogadro constant).

[1] However there are fluids with non-zero yield stress also like ketchups, collagen dispersions. The very nature of the yield stress has been discussed in literature with the conclusion that there is no real yield stress; that it looks like there is, depending on the timescale of the flow, more exactly on the Deborah number. From the modelling and practical point of view there is a yield stress, there are fluids with an activation mechanism.

The second level of description is the level of *Boltzmann equations and kinetic models.* The Boltzmann transport equation arises under suitable assumptions from the BBGKY hierarchy of $n-$particle functions by reduction to the one-particle function. The point of view is different: we follow the distribution function in the phase space of dimension $6n+1$ or in a seven dimensional space for a one-particle function. There are many variants of the Boltzmann model like the BGK equation, Fokker-Planck equation, Waldmann-Snider equation, and for our use important Vlasov-type equations, like Landau equation, (see Villani , 2012). Whereas the Boltzmann type equation assumes that the particles of the fluid interact only by short-range potentials or collisions, the Vlasov-type equations are suitable in situation of long range interactions, especially of the electromagnetic nature. This is useful for modelling plasmas in astrophysics and elsewhere. We can define Vlasov-Einstein, Vlasov-Maxwell, Vlasov-Yang-Mills and Vlasov-Poisson equations, (see Vedenyapin, Sinitsyn, Dulov , 2011). The equations are not time reversible, as it is shown by the celebrated Boltzmann H-theorem.

The third level then originates by integration of the Boltzmann-type equations over the momentum. We get evolution equations for density ϱ, velocity $\vec{u}$ and temperature ϑ as moments of the scaled Boltzmann equation in some limits. The equations are of *Navier-Stokes type* and when we start with Vlasov-Maxwell equations we obtain viscous *electro-magneto-hydrodynamical equations*, (see Arsénio, Saint-Raymond , 2016). Further averaging may be required for some models of turbulence.

The governing equations of motion in fluid mechanics have got the form of conservation laws: balance of mass, linear and angular momentum, energy and entropy. The general form of these balances is (Feireisl, Novotný , 2009)

$$\int_B d(t_2,x)\,\mathrm{d}x - \int_B d(t_1,x)\,\mathrm{d}x + \int_{t_1}^{t_2}\int_{\partial B} \vec{F}(t,x)\cdot\vec{n}(x)\,dS\,\mathrm{d}t = \int_{t_1}^{t_2}\int_B \varsigma(t,x)\,\mathrm{d}x\,\mathrm{d}t, \tag{1.1}$$

where d is the volumetric density of a quantity, $\vec{F}$ is its flow rate at the boundary of a "testing body" $B \in 2^{\Omega}$ and ς is its volumetric production rate. Of course, we assume that all the integrals in (1.1) exist, especially $\varsigma = \varsigma^+ - \varsigma^-$ is a signed measure, i. e. ς^+ and ς^- are non-negative regular Borel measures defined (at least) on $[0,T] \times \overline{\Omega}$. The equation (1.1) should hold for all times $t_1 \le t_2$ and all bodies B.

If we endow the conservation law (1.1) with some boundary conditions, like in a Neumann IBVP, we may formulate it in distributions. If now the measure ς is not absolutely continuous with respect to the Lebesgue measure, the quantity d is no longer weakly continuous in time and we can define its left and right limits in time as a measure in $\overline{\Omega}$ when we take the test functions as the product of a smooth, compactly supported test function in space and a continuously differentiable sequence converging to the shifted function $(1-H)(t)$, where $H(t)$ is the Heaviside function.

We review some simpler mathematical models than compressible viscous magnetohydrodynamics according to (Feireisl , 2010).

1.3 Incompressible Navier-Stokes equations

The field of partial differential equations was entered into by Leonhard Euler who derived a system for movement of fluids in 1757 (Euler , 1757)

$$\mathrm{div}_x \vec{u} = 0, \tag{1.2a}$$

$$\partial_t \vec{u} + \mathrm{div}_x(\vec{u} \otimes \vec{u}) + \nabla_x p = \vec{f}. \tag{1.2b}$$

Here $\vec{u}(t,x)$ is the velocity vector field and $p(t,x)$ is the pressure scalar field. The system (1.2a) – (1.2b) is nowadays known as the incompressible Euler equations. The equation (1.2a) expresses the incompressibility of the fluid under consideration, while (1.2b) is a particular form of Newton's law. Here, of course, t denotes time and $x \in \Omega$ spatial position with respect to a chosen origin of coordinates. The vector field $\vec{f}$ is assumed to be a known vector field of density of forces, usually external, which set or keep the fluid in motion.

It is well known that incompressible Euler equations (1.2a) – (1.2b) are conservative and hyperbolic. However, most real fluids are rather dissipative. Our experience with mixing a fluid tells us that when we stop mixing they come to a rest. Also, if a real fluid flows past an obstacle it does not move further unperturbed. This effect was for the first time quantified by Newton (Newton , 1687) who introduced his notion of viscosity — internal resistance of fluid against velocity changes. In the 19^{th} century this led to a reformulation of (1.2b) introducing a viscous damping force $\mathrm{div}_x \nu(\nabla_x \vec{u} + \nabla_x^T \vec{u})$ so that the renown incompressible Navier-Stokes equations are

$$\mathrm{div}_x \vec{u} = 0, \tag{1.3a}$$

$$\partial_t \vec{u} + \mathrm{div}_x(\vec{u} \otimes \vec{u}) + \nabla_x p - \mathrm{div}_x \nu(\nabla_x \vec{u} + \nabla_x^T \vec{u}) = \vec{f}, \tag{1.3b}$$

where ν is the kinematic viscosity. We shall assume that it is positive. The concept of viscosity is intrisically connected with statistical physics because it introduces time irreversibility which is not present in Newton's equation and cannot be simply generalized to general relativity without violating causality.

To have a complete mathematical formulation of a problem to be solved we have to endow the system (1.3a) – (1.3b) with initial and boundary conditions. Since the incompressible Navier-Stokes equations are differential equations of first order in time, we usually assume their initial conditions in the simplest case

$$\vec{u}(0,\cdot) = \vec{u}_0 \quad \text{in } \Omega, \tag{1.4}$$

where $\vec{u}_0$ is a spatial vector field of velocity of the fluid at a given time $t = 0$ and Ω is a domain in an ambient Euclidean space. For the boundary conditions we can choose from a wider set of conditions, so called no-slip boundary conditions

$$\vec{u} = \vec{0} \quad \text{on } \partial\Omega, \tag{1.5}$$

being still the most usual. The IBVP for incompressible Navier-Stokes equations (1.3a) – (1.3b), (1.4), (1.5) has been used extensively in many various numerical simulations. There are indications that it has got its central position in classical physics fairly: it is a canonical regularization of the incompressible Euler system (1.2a) – (1.2b) with the impermeability boundary condition

$$\vec{u} \cdot \vec{n} = 0 \quad \text{on } \partial\Omega, \tag{1.6}$$

where $\vec{n}$ is the unit outer vector field on the boundary $\partial\Omega$, and it can be recovered as a limit system of various more general models in fluid mechanics, especially of Boltzmann equation (Arsénio, Saint-Raymond , 2016) and as a low Mach number limit of compressible Navier-Stokes-Fourier system (Alazard , 2006). There is also a few examples of solutions to Navier-Stokes equations, but their stability and therefore physical relevance is largely unknown (cf. e. g. (Taylor, Green , 1937)).

The choice of proper physical boundary conditions for the incompressible Navier-Stokes system has not been settled in spite of serious recent effort. For various situations, especially for the analysis of pressure it may be suitable to select the Navier boundary conditions

$$\alpha\vec{u}\cdot\vec{\tau}+(1-\alpha)\vec{\tau}\cdot\mathbb{S}\vec{n}=0 \quad \text{on } \partial\Omega, \tag{1.7}$$

where we denoted the viscous part of the Cauchy stress tensor by $\mathbb{S}=2\nu\mathbb{D}$ with $\mathbb{D}:=\frac{1}{2}(\nabla_x\vec{u}+\nabla_x^T\vec{u})$ being the deformation rate tensor, $\alpha\in[0,1]$ a constant and $\vec{\tau}$ an arbitrary tangential vector field on the boundary $\partial\Omega$. Naturally, when (1.7) is combined with (1.6) the rôle of α is a switch between the no-slip boundary conditions (for $\alpha=1$) and the (complete) slip boundary conditions (for $\alpha=0$). The Navier boundary conditions can be understood as an expression for the velocity of the fluid at the boundary $\partial\Omega$; in that case α is linked to the friction coefficient. (Let us note that in the real world the friction coefficient should depend on the pressure at the boundary which seems to be out of reach of current mathematical fluid analysis and that one expects a plastic-like behaviour of the boundary condition, namely up to a certain threshold fluid it exhibits no-slip and after exceeding it partial or complete slip).

The issue of global well posedness for large data, which is usually required in mathematical modelling, is for (1.3a) – (1.3b), (1.4), (1.5) an outstanding long-term open problem, at least in three dimensions — we know functional settings where existence has been established and other functional settings where uniqueness holds. Historically, the question of existence of solutions led to the introduction of the notion of generalized solution, so called weak solutions by Leray (Leray , 1934). One of the main results on regularity of solutions to the incompressible Navier-Stokes system is due to Beale, Kato and Majda (Beale, Kato, Majda , 1984) which maintains that we have got a regular solution up to a time T for which

$$\|\vec{\omega}\|_{L^1(0,T;L^\infty(\Omega))}\leq c, \tag{1.8}$$

where $\vec{\omega}:=\mathrm{curl}_x\vec{u}$ is vorticity of the flow. This results entails that a corresponding 2D problem has got a smooth global solution. Moreover regularity criteria can be formulated in terms of Hölder continuity of the direction of the vorticity (see da Veiga , 2016).

The issue of regularity of the Navier-Stokes system is also intimately connected to the occurrence of singularities. The dimension of the singular set was estimated by the celebrated paper of Caffarelli, Kohn and Nirenberg (Caffarelli, Kohn, Nirenberg , 1982) for so called suitable weak solutions. The particular difficulty of the Navier-Stokes consists in an interplay between the nonlinear convective term $\mathrm{div}_x\vec{u}\otimes\vec{u}$, the non-local pressure term $\nabla_x p$ and the time derivative $\partial_t\vec{u}$. As was already mentioned in the book of Temam (Temam , 1977) we have lack of information on the regularity of the time derivative of the pressure. This was later carried out in more details by Wolf (Wolf , 2007) who split the pressure into terms corresponding to individual terms in the Navier-Stokes system (locally) and found out that pressure p can be decomposed $p = p_0 + \partial_t \tilde{p_h}$, where p_0 is a $L^{\frac{5}{3}}_{loc}((0,T)\times\Omega)$ function and $\tilde{p_h}$ a harmonic, zero-mean $C_{weak}([0,T];L^{\frac{5}{3}}_{loc}(\Omega))$ function — (see Wolf , 2007, Theorem 2.6, pp. 113 – 114).

The weak (variational) formulation by Leray (Leray , 1934) eliminates the pressure p choosing solenoidal vector fields as test functions:

$$\int_0^T \int_\Omega \vec{u}\cdot\nabla_x\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall\vec{\varphi}\in\mathcal{D}((0,T)\times\Omega;\mathbb{R}^3), \tag{1.9a}$$

$$\int_0^T \int_\Omega \left(\vec{u}\cdot\partial_t\vec{\varphi} + \vec{u}\otimes\vec{u}:\nabla_x\vec{\varphi} - \mathbb{S}:\nabla_x\vec{\varphi} + \vec{f}\cdot\vec{\varphi}\right)\mathrm{d}x\,\mathrm{d}t = \int_\Omega \vec{u}_0\cdot\vec{\varphi}(0,\cdot)\,\mathrm{d}x \tag{1.9b}$$

$$\forall\vec{\varphi}\in\left\{\vec{\psi}\in\mathcal{D}\left([0,T)\times\Omega;\mathbb{R}^3\right)\,:\,\mathrm{div}_x\vec{\psi}=0\right\}.$$

If now we assume that $\vec{u}_0 \in L^2(\Omega,\mathbb{R}^3)$, $\vec{f}\in L^2((0,T)\times\Omega,\mathbb{R}^3)$ we obtain from application of Grönwall's lemma upon testing the weak formulation (1.9b) with $\vec{\varphi} := \vec{u}$.

$$\|\vec{u}\|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} \le \|\vec{u}_0\|_{L^2(\Omega;\mathbb{R}^3)}\, e^{\frac{T}{2}} \quad \forall T\ge 0 \tag{1.10}$$

$$\left\|\nabla_x\vec{u}+\nabla_x^T\vec{u}\right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \le c\sqrt{1+e^T} \quad \forall T\ge 0 \tag{1.11}$$

with a constant c independent of the final time T, but dependent on the data $\vec{u}_0$ and $\vec{f}$.

These estimates determine the so called Leray-Hopf solution to (1.9a) – (1.9b) in the so called Ladyzhenskaya space $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))\cap L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$ by Korn's inequality and although the previous testing was only formal, in the existence proof of a solution to (1.9a) – (1.9b) by approximations we still obtain (1.10) – (1.11) for them, but in the limit we get the energetic inequality only

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\vec{u}(t,x)|^2\,\mathrm{d}x + \int_\Omega \mathbb{S}(t,x):\nabla_x\vec{u}(t,x)\,\mathrm{d}x \le \int_\Omega \vec{f}(t,x)\cdot\vec{u}(t,x)\,\mathrm{d}x, \tag{1.12}$$

by weak lower semicontinuity.

For the slip boundary conditions (1.7) with $\alpha = 0$ the weak formulation of (1.3b),(1.7), (1.4) is essentially the same as (1.9b) up to the choice of the space for test functions. This time it has to reflect (1.7) so that we can easily test (1.3b) with (approximations of) $\vec{u}$. The space is $\left\{\vec{\psi}\in\mathcal{D}\left([0,T)\times\overline{\Omega};\mathbb{R}^3\right)\,:\,\mathrm{div}_x\vec{\psi} = 0,\ \vec{\psi}\cdot\vec{n}\Big|_{\partial\Omega} = 0\right\}$. Using the famous Helmholtz decomposition $\vec{\psi} = H(\vec{\psi}) + \nabla_x\Phi$, where $H(\vec{\psi})$ is the solenoidal part of $\vec{\psi}$ and Φ satisfies the homogeneous Neumann problem for the Poisson equation $\triangle\Phi = \mathrm{div}_x\vec{\psi}$, we can test the weak formulation of (1.3b) with a non-soleinodal (formerly inadmissible) test function $\nabla_x\Phi$ which leads under the assumption of satisfaction of the compatibility condition $\mathrm{div}_x\vec{u}_0 = 0$ to the weak formulation of the equation for pressure in the case of slip boundary conditions

$$\int_\Omega p\,\triangle\vec{\psi}\,\mathrm{d}x = \int_\Omega (\mathbb{S}-\vec{u}\otimes\vec{u}):\nabla_x\nabla_x\vec{\psi}\,\mathrm{d}x - \int_\Omega \vec{f}\cdot\nabla_x\vec{\psi}\,\mathrm{d}x. \tag{1.13}$$

with $\vec{\psi}\cdot\vec{n}\Big|_{\partial\Omega}=0$. Here we can use the regularity theory for the Poisson equation and immediately conclude that $p\in L^{\frac{5}{3}}((0,T)\times\Omega)$ because $\vec{u}\in L^{\frac{10}{3}}((0,T)\times\Omega;\mathbb{R}^3)$ by the (isotropic) interpolation of the Ladyzhenskaya space.

The same summability of the pressure holds for the IBVP (1.3a)–(1.3b), (1.4), (1.5)(Solonnikov , 2002, pp. 356 – 362), but with a different proof based on the representation of the pressure by a compositions of the heat operator, double Riesz transform and the operator of divergence applied to a solution of a problem for heat equation in the spirit of Oseen. Let us finish this section noticing that the pressure function p in (1.3b) is not completely the same what we understand under a pressure in the simple physics. First of all, the function p is not a pressure, but rather a pressure divided by the density that is assumed in this case to be constant. Secondly, the pressure is not an absolute quantity, it is determined up to a constant and it can be negative, as well as positive. Finally, due to its low regularity, it may be unbounded and may not enjoy its trace on surfaces like $\partial\Omega$, so its coincidence with a measured pressure is doubtful. From the mathematical analysis it also yields that (the irregular part of) pressure may experience jumps in its time evolution.

1.4 Barotropic Compressible Navier-Stokes equations

From the discussion in the Section 1.3 we can see that the case of the incompressible Navier-Stokes equations is in a sense "singular", a limit case of zero compressibility or zero Mach number. Sometimes to solve the problems related to incompressible models we introduce a slight compressibility and let that compressibility tend to zero finally. That gives a clue we could embed incompressible Navier-Stokes equation in a broader class of barotropic compressible Navier-Stokes equation.

The barotropicity means that the pressure p is now a function of the density ρ which happens in the case of isothermal or adiabatic flows of an ideal gas. We study IBVP for barotropic compressible Navier-Stokes equations consisting of the balances of mass and momentum

$$\partial_t\varrho+\mathrm{div}_x\varrho\vec{u}=0, \tag{1.14a}$$

$$\partial_t\left(\varrho\vec{u}\right)+\mathrm{div}_x(\varrho\vec{u}\otimes\vec{u})+\nabla_x p-\mathrm{div}_x\mathbb{S}=\varrho\vec{f}, \tag{1.14b}$$

endowed with an initial conditions (1.4) and boundary conditions (1.7). We confine ourselves to the case where there is no inflow or outflow from the domain Ω, i. e. (1.6) holds. Let us note that for simplicity we took the right hand side of (1.14b) in the form $\varrho\vec{f}$ which covers the gravitational interaction, but does not cover e. g. the Lorentz force. Here also $\mathbb{S}$ is more general than in the incompressible case. In the simplest case of Newtonian compressible isotropic fluid $\mathbb{S}=\mu\left(\nabla_x\vec{u}+\nabla_x\vec{u}^T-\frac{2}{3}\mathrm{div}_x\vec{u}\,\mathbb{I}\right)+\eta\,\mathrm{div}_x\vec{u}\,\mathbb{I}$, where $\mu>0$ is the shear and $\eta\geq 0$ the bulk viscosity. This case is established from the hypotheses that $\mathbb{S}(\nabla_x\vec{u})$ should be a linear, symmetric and rotationally invariant function by Chorin and Marsden, (see Chorin, Marsden , 1979, pp. 32 – 33). The weak formulation of the problem (1.14a) – (1.14b) is analogous to (1.9a) – (1.9b) and is omitted for the sake of brevity (the MHD version is treated in Part 2).

Clearly, if we choose the no-slip boundary conditions ($\alpha = 1$ in (1.7)), then the total mass of fluid $||\varrho||_{L^1(\Omega)}$ is a constant of motion as stated by the law of mass conservation. If we formulate the problem (1.14a), (1.14b) and (1.7) with $\alpha = 0$ or $\alpha = 1$ weakly and test the weak formulation of (1.14b) with approximations of $\vec{u}$ and pass to the limit, we get an energy inequality, analogous to (1.12) in the incompressible case

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \left(\varrho(t,x)\,|\vec{u}(t,x)|^2 + 2H(\varrho(t,x))\right)\,\mathrm{d}x + \int_\Omega \mathbb{S}(t,x) : \nabla_x \vec{u}(t,x)\,\mathrm{d}x \leq \tag{1.15}$$
$$\int_\Omega \left(\varrho\vec{f}\right)(t,x)\cdot\vec{u}(t,x)\,\mathrm{d}x,$$

where $H(\varrho) = \varrho\int_1^\varrho \frac{p(x)}{x^2}\,\mathrm{d}x$ is a special expression for the Helmholtz free energy of the fluid and thanks to slip or no-slip boundary conditions combined with the impermeability (1.6) we have not got any energy flux accross the boundary (the energetical isolation).

The strategy of Lions (Lions , 1996, 1998) is to study problems in fluid dynamics first without possible boundary effects which can complicate for example the choice of suitable test functions. He studies problems in the flat torus $\mathcal{T}^3$ case (i. e. with periodic boundary conditions) and then in the $\mathbb{R}^3$ case where we need a certain decay conditions. The latter case needs a certain decay conditions "near infinity". One possible choice is a convergence towards a rest state with a constant density

$$\vec{u} \to \vec{0} \quad \varrho \to \overline{\varrho} \qquad \text{as } |x| \to \infty, \tag{1.16}$$

where $\overline{\varrho}$ is a non-negative constant.

Let us define an affinely shifted Helmholtz function

$$\mathcal{H}(\varrho) := H(\varrho) - (\varrho - \overline{\varrho})H_\varrho(\overline{\varrho}) - H(\overline{\varrho}). \tag{1.17}$$

This function is non-negative as $H(\varrho)$ is locally uniformly convex, if we assume that the pressure function $p(\varrho)$ is nondecreasing (a natural assumption except for processes undergoing structural changes as phase transitions), and then the energy inequality becomes

$$\frac{d}{dt}\int_\Omega \frac{1}{2}\left(\varrho(t,x)\,|\vec{u}(t,x)|^2 + \mathcal{H}(\varrho(t,x))\right)\,\mathrm{d}x + \int_\Omega \mathbb{S}(t,x) : \nabla_x \vec{u}(t,x)\,\mathrm{d}x \leq \tag{1.18}$$
$$\int_\Omega \left(\varrho\vec{f}\right)(t,x)\cdot\vec{u}(t,x)\,\mathrm{d}x,$$

where we emphasize the structure: the first term in the lhs is the kinetic energy of the fluid, the second one its Helmholtz free energy; their change is due to viscous dissipation (third term which is non-positive if transferred to the rhs) and the power of (external) forces — the first term in the rhs. This inequality should be understood in its weak formulation. The essential ingredient to get an analogue of the Ladyzhenskaya space for $\vec{u}$ in the compressible case is once again Korn's inequality, this time in the form

$$||\vec{u}||_{W^{1,p}(\Omega,\mathbb{R}^3)} \leq c\left(||\vec{u}||_{L^p(\Omega,\mathbb{R}^3)} + \left\|\nabla_x\vec{u} + \nabla_x\vec{u}^T - \frac{2}{3}\mathrm{div}_x\vec{u}\,\mathbb{I}\right\|_{L^p(\Omega,\mathbb{R}^{3\times 3})}\right) \tag{1.19}$$

which holds for $p \in (1, \infty)$ (cf. Reshetnyak , 1994, Theorem 3.2, p. 146 and Theorem 3.3, p. 149).

The pressure function p in (1.14b) is a sufficiently smooth function of the density ϱ, its particular form is given by an equation of state (EOS) for the fluid considered. A special attention has been given to the polytropic case, where

$$p(\varrho) = a\varrho^\gamma \qquad \gamma \geq 1 \tag{1.20}$$

holds true. Here $a > 0$ is a constant and γ denotes polytropic index, which in an isentropic (especially adiabatic reversible) process coincides with the adiabatic exponent. This is for ideal gases equal to the ratio of specific heat capacity at constant pressure and volume, respectively. For a monoatomic gas, like noble gases, it is $\gamma = \frac{5}{3} =: \gamma_1$. For gases with more degrees of freedom it is $\gamma < \gamma_1$. For the isothermal process of (the ideal) gas we have got from its EOS $\gamma = 1$. For the polytropic case (1.20) the a-priori estimates are a direct consequence of the energetic inequality (1.15), resp. (1.18) in the bounded, resp. unbounded case if we assume (1.16) with the help of Korn's inequality (1.19). We get

$$\varrho \in L^\infty(0, T; L^\gamma(\Omega)) \tag{1.21a}$$
$$\sqrt{\varrho}\vec{u} \in L^\infty(0, T; L^2(\Omega)) \tag{1.21b}$$
$$\vec{u} \in L^2(0, T; W^{1,2}(\Omega)) \tag{1.21c}$$

provided Ω is bounded. For unbounded domains Ω we get instead of (1.21a) an estimate for $\varrho - \overline{\varrho}$. Keeping in mind that an existence theorem for weak solutions of the IBVP (1.14a), (1.14b), (1.7) and replacements of the initial conditions (1.4) and the impermeability condition (1.6) will be obtained by the Banach-Alaoglu-Bourbaki theorem from a-priori estimates (1.21a)–(1.21c), we would like to estimate all the terms arising in weak formulation of the problem in weakly closed spaces, especially in terms of Lebesgue spaces in a $L^p((0, T) \times \Omega)$ for $p > 1$ and $T > 0$ arbitrary. Here the initial conditions have to be given for the density ϱ and for the momentum $\varrho\vec{u}$. The initial condition for ϱ will be implicitly given by the notion of the renormalized solution of (1.14a). The initial condition for the kinetic energy now follows from the initial conditions for density and momentum. Let us stress that this approach necessitates constant viscosities and strong convergence of the initial condition for density $\varrho_{0,n}$ in a Lebesgue space.

Next problem consists in identification of the weak limits, which will be denoted by overbars. This problem enforces by Hölder's inequality from the convective term that $\gamma > \frac{3}{2}$ and a need for an improved estimated of the pressure, since from (1.21a) we know

$$||p||_{L^1((0,T)\times\Omega)} \leq c \tag{1.22}$$

only. Let us notice that the problem of existence of weak solutions to the aforementioned IBVP would be simpler if it would have been known that the density ϱ enjoys a lower bound, i. e. no vacuum zones develop in time if they are not present in the initial condition for ϱ. In that case we might treat the case of non-constant, density dependent viscosity coefficients $\mu(\varrho)$ and $\eta(\varrho)$.

The essential invention needed for establishing existence of solutions to barotropic Navier-Stokes equations is the weak continuity for convergences implied by a-priori estimates (1.21a) – (1.21c) at hand of the quantity which Lions (Lions , 1996, 1998) called the effective viscous flux $evf := p\varrho - \varrho \mathrm{div}_x \triangle^{-1} \mathrm{div}_x \mathbb{S}$. Simple formal computation reveals that

$$evf = \varrho \triangle^{-1} \mathrm{div}_x \left(\varrho \vec{f}\right) - \partial_t \left[\varrho \mathrm{div}_x \triangle^{-1} \left(\varrho \vec{u}\right)\right] -$$

$$\mathrm{div}_x \left[\varrho \vec{u} \mathrm{div}_x \triangle^{-1} \left(\varrho \vec{u}\right)\right] + \varrho \vec{u} \cdot \nabla_x \triangle^{-1} \mathrm{div}_x \left(\varrho \vec{u}\right) - \varrho \mathrm{div}_x \triangle^{-1} \mathrm{div}_x \left(\varrho \vec{u} \otimes \vec{u}\right). \quad (1.23)$$

The rhs of (1.23) is weakly continuous in L^p-spaces with p sufficiently high (achievable by extra terms in a suitable approximation scheme).

The proof of weak continuity of the effective viscous flux leans on its precompactness established by compensated compactness — div-curl lemma, (cf. Málek, Nečas, Rokyta, Růžička, 1996, p. 158) for the 2D version. However we use its 4D version. For the convergence of approximative momenta $\varrho_n \vec{u}_n$ we choose the corresponding four-vectors as $\vec{U}_n := (\varrho_n, \varrho_n \vec{u}_n)$ and $\vec{V}_n^i := (u_n^i, 0, 0, 0)$, where $i = 1, \ldots, 3$ denotes the component of velocity vector field in cartesian coordinates. For $\gamma > \frac{3}{2}$ the hypotheses of the div-curl lemma are satisfied as $\mathrm{div}_{t,x} \vec{U}_n = 0$ along (1.14a), which is surely precompact, and $\mathrm{curl}_{t,x} V_n^i$ is precompact in $W^{-1,p}\left((0,T) \times \Omega\right)$ for $p \in [1, 4)$ independently of γ. Therefore

$$\varrho_n \vec{u}_n \rightharpoonup \varrho \vec{u} \qquad \text{in } L^{\frac{6\gamma}{4\gamma+3}}\left((0,T) \times \Omega\right). \quad (1.24)$$

This allows to pass to the limit in the weak formulation of (1.23): take an arbitrary $\varphi \in \mathcal{D}\left((0,T) \times \Omega\right)$ and partially integrate (1.23) for approximations indexed by n tested by φ

$$\int_0^T \int_{\mathbb{R}^3} evf_n \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_{\mathbb{R}^3} \varrho_n \varphi \triangle^{-1} \mathrm{div}_x \left(\varrho_n \vec{f}\right) \mathrm{d}x \, \mathrm{d}t + \quad (1.25)$$

$$\int_0^T \int_{\mathbb{R}^3} \varrho_n \mathrm{div}_x \triangle^{-1} \left(\varrho_n \vec{u}_n\right) \partial_t \varphi \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \left[\varrho_n \vec{u}_n \mathrm{div}_x \triangle^{-1} \left(\varrho_n \vec{u}_n\right) \cdot \nabla_x \varphi + \right.$$

$$\left. \varrho_n \varphi \vec{u}_n \cdot \nabla_x \triangle^{-1} \mathrm{div}_x \left(\varrho_n \vec{u}_n\right) - \nabla_x \triangle^{-1} \nabla_x \left(\varrho_n \varphi\right) : \varrho_n \vec{u}_n \otimes \vec{u}_n\right] \mathrm{d}x \, \mathrm{d}t.$$

Here we can use a weak time continuity in $L^p(\Omega)$ spaces which we can get from the system (1.14a) – (1.14b) by dual estimates of time derivatives of density and momentum $\partial_t \varrho_n$ and $\partial_t \left(\varrho_n \vec{u}_n\right)$ arguing like in (Boyer, Fabrie , 2013, Proposition V.1.7, p. 363). Thus, by virtue of (1.24), regularizing properties of the operator $\triangle^{-1} \mathrm{div}_x$ and the weak time continuity we can pass to the limit $n \to \infty$ in the first three terms on the rhs of (1.25).

The last two terms on the rhs of (1.25) can be rendered as a commutator of $\mathcal{I}$ and $\mathcal{R}$, where $\mathcal{I}$ is the identity operator and $\mathcal{R}$ is the pseudodifferential operator of double Riesz transform $\nabla_x \triangle^{-1} \nabla_x$ with its symbol $\frac{\xi_i \xi_j}{|\xi|}$. Then we can use once again the div-curl lemma as in the proof of Theorem 10.27, p. 350 in the book (Feireisl, Novotný , 2009, proof of Theorem 10.27, p. 350). Altogether the convergences above lead to the identity

$$\overline{evf} - \overline{p} \varrho + \mathrm{div}_x \triangle^{-1} \mathrm{div}_x \mathbb{S} \varrho = 0 \quad (1.26)$$

from the limit equation obtained from (1.14b). For the constant coefficient case (shear and bulk viscosities) we get

$$\mathrm{div}_x \triangle^{-1} \mathrm{div}_x \mathbb{S} = \left(\frac{4}{3}\mu + \eta\right) \mathrm{div}_x \vec{u} \quad (1.27)$$

and we can thus simplify (1.26) to

$$\overline{p(\varrho)\varrho} - \overline{p(\varrho)}\varrho = \left(\frac{4}{3}\mu + \eta\right)\left(\overline{\varrho \mathrm{div}_x \vec{u}} - \varrho \overline{\mathrm{div}_x \vec{u}}\right) \tag{1.28}$$

which can lead to the strong convergence of densities ϱ_n if the right hand side of (1.28) vanishes. However this could work for $\gamma \geq 2$ only because we have not got better summability of $\mathrm{div}_x \vec{u}$ than what follows from (1.21c).

The problem of $\varrho \notin L^2(\Omega)$ was successfully solved by Feireisl (Feireisl , 2001). One of the essential parts of the solution relies on renormalization of the continuity equation (1.14a); we talk about so called renormalized solutions. Whereas in the context of hyperbolic conservation laws we add to the original equation a set of inequalities for "entropies" which should select a physical unique solution, in the context of compressible Navier-Stokes equation we rather solve another equation than what was given (1.14a) so that we have not got a "source" term $-\varrho \mathrm{div}_x \vec{u}$ in the transport equation for ϱ. This is changed to $\beta(\varrho)\mathrm{div}_x \vec{u}$ instead with β a nonlinear, but $BC(\mathbb{R}_0^+)$ — a bounded and continuous function. The renormalized equation

$$\partial_t b(\varrho) + \mathrm{div}_x \left(b(\varrho)\vec{u}\right) - \left(b(\varrho) - \varrho b'(\varrho)\right) \mathrm{div}_x \vec{u} = 0 \tag{1.29}$$

should be valid in the sense of distributions (test functions not vanishing on the boundary $\partial\Omega$, nor at the initial time level $\{t = 0\}$, only at the final time level $\{t = T\}$). This formulation allows both for a weak formulation of the initial condition for density, and for the validity of the original (1.14a) when we can take suitable cut-off functions as the functions b and pass to the limit. Conversely, in (Feireisl, Novotný , 2009) it is proved that we can get (1.29) from (1.14a) for $\varrho \in L^\infty_{\mathrm{loc}}\left((0,T) \times \Omega\right)$ (and $\vec{u} \in L^1_{\mathrm{loc}}\left((0,T; W^{1,1}_{\mathrm{loc}}(\Omega; R^N)\right)\right)$. For the renormalized equation (1.29) we observe that the correct expression of the impermeability condition is now

$$\varrho \vec{u} \cdot \vec{n} = 0 \qquad \text{on } \partial\Omega \tag{1.30}$$

as $b'(\varrho)$ is not determined there.

Remark: The boundary condition (1.30) seems to be in a good agreement with the continuity of density ϱ across the boundary $\partial\Omega$ in a contact with vacuum established for the compressible Euler system (see Serre , 2015, page 3).

The next ingredient in the proof of existence is an improvement of the pressure estimates. This enabled Lions to prove existence for $\gamma \geq \frac{9}{5}$. The tool that made it possible in (Feireisl , 2001) to drop this condition is the oscillation defect measure. Finally, to get a strong convergence of ϱ_n to ϱ in $L^1\left((0,T) \times \Omega\right)$ we take as the renormalizing function b cut-off functions of $\varrho \log \varrho$ in (1.29) and assume strong convergence of the approximative initial conditions for density $\varrho_{0,n} \to \varrho_0$ in $L^q(\Omega)$ with $q > 1$. Formally, in the weak sense, we may use weak continuity of density and write identities for both approximative renormalized equation of continuity and for the renormalized equation for the limit:

$$\int_\Omega \left(\varrho_n \log \varrho_n\right)(t,x)\,\mathrm{d}x - \int_\Omega \left(\varrho_{0,n} \log \varrho_{0,n}\right)(x)\,\mathrm{d}x = -\int_0^t \int_\Omega \varrho_n \mathrm{div}_x \vec{u}_n \,\mathrm{d}x\,\mathrm{d}t \tag{1.31a}$$

$$\int_\Omega \left(\varrho \log \varrho\right)(t,x)\,\mathrm{d}x - \int_\Omega \left(\varrho_0 \log \varrho_0\right)(x)\,\mathrm{d}x = -\int_0^t \int_\Omega \varrho \mathrm{div}_x \vec{u}\,\mathrm{d}x\,\mathrm{d}t \tag{1.31b}$$

where $t \in \mathbb{R}^+$. Passing to the limit in (1.31a) and then substracting (1.31b), we obtain another expression for the rhs of (1.28)

$$\left(\frac{4}{3}\mu + \eta\right) \int_0^t \int_\Omega \left(\overline{\varrho \mathrm{div}_x \vec{u}} - \varrho \mathrm{div}_x \vec{u}\right) \,\mathrm{d}x \,\mathrm{d}t = \tag{1.32}$$

$$\left(\frac{4}{3}\mu + \eta\right) \int_\Omega \left(\varrho \log \varrho - \overline{\varrho \log \varrho}\right)(t, \cdot) \,\mathrm{d}x.$$

Now, as $\overline{p(\varrho)\varrho} \leq \overline{p(\varrho)}\varrho$, $\frac{4}{3}\mu + \eta > 0$ and $\varrho \log \varrho \leq \overline{\varrho \log \varrho}$ by weak lower semicontinuity of convex functions, we have got that the lhs of (1.32) is non-negative, while its rhs is non-positive, therefore we obtain identities

$$\overline{\varrho \log \varrho} = \varrho \log \varrho \tag{1.33a}$$

$$\overline{\varrho \,\mathrm{div}_x \vec{u}} = \varrho \,\mathrm{div}_x \vec{u} \tag{1.33b}$$

Thus, $\varrho_n \to \varrho$ in $L^1((0,T) \times \Omega)$ and interpolating with (1.21a) in the bounded Ω case we get

$$\varrho_n \to \varrho \quad \text{in } L^p\left((0,T) \times \Omega\right) \tag{1.34}$$

for $p \in [1, \gamma)$.

1.5 Compressible Navier-Stokes-Fourier equations

In the previous Sections 1.3 and 1.4 we have not followed what happens to the internal energy of the fluid if it is heated by internal friction due to nonvanishing viscosities. Naturally, we may say that the fluid is in a perfect equilibrium, e. g. in a contact with a thermostat. However, there are practical technical problems like a simple pipe (Hagen-Poisseuille) flow, where we can observe an increase of temperature $0.5°C/m$ which is not negligible. From the point of view of a mathematician, there are interesting problems of systems that are coupled. That means here that the change in temperature affects the flow. We may think of several mechanisms, how this happens, the temperature dependent viscosities is the most prominent one. Next, in meteorology, oceanography and astronomy we encounter so called Oberbeck-Bousinessq approximation wherein temperature influence density and through it buoyancy of the fluid. Also the heat (or entropy) transfer is affected by convection of the fluid, its viscous dissipation and pressure.

The term "Fourier" in the title refers to Fourier's law which states that the heat flux is proportional to the gradient of temperature

$$\vec{q} = -\kappa \nabla_x \vartheta. \tag{1.35}$$

This is by far the most used law in the field of heat transfer, although its rigorous derivation still poses a long-standing open problem for physicists in the field of statistical physics. However, the form of the law is the same as those of Darcy's and Fick's laws. Here ϑ is the absolute temperature, measured in Kelvins, $\vec{q}$ is the vector field of heat flux and κ is the non-negative coefficient of thermal conductivity which may depend on density, temperature and the temperature gradient, leading to a nonlinear version of Fourier's law. This option is not followed here. The thermal conductivity coefficient is allowed to depend on the magnetic field as well, which is termed sometimes as (Maggi)-Righi-Leduc effect in the case of electrons, sometimes as Senftleben-Beenakker effect in the case of neutral polyatomic paramagnetic or diamagnetic gases.

When we add the balance of energy to our balance of mass and balance of linear momentum, we obtain a complete system of conservation laws, as we assume the balance of linear momenta is automatically satisfied, because we choose a symmetric Cauchy stress tensor $\mathbb{T} = -p\mathbb{I} + \mathbb{S}$. The complete system in an inertial coordinate system reads

$$\partial_t \varrho + \mathrm{div}_x \varrho \vec{u} = 0, \tag{1.36a}$$

$$\partial_t (\varrho \vec{u}) + \mathrm{div}_x (\varrho \vec{u} \otimes \vec{u}) + \nabla_x p - \mathrm{div}_x \mathbb{S} = \varrho \vec{f}, \tag{1.36b}$$

$$\partial_t \left(\frac{1}{2} \varrho |\vec{u}|^2 \right) + \mathrm{div}_x \left(\frac{1}{2} \varrho |\vec{u}|^2 \vec{u} \right) + \mathrm{div}_x (p\vec{u}) - p \, \mathrm{div}_x \vec{u} = \mathrm{div}_x (\mathbb{S}\vec{u}) - \tag{1.36c}$$
$$\mathbb{S} : \nabla_x \vec{u} + \varrho \vec{f} \cdot \vec{u}.$$

The last equation (1.36c) arises when we formally multiply the momentum balance (1.36b) by $\vec{u}$ and it expresses the time change of the internal kinetic energy of the fluid. We now observe that when we integrate (1.36c) in Ω we get the rate of dissipation of the total kinetic energy $\frac{1}{2} \int_\Omega \varrho |\vec{u}|^2 \, \mathrm{d}x$ given by $- \int_\Omega \mathbb{T} : \mathbb{D} \, \mathrm{d}x \, \mathrm{d}t$ (viscous dissipation) and $\int_\Omega \vec{f} \cdot \vec{u} \, \mathrm{d}x \, \mathrm{d}t$ (power of (external) forces) as soon as the boundary conditions do not allow any flux of energy over the boundary $\partial\Omega$. The First Law of Thermodynamics claims that the total energy of such a closed system must be constant, therefore we introduce density of internal energy e such that the total energy (in our case sum of the internal and kinetic ones) is preserved. The equation for e has also got a form of a conservation law

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \vec{u}) + \mathrm{div}_x \vec{q} + p \, \mathrm{div}_x \vec{u} = r + \mathbb{S} : \nabla_x \vec{u}. \tag{1.37}$$

Here we have used our knowledge of thermodynamics that internal energy is linked to temperature and introduced into the equation both heat flux $\vec{q}$ and the (rate of the) heat sources r. Let us assume that the following assumption

$$r = 0 \tag{1.38}$$

and the following boundary conditions

$$\vec{u} = \vec{0}, \tag{1.39a}$$

$$\vec{q} \cdot \vec{n} = 0 \tag{1.39b}$$

are satisfied. If conditions (1.38), (1.39a) – (1.39b) are met then we say that our complete system (1.36a) – (1.36c) is mechanically and thermally isolated. For such a system we may form the balance of energy

$$\partial_t \left(\varrho e + \frac{1}{2} \varrho |\vec{u}|^2 \right) + \mathrm{div}_x \left(\varrho \left(e + \frac{1}{2} \varrho |\vec{u}|^2 \right) \vec{u} \right) + \mathrm{div}_x (\vec{q} - \mathbb{T}\vec{u}) = \varrho \vec{f} \cdot \vec{u} \tag{1.40}$$

and the total energy

$$\int_\Omega \left[\varrho\left(\frac{1}{2}|\vec{u}|^2+e\right)\right]\,\mathrm{d}x(t)=\int_0^t\int_\Omega \varrho\vec{f}\cdot\vec{u}\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega \left[\varrho\left(\frac{1}{2}|\vec{u}|^2+e\right)\right]\,\mathrm{d}x(0) \tag{1.41}$$

$$\text{a. e.}\quad t\in[0,T]$$

or in the case of a potential force field $\vec{f}=\nabla_x\Psi(x)$ in the form of sum of kinetic, potential and internal energy

$$\int_\Omega \left[\varrho\left(\frac{1}{2}|\vec{u}|^2-\Psi+e\right)\right]\,\mathrm{d}x(t)=\int_\Omega \left[\varrho\left(\frac{1}{2}|\vec{u}|^2-\Psi+e\right)\right]\,\mathrm{d}x(0) \tag{1.42}$$

$$\text{a. e.}\quad t\in[0,T],$$

where all the terms are actually non-negative for the case of a gravitational potential.

To be in agreement with thermodynamics to formulate correctly the complete system we have to take into account the Second Law of Thermodynamics. It states that the entropy in all physically admissible processes in a closed system does not decrease. We reformulate it into an inequality of the Clausius-Duhem type where the entropy production rate is non-negative. We will need a certain form of an equation of state (EOS) as in Section 1.4

$$p=p(\varrho,\vartheta). \tag{1.43}$$

The book (Feireisl, Novotný , 2009) treats forms of EOS that are near the EOS of the ideal gas. The symbol $p=p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e=e(\varrho,\vartheta)$ is the specific internal energy, interrelated through Maxwell's relation

$$\frac{\partial e}{\partial \varrho}=\frac{1}{\varrho^2}\left(p(\varrho,\vartheta)-\vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{1.44}$$

The entropy density function $s(\varrho,\vartheta)$ has to satisfy the Gibbs' relation

$$Ds=\vartheta^{-1}\left(De+pD\varrho^{-1}\right). \tag{1.45}$$

Dividing (1.37) by temperature ϑ leads to the entropy equality

$$\partial_t\left(\varrho s\right)+\mathrm{div}_x\left(\varrho s\vec{u}\right)+\mathrm{div}_x\frac{\vec{q}}{\vartheta}=\varsigma, \tag{1.46}$$

where the entropy production rate

$$\varsigma=\vartheta^{-1}\left(\mathbb{S}:\nabla_x\vec{u}-\vartheta^{-1}\vec{q}\cdot\nabla_x\vartheta\right)\geq 0, \tag{1.47}$$

from the choice of constitutive laws, namely Newton's rheological law (see an introduction in Section 1.4 with conditions on viscosities which are kept here even though these may not be constants anymore that make $\mathbb{S}:\nabla_x\vec{u}$ a non-negative quadratic form in terms of first space partial derivatives of $\vec{u}$) and Fourier's law (1.35)). Note that we may choose other constitutive laws and if we can enforce the condition $\varsigma\geq 0$ or even a stronger version of the entropy inequality, i. e. that the entropy production rate is maximal.

So far we have fulfilled the Zeroth, First and Second Laws of Thermodynamics. The Third Law is connected with the EOS and is violated in the case of ideal gas. This cannot be satisfied without quantum considerations. The system (1.36a), (1.36b), (1.37) with (1.38), (1.46) of balance laws of mass, momentum, energy without heat sources, entropy is certainly overdetermined. From the point of view of mathematical analysis we have got a problem with a viscous dissipation term $\mathbb{S} : \nabla_x \vec{u}$ because from the a-priori estimate (1.21c) it is in $L^1((0,T)\times\Omega)$ only. Therefore the balance of internal energy (1.37) we treat as a constraint and modify the entropy equality (1.46) to the inequality (1.48)

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\vec{u}) + \mathrm{div}_x\frac{\vec{q}}{\vartheta} \geq \varsigma, \tag{1.48}$$

if the entropy production rate is still given by (1.47). The explanation is that we cannot prove that the equality is satisfied unless the weak solution is smooth enough. If there are weak solutions such that entropy inequality (1.48) holds instead of entropy equality (1.46) it remains one of the major open problems of mathematical fluid dynamics. The weak formulation of (1.48) is standard (only take non-negative test function to preserve the inequality sign), once we assume there is an initial condition for entropy in the form

$$\varrho s(0,.) = (\varrho s)_0 \quad \text{in } \Omega. \tag{1.49}$$

It remains to specify an EOS and transport coefficients μ, η and κ. From the rational thermodynamics we can infer several requirements on the constitutive relations, but their particular form is not specified, because these are characteristics of a given material. We also sometimes do not know how the relations should look like in a particular situation, as relevant experimental data are missing or are insufficient, because mathematical theory typically needs bounds in extreme cases like very low or very high temperatures. Sometimes experiments are completely excluded like in many astrophysical situations. Finally, there is also a discrepancy between mathematics and physics. Available mathematical theory typically falls behind what is observed and needed in the physical practice. The theory of turbulence can serve as an example.

We require that our fluid is thermodynamical stable. This means that the following two inequalities hold for any state variables, a couple (ϱ, ϑ),

$$\partial_\varrho p > 0 \tag{1.50a}$$

$$\partial_\vartheta e > 0 \tag{1.50b}$$

Since $-\varrho\partial_p\frac{1}{\varrho} = \varrho^{-1}\left(\partial_\varrho p\right)^{-1} > 0$ is the (isothermal) compressibility of the fluid, the condition (1.50a) means that the fluid is always compressible. Moreover if we keep the entropy constant when taking the derivative of pressure with respect to density, we get square of the speed of sound in the fluid. Therefore (1.50a) means that the speed of sound in the fluid is always positive. The second condition (1.50b) means that the specific heat capacity (at constant volume) is always positive.

The mathematical theory of complete systems is essential divided into works dealing with local in time solutions or solutions not deviating too much from a stationary equilibrium, and works establishing the existence of global in time weak solutions. Let us focus on the latter. There we can discern three different approaches, cf. (Novotný , 2012; and references therein). The first one is due to Feireisl who split the pressure into its density-dependent part and a part depending on temperature in the linear way

$$p(\varrho, \vartheta) = p_c(\varrho) + \vartheta p_\vartheta(\varrho), \tag{1.51}$$

where both parts satisfy asymptotically power laws, p_ϑ with a sufficiently smaller growth with respect to the growth of p_c. This setting has to be accompanied with $\mu =$ const., $\eta =$ const., and $\kappa(\vartheta)$ having a suitable polynomial growth.

The second approach is due to Bresch and Desjardins, so called cold pressure. They use the same splitting (1.51), but this time $p_\vartheta(\varrho) = \varrho$ and $p_c(\varrho)$ has asymptotically power growth for large densities, but is singular for low densities. They are able to prove an estimate of ϱ in a Sobolev space if the viscosities are density dependent and linked by

$$2\left(\varrho + \frac{1}{3}\right)\mu' - \eta' = 2\mu. \tag{1.52}$$

The third approach, we use and focus on, is due to Feireisl and Novotný (Feireisl, Novotný , 2009). In this approach these authors generalized the expression for pressure of the ideal gas and augmented it with the radiative pressure due to the Stefan-Boltzmann law which helped them to get suitable a-priori bounds for temperature ϑ.

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \tag{1.53}$$

where $P : [0, \infty) \to [0, \infty)$ is a given function with the following properties:

$$P \in C^1([0, \infty)), \; P(0) = 0, \; P'(Z) > 0 \text{ for all } Z \geq 0, \tag{1.54}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{1.55}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{1.56}$$

and $a > 0$ is the Stefan-Boltzmann constant. According to Maxwell's relation (1.44), the specific internal energy e is

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{1.57}$$

and the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{1.58}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

The condition $(1.54)_3$ expresses a stronger version of the thermodynamical stability (1.50a), that is the compressibility at zero density (existing in the form of liminf at least) is still positive for any given temperature ϑ. We can also observe that there is a "degenerated region" of low temperatures ϑ and/or large densities ϱ, thus of large Z, where the inequality in (1.56) remains strict, that means our gas still possesses a certain internal energy density e even in the case of the hypothetical zero absolute temperature $\vartheta = 0$ and the expression for $M'(Z) \to 0-$ as $Z \to \infty$ thanks to (1.55). This means that our gas is assumed to have at least one component behaving like a Fermi-Dirac gas (in the case of plasmas this is usually an electron gas) except for other components which may follow the Bose-Einstein or Maxwell-Boltzmann statistics. In this case the Third Law of Thermodynamics is saved. Let us recall that in the case P is linear we have got the Boyle-Marriot law for ideal gases $p(\varrho, \vartheta) = \frac{R_m}{M_m} \varrho\vartheta$ with R_m the universal molar gas constant and M_m their (average) molecular weight does not satisfy (1.56). In this sense the current mathematical analysis cannot treat the isothermal ideal gas case and needs a Fermi contribution in the degenerate region.

The hypotheses for the transport coefficients are the following

$$c_1(1+\vartheta^\alpha) \le \mu(\vartheta) \le c_1^{-1}(1+\vartheta^\alpha),\ \mu'(\vartheta) < c_2,\ 0 \le \eta(\vartheta) \le c_3(1+\vartheta^\alpha), \tag{1.59}$$

$$c_1(1+\vartheta^3) \le \kappa(\vartheta) \le c_2(1+\vartheta^3) \tag{1.60}$$

with positive constants c_1, c_2, c_3 and an exponent $\alpha \in (\frac{2}{5}, 1]$. The third power in (1.60) corresponds to a contribution from the radiative heat transfer which is essential in astrophysical models. Actually, Feireisl, Novotný (2009) mentions that the coefficient in this contribution is proportional to the photon mean field path and the speed of light. Regarding a physical plausability of the range of the power α we may cite Sutherland's law which leads asymptotically to $\alpha = \frac{1}{2}$. Sutherland derived it from preliminary version of later Born-Green kinetic theory due to Maxwell (Sutherland , 1893). Mathur and Todos (Mathur, Thodos , 1963) have surveyed data from measurement of viscosities of certain gases up to 10 000 Kelvins. From their graph we may infer approximately the following values of α. Hydrogen — 0.86, helium — 0.88, nitrogen — 0.82, oxygen — 0.61, neon — 0.70. Other gases reviewed there are argon, krypton and xenon. Around 20 years later these values were corrected by Vargaftik and Vassilevskaya (Vargaftik, Vasilevskaya , 1984). Their values are: helium — 0.71, neon — 0.63, argon — 0.68, krypton — 0.69, xenon — 0.69. All are covered by the theory of Feireisl and Novotný.

To get a-priori estimates we introduce the shifted Helmholtz function, similar to (1.17), now associated to a given constant non-negative density $\overline{\varrho}$ and a positive temperature $\overline{\vartheta}$

$$\mathcal{H}(\varrho, \vartheta) := H(\varrho, \vartheta) - (\varrho - \overline{\varrho}) H_\varrho(\overline{\varrho}, \overline{\vartheta}) - H(\overline{\varrho}, \overline{\vartheta}). \tag{1.61}$$

Here the "Helmholtz free energy" is $H(\varrho, \vartheta) := \varrho e(\varrho, \vartheta) - \overline{\vartheta}\varrho s(\varrho, \vartheta)$. The shifted Helmholtz function has a number of nice properties ((see for proofs Feireisl, Novotný , 2009)), especially

1. $\mathcal{H}(\varrho, \vartheta)$ is non-negative
2. $\mathcal{H}(\varrho, \vartheta)$ attains its strict minimum at $(\overline{\varrho}, \overline{\vartheta})$
3. $\mathcal{H}(\varrho, \cdot)$ is decreasing on $(0, \overline{\vartheta})$ and increasing on $(\overline{\vartheta}, \infty)$
4. $\mathcal{H}(\cdot, \vartheta)$ is locally uniformly convex on $\mathbb{R}^+$

Let us choose for Ω bounded $\overline{\varrho}$ as the mean density $\overline{\varrho} := |\Omega|^{-1} \int_\Omega \varrho \, \mathrm{d}x$ which is constant in time due to (1.36a). The energy inequality written with the upper defined function is

$$\int_\Omega \left(\frac{1}{2}\, \varrho |\vec{u}|^2 + \mathcal{H}(\varrho, \vartheta) \right) \, \mathrm{d}x \, (t_2) + \ \overline{\vartheta} \varsigma \left[[t_1, t_2] \times \overline{\Omega} \right] \leq$$

$$\int_\Omega \left(\frac{1}{2}\, \varrho |\vec{u}|^2 + \mathcal{H}(\varrho, \vartheta) \right) \, \mathrm{d}x \, (t_1) \qquad \text{for a. e. } t_1 < t_2 \in [0, T]. \tag{1.62}$$

On the left hand side of (1.62) ς is a non-negative Radon measure and the symbol $\varsigma \left[[t_1, t_2] \times \overline{\Omega} \right]$ denotes the value of this measure over the set in the brackets.

Definition 1.5.1 *We say that $(\varrho, \vec{u}, \vartheta)$ is a weak solution of problem (1.36a) – (1.36b), (1.40), (1.39a), (1.39b) with (1.35), (1.60), (1.44) and $\mathbb{S}$ defined as in Subsection 1.4 with viscosities μ, η dependent on temperature ϑ subject to (1.59) iff*

$$\begin{gathered} \varrho \geq 0, \ \ \vartheta > 0 \ \textit{for a.a.}\ (t,x) \times \Omega, \\ \varrho \in L^\infty(0, T; L^{5/3}(\Omega)), \ \ \vartheta \in L^\infty(0, T; L^4(\Omega)), \\ \vec{u} \in L^2(0, T; W_0^{1, \frac{8}{5-\alpha}}(\Omega; \mathbb{R}^3)), \\ \vartheta \in L^2(0, T; W^{1,2}(\Omega)), \end{gathered}$$

and if ϱ, $\vec{u}$, ϑ satisfy the integral identities (1.63), (1.64), (1.65), (1.66):

$$\int_0^T \int_\Omega \left[\Big(\varrho + b(\varrho) \Big) \partial_t \varphi + \Big(\varrho + b(\varrho) \Big) \vec{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho) \varrho \Big) \mathrm{div}_x \vec{u} \, \varphi \right] \, \mathrm{d}x \ \mathrm{d}t \tag{1.63}$$

$$= - \int_\Omega \Big(\varrho_0 + b(\varrho_0) \Big) \varphi(0, \cdot) \ \mathrm{d}x$$

to be satisfied for any $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})$, and any $b \in C^\infty\left([0,\infty)\right)$, $b' \in C_c^\infty\left([0,\infty)\right)$,

$$\int_0^T \int_\Omega \left(\varrho \vec{u} \cdot \partial_t \vec{\varphi} + \varrho \vec{u} \otimes \vec{u} : \nabla_x \vec{\varphi} + p \, \mathrm{div}_x \vec{\varphi} \right) \ \mathrm{d}x \ \mathrm{d}t = \tag{1.64}$$

$$\int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x \vec{\varphi} - \varrho \vec{f} \cdot \vec{\varphi} \Big) \ \mathrm{d}x \ \mathrm{d}t - \int_\Omega (\varrho \vec{u})_0 \cdot \vec{\varphi}(0, \cdot) \ \mathrm{d}x$$

for any $\vec{\varphi} \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$,

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \right) \ \mathrm{d}x \, \mathrm{d}t \tag{1.65}$$

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \ \mathrm{d}x + \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} - \mathbb{S} : \nabla_x \vec{u} \right) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$,

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) \right) (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho \vec{f} \cdot \vec{u} \, \mathrm{d}x \, \mathrm{d}t = \tag{1.66}$$

$$\int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 \right) \ \mathrm{d}x \qquad \text{for a. a. } \tau \in (0, T).$$

With all this we can formulate the existence theorem due to Feireisl and Novotný (Feireisl, Novotný , 2009).

Theorem 1.5.1. *Let* $\Omega \in \mathbb{R}^3$ *be a bounded domain with* $\partial\Omega \in C^{2,\nu}$ *for* $\nu > 0$. *Let the hypotheses* (1.53)–(1.60) *hold. Let the initial data satisfy*

$$\varrho_0 \in L^{\frac{5}{3}}(\Omega), \tag{1.67}$$

$$\frac{|\vec{m}_0|^2}{\rho_0} \in L^1(\Omega), \tag{1.68}$$

$$\vartheta_0 \in L^\infty(\Omega) \quad ess \inf_{\Omega} \vartheta_0 > 0, \tag{1.69}$$

where $\vec{m}_0$ *is the initial value for the momentum. Then the IBVP for Navier-Stokes-Fourier system in the case of mechanical and thermal isolation* (1.36a), (1.36b), (1.48), (1.39a), (1.39b) *with initial conditions* $(\varrho_0, \vec{m}_0, \vartheta_0)$ *admits a weak solution* $(\varrho, \vec{u}, \vartheta)$ *in the sense of the Definition 1.5.1 satisfying*

$$\varrho \in C_w(0, T; L^{\frac{5}{3}}(\Omega)). \tag{1.70}$$

2. Global existence of a weak solution for a model in radiation magnetohydrodynamics

2.1 Introduction

There are a number of situations when stars can be described by compressible fluids and their dynamics is controlled by intense magnetic fields coupled with a simplified model of radiation. Following studies by (Ducomet, Feireisl, Nečasová , 2011) and (Ducomet, Feireisl , 2006) we consider a mathematical model of radiative flow where the motion of the fluid is described by the standard Galilean fluid mechanics giving an evolution of *the mass density* $\varrho = \varrho(t,x)$, the *velocity field* $\vec{u} = \vec{u}(t,x)$, and the *absolute temperature* $\vartheta = \vartheta(t,x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation is incorporated in the *radiative intensity* $I = I(t,x,\vec{\omega},\nu)$, depending on the director $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. This system of equations is coupled to a simplified Maxwell system of electrodynamics where we assume the quasineutrality of the plasma described and neglect the Maxwell displacement current. This system describes an evolution of *the magnetic induction* $\vec{B} = \vec{B}(t,x)$ and *the electric field* $\vec{E} = \vec{E}(t,x)$, resp. *the magnetic field* $\vec{H} = \vec{H}(t,x)$ and *the electric induction* $\vec{D} = \vec{D}(t,x)$. The collective effect of the radiation is then expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I: the radiation energy E_R is given as

$$E_R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu. \tag{2.1}$$

The time evolution of I is described by a transport equation with source terms $\tilde{S}$ depending on nonnegative quantities of the absolute temperature ϑ and frequency of radiation ν, while the effect of radiation on the macroscopic motion of the fluid is represented by an extra source term of radiative heating/cooling in the energy equation and an extra source term of acceleration/deceleration both evaluated in terms of $\tilde{S}$.

The Maxwell system of classical electrodynamics in our case reduces to *Faraday's law of induction*

$$\partial_t\vec{B} + \mathrm{curl}_x\vec{E} = 0, \tag{2.2}$$

together with *Gauss's law for magnetism*

$$\mathrm{div}_x\vec{B} = 0, \tag{2.3}$$

Ampère's law

$$\vec{J} = \mathrm{curl}_x\vec{H}, \tag{2.4}$$

Coulomb's law

$$\mathrm{div}_x\vec{D} = 0, \tag{2.5}$$

and (a nonlinear version of) *Ohm's law*

$$\vec{J} = \sigma(\vec{E} - \vec{B} \times \vec{u}), \tag{2.6}$$

where $\vec{B} = \zeta\vec{H}$, $\vec{D} = \tilde{\varepsilon}\vec{E}$, σ is the (nonlinear) electrical conductivity, $\zeta = \zeta\left(\left|\vec{H}\right|\right)$ and $\tilde{\varepsilon}$ is the dielectric permittivity. All the material properties are assumed to be scalars.

This gives us from (2.2)

$$\partial_t\vec{B} + \mathrm{curl}_x(\vec{B} \times \vec{u}) + \mathrm{curl}_x(\frac{1}{\sigma}\mathrm{curl}_x(\frac{1}{\zeta}\vec{B})) = 0. \tag{2.7}$$

Following (Blanc, Ducomet , 2015) we will denote

$$\mathcal{M}(s) = \int_0^s \tau\partial_\tau(\tau\zeta(\tau))d\tau, \tag{2.8}$$

and rewrite the equation (2.2) as a version of the Poynting theorem

$$\partial_t\mathcal{M}(|\vec{H}|) + \vec{J} \cdot \vec{E} = \mathrm{div}_x(\vec{H} \times \vec{E}). \tag{2.9}$$

Together with the principles of continuum mechanics, the magnetofluid (Cabannes , 1970, Kulikovskiy, Lyubimov , 1965) with radiation effects (Pomraning , 2005) problem can be described by the system of equations

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \text{ in } (0,T) \times \Omega; \tag{2.10}$$

$$\begin{gathered}\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u} \otimes \vec{u}) + \nabla_x p(\varrho,\vartheta) = \\ \mathrm{div}_x\mathbb{S} - \vec{S}_F + \varrho\nabla_x\Psi + \zeta\vec{J} \times \vec{H} \text{ in } (0,T) \times \Omega;\end{gathered} \tag{2.11}$$

$$\begin{gathered}\partial_t\left[\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right) + \mathcal{M}(|\vec{H}|)\right] + \mathrm{div}_x\left[\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta) + \frac{p(\varrho,\vartheta)}{\varrho}\right)\vec{u}\right] = \\ \varrho\nabla_x\Psi \cdot \vec{u} - \mathrm{div}_x\left(\vec{q} - \mathbb{S}\vec{u} + \frac{\zeta}{\tilde{\varepsilon}}\vec{D} \times \vec{H}\right) - S_E \text{ in } (0,T) \times \Omega;\end{gathered} \tag{2.12}$$

$$\partial_t I + c\vec{\omega} \cdot \nabla_x I = c\tilde{S} \text{ in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2. \tag{2.13}$$

Note that, contrary to the model studied in (Ducomet, Nečasová , 2014), a radiation term appears in the momentum equation in spite of this term may be small. The electrical conductivity σ hidden in (2.11) can depend on the density ϱ, on the temperature ϑ and the magnetic field $\vec{H}$ (cf. (2.34)) of the magnetofluid.

The symbol $p = p(\varrho,\vartheta)$ denotes the (equilibrium) thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through *Maxwell's relation*

$$\frac{\partial e}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{2.14}$$

Observe that both the pressure and the internal energy involve both a radiation and a thermal term. The meaning of this splitting is that there is a part of the photon gas in the equilibrium with plasma whereas another part is not. The latter is described by the transport equation (2.13) and is caused mainly by inverse Compton scattering and synchrotron radiation (Kolb , 2008). Naturally, our description is somehow "mixed" since we use classical thermodynamics and classical electrodynamics for the description of matter while the radiation is described by geometrical optics.

Furthermore, $\mathbb{S}$ is the viscous part of the stress tensor determined by *Newton's rheological law*

$$\mathbb{S} = \mu \left(\vartheta, \left|\vec{H}\right|\right) \left(\nabla_x \vec{u} + \nabla_x^T \vec{u} - \frac{2}{3}\mathrm{div}_x \vec{u}\,\mathbb{I}\right) + \eta \left(\vartheta, \left|\vec{H}\right|\right) \mathrm{div}_x \vec{u}\ \mathbb{I}, \tag{2.15}$$

where the shear viscosity coefficient $\mu > 0$ and the bulk viscosity coefficient $\eta \geq 0$ are effective functions of the absolute temperature and the magnitude of the magnetic field. Once again, we tacitly assume isotropy of the considered medium (without the presence of a magnetic field). Similarly, $\vec{q}$ is the heat flux given by *Fourier's law*

$$\vec{q} = -\left(\kappa_R \vartheta^3 + \kappa_M \left(\varrho, \vartheta, \left|\vec{H}\right|\right)\right) \nabla_x \vartheta, \tag{2.16}$$

with the constant radiative heat conductivity coefficient $\kappa_R > 0$ and with a molecular heat conductivity coefficient $\kappa_M > 0$.

Further the source term of radiation is due to absorption/emission and scattering of light

$$\tilde{S} = S_{a,e} + S_s, \tag{2.17}$$

where

$$S_{a,e}(t, x, \vec{\omega}, \nu) = \sigma_a(\nu, \vartheta)\Big(\mathfrak{B}(\nu, \vartheta) - I(t, x, \vec{\omega}, \nu)\Big), \tag{2.18}$$

$$S_s(t, x, \vec{\omega}, \nu) = \sigma_s(\nu, \vartheta) \left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(t, x, \vec{\omega}, \nu)\ \mathrm{d}\vec{\omega} - I(t, x, \vec{\omega}, \nu)\right), \tag{2.19}$$

$$S_E(t, x) = \int_{\mathcal{S}^2}\int_0^\infty \tilde{S}(t, x, \vec{\omega}, \nu)\ \mathrm{d}\vec{\omega}\,\mathrm{d}\nu\ , \tag{2.20}$$

and

$$\vec{S}_F(t, x) = c^{-1}\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\tilde{S}(t, x, \vec{\omega}, \nu)\ \mathrm{d}\vec{\omega}\,\mathrm{d}\nu, \tag{2.21}$$

with the absorption coefficient $\sigma_a = \sigma_a(\nu, \vartheta) \geq 0$, and the scattering coefficient $\sigma_s = \sigma_s(\nu, \vartheta) \geq 0$. Here $\mathfrak{B}(\nu, \vartheta)$ denotes (equilibrial) black body radiation. According to Planck's law we recall

$$\mathfrak{B}(\nu, \vartheta) = \frac{2h\vartheta^3 c^{-2}}{e^{\frac{h\nu}{k_B\vartheta}} - 1}. \tag{2.22}$$

More restrictions on the structural properties of constitutive relations will be imposed in Section 2.2 below.

System (2.10) – (2.22) is supplemented with the boundary conditions modelling the mechanical and heat isolation combined with no-slip and transparency (radiation does not reflect back to the domain Ω) at the boundary:

$$\vec{u}|_{\partial\Omega} = \vec{0},\ \ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0; \tag{2.23}$$

$$I(t,x,\vec{\omega},\nu)=0 \text{ for } x\in\partial\Omega,\ \vec{\omega}\cdot\vec{n}\leq 0, \tag{2.24}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

For the electromagnetic fields we adopt boundary conditions of a perfect conductor (assuming outside Ω there are zero fields and using the continuity of the following components of the fields across $\partial\Omega$)

$$\vec{E}\times\vec{n}|_{\partial\Omega}=\vec{0},\ \vec{B}\cdot\vec{n}|_{\partial\Omega}=0. \tag{2.25}$$

It remains to complement the system with the Poisson equation for the self-gravitational potential Ψ from the right-hand side of (2.11)

$$-\Delta\Psi=4\pi G\varrho, \tag{2.26}$$

where G is Newton's gravitational constant.

System (2.10) – (2.26) can be viewed as a simplified model in radiation hydrodynamics, the physical foundations of which were described by (Pomraning , 2005) and (Mihalas, Weibel-Mihalas , 1984) in the framework of the theory of special relativity. Similar systems have been investigated more recently in astrophysics and laser applications (in the relativistic and inviscid case) by (Lowrie, Morel, Hittinger , 1999), (Buet, Després , 2004), with a special attention to asymptotic regimes, see also (Dubroca, Feugeas , 1999), (Lin , 2007) and (Lin, Coulombel, Goudon , 2006) for related numerical issues.

The *existence* of local-in-time solutions and sufficient conditions for blow up of classical solutions in the non-relativistic inviscid case were obtained by (Zhong, Jiang , 2007), see also the recent papers (Jiang, Wang , 2009; 2012) for related one-dimensional "Euler-Boltzmann" type models. Moreover, a simplified version of the system has been investigated by (Golse, Perthame , 1986), where the global existence was proved by means of the theory of nonlinear semigroups.

Concerning *viscous* fluids, a number of similar results have been considered in the recent past in the one-dimensional geometry (Amosov , 1985, Ducomet, Nečasová , 2010a;b; 2012; 2013) and a global existence result has also recently been proved in the 3D setting in (Ducomet, Feireisl, Nečasová , 2011) under some hypotheses on transport coefficients, for the "complete system" (when a radiative source appears only in the right-hand side of (2.11)).

Our goal in the present paper is to show that the existence theory developed in (Ducomet, Feireisl, Nečasová , 2011) and (Ducomet, Feireisl , 2006) relying on previous works (Feireisl , 2004), (Feireisl , 2001) and (Feireisl, Novotný , 2009; Chapter 3), can be adapted to the problem (2.10) – (2.26).

As stressed in (Ducomet, Feireisl, Nečasová , 2011), a complete proof of existence is now well understood ((see Feireisl, Novotný , 2009; Chapter 3)) therefore we focus as in (Ducomet, Feireisl, Nečasová , 2011) on the property of *weak sequential stability* for problem (2.10) – (2.26) in the framework of the weak solutions introduced in (Ducomet, Feireisl , 2006). More specifically, we introduce a concept of finite energy weak solution in the spirit of (Ducomet, Feireisl , 2006) and show that any sequence $\{\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon,\vec{H}_\varepsilon,I_\varepsilon\}_{\varepsilon>0}$ of solutions to problem (2.10) – (2.26), bounded in the natural energy norm, possesses a subsequence converging to (another) weak solution of the same problem. Such a property highlights the essential ingredients involved in the "complete" proof of existence that may be carried over by means of the arguments delineated in (Feireisl, Novotný , 2009; Chapter 3).

The essential contribution to the proof comes from the entropy inequality. Due to a relevant "radiative" contribution one faces a similar situation encountered in (Ducomet, Feireisl, Nečasová , 2011), namely that the total entropy production has not a "definite sign" and, accordingly, we can establish the strong convergence of the radiative contribution with the help of regularity of velocity averages. This is also connected to the fact that we do not introduce radiation entropy in the total entropy inequality.

The paper is organized as follows. In Section 2.2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (2.10) – (2.26), and state the main result. Uniform bounds imposed on weak solutions by the data are derived in Section 2.3.1. The property of *weak sequential stability* of a bounded sequence of weak solutions is established in Section 2.3.2. Finally, we introduce a suitable approximation scheme and discuss the main steps of the proof of existence in Section 2.3.3.

2.2 Hypotheses and main results

Hypotheses imposed on constitutive relations and transport coefficients are motivated by the general *existence theory* for the Navier-Stokes-Fourier system developed in (Feireisl, Novotný , 2009; Chapter 3) and reasonable physical assumptions (Pomraning , 2005).

Firstly, we consider the pressure in the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{2.27}$$

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties:

$$P \in C^1([0,\infty)), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.28}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.29}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.30}$$

The component $\frac{a}{3}\vartheta^4$ represents the effect of the "equilibrium" radiation pressure ((see Ducomet, Feireisl, Nečasová , 2011) for motivations and (Feireisl, Novotný , 2009) for details). Essentially, these hypotheses are implications of general principles of thermodynamical stability and the assumption that there is at least one component in the plasmatic mixture behaving in the degenerate regime as a Fermi gas (we may think of it in most cases as of an electron gas). The constant a is the Stefan-Boltzmann constant.

According to Maxwell's relation (2.14) and statistical kinetic theory, the internal energy density e is

$$e(\varrho,\vartheta) = \frac{3}{2}\varrho^{-1}\vartheta^{5/2} P\left(\varrho\vartheta^{-3/2}\right) + a\vartheta^4\varrho^{-1}, \tag{2.31}$$

and the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\varrho\vartheta^{-3/2}\right) + \frac{4a}{3}\vartheta^3\varrho^{-1}, \tag{2.32}$$

with a function M satisfying by (2.29)

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

Additional entropy for the photon gas out of equilibrium is not introduced.

The transport coefficients μ, η, and κ_M are continuously differentiable functions of their respective variables admitting a common temperature scaling such that there exist $c_1, c_2, c_3, c > 0$

$$\mu'_\vartheta\left(\vartheta, \left|\vec{H}\right|\right) < c_3, \tag{2.33}$$

$$c_1(1+\vartheta) \le \eta\left(\vartheta, \left|\vec{H}\right|\right), \sigma^{-1}(\varrho, \vartheta, \vec{B}), \mu\left(\vartheta, \left|\vec{H}\right|\right) \le c_2(1+\vartheta), \tag{2.34}$$

$$\kappa_M\left(\varrho, \vartheta, \left|\vec{H}\right|\right) \le c(1+\vartheta^3), \tag{2.35}$$

for any $\vartheta \ge 0$. We consider the magnetic permeability ζ satisfying the following property

$$\underline{c_k} s(1+s)^{-k} \le \partial_s^k(s\zeta(s)) \le \overline{c}_k s(1+s)^{-k}, \tag{2.36}$$

for any $s \ge 0$ and for $k = 0, 1$ with $\underline{c}_k, \overline{c}_k > 0$. Moreover, we assume that σ_a, σ_s are continuous functions of ν, ϑ such that there exist $c_4, c_5, c_6 > 0$ and $h \in L^1(0, \infty)$ and it holds

$$0 \le \sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \le c_4, \; 0 \le \sigma_a(\nu, \vartheta)\mathfrak{B}(\nu, \vartheta) \le c_5, \tag{2.37}$$

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta), \sigma_a(\nu, \vartheta)\mathfrak{B}(\nu, \vartheta) \le h(\nu), \tag{2.38}$$

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \le c_6\vartheta, \tag{2.39}$$

for all $\nu \ge 0$, $\vartheta \ge 0$. Relations (2.37) – (2.39) represent "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν and small temperatures ϑ.

The relation (2.39) is similar to coefficients which were derived by (Ripoll, Dubroca, Duffa , 2001). The coefficients are deduced by averanging a specific radiative intensity over the space of frequencies and are generalized versions of the Planck mean (Ripoll, Dubroca, Duffa , 2001). The assumption (2.38) was used in the work of Golse et al. (see Golse, Perthame , 1986; an assumption (H4)) which gives us the boundedness with respect ν. It is not any contradiction to (2.39).

We just recall the definitions introduced in (Ducomet, Feireisl, Nečasová , 2011). In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (2.10) is replaced by its *renormalized* version introduced in (DiPerna, Lions , 1989) represented by the family of integral identities

$$\int_0^T \int_\Omega \left[\Big(\varrho + b(\varrho)\Big)\partial_t\varphi + \Big(\varrho + b(\varrho)\Big)\vec{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\,\varphi\right] \,\mathrm{d}x\,\mathrm{d}t = \tag{2.40}$$

$$-\int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot)\ \mathrm{d}x,$$

to be satisfied for any $\varphi \in C_c^\infty([0,\infty)\times\overline{\Omega})$, and any $b \in C^\infty([0,\infty))$, $b' \in C_c^\infty([0,\infty))$, where (2.40) implicitly includes the initial condition

$$\varrho(0,\cdot) = \varrho_0.$$

Similarly, the momentum equation (2.11) is replaced by its weak version

$$\int_0^T \int_\Omega \left(\varrho\vec{u}\cdot\partial_t\vec{\varphi} + \varrho\vec{u}\otimes\vec{u} : \nabla_x\vec{\varphi} + p\,\mathrm{div}_x\vec{\varphi}\right)\ \mathrm{d}x\ \mathrm{d}t \tag{2.41}$$

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x\vec{\varphi} - \varrho\nabla_x\Psi\cdot\vec{\varphi} + \vec{S}_F\cdot\vec{\varphi} - \zeta(\vec{J}\times\vec{H})\cdot\vec{\varphi}\ \mathrm{d}x\ \mathrm{d}t$$

$$-\int_\Omega (\varrho\vec{u})_0\cdot\vec{\varphi}(0,\cdot)\ \mathrm{d}x,$$

for any $\vec{\varphi} \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. For (2.41) to make sense, especially the term $\int_0^T\int_\Omega \mathbb{S} : \nabla_x\varphi\ \mathrm{d}x\ \mathrm{d}t$, the field $\vec{u}$ must belong to a certain Bochner space with a Sobolev space with respect to the spatial variable and we require that

$$\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.42}$$

where (2.42) already includes the no-slip boundary conditions $(2.23)_1$. Gravitational potential Ψ is given by (2.26) considered on the whole space $\mathbb{R}^3$, where ϱ was extended to be zero outside Ω.

As the term $\mathbb{S}\vec{u}$ in the total energy balance (2.12) is not controlled on the (hypothetical) vacuum zones of vanishing density, we replace (2.12) by the internal energy equation as in (Feireisl, Novotný , 2009)

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\vec{u}) + \mathrm{div}_x\vec{q} = \mathbb{S} : \nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} - S_E + \vec{u}\cdot\vec{S}_F + \frac{1}{\sigma}\left|\mathrm{curl}_x\vec{H}\right|^2. \tag{2.43}$$

Furthermore, dividing (2.43) by ϑ and using Maxwell's relation (2.14), we may rewrite (2.43) as the entropy equation

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s\vec{u}) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = r, \tag{2.44}$$

where the entropy production rate r is

$$r = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{1}{\sigma}\left|\mathrm{curl}_x\vec{H}\right|^2\right) + \frac{\vec{u}\cdot\vec{S}_F - S_E}{\vartheta}, \tag{2.45}$$

where the first term $r_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{1}{\sigma}\left|\mathrm{curl}_x\vec{H}\right|^2\right)$ is the (non-negative) matter entropy production by virtue of the constitutive laws (2.15) and (2.16). The second term on the right-hand side of (2.45) is due to the radiative entropy rate which has not got a definite sign since it corresponds to the radiative heating/cooling.

For the smooth fields we can get an evolution equation for the sum of the density of the kinetic energy $\frac{1}{2}\varrho\left|\vec{u}\right|^2$ and the magnetic energy $\mathcal{M}(\vec{H})$ subtracting

(2.43) from (2.12). However, generally for weak solutions we cannot exclude that the entropy dissipation rate due to a heat exchange, internal viscous friction and Foucault eddy currents is larger than r in compliance with the Second law of Thermodynamics and equation (2.44) has to be replaced in the weak formulation by the inequality

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.46}$$

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0,\cdot) \, \mathrm{d}x + \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} - \mathbb{S} : \nabla_x \vec{u} - \frac{1}{\sigma} \left| \mathrm{curl}_x \vec{H} \right|^2 - \vec{u} \cdot \vec{S}_F \right.$$
$$\left. + S_E \right) \varphi \, \mathrm{d}x \, \mathrm{d}t,$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$.

Since replacing *equation* (2.12) by *inequality* (2.46) would certainly result in a formally underdetermined problem, system (2.40), (2.41), (2.46) must be supplemented with the total energy balance

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \varrho(t,x) \left| \vec{u}(t,x) \right|^2 + \varrho e(\varrho(t,x), \vartheta(t,x)) + c^{-1} \int_{\mathcal{S}^2} \int_0^\infty I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu \right.$$
$$\left. + \mathcal{M}(|\vec{H}|) - \frac{1}{2} \varrho(t,x) \Psi(t,x) \right) \mathrm{d}x = \int_\Omega \int_{\mathcal{S}^2} \int_0^\infty \mathrm{div}_x \left(\vec{\omega} I(t,x,\vec{\omega},\nu) \right) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu \, \mathrm{d}x, \tag{2.47}$$

which can be rephrased as follows

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho,\vartheta) + E_R + \mathcal{M}(|\vec{H}|) - \frac{1}{8\pi G} \left| \nabla \Psi \right|^2 \right) (\tau,\cdot) \, \mathrm{d}x \tag{2.48}$$

$$+ \int_0^\tau \iint_{\substack{\partial\Omega \times \mathcal{S}^2 \\ \vec{\omega}\cdot\vec{n} \geq 0}} \int_0^\infty I(t,x,\vec{\omega},\nu) \, \vec{\omega} \cdot \vec{n} \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} + \mathcal{M}(|\vec{H}|)(0,\cdot) - \frac{1}{8\pi G} \left| \nabla \Psi_0 \right|^2 \right) \, \mathrm{d}x,$$

$$\text{for a. a. } \tau \in (0,T),$$

where E_R is given by (2.1), and

$$E_{R,0} = c^{-1} \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu.$$

The transport equation (2.13) can be extended to the whole physical space $\mathbb{R}^3$ provided we set

$$\sigma_a(x,\nu,\vartheta) = \mathbb{I}_\Omega \sigma_a(\nu,\vartheta), \;\; \sigma_s(x,\nu,\vartheta) = \mathbb{I}_\Omega \sigma_s(\nu,\vartheta),$$

and take the initial distribution $I_0(x,\vec{\omega},\nu)$ to be zero for $x \in \mathbb{R}^3 \setminus \Omega$. Accordingly, for any fixed $\vec{\omega} \in \mathcal{S}^2$, equation (2.13) can be considered a linear transport equation defined in $(0,T) \times \mathbb{R}^3$, with a right-hand side $c\tilde{S}$. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$. Then the gravitational potential Ψ is defined on the whole $\mathbb{R}^3$ by the Newtonian potential.

Remark: Let us mention at this point that such an extension of the transport equation is only valid in the context of weak solutions. It would not be the case for strong solutions for which construction of the solution using method of characteristics would lead to a contradiction between the two problems due to boundary conditions. (We know that $I = 0$ on the part of boundary where $\omega \cdot n \leq 0$.) We can refer to the work of Bardos and it can be find in Lecture Notes of Golse et al. (see Allaire et al. , 2015; Lemma 2.3.3).

Finally, we need to state the weak formulation of the reduced Maxwell system (2.2), (2.6)

$$\int_0^T \int_\Omega \left(\vec{B} \cdot \partial_t \vec{\varphi} - \left(\vec{B} \times \vec{u} + \frac{1}{\sigma} \mathrm{curl}_x \vec{H} \right) \cdot \mathrm{curl}_x \vec{\varphi} \right) \, \mathrm{d}x \, \mathrm{d}t = \tag{2.49}$$

$$- \int_\Omega \vec{B}_0 \cdot \vec{\varphi}(0, \cdot) \, \mathrm{d}x,$$

holds for any $\vec{\varphi} \in \mathcal{D}([0,T) \times \mathbb{R}^3, \mathbb{R}^3)$.

In accordance with the boundary conditions (2.24), (2.25) one also take

$$\vec{B}_0 \in L^2(\Omega), \ \mathrm{div}_x \vec{B}_0 = 0 \text{ in } D'(\Omega), \ \vec{B}_0 \cdot n|_{\partial\Omega} = 0. \tag{2.50}$$

Definition 2.2.1 *We say that $(\varrho, \vec{u}, \vartheta, \vec{B}, I)$ is a weak solution of problem (2.10) – (2.26) iff*

$$\begin{aligned} \varrho_0 \geq 0 \ \textit{a.e. in } \Omega, \ \varrho_0 \in L^{\frac{5}{3}}(\Omega), \ \frac{(\varrho\vec{u})_0}{\sqrt{\rho_0}} \in L^2(\Omega, \mathbb{R}^3), \\ (\varrho e(\varrho, \vartheta))_0 = \varrho_0 e(\varrho_0, \vartheta_0) \in L^1(\Omega), \ \vartheta_0 > 0 \ \textit{a.e. in } \Omega, \ \ \vartheta_0 \in L^\infty(\Omega), \\ \Psi_0 = G(-\Delta)^{-1} \mathbb{I}_\Omega \varrho_0, \ \vec{B}_0 \in L^2(\Omega, \mathbb{R}^3), \ \mathrm{div}_x \vec{B}_0 = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \ \vec{B}_0 \cdot \vec{n} \Big|_{\partial\Omega} = 0, \\ I_0 \geq 0 \ \textit{a.e. in } \Omega \times \mathcal{S}^2 \times (0, \infty), \\ I_0 \in L^1(\mathbb{R}^3 \times \mathcal{S}^2 \times (0,\infty)) \cap L^\infty(\mathbb{R}^3 \times \mathcal{S}^2 \times (0, \infty)), \\ (\varrho s(\varrho, \vartheta))_0 = \varrho_0 s(\varrho_0, \vartheta_0) \in L^1_{\mathrm{loc}}(\Omega), \\ \varrho \geq 0, \ \vartheta > 0 \textit{ for a.a. } (t,x) \times \Omega, \ I \geq 0 \ \textit{a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0, \infty), \\ \varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \\ \vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \ \vec{B} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)), \\ \mathcal{M}\left(\left| \vec{H} \right| \right) \in L^\infty(0,T; L^1(\Omega)), \mathrm{div}_x \vec{B}(t) = 0, \ \vec{B}(t) \cdot n|_{\partial\Omega} = 0, \ t \in (0,T), \\ I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0, \infty)), \end{aligned}$$

and $(\varrho, \ \vec{u}, \ \vartheta, \ \vec{B}, \ I)$ satisfy the integral identities (2.40), (2.41), (2.46), (2.48), and (2.49) together with the transport equation (2.13) and boundary conditions (2.23) – (2.25) at least in the sense of traces.

The main result of the present paper can be stated as follows. Weak limits are generally denoted with an overbar.

Theorem 2.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$ for an $\alpha > 0$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.27) – (2.32), and that the transport coefficients η, κ_M, ζ, σ, μ, σ_a, and σ_s comply with (2.33) – (2.39).*

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (2.10) – (2.26) in the sense of Definition 2.2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{2.51}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon) + E_{R,\varepsilon} + \mathcal{M}\left(\left|\vec{H}\right|\right) - \frac{1}{2}\varrho\Psi \right)(0,\cdot)\,\mathrm{d}x \tag{2.52}$$

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}}|(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} + \mathcal{M}\left(\left|\vec{H}\right|\right)_{0,\varepsilon} - \frac{1}{2}(\varrho\Psi)_{0,\varepsilon} \right)\,\mathrm{d}x \le E_0,$$

$$\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)(0,\cdot)\,\mathrm{d}x \equiv \int_\Omega (\varrho s)_{0,\varepsilon}\,\mathrm{d}x \ge S_0,$$

and

$$0 \le I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \le I_0,\ |I_{0,\varepsilon}(x,\vec{\omega},\nu)| \le h(\nu) \text{ for a certain } h \in L^1(0,\infty). \tag{2.53}$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T];L^{5/3}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$
$$\vec{B}_\varepsilon \to \vec{B} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)) \cap L^\infty(0,T;L^2(\Omega)),$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, \vec{B}, I\}$ is a weak solution of problem (2.10) – (2.26).

Note that a *strong convergence* is required only for the initial distribution of the densities $\{\varrho_{\varepsilon,0}\}_{\varepsilon>0}$ (see (2.51)), while the remaining initial data are only bounded in suitable norms. This is due to the fact that the evolution of the density is governed by continuity equation (2.10) having a hyperbolic character without any smoothing effect incorporated.

2.3 Proof of Theorem 2.2.1

Following (Ducomet, Feireisl, Nečasová , 2011), the proof consists of three steps. We establish uniform estimates on the family $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ independent of $\varepsilon \to 0+$ first. Secondly, we observe that the extra forcing terms in (2.41), (2.46) due to radiation are bounded in suitable Lebesgue norms. In particular, the analysis of the macroscopic variables ϱ_ε, $\vec{u}_\varepsilon$, ϑ_ε is essentially the same as in the case of the Navier-Stokes-Fourier system presented in (Feireisl, Novotný , 2009). Consequently, as in (Ducomet, Feireisl, Nečasová , 2011) the proof of Theorem 2.2.1 reduces to the study of the transport equation (2.13) governing the time evolution of the radiation intensity I_ε and Maxwell's system (2.2) and (2.6). In the last step we introduce an approximation scheme similar to that used in (Feireisl, Novotný , 2009; Chapter 3) and sketch the main ideas of a complete proof of the existence of global-in-time weak solutions to problem (2.10) – (2.26).

2.3.1 Uniform bounds

Uniform (*a-priori*) bounds follow immediately from the total energy balance and the entropy production equation.

The total energy balance (2.47), combined with hypotheses of Theorem 2.2.1 gives

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{2.54}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)\|_{L^1(\Omega)} \leq c, \tag{2.55}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|E_{R,\varepsilon}\|_{L^1(\Omega)} \leq c, \tag{2.56}$$

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\|\mathcal{M}\left(\left|\vec{H}_\varepsilon\right|\right)\right\|_{L^1(\Omega)} \leq c, \tag{2.57}$$

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vec{B}_\varepsilon\|_{L^2(\Omega)} \leq c. \tag{2.58}$$

Thus, as the internal energy contains the radiation component proportional to ϑ^4 (cf. (2.27)), we deduce from (2.55) that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vartheta_\varepsilon\|_{L^4(\Omega)} \leq c, \tag{2.59}$$

and, by virtue of hypotheses (2.27) – (2.30),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \leq c. \tag{2.60}$$

This crucial uniform estimate we get "for free" by the proportionality of pressure and internal energy density of the fluid part of the internal energy and by the assumption that we deal with a component behaving like a Fermi gas (cf. (Feireisl, Novotný , 2009; Chapter 2))

$$\varrho e \geq a\vartheta^4 + \frac{3}{2}p_\infty \varrho^{\frac{5}{3}}. \tag{2.61}$$

Since the quantity I_ε is non-negative, we have from (2.13)

$$\frac{1}{c}\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon \leq \sigma_a(\nu,\vartheta_\varepsilon)\mathfrak{B}(\nu,\vartheta_\varepsilon) + \sigma_s(\nu,\vartheta_\varepsilon)\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega},\cdot)\ \mathrm{d}\vec{\omega} \leq \tag{2.62}$$

$$c_5 + c_4\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega},\cdot)\ \mathrm{d}\vec{\omega},$$

as the coefficients σ_s, σ_a are non-negative and bounded by (2.37). Thus we deduce a uniform bound

$$0 \leq I_\varepsilon(t,x,\nu,\vec{\omega}) \leq c(T)(1+\sup_{x\in\Omega,\ \nu\geq 0,\vec{\omega}\in\mathcal{S}^2} I_{0,\varepsilon}) \leq c(T)(1+I_0) \text{ for any } t\in[0,T] \tag{2.63}$$

by (2.53) with a certain non-negative function $c(t)$.

Cut-off hypothesis (2.38) together with (2.63) yield

$$\|S_{E,\varepsilon}\|_{L^\infty((0,T)\times\Omega)} + \|\vec{S}_{F,\varepsilon}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \tag{2.64}$$

Moreover, due to (2.39) it holds

$$\left\|\frac{S_{E,\varepsilon}}{\vartheta_\varepsilon}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\frac{\vec{S}_{F,\varepsilon}}{\vartheta_\varepsilon}\right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \tag{2.65}$$

As the viscosity coefficients satisfy (2.33) – (2.34), we get

$$\|\varrho_\varepsilon\vec{u}_\varepsilon\|^2_{L^2((0,T),L^1(\Omega))} + \int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\mathbb{S}_\varepsilon : \nabla_x\vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \geq$$

$$c_1\left\|\nabla_x\vec{u}_\varepsilon + \nabla_x^T\vec{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x\vec{u}_\varepsilon\mathbb{I}\right\|^2_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})} + \|\varrho_\varepsilon\vec{u}_\varepsilon\|^2_{L^2((0,T),L^1(\Omega))}$$

$$\geq c_7\|\vec{u}_\varepsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where we have used a variant of the Korn-Poincaré inequality ((see Feireisl, Novotný , 2009; Chapter 2, Proposition 2.1)) and c_7 depends only on the uniform bounds of ϱ and c_1.

On the other hand, in accordance with (2.65) by Hölder's inequality

$$\left|\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\vec{S}_{F,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \leq c\|\vec{u}_\varepsilon\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)}.$$

Then the entropy inequality (2.46) (with positive production terms, the rest estimated by (2.65)) yields the uniform bounds *for Ω bounded*

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} \leq c, \tag{2.66}$$

$$\|\nabla_x\vartheta_\varepsilon\|_{L^2((0,T)\times\Omega)} \leq c, \tag{2.67}$$

$$\left\|\frac{1}{\vartheta_\varepsilon\sigma_\varepsilon}\left|\mathrm{curl}_x\vec{H}_\varepsilon\right|^2\right\|_{L^1((0,T)\times\Omega)} \leq c, \tag{2.68}$$

upon testing with approximations of the test function $\varphi = \mathbb{I}_\Omega(x)\mathbb{I}_{[0,T]}(t)$ and passing to the limit. Moreover we have

$$\left\|\mathrm{curl}_x\vec{H}_\varepsilon\right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \tag{2.69}$$

Finally we summarize the lemmas from (Duvaut, Lions , 1976; Chapter 7) concerning the (linearized) equation for the evolution of magnetic field $\vec{B}$ (2.49):

- The boundary condition expressing continuity of the tangential component of the electric field $(2.25)_1$ is automatically satisfied by the weak formulation (2.49).
- The boundary condition expressing continuity of the normal component of the magnetic field $(2.25)_2$ is satisfied once we choose $\vec{B}_{\varepsilon,0} \cdot \vec{n} = 0$.
- The same is true for the condition of solenoidality (2.3) once we guarantee $\mathrm{div}_x \vec{B}_{\varepsilon,0} = 0$ in $\mathcal{D}'(\Omega)$.
- We have got a Hodge-type estimate

$$\left\|\vec{B}_\varepsilon\right\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le c\left(\left\|\mathrm{curl}_x \vec{B}_\varepsilon\right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\|\mathrm{div}_x \vec{B}_\varepsilon\right\|_{L^2(\Omega)} + \right.$$

$$\left.\left\|\vec{B}_\varepsilon \cdot \vec{n}\right\|_{W^{\frac{1}{2},2}(\partial\Omega)}\right) \le c\left\|\mathrm{curl}_x \vec{B}_\varepsilon\right\|_{L^2(\Omega,\mathbb{R}^3)} \le c,$$

meaning that we have got a uniform estimate of the magnetic induction

$$\left\|\vec{B}_\varepsilon\right\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))} \le c, \tag{2.70}$$

and also for the magnetic field

$$\left\|\vec{H}_\varepsilon\right\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))} \le c. \tag{2.71}$$

To estimate the pressure functions $p(\varrho_\varepsilon, \vartheta_\varepsilon)$ globally we start with the observation that estimates (2.60), (2.66) imply that the sequences $\{\varrho_\varepsilon \vec{u}_\varepsilon\}_{\varepsilon>0}$, $\{\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^p((0,T)\times\Omega)$ for a certain $p > 1$, namely $p = \frac{45}{43}$. Similarly, combining (2.59), (2.66), (2.67), we get

$$\{\mathbb{S}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^p((0,T)\times\Omega;\mathbb{R}^{3\times3}) \text{ for a certain } p > 1, \text{namely } p = \frac{34}{23}.$$

Moreover, $\{\varrho_\varepsilon \nabla_x \Psi_\varepsilon\}_{\varepsilon>0}, \{\vec{J}_\varepsilon \times \vec{B}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^p((0,T)\times\Omega)$ for a certain $p > 1$. Now, repeating the arguments of (Feireisl, Petzeltová , 2000), we observe that the quantities

$$\varphi(t,x) = \tilde{\psi}(t)\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha,\ \tilde{\psi} \in \mathcal{D}(0,T) \text{ for sufficiently small parameters } \alpha,\ \omega > 0$$

may be used as test functions in the momentum equation (2.41), where $\mathcal{B}[v]$ is a suitable branch of solutions to the boundary value problem

$$\mathrm{div}_x\Big(\mathcal{B}[v]\Big) = v - \frac{1}{|\Omega|}\int_\Omega v \,\mathrm{d}x,\ \mathcal{B}[v]|_{\partial\Omega} = 0. \tag{2.72}$$

Here $\mathcal{B}$ is the Bogovskii operator and α denotes a convolution parameter in time since we need to test with a continuously differentiable function.

Next we get estimates of $\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$ in $L^q(0,T;W^{1,p}(\Omega,\mathbb{R}^3))$ for all $q \in [1,\infty]$ and $p \in (1,\infty)$ by $\{\varrho_\varepsilon^\omega\}^\alpha$ in $L^q(0,T;L^p(\Omega,\mathbb{R}^3))$ and of $\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$ setting the renormalization function in (2.40) $b(\varrho_\varepsilon) := \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$. This leads to an estimate of the term $\int_0^T \tilde{\psi}(t) \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon)\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t$ by eight integrals resulting from the momentum equation (2.41). We omit most details which can the reader find in (Feireisl, Novotný , 2009; Chapter 2). Let us just note that the "worst" term arises from the time derivative which we a-priori do not control in a Lebesgue space and therefore we partially integrate in time and estimate it as follows

$$\left| - \int_0^T \tilde{\psi}(t) \int_\Omega \varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t \right| \le \left\| \tilde{\psi} \right\|_{C(0,T)} \|\varrho_\varepsilon \vec{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega,\mathbb{R}^3))} \times \tag{2.73}$$

$$\|\partial_t \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\|_{L^1(0,T;L^5(\Omega,\mathbb{R}^3))} \le c \left\| \tilde{\psi} \right\|_{C(0,T)} \|\varrho_\varepsilon \vec{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega,\mathbb{R}^3))} \times$$

$$\Big\{ \|\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \vec{u}_\varepsilon\|_{L^1(0,T;L^5(\Omega,\mathbb{R}^3))} +$$

$$\left\| \left[\varrho_\varepsilon \left(\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\right)' - \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\right] \mathrm{div}_x \vec{u}_\varepsilon \right\|_{L^1(0,T;L^{\frac{15}{8}}(\Omega,\mathbb{R}^3))} \Big\} \le c \quad \text{for} \quad \omega \le \frac{11}{18},$$

by (2.60), (2.54) and (2.66) and a repeated use of Hölder's inequality and Sobolev embedding. "New terms" in comparison to (Feireisl, Novotný , 2009; Chapter 2) are estimated as follows

$$\left| - \int_0^T \tilde{\psi}(t) \int_\Omega \varrho_\varepsilon \nabla_x \Psi_\varepsilon \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t \right| \le c \left\| \tilde{\psi} \right\|_{C(0,T)} \|\nabla \psi_\varepsilon\|_{L^\infty(0,T;L^{\frac{15}{4}}(\Omega))} \times \tag{2.74}$$

$$\|\varrho_\varepsilon^\omega\|_{L^1(0,T;L^{\frac{15}{7}}(\Omega))} \|\varrho_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \le c \quad \text{for} \quad \omega \le \frac{7}{9},$$

$$\left| \int_0^T \tilde{\psi}(t) \int_\Omega \vec{S}_{F,\varepsilon} \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t \right| \le c \left\| \tilde{\psi} \right\|_{C(0,T)} \left\| \vec{S}_{F,\varepsilon} \right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \times \tag{2.75}$$

$$\|\varrho_\varepsilon^\omega\|_{L^1(0,T;L^{1+\delta}(\Omega))} \le c \quad \text{for small } \delta > 0 \text{ and } \omega < \frac{5}{3}$$

and

$$\left| - \int_0^T \tilde{\psi}(t) \int_\Omega \sigma(\varrho_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon) \left(\vec{E}_\varepsilon + \vec{u}_\varepsilon \times \vec{B}_\varepsilon \right) \times \vec{B}_\varepsilon \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ dt \right| \le c \times \tag{2.76}$$

$$\left\| \tilde{\psi} \right\|_{C(0,T)} \left\| \mathrm{curl}_x \vec{H}_\varepsilon \right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \|\varrho_\varepsilon^\omega\|_{L^{\frac{15}{8}}(0,T;L^{\frac{15}{8}}(\Omega))} \left\| \vec{B}_\varepsilon \right\|_{L^{\frac{10}{3}}((0,T)\times\Omega;\mathbb{R}^3)} \le c$$

$$\text{for} \quad \omega \le \frac{8}{9}.$$

The resulting (uniform in ε) estimate reads

$$\int_0^T \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon) \varrho_\varepsilon^\omega \ \mathrm{d}x \ \mathrm{d}t < c, \tag{2.77}$$

where $\omega \le \min\left\{\frac{5}{3}, \frac{8}{9}, \frac{7}{9}, \frac{55}{102}, \frac{11}{18}, \frac{2}{27}\right\} = \frac{2}{27}$, in particular, we can arrive at the following regularity by upper bounds for pressure in the non-degenerate region (for small Z in the sense of (2.28)) and in the degenerate region, respectively and homogeneous regularity of temperature by (2.59) $\|\vartheta_\varepsilon\|_{L^{\frac{17}{3}}((0,T)\times\Omega)} \le c$

$$\{p(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^p((0,T)\times\Omega) \text{ for a } \ p > 1, \text{namely } p = \frac{47}{45}. \tag{2.78}$$

2.3.2 Weak sequential stability

We sketch the principal part of the proofs and focus mainly on the issues related to weak sequential stability of quantities related to radiation and magnetic field that require new ideas. In particular, we examine in details the extra terms in the entropy balance equation (2.46).

2.3.2.1 Weak (sequential) stability of macroscopic thermodynamic quantities

After the uniform estimates on the radiation forcing terms $S_{E,\varepsilon}$ and $\vec{S}_{F,\varepsilon}$ in (2.64), a strong (pointwise) convergence of the macroscopic thermodynamic quantities $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be shown exactly as in (Ducomet, Feireisl , 2006).

To begin, using the uniform bounds established in Section 2.3.1 we observe that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{w}}([0,T]; L^{\frac{5}{3}}(\Omega)), \tag{2.79}$$

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)), \tag{2.80}$$

and

$$\log(\vartheta_\varepsilon) \to \overline{\log(\vartheta)} \text{ weakly in } L^2((0,T)\times\Omega), \tag{2.81}$$

and

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.82}$$

possibly passing to suitable subsequences. Moreover, since the (weak) time derivative $\partial_t\left(\varrho_\varepsilon\vec{u}_\varepsilon\right)$ of momenta can be expressed by means of momentum balance (2.41) (Lorentz force density bounded as in (2.76)), we have got

$$\varrho_\varepsilon\vec{u}_\varepsilon \to \varrho\vec{u} \text{ in } C_{\mathrm{w}}([0,T]; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \tag{2.83}$$

Since our system contains quantities depending on ϱ and ϑ in a general nonlinear way, pointwise (resp. a. e.) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ must be established in order to perform the limit $\varepsilon \to 0+$. This step is apparently easier to carry out for the temperature because of the uniform bounds available for $\nabla_x\vartheta_\varepsilon$.

2.3.2.2 Pointwise convergence of temperature

A. e. convergence of the sequence $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be established essentially by the monotonicity arguments. The main problem are possible uncontrollable time oscillations in hypothetical zones of vacuum, here eliminated by the presence of radiation component in the entropy inequality. More specifically, our goal is to show that

$$\int_0^T\int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta)\Big)\Big(\vartheta_\varepsilon - \vartheta\Big)\ \mathrm{d}x\ \mathrm{d}t \to 0 \text{ as } \varepsilon\to 0+, \tag{2.84}$$

which, in accordance with hypothesis (2.32), implies the desired conclusion

$$\vartheta_\varepsilon \to \vartheta \text{ in } L^4((0,T)\times\Omega), \text{ in particular, } \vartheta_{\varepsilon_k} \to \vartheta \text{ a. e. in } (0,T)\times\Omega. \tag{2.85}$$

In order to see (2.84), we first observe that

$$\int_0^T\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)\Big(\vartheta_\varepsilon-\vartheta\Big)\ \mathrm{d}x\ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+\,.$$

Indeed this follows by means of the Aubin-Lions compactness lemma as

$$\vartheta_\varepsilon - \vartheta \to 0 \text{ weakly in } L^2(0,T;W^{1,2}(\Omega)),$$

and the (weak) time derivative $\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon))$ can be expressed by means of the entropy inequality (2.46).

Consequently, it remains to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta)\Big(\vartheta_\varepsilon - \vartheta\Big) \, \mathrm{d}x \, \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+ . \tag{2.86}$$

To see (2.86), we combine the bounds imposed on $\partial_t b(\varrho_\varepsilon)$ by the renormalized equation (2.40), with the estimates on the temperature gradient (2.67), to deduce that

$$\nu_{t,x}[\varrho_\varepsilon, \vartheta_\varepsilon] = \nu_{t,x}[\varrho_\varepsilon] \otimes \nu_{t,x}[\vartheta_\varepsilon] \ \text{ a. e. in } \ (0,T) \times \Omega, \tag{2.87}$$

where the symbol $\nu[\varrho_\varepsilon, \vartheta_\varepsilon]$ denotes a Young measure associated to the family $\{\varrho_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$, while $\nu[\varrho_\varepsilon]$, $\nu[\vartheta_\varepsilon]$ stand for Young measures generated by $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, respectively. In order to conclude, we use the following result frequently called *Fundamental theorem of the theory of Young measures* ((see Pedregal , 1997; Chapter 6, Theorem 6.2)):

Theorem 2.3.1. *Let* $\{\vec{v}_n\}_{n=1}^\infty$, $\vec{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ *be a sequence of functions bounded in* $L^1(Q;\mathbb{R}^M)$, *where* Q *is a domain in* $\mathbb{R}^N$.

Then there exist a subsequence (not relabeled) and a parametrized family $\{\nu_y\}_{y\in Q}$ *of probability measures on* $\mathbb{R}^M$ *depending measurably on* $y \in Q$ *with the following property:*

For any Caratheodory function $\Phi = \Phi(y,z)$, $y \in Q$, $z \in \mathbb{R}^M$ *such that*

$$\Phi(\cdot, \vec{v}_n) \to \overline{\Phi} \textit{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y,z) \, \mathrm{d}\nu_y(z) \textit{ for a. a. } y \in Q.$$

In virtue of Theorem 2.3.1, relation (2.87) implies (2.86). We have proved the almost everywhere convergence of the temperature claimed in (2.85). Note that this step leans heavily on the presence of the *radiative entropy flux.*

2.3.2.3 Pointwise convergence of density

Although the pointwise convergence of the family of densities $\{\varrho_\varepsilon\}_{\varepsilon>0}$ represents one of the most delicate questions of the existence theory for the compressible Navier-Stokes system, this step is nowadays well understood. The idea is to use the quantities

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x \Delta^{-1}[\mathbb{I}_\Omega T_k(\varrho_\varepsilon)]$$

as test functions in the weak formulation of momentum equation (2.41). Similarly, we can let $\varepsilon \to 0+$ in (2.41) and test the resulting expression by

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x \Delta^{-1}[\mathbb{I}_\Omega \overline{T_k(\varrho_\varepsilon)}],$$

where $\psi \in C_c^\infty(0,T)$, $\phi \in C_c^\infty(\Omega)$, and T_k is a cut-off function,

$$T_k(z) = \min\{z, k\}.$$

In the limit for $\varepsilon \to 0+$, this procedure yields the celebrated relation for the *effective viscous pressure* discovered by (Lions , 1996, 1998), relating last two expressions whose weak limits have not been identified yet, which reads in the present setting as

$$\int_0^T \int_\Omega \psi\phi \left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.88}$$

$$= \int_0^T \int_\Omega \psi\phi \left(\overline{\mathbb{S} : \mathcal{R}[\mathbb{I}_\Omega T_k(\varrho)]} - \mathbb{S} : \mathcal{R}[\mathbb{I}_\Omega \overline{T_k(\varrho)}] \right) \, \mathrm{d}x \, \mathrm{d}t,$$

where the ovebars denote weak limits of the composed quantities (in the appropriate Lebesgue spaces $L^p((0,T)\times\Omega)$, for $p = \frac{47}{45}$ and $p = \frac{34}{23}$, respectively, thus in $L^1((0,T)\times\Omega)$ as well) and where $\mathcal{R} = \mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j}$ is a pseudo-differential operator with its symbol

$$R = \frac{\xi \otimes \xi}{|\xi|^2}$$

((see Feireisl, Novotný , 2009; Section 3.6)). Note that the presence of the extra term $\vec{S}_F$ in (2.41) does not present any additional difficulty as

$$\int_0^T \int_\Omega \psi \vec{S}_{F,\varepsilon} \cdot \phi \nabla_x \Delta^{-1}[T_k(\varrho_\varepsilon)] \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \psi \vec{S}_F \cdot \phi \nabla_x \Delta^{-1}[\overline{T_k(\varrho)}] \, \mathrm{d}x \, \mathrm{d}t.$$

The same applies to the Lorentz force density as we can apply the Aubin-Lions lemma due to (2.70), $\partial_t \vec{B}_\varepsilon \in L^2(0,T;[W^{1,4}(\Omega)]^*)$ by (2.58), (2.66), (2.34) and (2.68) to obtain

$$\vec{B}_\varepsilon \to \vec{B} \quad \text{in } L^2\left((0,T)\times\Omega, \mathbb{R}^3\right) \tag{2.89}$$

which together with once again (2.68) leads to the identification of the weak limit in $L^{\frac{5}{4}}((0,T)\times\Omega)$ of the term involving the magnetic induction $\vec{B}$ in (2.41).

The following *commutator lemma* is in the spirit of Coifman and Meyer (Coifman, Meyer , 1975):

Lemma 2.3.1. Let $w \in W^{1,2}(\mathbb{R}^3)$ and $\vec{Z} \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ be given, with $6/5 < p < \infty$.

Then, for any $1 < s < 6p/(6+p)$,

$$\left\| \mathcal{R}[w\vec{Z}] - w\mathcal{R}[\vec{Z}] \right\|_{W^{\beta,s}(\mathbb{R}^3;\mathbb{R}^3)} \le c\|w\|_{W^{1,2}(\mathbb{R}^3)} \|\vec{Z}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)},$$

where $0 < \beta = \frac{3}{s} - \frac{6+p}{6p}$, and $c = c(p)$ are positive constants.

Applying Lemma 2.3.1 to the expression on the right-hand side of (2.88) and using the (weak) compactness in time of $T_k(\varrho_\varepsilon)$ following[1] from the renormalized equation (2.40), we obtain

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\mu\left(\vartheta, \left|\vec{H}\right|\right) + \eta\left(\vartheta, \left|\vec{H}\right|\right) \right) \overline{T_k(\varrho)\mathrm{div}_x \vec{u}} \tag{2.90}$$

$$= \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \left(\frac{4}{3}\mu\left(\vartheta, \left|\vec{H}\right|\right) + \eta\left(\vartheta, \left|\vec{H}\right|\right) \right) \overline{T_k(\varrho)}\mathrm{div}_x \vec{u},$$

[1]Naturally the T_k function has to be approximated by differentiable functions first.

cf. (Feireisl, Novotný , 2009; Section 3.7.4) with the help of (2.85) and (2.89).

Now, introducing the functions

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\ \mathrm{d}z,$$

we deduce from renormalized equation (2.40) that

$$\int_0^T \int_\Omega \left(\overline{\varrho L_k(\varrho)} \partial_t \varphi + \overline{\varrho L_k(\varrho)} \vec{u} \cdot \nabla_x \varphi - \overline{T_k(\varrho) \mathrm{div}_x \vec{u}} \varphi \right)\ \mathrm{d}x\ \mathrm{d}t$$

$$= - \int_\Omega \varrho_0 L_k(\varrho_0) \varphi(0, \cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$. It follows from (2.90) that

$$\mathbf{osc}_q [\varrho_\varepsilon \to \varrho] \left((0,T) \times \Omega \right) \tag{2.91}$$

$$\equiv \sup_{k \geq 1} \left(\limsup_{\varepsilon \to 0} \int_0^T \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q\ \mathrm{d}x\ \mathrm{d}t \right) < \infty,\ \forall q \in \left(2, \frac{8}{3} \right),$$

where **osc** is the oscillation defect measure introduced in (Feireisl , 2001). In particular, relation (2.91) implies that the limit functions ϱ, $\vec{u}$ satisfy renormalized equation (2.40) ((see Feireisl, Novotný , 2009; Lemma 3.8)); whence

$$\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho) \right)(\tau)\ \mathrm{d}x + \int_0^\tau \int_\Omega \left(\overline{T_k(\varrho) \mathrm{div}_x \vec{u}} - \overline{T_k(\varrho)} \mathrm{div}_x \vec{u} \right)\ \mathrm{d}x \tag{2.92}$$

$$= \int_0^\tau \int_\Omega \left(T_k(\varrho) \mathrm{div}_x \vec{u} - \overline{T_k(\varrho)} \mathrm{div}_x \vec{u} \right)\ \mathrm{d}x\ \mathrm{d}t \text{ for any } \tau \in [0,T].$$

Using once more (2.91), we can let $k \to \infty$ in (2.92) to obtain the desired conclusion

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho),$$

particularly,

$$\varrho_\varepsilon \to \varrho \text{ in } L^1((0,T) \times \Omega), \tag{2.93}$$

(see for details Feireisl, Novotný , 2009; Section 3.7.4).

Relations (2.79) – (2.82), (2.85), (2.93), and by means of an interpolation from (2.89) and (2.70)

$$\vec{B}_\varepsilon \to \vec{B} \text{ in } L^2((0,T); L^q(\Omega, \mathbb{R}^3)), \quad 1 \leq q < 6. \tag{2.94}$$

together with the previous uniform bounds allow us to pass to the limit in the weak formulation of the Navier-Stokes-Fourier system and the simplified Maxwell's system, as soon as we show convergence of the sequence $\{I_\varepsilon\}_{\varepsilon > 0}$. This will be accomplished in the forthcoming section.

2.3.2.4 Convergence of radiation intensity

Our ultimate goal is to establish convergence of the quantities arising in the entropy production rate by radiation

$$\frac{1}{\vartheta_\varepsilon} S_{E,\varepsilon} = \frac{1}{\vartheta_\varepsilon} \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon) \left[\int_{\mathcal{S}^2} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon) - I_\varepsilon \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu +$$

$$\frac{1}{\vartheta_\varepsilon} \int_0^\infty \sigma_s(\nu, \vartheta_\varepsilon) \int_{\mathcal{S}^2} \left[\frac{1}{4\pi} \int_{\mathcal{S}^2} I_\varepsilon(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} - I_\varepsilon(t, x, \vec{\omega}, \nu) \right] \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu =$$

$$\frac{1}{\vartheta_\varepsilon(t, x)} \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon(t, x)) \left[\int_{\mathcal{S}^2} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon(t, x)) - I_\varepsilon(t, x, \vec{\omega}, \nu) \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu$$

and

$$\frac{1}{\vartheta_\varepsilon} \vec{S}_{F,\varepsilon} \cdot \vec{u}_\varepsilon =$$

$$\frac{1}{c\vartheta_\varepsilon} \vec{u}_\varepsilon \cdot \int_0^\infty \sigma_s(\nu, \vartheta_\varepsilon) \int_{\mathcal{S}^2} \vec{\omega} \left[\frac{1}{4\pi} \int_{\mathcal{S}^2} I_\varepsilon(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} - I_\varepsilon(t, x, \vec{\omega}, \nu) \right] \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu +$$

$$\frac{1}{c\vartheta_\varepsilon(t, x)} \vec{u}_\varepsilon \cdot \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon(t, x)) \left[\int_{\mathcal{S}^2} \vec{\omega} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon(t, x)) - I_\varepsilon(t, x, \vec{\omega}, \nu) \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu$$

Since $\vartheta_\varepsilon \to \vartheta$ a. e. in $(0, T) \times \Omega$, the desired result follows from compactness of the velocity averages over the sphere $\mathcal{S}^2$ established by Golse et al. (Golse, Lions, Perthame, Sentis , 1988, Golse, Perthame, Sentis , 1985), see also (Bournaveas, Perthame , 2001), and hypothesis (2.38). Specifically, we use the following result ((see Golse, Lions, Perthame, Sentis , 1988)):

Proposition 2.3.1. Let $I \in L^q([0, T] \times \mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$, $\partial_t I + c\omega \cdot \nabla_x I \in L^q([0, T] \times \mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$ for a certain $q > 1$. In addition, let $I_0 \equiv I(0, \cdot) \in L^\infty(\mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$.

Then

$$\tilde{I}(t, x, \nu) \equiv \int_{\mathcal{S}^2} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega}$$

belongs to the space $W^{s,q}([0, T] \times \mathbb{R}^n \times \mathbb{R})$ for any s, $0 < s < \inf\{1/q, 1 - 1/q\}$, and

$$\|\tilde{I}\|_{W^{s,q}} \leq c(I_0)(\|I\|_{L^q} + \|\partial_t I + c\omega \cdot \nabla I\|_{L^q}).$$

As the radiation intensity I_ε satisfies the transport equation (2.13), by virtue of hypotheses (2.9) and (2.10) where $\tilde{S}$ is bounded in $(L^q \cap L^\infty)\,([0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R})$, a direct application of Proposition 2.3.1 yields the desired conclusion

$$\int_{\mathcal{S}^2} I_\varepsilon(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \text{ in } L^2((0, T) \times \Omega),$$

and

$$\int_{\mathcal{S}^2} \vec{\omega} I_\varepsilon(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \to \int_{\mathcal{S}^2} \vec{\omega} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \text{ in } L^2((0, T) \times \Omega),$$

for any fixed $\nu > 0$. Consequently

$$\frac{1}{\vartheta_\varepsilon} S_{E,\varepsilon} \to \frac{1}{\vartheta} S_E, \tag{2.95}$$

and

$$\frac{1}{\vartheta_\varepsilon}\vec{S}_{F,\varepsilon}\cdot\vec{u}_\varepsilon \to \frac{1}{\vartheta}\vec{S}_F\cdot\vec{u} \tag{2.96}$$

as required, and Theorem 2.2.1 is proved by convergences (2.93), (2.82), (2.85), (2.94), (2.95) and (2.96). The entropy inequality (2.46) needsadditionally the convergence of the initial total entropy of approximations to the initial entropy of its limit and weak upper semicontinuity of the right-hand side of (2.46) due to (2.16), (2.15), (2.82), (2.80). Moreover we need the positivity of the absolute temperature ϑ. This is due to the convergences (2.81) and (2.85).

2.3.2.5 The Maxwell equation

The Maxwell system is represented by weak formulation (2.49).

We have the following convergences:

- $\vec{B}_\varepsilon \to \vec{B}$ weakly in $L^2(0,T,W^{1,2}(\Omega))$,
- $\vec{B}_\varepsilon \to \vec{B}$ strongly in $L^2(0,T,L^2(\Omega))$,
- $\vec{J}_\varepsilon \times \vec{B}_\varepsilon \to \vec{J}\times\vec{B}$ weakly in $L^p((0,T)\times\Omega;\mathbb{R}^3) \quad \forall p\in\left[1,\frac{5}{4}\right]$.

Here we factor the only nonlinear term as follows to use the uniform bound (2.68)

$$\frac{1}{\sigma}\mathrm{curl}_x\vec{H}_\varepsilon = \sqrt{\vartheta_\varepsilon\sigma^{-1}(\varrho_\varepsilon,\vartheta_\varepsilon,\vec{B}_\varepsilon)}\sqrt{\frac{1}{\vartheta_\varepsilon\sigma(\varrho_\varepsilon,\vartheta_\varepsilon,\vec{B}_\varepsilon)}}\mathrm{curl}_x\vec{H}_\varepsilon, \tag{2.97}$$

and get the uniform bound in a reflexive Banach space

$$\left\|\sigma^{-1}(\varrho_\varepsilon,\vartheta_\varepsilon,\vec{B}_\varepsilon)\mathrm{curl}_x\vec{H}_\varepsilon\right\|_{L^{\frac{34}{23}}((0,T)\times\Omega)} \le c. \tag{2.98}$$

This suffices to pass to the limit in (2.49). □

2.3.3 Approximating scheme and global-in-time existence

We conclude the paper by proposing an approximation scheme to be used to prove the existence of global-in-time weak solutions to problem (2.10) – (2.26). The scheme is essentially the same as in (Feireisl, Novotný , 2009; Chapter 3), the extra terms are put in { }. The dependence of approximate solutions on the parameters of approximation δ and d has been in notation suppressed.

- The continuity equation (2.10) is replaced by an "artificial viscosity" approximation

  $$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = \{d\Delta\varrho\},\ d>0, \tag{2.99}$$

  to be satisfied on $(0,T)\times\Omega$, and supplemented by the homogeneous Neumann boundary conditions

  $$\nabla_x\varrho\cdot\vec{n}|_{\partial\Omega} = 0. \tag{2.100}$$

  The initial distribution of the approximate densities is given through

  $$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{2.101}$$

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}),\ \nabla_x \varrho_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0,\ \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0, \tag{2.102}$$

with a positive parameter $\delta > 0$. We recall the requirement of the strong convergence of initial approximations of density (2.51) supplemented additionally with the condition

$$|\{\varrho_{0,\delta} < \varrho_0\}| \to 0+ \quad \text{as } \delta \to 0+\,. \tag{2.103}$$

- The momentum equation is replaced by a Faedo-Galerkin approximation:

$$\int_0^T \int_\Omega \Big(\varrho\vec{u}\cdot\partial_t\vec{\varphi} + \varrho\vec{u}\otimes\vec{u} : \nabla_x\vec{\varphi} + \big(p + \{\delta\left(\varrho^\Gamma + \varrho^2\right)\}\big)\,\mathrm{div}_x\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t =$$

$$\int_0^T \int_\Omega \Big(\{d(\nabla_x\varrho\nabla_x\vec{u})\}\cdot\vec{\varphi} + \mathbb{S}_\delta : \nabla_x\vec{\varphi} - \varrho\nabla_x\Psi\cdot\vec{\varphi} - \vec{S}_F\cdot\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t -$$

$$\int_\Omega (\varrho\vec{u})_0\cdot\vec{\varphi}\;dx, \tag{2.104}$$

to be satisfied for any test function $\vec{\varphi} \in C_c^1([0,T), X_n)$, where

$$X_n \subset C^{2,\nu}(\overline{\Omega};\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3) \tag{2.105}$$

is a finite-dimensional space of functions satisfying the no-slip boundary conditions

$$\vec{\varphi}|_{\partial\Omega} = \vec{0}. \tag{2.106}$$

The space X_n is endowed with the Hilbert structure induced by the scalar product of the Lebesgue space $L^2(\Omega; R^3)$.

We set

$$\mathbb{S}_\delta := \mathbb{S}_\delta\left(\nabla_x\vec{u},\vartheta,\vec{H}\right) = \left(\mu\left(\vartheta,\left|\vec{H}\right|\right) + \delta\vartheta\right)\left(\nabla_x\vec{u} + \nabla_x^T\vec{u} - \tfrac{2}{3}\mathrm{div}_x\vec{u}\,\mathbb{I}\right) +$$

$$\left(\eta\left(\vartheta,\left|\vec{H}\right|\right)\mathrm{div}_x\vec{u}\right)\mathbb{I}. \tag{2.107}$$

- We replace the energy equation (2.12) with a modified internal energy balance

$$\partial_t(\varrho e + \{\delta\varrho\vartheta\}) + \mathrm{div}_x\Big((\varrho e + \{\delta\varrho\vartheta\})\vec{u}\Big) - \tag{2.108}$$

$$\mathrm{div}_x\left(\kappa_M(\varrho,\vartheta,\left|\vec{B}\right|) + \kappa_R\vartheta^3 + \{\delta\left(\vartheta^\Gamma + \vartheta^{-1}\right)\}\nabla_x\vartheta\right) =$$

$$\mathbb{S}_\delta : \nabla_x\vec{u} + \frac{1}{\sigma}\left|\mathrm{curl}_x\vec{H}\right|^2 + \vec{u}\cdot\vec{S}_F + \Big\{ d\delta(\Gamma|\varrho|^{\Gamma-2} + 2)|\nabla_x\varrho|^2 +$$

$$\delta\vartheta^{-2} - d\vartheta^5 + 2\delta\vartheta \left[\left| \frac{\nabla_x \vec{u} + \nabla_x \vec{u}^T}{2} \right|^2 - \frac{1}{3} (\mathrm{div}_x \vec{u})^2 \right] \Bigg\} - p\ \mathrm{div}_x \vec{u} - S_E,$$

to be satisfied in $(0,T) \times \Omega$, together with the no-heat-flux boundary condition

$$\nabla_x \vartheta \cdot \vec{n}|_{\partial\Omega} = 0. \tag{2.109}$$

The initial condition reads

$$\varrho(e + \delta\vartheta)(0,\cdot) = \varrho_{0,\delta}(e(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta\vartheta_{0,\delta}), \tag{2.110}$$

where the (approximate) temperature distribution satisfies

$$\vartheta_{0,\delta} \in C^1(\overline{\Omega}),\ \nabla_x \vartheta_{0,\delta} \cdot \vec{n}|_{\partial\Omega} = 0,\ \inf_{x\in\Omega} \vartheta_{0,\delta}(x) > 0. \tag{2.111}$$

- We add the equation for the radiative transfer

$$\frac{1}{c}\partial_t I + \vec{\omega} \cdot \nabla_x I = \tilde{S} \text{ in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty), \tag{2.112}$$

  together with the transparency condition (2.24).

- Finally we require satisfaction of the unmodified equation for the magnetic induction $\vec{B}$ (2.49), the solenoidality condition (2.3) with approximate initial conditions

$$\vec{B}(0,\cdot) = \vec{B}_{0,\delta} \tag{2.113}$$

  with $\vec{B}_{0,\delta} \in \mathcal{D}(\Omega, \mathbb{R}^3)$, $\mathrm{div}_x \vec{B}_{0,\delta} = 0$ and

$$\vec{B}_{0,\delta} \to \vec{B}_0 \quad \text{in } L^2(\Omega, \mathbb{R}^3) \text{ as } \delta \to 0+\,. \tag{2.114}$$

Given a family of approximate solutions $\{\varrho_{d,\delta}, \vec{u}_{d,\delta}, \vartheta_{d,\delta}, \vec{B}_{d,\delta}, I_{d,\delta}\}_{d>0,\delta>0}$, we may construct a weak solution of system (2.1) – (2.26) letting successively $d \to 0$, $\delta \to 0$ and using compactness arguments delineated in the previous part of this paper. The reader may consult (Feireisl, Novotný , 2009; Chapter 3) for all technical details. The approximate solutions can be constructed by means of a fixed point argument applied to the couple $\vec{u}$, I, similarly to (Feireisl, Novotný , 2009; Chapter 3, Section 3.4).

3. Introduction to limits

3.1 Introduction to (singular) limits in fluid mechanics

In this introduction we closely follow the book of Feireisl and Novotný (Feireisl, Novotný , 2009) and the paper of Feireisl (Feireisl , 2010).

Typical limit problems in mathematical fluid mechanics are limit transitions from compressible to incompressible equations when the Mach number Ma convergences to zero, MHD limits when Ma or Alfvén number Al tend to zero, however often systems of mechanics are formal asymptotical limits of more complex systems (e. g. *incompressible fluids are fluids where the typical speed U_{ref} is very small with respect to the speed of sound $c : U_{ref} \ll c$.*) Simplified systems are often more convenient for numerical solutions etc., but they may sometimes have also worse mathematical properties than more complex systems. The prevailing majority of existing literature on system scalings deal with some formal asymptotic analyses that can be still useful as we obtain for example a candidate of a limit system for a further rigorous analysis. The class of weak solutions is sufficiently large to capture singularities that may develop in a finite time, even though it may be too large. Feireisl, Novotný (2009) discusses several singular limits - as tools for convergence it discusses the *Lighthill acoustic analogy* known from books on engineering and the method of decomposition to the essential and residual part which vanishes in the limit. *Lowly stratified fluids are fluids where Ma tends to zero and simultaneously the Froude number Fr is strongly dominated by the Mach number Ma.* In this situation from the **dissipation inequality** (Second Law of Thermodynamics) and the hypotheses of **thermodynamical stability** by means of **solution of acoustic equations with a right hand side expressed by a non-negative Borel measure** we get the *Oberbeck-Boussinesq approximation* even in the case of **ill-prepared data** when we have to get prepared to the propagation of acoustic waves.

The book (Feireisl, Novotný , 2009) studies models of strongly stratified fluids with applications in astrophysics and meteorology, e. g. a model of *stellar radiative zones.* Strong damping of acoustic waves takes place in the case of viscous equations with the no-slip boundary conditions. There are also such geometrical conditions established that the velocity $\vec{u}$ does not exhibit oscillations in time in the case of low Mach number. Authors are also able to get strong convergence of $\vec{u}$ when $Ma \to 0+$ if the domain Ω is sufficiently large (especially unbounded) with the help of estimates of Strichartz type. They also identify the singular limits obtained by the acoustic analogy.

In the applications of fluid mechanics we usually meet compressible fluids, but most of the theory is still devoted to incompressible models, as they are compelling from the point of view of nonlinear differential equations. The general strategy of the study of the singular limits is to scale the original primitive equations expressing conservation laws of mass, momentum and energy, for example the Navier-Stokes-Fourier system, to find out what singular regime is applicable to a given flow, if any, to take the limit formally and identify the target system and to take the limit rigourously knowing it; sometimes the qualitative properties of the target system can be applied to the primitive system if we are in a sense "close to the target system" — an example is the result of Hagstrom and Lorenz (Hagstrom, Lorenz , 2002), whose target system under vanishing Mach number limit was the two dimensional incompressible Navier-Stokes system for which we know regularity and uniqueness due to Kiselev and Ladyzhenskaya. They were able to get the global existence of smooth solutions of slightly compressible heat-conducting fluids (deviation of temperature and density from constants need to be small, as well as the initial $\mathrm{div}_x \vec{u}_0$, while $\vec{u}$ may be otherwise arbitrary).

In the study of the limit process solvability of the primitive equations, its stability with respect to singular parameters and convergence towards the target system should be addressed with a focus on possible instabilities that are important from the point of view of both mathematical and numerical analysis and simulations. Let us emphasize that there are two different approaches to the limits in fluid mechanics — the first one, more classical, depends on the notion and properties of strong (or classical) solutions which need not to exist globally in time, whereas the second one starts with weak solutions that may be globally well-posed, but in the latter case we need initial conditions in the vicinity of the thermodynamical equilibrium, which is not feasible for some applications.

3.1.1 Navier-Stokes-Fourier system

Let us assume that we have got a monoatomic gas satisfying the usual assumption on the equations of state $p(\varrho, \vartheta)$, $e(\varrho, \vartheta)$ and the associated entropy function $s(\varrho, \vartheta)$ (1.53)–(1.58).

Feireisl and Petzeltová (Feireisl, Petzeltová , 1998) studied the equilibrium figures of such a gas under the assumptions of (strong) thermodynamical stability (1.50a), (1.50b), $(1.53)_3$ and established that any equilibrium is given by the total mass $M := \int_\Omega \varrho(x)\, dx$ and the total energy $E = \int_\Omega [\varrho e(\varrho, \vartheta) - \varrho \Psi(x)]\, dx$ if the fluid occupies a domain Ω and is subjected to a potential force with a static potential Ψ as the balance of linear momentum (1.36b) reduces for $\vec{u} \equiv \vec{0}$ to the equlibrium of forces — an equation for the equilibrium density distribution $\tilde{\varrho}$

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x \Psi \qquad \text{in } \Omega, \tag{3.1}$$

where $\overline{\vartheta}$ is a constant temperature, with the help of the maximization of the total entropy thanks to the Second Law of Thermodynamics.

The equilibrial equation (3.1) may be easily integrated with the help of the Helmholtz function; especially in the non-physical case $\overline{\vartheta} \equiv 0$ we get from the equation of state the so called cold pressure $p_c(\varrho) = p_\infty \tilde{\varrho}^{\frac{5}{3}}$ with p_∞ from (1.56) and the solution

$$\tilde{\varrho}^{\frac{2}{3}}(x) = \frac{2}{5p_\infty} \Psi(x) + c, \tag{3.2}$$

where we remind Ψ is bounded from above, $p_\infty > 0$ for Fermi gases and c is constant satisfying the bound

$$c \leq \frac{5p_\infty}{2} \left(\frac{M}{|\Omega|} \right)^{\frac{2}{3}} - \|\Psi\|_1 |\Omega|^{-1} \tag{3.3}$$

whose rhs is negative at least for M small enough and therefore there has to be vacuum in an equilibrium figure of a sufficiently diluted fluid under the condition of absolute zero temperature.

Kinematics of the fluid motion is described by the usual conservation laws in the Eulerian description, since the Lagrangian desciption is usually unfeasible as it requires $\vec{u} \in L^\infty(0,T;C^{0,1}_{loc}(\Omega))$ regularity of the velocity field which is generally not known. For incompressible Navier-Stokes equation the existence and uniqueness of Lagrangian trajectories was established by Robinson and Sadowski (Robinson, Sadowski , 2009) under a milder assumption $\vec{u} \in L^{\frac{6}{5}}(0,T;L^\infty(\Omega))$ in three dimensions.

Since the regularity needed for uniqueness of streamlines in the Lagrangian description is not known, we may turn to the question of unique solvability of equation of continuity (1.36a) — here the formal testing with 2ϱ and an application of the Grönwall lemma necessitates that the divergence of the velocity fields should satisfy

$$\mathrm{div}_x \vec{u} \in L^1(0,T;L^\infty(\Omega)) \tag{3.4}$$

even in the incompressible case (see Crippa, Spinolo , 2009; p. 5), (cf. Feireisl, Novotný , 2009; Appendix); this is an important regularity criterion. The assumptions on the vector field $\vec{u}$ itself can be relaxed from e. g. , (see Feireisl, Novotný , 2009), $\vec{u} \in L^1\left(0,T;W^{1,\frac{5}{2}}(\Omega,\mathbb{R}^N)\right)$ to $\vec{u} \in L^1\left(0,T;BV(\Omega,\mathbb{R}^N)\right)$ by fine properties of BV functions and the geometric measure theory or even to fields $\vec{u} \in BV_{loc}\left((0,T)\times\mathbb{R}^N;\mathbb{R}^N\right) \cap C\left((0,T;L^1_{loc}(\mathbb{R}^N);\mathbb{R}^N\right)$, (see Crippa, Spinolo , 2009; pp. 6–7).

Let us recall the total energy balance in the case of a static potential force (3.5), where we can discern convective (the second term) and diffusive energy fluxes (the third term), the latter being a result of random microscopic forces and a source of time irreversibility (cf. entropy production rate ς in (1.47))

$$\partial_t\left(\varrho\left(e+\frac{1}{2}|\vec{u}|^2-\Psi\right)\right)+\mathrm{div}_x\left(\varrho\left(e+\frac{1}{2}\varrho|\vec{u}|^2+p\right)\vec{u}\right)+\mathrm{div}_x\left(\vec{q}-\mathbb{S}\vec{u}\right)=0. \tag{3.5}$$

Therefore, the complete Navier-Stokes-Fourier system is energetically isolated if there holds

$$\varrho\left(\frac{|\vec{u}|^2}{2}+e+p\right)\vec{u}\cdot\vec{n}+\vec{q}\cdot\vec{n}-\mathbb{S}\vec{u}\cdot\vec{n}\Big|_{\partial\Omega}=0, \tag{3.6}$$

which in the case of the mechanical isolation (1.6) and Navier boundary conditions (1.7) amounts to

$$\vec{q}\cdot\vec{n}+\frac{\alpha}{1-\alpha}|\vec{u}|^2\Big|_{\partial\Omega}=0 \tag{3.7}$$

The boundary condition (3.7) holds true for partial and complete slip ($\alpha \in [0,1)$); in the case of the no-slip conditions (1.5) ($\alpha = 1$) the condition has to be changed to just

$$\vec{q}\cdot\vec{n}|_{\partial\Omega}=0 \tag{3.8}$$

as we already know.

3.1.2 Scaled Navier-Stokes-Fourier system

As we have already mentioned in the last subsection, before we take any singular limit, we scale our system, so that characteristic numbers of flows appear; we distinguish two options: *the constitutive scaling* where we think that we observe flows of fluids whose material properties which are bound by thermodynamic relations attain some extreme values in the limit and *the process scaling* where we scale kinematic variables like typical length or time scale or the characteristic speed of the flow; the scaled NSF system (SNSF) is

$$Sr\ \partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0, \tag{3.9}$$

$$Sr\ \partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \frac{1}{Ma^2}\ \nabla_x p(\varrho, \vartheta) = \frac{1}{Re}\ \mathrm{div}_x \mathbb{S} + \frac{1}{Fr^2} \varrho \nabla \Psi. \tag{3.10}$$

$$Sr \partial_t \left(\varrho s\right) + \mathrm{div}_x \left(\varrho s \vec{u}\right) + \frac{p_{ref}}{\varrho_{ref} e_{ref}} \frac{1}{Pe}\ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta}\right) = \frac{p_{ref}}{\varrho_{ref} e_{ref}} \varsigma, \tag{3.11}$$

with

$$\varsigma = \frac{1}{\vartheta}\left(\frac{Ma^2}{Re} \mathbb{S} : \nabla_x \vec{u} - \frac{1}{Pe} \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta}\right), \tag{3.12}$$

and the scaled equation for the total energy gives finally the total energy balance

$$Sr\ \frac{d}{dt} \int_\Omega \left(\frac{Ma^2}{2}\ \varrho |\vec{u}|^2 + \frac{p_{ref}}{\varrho_{ref} e_{ref}} \varrho e - \frac{Ma^2}{Fr^2} \varrho \Psi\right)\ dx = 0 \tag{3.13}$$

where the non-dimensional numbers are defined by referential length L_{ref}, the referential time T_{ref}, the referential speed U_{ref}, the referential pressure p_{ref}, the referential density ϱ_{ref}, the referential shear viscosity μ_{ref}, the referential value of the potential Ψ_{ref}, the referential temperature ϑ_{ref}, and the referential thermal conductivity κ_{ref} : the Strouhal number $Sr := L_{ref} T_{ref}^{-1} U_{ref}$, the Mach number $Ma := U_{ref} p_{ref}^{-\frac{1}{2}} \varrho_{ref}^{\frac{1}{2}}$, the Reynolds number $Re := \varrho_{ref} U_{ref} L_{ref} \mu_{ref}^{-1}$, the Froude number $Fr := U_{ref} \Psi_{ref}^{-\frac{1}{2}}$ and the Péclet number $Pe := p_{ref} L_{ref} U_{ref} \vartheta_{ref}^{-1} \kappa_{ref}^{-1}$.

3.1.3 Incompressible limits $Ma = \varepsilon$

3.1.3.1 Main ideas

The incompressible limits arise when we assume that the referential speed U_{ref} is small with respect to the speed of sound, i. e. Ma is small of order $\varepsilon \to 0+$, Fr may be small as well: $Fr = \varepsilon^\beta, \beta \in [0,1]$ and other non-dimensional numbers (Sr, Re, Pe) are of order 1 and essentially use the scaled total dissipation (energy-entropy) balance

$$\int_\Omega \left(\frac{\varepsilon^2}{2} \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \mathcal{H}_{\tilde{\varrho}_\varepsilon, \overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon)\right)\ dx\,(t_2) +\ \overline{\vartheta} \varsigma_\varepsilon \left[[t_1, t_2] \times \overline{\Omega}\right] \leq \tag{3.14}$$

$$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \mathcal{H}_{\tilde{\varrho}_\varepsilon, \overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon)\right)\ dx\,(t_1),$$

where the shifted Helmholtz function associated with the attracting static solution $(\tilde{\varrho}_\varepsilon, \overline{\vartheta}, 0)$ is

$$\mathcal{H}_{\tilde{\varrho}_\varepsilon, \overline{\vartheta}}(\varrho, \vartheta) := H(\varrho, \vartheta) - (\varrho - \tilde{\varrho}_\varepsilon) H_\varrho(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) - H(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) \tag{3.15}$$

where $\overline{\vartheta}$ is a constant temperature due to (3.12), Fourier's law and a boundedness of initial conditions (in (3.14)) (see below), $\tilde{\varrho}_\varepsilon$ solves the simplified (3.10)

$$\nabla_x p(\tilde{\varrho}_\varepsilon, \overline{\vartheta}) = \varepsilon^{2-2\beta}\tilde{\varrho}_\varepsilon \nabla_x \Psi \qquad \text{in } \Omega, \tag{3.16}$$

for almost all $0 \le t_1 \le t_2 \le T$.

In the sequel we assume that $p_{ref}\varrho_{ref}^{-1}e_{ref}^{-1} = 1$ in (3.11), (3.13), cf. the equation for internal energy density for a monoatomic gas $p = \frac{2}{3}\varrho e_{ref}$ and assume we deal with prepared data

$$\varrho(0,\cdot) = \varrho_{0,\varepsilon} = \tilde{\varrho}_\varepsilon + \varepsilon\varrho_{0,\varepsilon}^{(1)} \tag{3.17}$$

$$\vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta_{0,\varepsilon}^{(1)} \tag{3.18}$$

that are either *ill-prepared* if we assume uniform boundedness of correctors/deviations $\varrho_{0,\varepsilon}^{(1)}, \vartheta_{0,\varepsilon}^{(1)}$ like

$$\left\|\varrho_{0,\varepsilon}^{(1)}\right\|_{L^p(\Omega)} \le c, \tag{3.19a}$$

$$\left\|\vartheta_{0,\varepsilon}^{(1)}\right\|_{L^p(\Omega)} \le c \tag{3.19b}$$

for all $p \ge 2$ or *well-prepared* if we assume (moreover) the convergence of the correctors to zero

$$\varrho_{0,\varepsilon}^{(1)} \to 0, \tag{3.20a}$$

$$\vartheta_{0,\varepsilon}^{(1)} \to 0 \tag{3.20b}$$

in the same spaces and optionally

$$\operatorname{div}_x \vec{u}_{0,\varepsilon} = 0; \tag{3.21}$$

the ill-prepared data present obstacles to the convergence in the convective term due to propagation of acoustic waves as can be seen from Lighthill's acoustic analogy that reads for the easier case of $\Psi \equiv 0$ (thus $\tilde{\varrho} =: \overline{\varrho}$ is a constant)

$$\varepsilon\partial_t r_\varepsilon + \omega \operatorname{div}_x \vec{Q}_\varepsilon = \varepsilon a_\varepsilon, \tag{3.22a}$$

$$\varepsilon\partial_t \vec{Q}_\varepsilon + \nabla_x r_\varepsilon = \varepsilon b_\varepsilon, \tag{3.22b}$$

where the right hand sides of the wave equation written as a system ((see Layton, Novotný , 2010) for the barotropic case) are

$$a_\varepsilon := \frac{2}{3}\overline{\vartheta}\left\{\operatorname{div}_x\left[\frac{s(\overline{\varrho},\overline{\vartheta}) - s(\varrho,\vartheta)}{\varepsilon}\varrho\vec{u} + \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\frac{\nabla_x\vartheta_\varepsilon}{\varepsilon}\right] + \frac{\varsigma}{\varepsilon}\right\}, \tag{3.23a}$$

$$b_\varepsilon := \operatorname{div}_x\left(\mathbb{S}_\varepsilon - \varrho_\varepsilon\vec{u}_\varepsilon\vec{u}_\varepsilon\right) + \nabla_x\left\{\frac{2}{3}\overline{\vartheta}\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) - \overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2} + \right. \tag{3.23b}$$

$$\left.\frac{\overline{\vartheta}}{3}\left[\frac{5\overline{\vartheta}^{\frac{3}{2}}}{\overline{\varrho}}P\left(\frac{\overline{\varrho}}{\overline{\vartheta}^{\frac{3}{2}}}\right) - 2M\left(\frac{\overline{\varrho}}{\overline{\vartheta}^{\frac{3}{2}}}\right)\right]\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon^2} - \frac{p(\varrho_\varepsilon,\vartheta_\varepsilon) - p(\overline{\varrho},\overline{\vartheta})}{\varepsilon^2}\right\}$$

and

$$r_\varepsilon := \frac{2}{3}\overline{\vartheta}\frac{\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)-\overline{\varrho}s(\overline{\varrho},\overline{\vartheta})}{\varepsilon}+\frac{\overline{\vartheta}}{3}\left[\frac{5\overline{\vartheta}^{\frac{3}{2}}}{\overline{\varrho}}P\left(\frac{\overline{\varrho}}{\overline{\vartheta}^{\frac{3}{2}}}\right)-2M\left(\frac{\overline{\varrho}}{\overline{\vartheta}^{\frac{3}{2}}}\right)\right]\frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}, \tag{3.24a}$$

$$\vec{Q}_\varepsilon := \varrho_\varepsilon\vec{u}_\varepsilon, \tag{3.24b}$$

and the square of the speed of propagation

$$\omega := \frac{5}{3}\frac{\overline{\vartheta}^{\frac{5}{2}}}{\overline{\varrho}}P\left(\frac{\overline{\varrho}}{\overline{\vartheta}^{\frac{3}{2}}}\right). \tag{3.25}$$

In order to get a formula for the solution of the wave equation by Duhamel's formula, we modify the definition of the combination of the entropy and density correctors to involve the non-negative measure ς_ε on the left hand side of (3.22a) and change the right hand side of (3.22b) accordingly

$$\tilde{r_\varepsilon} := r_\varepsilon + \frac{2}{3}\overline{\vartheta}\frac{\Sigma_\varepsilon}{\varepsilon} \tag{3.26a}$$

$$\tilde{a_\varepsilon} := a_\varepsilon - \frac{2}{3}\overline{\vartheta}\frac{\varsigma_\varepsilon}{\varepsilon} \tag{3.26b}$$

$$\tilde{b_\varepsilon} := b_\varepsilon + \frac{2}{3}\overline{\vartheta}\frac{\Sigma_\varepsilon}{\varepsilon^2} \tag{3.26c}$$

where Σ_ε is a measure-valued function such that $\partial_t\Sigma_\varepsilon = -\varsigma_\varepsilon$ and split the momentum sequence $\vec{Q}_\varepsilon$ into a solenoidal and gradient part by the standard Helmholtz decomposition $\vec{Q}_\varepsilon = H(\vec{Q}_\varepsilon) + \nabla_x\Psi_\varepsilon$, where the "potential" Ψ_ε satisfies the Neumann problem

$$\triangle\Psi_\varepsilon = \mathrm{div}_x\vec{Q}_\varepsilon \qquad \text{in } \Omega, \tag{3.27a}$$

$$\frac{\partial\Psi_\varepsilon}{\partial\vec{n}} = 0 \qquad \text{on } \partial\Omega, \tag{3.27b}$$

by definition and the wave equation in $(0,T)\times\Omega$ we get when we project the equation (3.22b) on gradients by the orthogonal complement of the Helmholtz projection $H^\perp$

$$\varepsilon\partial_t\tilde{r_\varepsilon} + \omega\,\triangle\Psi_\varepsilon = \varepsilon\tilde{a_\varepsilon}, \tag{3.28a}$$

$$\varepsilon\nabla_x\partial_t\Psi_\varepsilon + \nabla_x\tilde{r_\varepsilon} = \varepsilon H^\perp\left(\tilde{b_\varepsilon}\right). \tag{3.28b}$$

The right hand side of (3.28b) makes a good sense if $\tilde{b_\varepsilon}$ and $\tilde{r_\varepsilon}$ satisfy certain compatibility conditions, for example for the complete slip ($\alpha = 0$ in (1.7)); in such a case we may write an explicit Duhamel's formula for the potential Ψ_ε in terms of its initial condition $\Psi_\varepsilon(0,\cdot)$, initial condition for the corrector $\tilde{r_\varepsilon}(0,\cdot)$, the right hand sides $\tilde{a_\varepsilon}$ and $H^\perp\left(\tilde{b_\varepsilon}\right)$; its time dependence is due to epsilons in (3.28a)–(3.28b) of the type $\exp(iA_\varepsilon t)$, where A_ε is a certain non-negative self-adjoint operator on $L^2(\Omega)$, in the case of zero potential $\Psi = 0$ it is

$$A_\varepsilon := \varepsilon^{-1}\sqrt{-\omega\,\triangle_N}, \tag{3.29}$$

where ω is a constant defined in (3.25) and $-\omega \triangle_N$ is a multiple of the Laplace operator with the homogeneous Neumann boundary condition which may have positive eigenvalues making the potential Ψ_ε, that is the gradient part of the momenta $\vec{Q}_\varepsilon$, an oscillating function in time with frequencies of order ε^{-1}.

The rapid oscillations of $\vec{Q}_\varepsilon$ may spoil its strong convergence, however the limit passage for small Ma is still achieved, since we extract from thermodynamical stability of the fluid under consideration (1.50a)–(1.50b) the inequalities for the shifted Helmholtz function (3.15)

$$C_1(\overline{\varrho},\overline{\vartheta})\left(|\varrho-\overline{\varrho}|^2+|\vartheta-\overline{\vartheta}|^2\right) \le \mathcal{H}_{\overline{\varrho},\overline{\vartheta}}(\varrho,\vartheta) \tag{3.30}$$

for all $(\varrho,\vartheta)\in \mathcal{O}^H_{ess}$,

$$C_2(\overline{\varrho},\overline{\vartheta})\left(\varrho e(\varrho,\vartheta)+\varrho|s(\varrho,\vartheta)|\right) \le \mathcal{H}_{\overline{\varrho},\overline{\vartheta}}(\varrho,\vartheta), \tag{3.31}$$

for all $(\varrho,\vartheta)\in \mathcal{O}^H_{res}$, where $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values is defined as

$$\mathcal{O}^H_{ess} := \left\{(\varrho,\vartheta)\in\mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2}<\varrho<2\overline{\varrho},\ \frac{\overline{\vartheta}}{2}<\vartheta<2\overline{\vartheta}\right\}, \tag{3.32}$$

and its residual counterpart

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2\backslash\mathcal{O}^H_{ess}. \tag{3.33}$$

Let $\{\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon\}_{\varepsilon>0}$ be a family of solutions of the scaled Navier-Stokes system. We call $\mathcal{M}^\varepsilon_{ess}\subset(0,T)\times\Omega$ the set

$$\mathcal{M}^\varepsilon_{ess} = \left\{(t,x)\in(0,T)\times\Omega \ : \ (\varrho_\varepsilon(t,x),\vartheta_\varepsilon(t,x))\in\mathcal{O}_{ess}\right\},$$

and $\mathcal{M}^\varepsilon_{res}=(0,T)\times\Omega\backslash\mathcal{M}^\varepsilon_{ess}$ the corresponding residual set. To any measurable function h we associate its decomposition into essential and residual parts

$$h=[h]_{ess}+[h]_{res},$$

where $[h]_{ess}=h\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}$ and $[h]_{res}=h\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{res}}$. and using coercivity bounds (3.30), (3.31) in (3.14) we infer

$$\begin{aligned}\operatorname*{ess\,sup}_{t\in(0,T)} \Big\{ &\|[\varrho_\varepsilon-\overline{\varrho}]_{ess}(t)\|^2_{L^2(\Omega)}+\|[\vartheta_\varepsilon-\overline{\vartheta}]_{ess}(t)\|^2_{L^2(\Omega)}+\|[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}+\\ &\|[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}\Big\}\le C\varepsilon^2,\end{aligned} \tag{3.34}$$

$$\varsigma_\varepsilon\left[[0,T]\times\overline{\Omega}\right]\le C\varepsilon^2, \tag{3.35}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)}\|\sqrt{\varrho_\varepsilon}\,\vec{u}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{R}^3)}\le C, \tag{3.36}$$

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))}\le C, \tag{3.37}$$

$$\left\|\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}\right\|_{L^2(0,T;W^{1,2}(\Omega))}\le C \tag{3.38}$$

with C independent of ε, that is why we obtain for subsequences almost everywhere convergence in $(0,T)\times\Omega$ of both ϱ_ε and ϑ_ε.

Because densities and temperatures converge almost everywhere, it remains to decide if this is the case also for $\vec{u}_\varepsilon$ - this need not be the case because of acoustic waves; the convergence proof is stated e. g. in (Layton, Novotný , 2010; pp. 20–23) and briefly is as follows: we use the Helmholtz decomposition on both $\vec{Q}_\varepsilon$ and $\vec{u}_\varepsilon$

$$\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon = H^\perp(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes H(\vec{u}_\varepsilon) + H^\perp(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes H^\perp(\vec{u}_\varepsilon) + H(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes \vec{u}_\varepsilon. \tag{3.39}$$

The last term in (3.39) converges to $\overline{\varrho}\vec{u} \otimes \vec{u}$ weakly in $L^1((0,T) \times \Omega; \mathbb{R}^{3\times 3})$ as $\vec{u}_\varepsilon$ converges weakly to $\vec{u}$ in $L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^3))$ along (3.37) and $H(\varrho_\varepsilon \vec{u}_\varepsilon)$ converges strongly to $\overline{\varrho}\vec{u}$ in $L^2\left(0,T; [W^{1,2}(\Omega;\mathbb{R}^3)]^*\right)$ by the Arzelà-Ascoli theorem from (3.10) and the compact embedding $L^5(\Omega;\mathbb{R}^3) \hookrightarrow\hookrightarrow [W^{1,2}(\Omega;\mathbb{R}^3)]^*$. The first term is treated by the weak convergence $H^\perp(\vec{u}_\varepsilon) \rightharpoonup 0$ in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$ by (3.37) and the limit in (3.9) as soon as it is rewritten as

$$H^\perp(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes H(\vec{u}_\varepsilon) = H^\perp((\varrho_\varepsilon - \overline{\varrho})\vec{u}_\varepsilon) \otimes H(\vec{u}_\varepsilon) + \overline{\varrho} H^\perp(\vec{u}_\varepsilon) \otimes H(\vec{u}_\varepsilon). \tag{3.40}$$

The first term on the right hand side of (3.40) converges to 0 as $H^\perp((\varrho_\varepsilon - \overline{\varrho})\vec{u}_\varepsilon) \to 0$ in $L^2(0,T; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3))$ not only weakly, but strongly as well by the strong convergence of ϱ_ε. The second term on the right hand side of (3.40) in fact needs the strong convergence of $H(\vec{u}_\varepsilon)$. This is indeed the case because

$$\overline{\varrho}(H(\vec{u}_\varepsilon))^2 = H((\overline{\varrho} - \varrho_\varepsilon)\vec{u}_\varepsilon) \cdot H(\vec{u}_\varepsilon) + H(\varrho_\varepsilon \vec{u}_\varepsilon) \cdot H(\vec{u}_\varepsilon) \rightharpoonup \overline{\varrho}\vec{u}^2 \tag{3.41}$$

$$\text{in } L^1((0,T) \times \Omega)$$

and by the de la Vallé-Poussin criterion

$$H(\vec{u}_\varepsilon) \to \vec{u} \qquad \text{in } L^1((0,T) \times \Omega; \mathbb{R}^3) \tag{3.42}$$

and by the Kadets-Klee property of the L^2-norm from (3.41) and $H(\vec{u}_\varepsilon) \rightharpoonup \vec{u}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. It implies

$$H(\vec{u}_\varepsilon) \to \vec{u} \qquad \text{in } L^2((0,T) \times \Omega; \mathbb{R}^3) \tag{3.43}$$

Thus,

$$\overline{\varrho} H^\perp(\vec{u}_\varepsilon) \otimes H(\vec{u}_\varepsilon) \to 0 \qquad \text{in } L^1((0,T) \times \Omega; \mathbb{R}^{3\times 3}). \tag{3.44}$$

It remains to settle the weak convergence of the second term in (3.39) $H^\perp(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes H^\perp(\vec{u}_\varepsilon)$. In fact, this limit can be characterized by a measure. For the limit it suffices to prove that $\mathrm{div}_x H^\perp(\varrho_\varepsilon \vec{u}_\varepsilon) \otimes H^\perp(\vec{u}_\varepsilon)$ converges to a gradient, since this can be immersed into the pressure term of the incompressible Navier-Stokes system.

For the isentropic case treated in (Layton, Novotný , 2010) with $\gamma = \frac{5}{3}$ in (1.20) there is $\tilde{a}_\varepsilon \equiv 0$ and for the bounded Ω case we use the Galerkin method for the system (3.22a)–(3.22b) with a special othonormal bases in $L^2(\Omega)$, namely the complete system of orthonormal eigenfunctions of the Neumann problem for the Laplace operator

$$-\triangle \Psi_{j,m} = \Lambda_j \Psi_{j,m} \qquad \text{in } \Omega, \tag{3.45a}$$

$$\frac{\partial \Psi_{j,m}}{\partial \vec{n}} = 0 \qquad \text{on } \partial\Omega, \tag{3.45b}$$

$\{\Psi_{j,m}\}_{j=0,m=1}^{\infty,m_j}$ where m numbers the multiplicity of eigenvalues Λ_j that are ordered to form an increasing sequence of non-negative numbers. From the spectral theory it is known that the least eigenvalue $\Lambda_0 = 0$ and the corresponding eigenspace spanned by $\nabla_x \Psi_{0,1}$ is the space of solenoidal functions $H\left(L^2\left(\Omega, \mathbb{R}^3\right)\right)$. Therefore the space $H^{\perp}\left(L^2\left(\Omega, \mathbb{R}^3\right)\right)$ we decompose into the direct sum of eigenspaces $\overline{\oplus_{j=0,m=1}^{\infty,m_j} \mathrm{span}\, \Lambda_j^{-\frac{1}{2}} \nabla_x \Psi_{j,m_j}}$, where the overline denotes the closure in the space $L^2\left(\Omega, \mathbb{R}^3\right)$. Said briefly, we represent $\tilde{r_\varepsilon}$ as a (Fourier) series in $\Psi_{j,m}$ and $\vec{Q}_\varepsilon$ as a series in $\nabla_x \Psi_{j,m}$ (of course without the first term, a multiple of $\Psi_{0,1}$, resp. $\nabla_x \Psi_{0,1}$) and denote by $\{\cdot\}_M$ a linear combination formed from the series by the truncation of the series involving eigenvalues Λ_j not greater than $M > 0$. With this we can split $H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) \otimes H^{\perp}\left(\vec{u}_\varepsilon\right)$ into four parts and use that in the smooth enough domains Ω the truncated series are smooth as well

$$
\begin{gathered}
H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) \otimes H^{\perp}\left(\vec{u}_\varepsilon\right) = \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M \otimes \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M + \\
\left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M \otimes \left\{H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M\right\} + \\
\left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M\right\} \otimes \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M + \\
\left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M\right\} \otimes \left\{H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M\right\}.
\end{gathered} \tag{3.46}
$$

The rest terms $H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M$ and $H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M$ are small with respect to M, but with respect to ε as well. This follows from the reduction of the latter to the former

$$
\begin{gathered}
H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M = \\
H^{\perp}\left(\left(\varrho_\varepsilon - \overline{\varrho}\right) \vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\left(\varrho_\varepsilon - \overline{\varrho}\right) \vec{u}_\varepsilon\right)\right\}_M + \overline{\varrho}\left[H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M\right] \rightarrow \\
\overline{\varrho} \lim_{\varepsilon \rightarrow 0+} \left[H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M\right]
\end{gathered} \tag{3.47}
$$

in $L^2\left(0, T; L^{\frac{5}{4}}(\Omega; \mathbb{R}^3)\right)$ because of the convergence after (3.40). The other rest term can be made arbitrarily small in $\left[L^2\left((0,T) \times \Omega\right)\right]^3$ by choosing M large enough in the estimate

$$
\begin{gathered}
\left\|H^{\perp}\left(\vec{u}_\varepsilon\right) - \left\{H^{\perp}\left(\vec{u}_\varepsilon\right)\right\}_M\right\|_{L^2(\Omega)}^2 = \sum_{\substack{j>0 \\ \Lambda_j > M}} \Lambda_j^{-1} \sum_{m=1}^{m_j} \Lambda_j \frac{1}{\Lambda_j} \left(\vec{u}_\varepsilon, \nabla_x \Psi_{j,m}\right)_{L^2(\Omega)} \leq \\
\frac{1}{M} \left\|\mathrm{div}_x \vec{u}_\varepsilon\right\|_{L^2(\Omega)}^2 .
\end{gathered} \tag{3.48}
$$

The remaining first term in the right hand side of (3.46) tends to zero with $\varepsilon \rightarrow 0+$ when tested with a gradient of any solenoidal smooth compactly supported (in Ω) test function $\vec{\varphi}$ (converges to a gradient). This is implied by the direct computation supported by the non-homogeneous wave equation for $\nabla_x \Phi_\epsilon = \left\{H^{\perp}\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M$. The finite linear combination for these particular functions satisfies (by (3.28a)–(3.28b)) the system

$$
\varepsilon \partial_t d_\varepsilon + \omega \,\triangle \Phi_\varepsilon = 0, \tag{3.49a}
$$

$$
\varepsilon \partial_t \nabla_x \Phi_\varepsilon + \nabla_x d_\varepsilon = \varepsilon \left\{H^{\perp}\left(\tilde{b_\varepsilon}\right)\right\}_M . \tag{3.49b}
$$

$$
\begin{aligned}
&\frac{1}{\overline{\varrho}}\int_0^T\int_\Omega \left\{H^\perp\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M \otimes \left\{H^\perp\left(\left(\varrho_\varepsilon+\overline{\varrho}-\varrho_\varepsilon\right)\vec{u}_\varepsilon\right)\right\}_M : \nabla_x\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \\
&O(\varepsilon)+\frac{1}{\overline{\varrho}}\int_0^T\int_\Omega \left\{H^\perp\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M \otimes \left\{H^\perp\left(\varrho_\varepsilon \vec{u}_\varepsilon\right)\right\}_M : \nabla_x\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t = \\
&\frac{1}{\overline{\varrho}}\sum_{j=1}^3\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Phi_\varepsilon\partial_j\Phi_\varepsilon\partial_j\varphi_k\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon) = \\
&-\frac{1}{2\overline{\varrho}}\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\left|\nabla_x\Phi_\varepsilon\right|^2\varphi_k\,\mathrm{d}x\,\mathrm{d}t - \frac{1}{\overline{\varrho}}\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\Phi_\varepsilon\,\triangle\Phi_\varepsilon\varphi_k\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon) = \\
&-\frac{1}{2\overline{\varrho}}\sum_{k=1}^3\int_0^T\int_\Omega \partial_k\left|\nabla_x\Phi_\varepsilon\right|^2\varphi_k\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{\overline{\varrho}\omega}\sum_{k=1}^3\int_0^T\int_\Omega \partial_t\left(d_\varepsilon\partial_k\Phi_\varepsilon\right)\varphi_k\,\mathrm{d}x\,\mathrm{d}t - \\
&\frac{\varepsilon}{\overline{\varrho}\omega}\sum_{k=1}^3\int_0^T\int_\Omega d_\varepsilon\partial_t\partial_k\Phi_\varepsilon\varphi_k\,\mathrm{d}x\,\mathrm{d}t + O(\varepsilon) = \frac{1}{2\overline{\varrho}}\sum_{k=1}^3\int_0^T\int_\Omega \left|\nabla_x\Phi_\varepsilon\right|^2\partial_k\varphi_k\,\mathrm{d}x\,\mathrm{d}t - \\
&\frac{\varepsilon}{\overline{\varrho}\omega}\sum_{k=1}^3\int_0^T\int_\Omega d_\varepsilon\partial_k\Phi_\varepsilon\partial_t\varphi_k\,\mathrm{d}x\,\mathrm{d}t - \frac{\varepsilon}{\overline{\varrho}\omega}\int_0^T\int_\Omega d_\varepsilon\left\{H^\perp\left(\tilde{b_\varepsilon}\right)\right\}_M\cdot\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t + \\
&\frac{1}{\overline{\varrho}\omega}\sum_{k=1}^3\int_0^T\int_\Omega d_\varepsilon\partial_k d_\varepsilon\varphi_k\,\mathrm{d}x\,\mathrm{d}t - \frac{\varepsilon}{\overline{\varrho}\omega}\sum_{k=1}^3\int_\Omega d_\varepsilon(0)\partial_k\Phi_\varepsilon(0)\varphi_k(0)\,\mathrm{d}x + O(\varepsilon) = \\
&\frac{1}{2\overline{\varrho}\omega}\int_0^T\int_\Omega \left(\omega\left|\nabla_x\Phi_\varepsilon\right|^2 - \left|d_\varepsilon\right|^2\right)\mathrm{div}_x\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \frac{\varepsilon}{\overline{\varrho}\omega}\int_0^T\int_\Omega d_\varepsilon\nabla_x\Phi_\varepsilon\cdot\partial_t\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \\
&\frac{\varepsilon}{\overline{\varrho}\omega}\int_0^T\int_\Omega d_\varepsilon\left\{H^\perp\left(\tilde{b_\varepsilon}\right)\right\}_M\cdot\vec{\varphi}\,\mathrm{d}x\,\mathrm{d}t - \frac{\varepsilon}{\overline{\varrho}\omega}\int_\Omega d_\varepsilon(0)\nabla_x\Phi_\varepsilon(0)\cdot\vec{\varphi}(0)\,\mathrm{d}x + O(\varepsilon) = \\
&O(\varepsilon)
\end{aligned}
$$

as, say, $\|\Phi_\varepsilon\|_{W^{1,2}(0,T;C^2(\overline{\Omega}))} + \|d_\varepsilon\|_{W^{1,\infty}(0,T;C^2(\overline{\Omega}))} \le c$ by the Galerkin method with the right hand side $\left\|\left\{H^\perp\left(\tilde{b_\varepsilon}\right)\right\}_M\right\|_{L^2(0,T;C^2(\overline{\Omega}))} \le c$. The desired convergence of the convective term is achieved after taking the limit $\lim_{M\to\infty}\lim_{\varepsilon\to 0+}$. Sofar the demonstration of this limit procedure in (Layton, Novotný , 2010) for Ω bounded.

On large domains however the acoustic waves may be spread and damped by dispersion, especially if the domain Ω is so large that their interference with waves reflected from the boundary $\partial\Omega$ is negligible.

This is expressed mathematically by the assumption motivated by Kukučka (Kukučka , 2014) for external domains

$$
\Omega = \cup_{i=1}^\infty \Omega_\varepsilon, \quad \partial\Omega \subset \partial\Omega_\varepsilon, \qquad \lim_{\varepsilon\to 0+} \varepsilon\; inf_{x\in\mathbb{R}^n\setminus\Omega}\,\mathrm{dist}[x,\partial\Omega_\varepsilon\setminus\partial\Omega] = \infty \quad (3.50)
$$

where we may think of the domains Ω_ε as of intersections of Ω with closed balls of radii growing faster than ϵ^{-1}. However a theory has been elaborated that the limit in the convective term can be achieved for larger classes of unbounded domains.

In the unbounded case by intersection with balls above Feireisl (Feireisl , 2010) was able to combine the uniform bound (3.37) with the RAGE Theorem applied to the evolution semigroup generated by the self-adjoint, densely defined in $L^2(\Omega)$ linear operator A_ε with a compact operator given by a localization of $-\triangle_N$ iff its point spectrum in Ω is void. Then $\nabla_x\Psi_\varepsilon = H^\perp(\vec{Q}_\varepsilon) \to 0$ in $L^2((0,T)\times\Omega_\varepsilon;\mathbb{R}^3)$ in (3.28a)–(3.28b).

An alternative method is based on a result of Kato and enables to compute the rate of the decay of local acoustic energy by a particular norm of the resolvent of the operator $\varepsilon A_\varepsilon$.

The rate of the decay of $\nabla_x \Psi_\varepsilon$ is given by the spectrum of that operator and the speed of propagation of acoustic waves; e. g. in the case of an exterior domain Ω one can split the rate into a part where the so called limiting absorption principle for the operator $-\triangle_N$ can be applied and a part where we can utilize a cosine Fourier transform (applied to a localized function from $\mathcal{D}(\Omega)$).

3.1.3.2 Some results from the literature

The fifth chapter of the book by Feireisl and Novotný (Feireisl, Novotný , 2009) treats the limit case of *lowly stratified fluids* where in SNSF (3.9)–(3.13) $Ma = \varepsilon$, $Fr = \sqrt{\varepsilon}$, $Sr = Re = Pe = 1$ is set. They prove under some assumptions on the data that the target system obtained as the limit of SNSF as $\varepsilon \to 0+$ is the renown and heavily used *Oberbeck-Boussinesq approximation* in the case of the perfect slip boundary conditions

$$\operatorname{div}_x \vec{u} = 0, \tag{3.51a}$$

$$\partial_t \vec{u} + \operatorname{div}_x \vec{u} \otimes \vec{u} + \nabla_x \Pi = \frac{\mu}{\overline{\varrho}} \triangle \vec{u} - \alpha \Theta \nabla_x \Psi, \tag{3.51b}$$

$$\overline{\varrho} c_p \left(\partial_t \Theta + \vec{u} \cdot \nabla_x \Theta \right) - \kappa \triangle \Theta = \overline{\varrho} \alpha \overline{\vartheta} \nabla_x \Psi \cdot \vec{u}, \tag{3.51c}$$

where $\alpha := \overline{\varrho}^{-1} \frac{p_\vartheta}{p_\varrho}(\overline{\varrho}, \overline{\vartheta})$ is the coefficient of (volumetric) thermal expansion, $c_p := \left(e_\vartheta + \overline{\varrho}^{-1} \overline{\vartheta} \alpha p_\vartheta\right)(\overline{\varrho}, \overline{\vartheta})$ is the specific heat of the fluid at constant pressure both evaluated at an equilibrium $(\overline{\varrho}, \overline{\vartheta})$ (as are the transport coefficients μ and κ), Π is an "incompressible pressure", $\Theta := \lim_{\varepsilon \to 0+} \vartheta_\varepsilon^{(1)}$ is the corrector of temperature $\overline{\vartheta}$ and the potential Ψ is assumed to be Lipschitz continuous and time independent.

In astrophysics, in stellar interiors, we encounter plasmas showing various phenomena at many length and time scales; for small length scales we may get that the Péclet number Pe measuring the rate of thermal conduction vs. thermal convection is very small. The sixth chapter of (Feireisl, Novotný , 2009) then establishes the link between the primitive system SNSF (3.9)–(3.13) with $Ma = \varepsilon$, $Fr = \varepsilon$, $Pe = \varepsilon^2$, $Sr = Re = 1$ and the *anelastic approximation*; the case where $Ma = Fr$ holds for the so called *strong stratification*. The anelastic system contains a non-constant limit density $\tilde{\varrho}$, a second corrector to temperature $\Theta := \lim_{\varepsilon \to 0+} \vartheta_\varepsilon^{(2)}$ and the limit velocity $\vec{u}$ and therefore has a non-constant speed of sound

$$\operatorname{div}_x \tilde{\varrho} \vec{u} = 0, \tag{3.52a}$$

$$\begin{aligned} \partial_t \left(\tilde{\varrho} \vec{u} \right) + \operatorname{div}_x \tilde{\varrho} \vec{u} \otimes \vec{u} + \tilde{\varrho} \nabla_x \Pi = \mu \left(\nabla_x \vec{u} + \nabla_x \vec{u}^T - \frac{2}{3} \operatorname{div}_x \vec{u} \mathbb{I} \right) + \\ \eta \nabla_x \operatorname{div}_x \vec{u} - \frac{\tilde{\varrho}}{\overline{\vartheta}} \Theta \nabla_x \Psi, \end{aligned} \tag{3.52b}$$

$$\nabla_x p(\tilde{\varrho}, \overline{\vartheta}) = \tilde{\varrho} \nabla_x \Psi, \tag{3.52c}$$

$$-\kappa \triangle \Theta = \tilde{\varrho} \nabla_x \Psi \cdot \vec{u}. \tag{3.52d}$$

Donatelli and Marcati (Donatelli, Marcati , 2016) have recently proved the convergence of weak solutions of the rescaled quantum hydrodynamics system

$$\partial_t \varrho + \operatorname{div}_x \vec{J} = 0, \tag{3.53a}$$

$$\partial_t \vec{J} + \operatorname{div}_x \left(\varrho^{-1} \vec{J} \otimes \vec{J} \right) + \frac{\nabla_x p(\varrho)}{Ma^2} = \operatorname{div}_x \left(\varrho \nabla_x^2 \log \varrho \right) \tag{3.53b}$$

to the unique strong local in time solution to the incompressible Euler equations (IE) as the Mach number $Ma \to 0+$ for the IBVP in the periodic setting for ill-prepared, but strongly convergent initial data for the charge density ϱ, current density $\vec{J}$ and the "kinetic energy" $\left|\vec{J}\right|^2 \varrho^{-1}$. The convergence of the kinetic energy is weak in its natural function space from a-priori estimates, but strong in its solenoidal part $H\left(\left|\vec{J}\right|^2 \varrho^{-1}\right)$ and in the convergence of ϱ_ε.

In the context of strong solution Alazard (Alazard , 2005) establishes hypotheses under which the Cauchy problem for SNSF with $Sr = 1, \Psi \equiv 0$ is locally well-posed in Sobolev spaces $W^{s,2}(\mathbb{R}^N)$ uniformly in Pe, Re, Ma, where $s > (N+2)/2$, $N \geq 3$ and the initial conditions have to be bounded in the space $W^{s+1,2}(\mathbb{R}^N)$. He works with special norms for $p, \vec{u}, \vartheta$ in those spaces, so that his result holds for $Pe = 0$ or $Re = 0$ as well.

Similar problems as for the SNSF system arise also in the **inviscid case**. Here the primitive system are the *compressible Euler equations* (CE). The incompressible limit of CE can be regarded as the limit $Ma = \varepsilon \to 0+$ or the limit of the large adiabatic exponent $\gamma \to \infty$. Chen, Huang, Wang and Xiang (Chen et al. , 2016) have recently proved that there is a compactness framework in the steady CE and found that strong $C^{1,\beta}$, $\beta \in (0, 1)$ solutions of steady CE converge to solutions of incompressible Euler equations (IE) as $\gamma \to \infty$ in the case of full (i. e. compressible Euler system with an equation for internal energy e) Euler system in an infinitely long two-dimensional nozzle for a sufficiently small flow rate. The convergence is strong, almost everywhere and towards weak solutions of the inhomogeneous IE. In the three-dimensional, axially symmetric case they prove the convergence of strong C^1 solutions of the steady homentropic CE system towards the weak solutions of homogeneous IE in an infinitely long nozzle for a sufficiently small flow rate as $\gamma \to \infty$ once again. The convergence is strong and almost everywhere.

The problem of propragation of acoustic waves in the incompressible limit for full CE system is more difficult because the resulting wave equation $\varepsilon^2 \partial_t \left(a_\varepsilon \partial_t u_\varepsilon\right) - \operatorname{div}_x \left(b_\varepsilon \nabla_x u_\varepsilon\right) = r_\varepsilon$ has not got constant coefficients. Métivier and Schochet (Métivier, Schochet , 2001) gave the form of a spatial decay of $a_\varepsilon > 0$ and $b_\varepsilon > 0$ both converging in $C\left([0, T]; W^{s,2}_{loc}(\mathbb{R}^N)\right)$ with $s > (N+2)/2$ as above so that if $r_\varepsilon \to 0$ in $L^2\left((0, T) \times \mathbb{R}^N\right)$ and u_ε is bounded in $C\left([0, T]; W^{2,2}_{loc}(\mathbb{R}^N)\right)$ then $u_\varepsilon \to 0$ in $L^2\left(0, T; L^2_{loc}(\mathbb{R}^N)\right)$. This enables to take the incompressible limit in the context of local in time strong solutions. The result of Métivier and Schochet was later extended to the case of an exterior domain Ω with a compact smooth boundary by Alazard (Alazard , 2005) who therefore got the convergence of classical solutions to CE towards the solutions of IE with ill-prepared data in the interior of the domain Ω. The essential point in the proof is that the convergence is actually strong.

3.1.4 Hydrodynamic limits $Kn = \varepsilon$

As we already noticed in the introduction to this thesis, hydrodynamic models as NSF or Euler equations can be thought of as limits of kinetic models based on variants of the Boltzmann equation.

In this context, one of the most important limits is the limit of small Knudsen number Kn. The Knudsen number is the ratio of the mean free path l in the fluid and the typical length L_{ref}

$$Kn := \frac{l}{L_{ref}}. \tag{3.54}$$

The limit $Kn = \varepsilon$ clearly means that the interactions between individual particles of the fluid are more and more common and that they become localized at the same time. In the limit $\varepsilon \to 0+$ we expect to arrive at a hydrodynamical model such as the incompressible Navier-Stokes-Fourier system for the moments of the original rescaled Boltzmann equation.

3.1.4.1 Some results from the literature

Dimarco and Motsch (Dimarco , 2016) have recently carried out this limit for the linearized Boltzmann equation with its right hand side given by a Bhatnagar-Gross-Crook (BGK) operator describing random jumps of particles with their jump rate given as a compound Poisson process in time and a new direction as a probability density function centered at the average direction at the place of the jump (alignment of the particle to the flow of others). The limit system is up to multiplicative constants and projection of the force due to pressure on the hyperplane perpendicular to the velocity of flow the homentropic CE system with the polytropic index $\gamma = 1$.

On the other hand Briant (Briant , 2015) has studied the limit regime $Sr = Kn = \varepsilon$ of the linearized Boltzmann equation with Grad's angular cut-off and a hard potential and proved under the assumption of smallness of initial data (with bounds independent of ε) and a special structure of the perturbation of the rhs of the Boltzmann equation (with respect to the Maxwellian) that the limit system is the Oberbeck-Boussinesq approximation ((3.51a)–(3.51c) with $\Psi \equiv 0$, $\overline{\varrho} = 1$, $\alpha = 1$, $c_p = 1$) in the periodic boundary condition case. The convergence is weak, but under an additional assumption on the regularity of the perturbation it is also strong in a certain norm with the rate $|\varepsilon \log \varepsilon|$ iff the data are well-prepared.

3.1.5 Inviscid limits $Re = \frac{1}{\varepsilon}$

Inviscid limits are one of the most prominent topics ín the mathematical fluid dynamics. The typical question is if we can rigorously substantiate the limit transition from the Navier-Stokes equations to the Euler system when the dynamical viscosity vanishes. This problem is one of the singularly perturbed problems as the viscosity multiplies the term with the highest derivative in the Navier-Stokes equations; moreover the no-slip boundary conditions (1.5) or more generally the Navier boundary conditions (1.7) in the limit change to the impermeability condition (1.6). This is usually accompanied by the appearance of the boundary layer of the width of order $Re^{-\frac{1}{2}}$. The fluid motion in the boundary layer is governed by the Prandtl equations, but they are ill-posed at least in two spatial dimensions in the Sobolev spaces after linearization around the shear profile given by the solution to a heat equation (Gérard-Varet , 2009).

3.1.5.1 Some results from the literature

Li (Li , 2016) has recently performed the vanishing viscosity, quasineutral limit $Re \to \infty$, $\lambda_D \to 0+$ in the one-dimensional bipolar compressible Navier-Stokes-Poisson system. This system consists of the barotropic compressible Navier-Stokes equations with the force on the right hand side given by the electrostatic part of the Lorentz force (magnetic induction neglected) written for two species (with a possible application in plasmas: electrons and ions) together with Gauss's law

$$\lambda_D^2 \mathrm{div}_x \vec{E} = \varrho_1 - \varrho_2, \tag{3.55}$$

where $\lambda_D \geq 0$ is the Debye shielding length, and $\vec{E}$ is the vector electric field. Of course divergence operator reduces to the derivative with respect to the spatial variable (x). The author proves that we can choose well-prepared data — an approximate rarefaction wave given by the solution to the Burgers equation with a special initial condition depending on $\lambda_D = \varepsilon^a$ connecting vacuum with a constant state $(\varrho_+, \vec{u}_+)$ such that there is a global smooth solution converging with $\varepsilon \to 0+$ towards a rarefaction wave solution of the compressible Euler system pointwise (except for the origin) and uniformly except for the initial layer. Here $a > 0$ is a suitable power dependent on the polytropic index $\gamma > 1$, Re is selected as $\varepsilon^{-a} |\log \varepsilon|^{-1}$ and for the electric field Li chooses the initial condition $\varepsilon^{-1}\overline{E}_0$ and $\lim_{x\to\infty} \overline{E}(x,t) = 0$. The rescaled electric field $\overline{E}$ converges uniformly to 0 except for the initial layer as well.

Sueur (Sueur , 2014) used the relative energy estimates for the NSF system to prove the strong convergence in natural energy norms from a-priori estimates of weak solutions to the barotropic compressible Navier-Stokes equations towards local-in-time continuously differentiable solution of the homentropic CE system as $Re = \varepsilon \to 0+$ if the initial conditions converge strongly in some similar norms and if the viscous dissipation rate in a boundary layer whose width is proportional to the width of Prandtl's boundary layer tends to zero. This condition is needed for the no-slip boundary conditions (1.5), but for the Navier boundary conditions (1.7) it is replaced by the stipulation that the rescaled friction coefficient $\varepsilon^{\frac{\alpha}{1-\alpha}}$ tends to 0 where α is the coefficient from (1.7).

4. Low Mach and Péclet number limit for a model of stellar tachocline and upper radiative zones

4.1 Introduction

Our motivation in this work is the rigorous analysis of the equations describing parts of stars called radiative zones which are one of the most basic structures constituting stars among cores, convection zones, photospheres and atmospheres. Our model can be also applied to tachoclines which are transition layers between convection and radiative zones of stars. In this context it is conjectured that magnetic field of stars arises when the poloidal orientation of magnetic fields changes to toroidal and that a dynamo effect is present in tachoclines (Shore , 1992). Tachoclines are not homogeneous and stable structures and they move steadily. In their upper parts the Péclet numbers are high (of the order 600), but in the vicinity of the radiative part they drop below 1. Their distinctive feature concerns rotation, naïvely speaking the convective zone behaves in this respect as a fluid and rotates differentially, whereas the radiative zone more like a solid and rotates as a rigid body. The origin of these rotational changes has preocuppied astrophysicists and astronomes, particularly in connection with helioseismological observations (Broomhall et al. , 2014)

Gravitational forces in these regions are high, however the fluid is no longer strongly stratified as show non-dimensional numbers associated to the **solar tachocline**. Namely *the Froude number Fr* measuring the strength of gravitational interactions (see Section 4.3 below for precise definitions) is $Fr = 3.11 \times 10^{-3}U$, where U is the referential speed of flow in SI units. *The Mach number Ma* measuring the compressibility is $Ma = 1.49 \times 10^{-7}U$, i. e. the fluid is almost incompressible for sufficiently slow motions and one has $Fr^2 \sim Ma$ (Mach number is due to high temperatures when radiation dominates). Finally *Péclet number Pe* need not to be sufficiently small in the solar tachocline (we assume $Ma^2 = Pe$), but thermal diffusivity in **giant stars** can be seven orders of magnitude larger than that of the Sun (see Garaud, Kulenthirarajah , 2015; page 22).

Notice in conclusion that our low stratification model can be applied to other compact stellar objects, as the fraction of Fr and Ma depends on the ratio of temperature, density and is inversely proportional to the square of characteristic length. Therefore white dwarfs are too cold to be described by low stratification models, but neutron stars, especially newly born, are not. Validity of classical MHD may be restricted to their (outer) crusts though; in their superfluid cores a quantum description is inevitable.

Let us complete this physical introduction by drawing the reader's attention to the fact that models in stellar physics are computationally time consuming. Rieutord (2014) has estimated for example that modelling a single supergranule on the Sun would require having more than power of the Sun at our disposal!

That is why (Lignières , 1999) has initiated studies of models at small Péclet number as through the Boussinesq-Oberbeck approximation density variations with temperature enter through the buoyancy force only and moreover temperature can be expressed by the velocity field.

In our previous work (Donatelli, Ducomet, Nečasová , 2015) we analyzed a thick disk model for the Mach number of order ε, $\varepsilon \to 0$ whereas the Peclet number was of order 1. Instead as in (Novotný, Růžička, Thäter , 2011), in the present one we consider a model where the Peclet number is of order ε^2 and the domain is general.

The mathematical model we consider is the compressible heat conducting MHD system (Ducomet, Feireisl , 2006) describing the motion of a viscous plasma confined in Ω, a 3D domain, moreover as we suppose a global rotation of the system, some new terms appear due to the change of frame and we also suppose that the fluid exchanges energy with radiation through radiative cooling/heating (see (Ducomet, Feireisl , 2006), (Ducomet, Kobera, Nečasová , 2014)), but neglect radiative accelerations.

More precisely, the non-dimensional system of equations giving the evolution of the mass density $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, the (divergence-free) magnetic field $\vec{B} = \vec{B}(x,t)$, and the radiative intensity $I = I(x,t,\vec{\omega},\nu)$ as functions of the time $t \in (0,T)$, the spatial coordinate $x = (x_1,x_2,x_3) \in \Omega \subset \mathbb{R}^3$, and (for I) the angular and frequency variables $(\vec{\omega},\nu) \in \mathcal{S}^2 \times \mathbb{R}_+$, reads as follows

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \quad \text{in } (0,T) \times \Omega, \tag{4.1}$$

$$\partial_t(\varrho \vec{u}) + \mathrm{div}_x(\varrho \vec{u} \otimes \vec{u}) + \nabla_x p + 2\varrho \vec{\chi} \times \vec{u} =$$

$$\mathrm{div}_x \mathbb{S} + \varrho \nabla \Psi + \frac{1}{2} \varrho \nabla_x |\vec{\chi} \times \vec{x}|^2 + \vec{j} \times \vec{B} \qquad \text{in } (0,T) \times \Omega, \tag{4.2}$$

$$\partial_t (\varrho e) + \mathrm{div}_x (\varrho e \vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \, \mathrm{div}_x \vec{u} + \vec{j} \cdot \vec{E} - S_E \quad \text{in } (0,T) \times \Omega, \tag{4.3}$$

$$\frac{1}{c} \partial_t I + \vec{\omega} \cdot \nabla_x I = S \quad \text{in } (0,T) \times \Omega \times (0,\infty) \times \mathcal{S}^2. \tag{4.4}$$

$$\partial_t \vec{B} + \mathrm{curl}_x(\vec{B} \times \vec{u}) + \mathrm{curl}_x(\lambda \, \mathrm{curl}_x \vec{B}) = 0 \qquad \text{in } (0,T) \times \Omega. \tag{4.5}$$

$$-\Delta \Psi = 4\pi G(\tilde{\eta} \varrho + g) \quad \text{in } (0,T) \times \Omega. \tag{4.6}$$

In the electromagnetic source terms, electric current $\vec{j}$ and electric field $\vec{E}$ are interrelated by *Ohm's law*

$$\vec{j} = \sigma(\vec{E} + \vec{u} \times \vec{B}),$$

and *Ampère's law*

$$\zeta \vec{j} = \mathrm{curl}_x \vec{B},$$

where $\zeta > 0$ is the (constant) magnetic permeability. Moreover in (4.5) $\lambda = \lambda(\vartheta) > 0$ is the magnetic diffusivity of the fluid.

In (4.6) Ψ is the gravitational potential and the corresponding source term in (4.2) is the Newton force $\varrho \nabla \Psi$. G is the Newton constant and g is a given function, modelling an external gravitational effect. Supposing that ϱ is extended by 0 outside Ω and solving (4.6), we have

$$\Psi(t,x) = G \int_\Omega K(x-y)(\eta \varrho(t,y) + g(y)) \, dy,$$

where $K(x) = \frac{1}{|x|}$, and the parameter $\tilde{\eta}$ may take the values 0 or 1: for $\tilde{\eta} = 1$ selfgravitation is present and for $\tilde{\eta} = 0$ gravitation only acts as an external field (the attraction by a given massive central object, modeled by g, may prevail over the selfgravitation, cf. (Padmanabhan , 2001)).

We also assume that the system is globally rotating at a uniform velocity χ around the vertical direction $\vec{e}_3$ and we denote $\vec{\chi} = \chi\vec{e}_3$. Then Coriolis acceleration term $2\varrho\vec{\chi}\times\vec{u}$ appears in the system, together with the centrifugal force term $\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2$ (see Choudhuri , 1998).

We consider here the simplified model studied in (Ducomet, Nečasová , 2014) where radiation does not appear in the momentum equation (see also Teleaga et al. , 2006): only the source term S_E is present in the energy equation

$$S_E(t,x) = \int_{\mathcal{S}^2}\int_0^\infty S(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu.$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through Maxwell's relation

$$\frac{\partial e}{\partial \varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial \vartheta}\right). \tag{4.7}$$

Furthermore, $\mathbb{S}$ is the Newtonian viscous stress tensor determined by

$$\mathbb{S} = \mu\left(\nabla_x\vec{u} + \nabla_x^T\vec{u} - \frac{2}{3}\mathrm{div}_x\vec{u}\,\mathbb{I}\right) + \eta\ \mathrm{div}_x\vec{u}\ \mathbb{I}, \tag{4.8}$$

where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ and the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature. Similarly, $\vec{q}$ is the heat flux given by Fourier's law

$$\vec{q} = -\kappa\nabla_x\vartheta, \tag{4.9}$$

with a heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$. Finally,

$$S = S_{a,e} + S_s, \tag{4.10}$$

where

$$S_{a,e} = \sigma_a\Big(\mathfrak{B}(\nu,\vartheta) - I\Big),\ S_s = \sigma_s\left(\tilde{I} - I\right). \tag{4.11}$$

In this formula $\tilde{I} := \frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega}$ and $\mathfrak{B}(\nu,\vartheta) = 2h\nu^3c^{-2}\left(e^{\frac{h\nu}{k\vartheta}} - 1\right)^{-1}$ is the radiative equilibrium function where h and k are the Planck and Boltzmann constants, $\sigma_a = \sigma_a(\nu,\vartheta) \geq 0$ is the absorption coefficient and $\sigma_s = \sigma_s(\nu,\vartheta) \geq 0$ is the scattering coefficient. More restrictions on these structural properties of constitutive quantities will be imposed in Section 4.2 below.

System (4.1) – (4.6) is supplemented with the "no-slip, thermal isolation, perfect conductor, no reflection, no radiative entropy flux" boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0,\ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{E}\times\vec{n}|_{\partial\Omega} = 0, \tag{4.12}$$

$$I(t,x,\nu,\vec{\omega}) = 0 \text{ on } \Gamma_-,\ \vec{q}^R\cdot\vec{n}(x) = 0 \text{ for } x\in\partial\Omega, \tag{4.13}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$, $\Gamma_- := \{(x, \vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 : \vec{\omega} \cdot \vec{n}_x \leq 0\}$ and the radiative entropy flux $\vec{q}^R$ will be defined in the next Section. Similarly we define $\Gamma_+ := \partial\Omega \times \mathcal{S}^2 \setminus \Gamma_-$.

Let us mention that previous works have been achieved in the previous framework but, to our knowledge, not in the case of rotating fluid with radiation (with an exception of (Donatelli, Ducomet, Nečasová , 2015)). Among themKukučka (2011) studied the case when Mach and Alfvén number go to zero in the case of a bounded domain and Novotný and collaborators (Novotný, Růžička, Thäter , 2011) investigated the problem in the case of strong stratification. Let us also mention the works of Trivisa et al. (Kwon, Trivisa , 2011) and Wang et al. (Hu, Wang , 2009), and related articles of Jiang et al. (Jiang, Ju, Li , 2012; 2010, Jiang et al. , 2014).

Our work differs from theirs in that we take a larger Froude number and add radiation and non-inertial effects.

This paper is organized as follows.

In Section 4.2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (4.1) – (4.13), and state the existence result for our model. In Section 4.3 we compute the formal asymptotics of the problem. Uniform bounds imposed on weak solutions by the data are derived in Section 4.4. The convergence theorem is proved in Section 4.5. Existence of a solution for the target system is briefly given in the Appendix.

4.2 Hypotheses and stability result

As in (Ducomet, Feireisl, Nečasová , 2011) we consider a pressure law in the form

$$p(\varrho, \vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{4.14}$$

where $P : [0, \infty) \to [0, \infty)$ is a given function with the following properties:

$$P \in C^2([0, \infty)), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{4.15}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{4.16}$$

$$\lim_{Z \to \infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{4.17}$$

According to Maxwell's relation (4.7), the specific internal energy e is

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \frac{\vartheta^{3/2}}{\varrho} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{4.18}$$

and the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{4.19}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

To ensure positivity of the total entropy production rate, as in (Donatelli, Ducomet, Nečasová , 2015), in this paper we explicitly introduce the entropy for the photon gas in the sequel.

The transport coefficients μ, η, κ and λ are continuously differentiable and Lipschitz functions of the absolute temperature with the properties,

$$c_1(1+\vartheta) \leq \mu(\vartheta),\ \mu'(\vartheta) < c_2,\ 0 \leq \eta(\vartheta) \leq c_3(1+\vartheta), \tag{4.20}$$

$$c_1(1+\vartheta^r) \leq \kappa(\vartheta) \leq c_2(1+\vartheta^r) \tag{4.21}$$

$$c_5(1+\vartheta) < \lambda(\vartheta) \leq c_4(1+\vartheta^p) \tag{4.22}$$

for any $\vartheta \geq 0$, for a $1 \leq p < \frac{17}{6}$ and $r = 3$.

Moreover we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 < \sigma_a(\nu,\vartheta) \leq c_1, 0 \leq \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \leq c_1, \tag{4.23}$$

$$0 \leq \sigma_a(\nu,\vartheta)B(\nu,\vartheta), |\partial_\vartheta\{\sigma_a(\nu,\vartheta)B(\nu,\vartheta)\}| \leq c_2, \tag{4.24}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \leq h(\nu),\ h \in L^1(0,\infty). \tag{4.25}$$

for all $\nu \geq 0$, $\vartheta \geq 0$, where $c_{1,2,3,4,5}$ are positive constants.

Let us recall some definitions introduced in (Ducomet, Feireisl, Nečasová , 2011).

- In the weak formulation of the Navier-Stokes-Fourier system the *equation of continuity* (4.1) is replaced by its (weak) renormalized version (DiPerna, Lions , 1989) represented by the family of integral identities

$$\int_0^T \int_\Omega \Big[\Big(\varrho + b(\varrho)\Big)\partial_t \varphi + \Big(\varrho + b(\varrho)\Big)\vec{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\ \varphi\Big]\,\mathrm{d}x\,\mathrm{d}t$$

$$= -\int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot)\,\mathrm{d}x \tag{4.26}$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty([0,\infty))$, $b' \in C_c^\infty([0,\infty))$, where (4.26) implicitly includes the initial condition $\varrho(0,\cdot) = \varrho_0$.

- Similarly, the *momentum equation* (4.2) is replaced by

$$\int_0^T \int_\Omega \Big((\varrho\vec{u})\cdot\partial_t\vec{\varphi} + (\varrho\vec{u}\otimes\vec{u}) : \nabla_x\vec{\varphi} + p\ \mathrm{div}_x\vec{\varphi} + 2\varrho\vec{\chi}\times\vec{u}\cdot\vec{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t \tag{4.27}$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\vec{\varphi} - \varrho\nabla_x\Psi\cdot\vec{\varphi} - \vec{j}\times\vec{B}\cdot\vec{\varphi} - \frac{1}{2}\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2\cdot\vec{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t - \int_\Omega (\varrho\vec{u})_0\cdot\vec{\varphi}(0,\cdot)\ \mathrm{d}x$$

for any $\vec{\varphi} \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. As usual, for (4.27) to make sense, the field $\vec{u}$ must belong to a certain Sobolev space with respect to the spatial variable and we require that

$$\vec{u} \in L^2\left(0,T; W_0^{1,2}\left(\Omega;\mathbb{R}^3\right)\right), \tag{4.28}$$

where (4.28) already includes the no-slip boundary conditions (4.12).

- The *magnetic equation* (4.5) is replaced by

$$\int_0^T \int_\Omega \Big(\vec{B}\cdot\partial_t\vec{\varphi} - (\vec{B}\times\vec{u} + \lambda\mathrm{curl}_x\vec{B})\cdot\mathrm{curl}_x\vec{\varphi}\Big)\ \mathrm{d}x\ \mathrm{d}t + \int_\Omega \vec{B}_0\cdot\vec{\varphi}(0,\cdot)\ \mathrm{d}x = 0, \tag{4.29}$$

to be satisfied for any vector field $\vec{\varphi} \in \mathcal{D}([0,T) \times \Omega; \mathbb{R}^3)$.

Here, according to the boundary conditions, one has to take

$$\vec{B}_0 \in L^2(\Omega),\ \mathrm{div}_x \vec{B}_0 = 0 \text{ in } \mathcal{D}'(\Omega),\ \vec{B}_0 \cdot \vec{n}|_{\partial\Omega} = 0. \tag{4.30}$$

Temam (Following 1977; Theorem 1.4), $\vec{B}_0$ belongs to the closure of all solenoidal functions from $\mathcal{D}(\Omega)$ with respect to the L^2-norm.

Anticipating (see (4.42) below) we see that

$$\vec{B} \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)),\ \mathrm{curl}_x \vec{B} \in L^2(0,T; L^2(\Omega;\mathbb{R}^3))$$

and we deduce from (4.29) that

$$\mathrm{div}_x \vec{B}(t) = 0 \text{ in } \mathcal{D}'(\Omega),\ \vec{B}(t) \cdot \vec{n}|_{\partial\Omega} = 0 \text{ for a. a. } t \in (0,T).$$

In particular, using (Duvaut, Lions , 1976; Theorem 6.1), we conclude that

$$\vec{B} \in L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^3)),\ \mathrm{div}_x \vec{B}(t) = 0,\ \vec{B} \cdot \vec{n}|_{\partial\Omega} = 0 \text{ for a. a. } t \in (0,T). \tag{4.31}$$

- From (4.2) and (4.3) we find the *energy conservation law*

$$\partial_t \Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{1}{2\zeta}|\vec{B}|^2 \Big) + \mathrm{div}_x \Big((\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + p)\vec{u} + \vec{E} \times \vec{B} - \mathbb{S}\vec{u} + \vec{q} \Big)$$

$$= \varrho \nabla_x \Psi \cdot \vec{u} + \frac{1}{2}\varrho \nabla_x |\vec{\chi} \times \vec{x}|^2 \cdot \vec{u} - S_E. \tag{4.32}$$

As the gravitational potential Ψ is determined by equation (4.6) considered on the whole space $\mathbb{R}^3$, the density ϱ being extended to be zero outside Ω we observe that

$$\int_\Omega \varrho \nabla_x \Psi \cdot \vec{u} \ \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \varrho \Psi \ \mathrm{d}x,$$

in the same stroke

$$\frac{1}{2} \int_\Omega \varrho \nabla_x |\vec{\chi} \times \vec{x}|^2 \cdot \vec{u} \ \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{2} \int_\Omega \varrho |\vec{\chi} \times \vec{x}|^2 \ \mathrm{d}x.$$

Denoting now by E^R the radiative energy given by

$$E^R(t,x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(t,x,\vec{\omega},\nu) \ \mathrm{d}\vec{\omega} \ \mathrm{d}\nu, \tag{4.33}$$

and integrating the radiative transfer equation (4.4), we get

$$\partial_t \int_\Omega E^R \ \mathrm{d}x + \int \int_{\partial\Omega \times \mathcal{S}^2,\ \vec{\omega}\cdot\vec{n} \geq 0} \int_0^\infty I(t,x,\vec{\omega},\nu)\, \vec{\omega} \cdot \vec{n} \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega} \ \mathrm{d}S_x \ \mathrm{d}t = \int_\Omega S_E \ \mathrm{d}x.$$

so, by using boundary conditions, we can rearrange (4.32), as follows,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{1}{2\zeta}|\vec{B}|^2 - \frac{1}{2}\varrho\Psi - \frac{1}{2}\varrho|\vec{\chi} \times \vec{x}|^2 + E^R \Big) \ \mathrm{d}x$$

$$+ \int \int_{\Gamma_+} \int_0^\infty I(t,x,\vec{\omega},\nu)\, \vec{\omega} \cdot \vec{n} \ \mathrm{d}\nu \ \mathrm{d}\vec{\omega} \ \mathrm{d}S_x = 0. \tag{4.34}$$

- Finally, dividing (4.3) by ϑ and using Maxwell's relation (4.7), we obtain the *entropy equation*

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma, \tag{4.35}$$

where

$$\varsigma = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\zeta} |\mathrm{curl}_x \vec{B}|^2 \right) - \frac{S_E}{\vartheta}, \tag{4.36}$$

where the first term $\varsigma_m := \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\zeta} |\mathrm{curl}_x \vec{B}|^2 \right)$ is the (non-negative) electromagnetic matter entropy production.

In order to identify the second term in (4.36), let us recall from (Balian , 2007) the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] d\vec{\omega}\, d\nu, \tag{4.37}$$

where $n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2} \int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left[n \log n - (n+1) \log(n+1) \right] \vec{\omega}\; d\vec{\omega}\, d\nu, \tag{4.38}$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x \vec{q}^R = -\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \log \frac{n}{n+1}\; S\; d\vec{\omega} d\nu =: \varsigma^R. \tag{4.39}$$

With the identity $\log \frac{n(B)}{n(B)+1} = -\frac{h\nu}{k\vartheta}$ with $\mathfrak{B} = \mathfrak{B}(\vartheta, \nu)$ denoting Planck's function, and using the definition of S, the right-hand side of (4.39) rewrites

$$\varsigma^R = \frac{S_E}{\vartheta} - \frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\mathfrak{B})}{n(\mathfrak{B})+1} \right] \sigma_a (\mathfrak{B} - I)\; d\vec{\omega}\, d\nu$$

$$-\frac{k}{h} \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (\tilde{I} - I)\; d\vec{\omega} d\nu,$$

where we used the hypothesis that the transport coefficients $\sigma_{a,s}$ do not depend on $\vec{\omega}$. So we obtain finally

$$\partial_t \left(\varrho s + s^R \right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R \right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = \varsigma + \varsigma^R. \tag{4.40}$$

and equation (4.35) is replaced in the weak formulation by the inequality

$$\int_0^T \int_\Omega \left((\varrho s + s^R) \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \left(\frac{\vec{q}}{\vartheta} + \vec{q}^R \right) \cdot \nabla_x \varphi \right) \mathrm{d}x\ \mathrm{d}t \tag{4.41}$$

$$\leq - \int_\Omega (\varrho s + s^R)_0 \varphi(0, \cdot)\ \mathrm{d}x - \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\zeta} |\mathrm{curl}_x \vec{B}|^2 \right) \varphi\ \mathrm{d}x\ \mathrm{d}t$$

$$-\frac{k}{h} \int_0^T \int_\Omega \left[\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\mathfrak{B})}{n(\mathfrak{B})+1} \right] \sigma_a (\mathfrak{B} - I)\; d\vec{\omega}\, d\nu \right.$$

$$+\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s(\tilde{I} - I)\, d\vec{\omega}\, d\nu \Bigg] \varphi \, \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, where the sign of all the terms in the right hand side may be controlled.

- Since replacing equation (4.3) by inequality (4.41) would result in a formally under-determined problem, system (4.26), (4.27), (4.41) must be supplemented with the *total energy balance*

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2\zeta} |\vec{B}|^2 - \frac{1}{2} \varrho \Psi - \frac{1}{2} \varrho |\vec{\chi} \times \vec{x}|^2 + E^R \right) (\tau, \cdot) \, \mathrm{d}x \tag{4.42}$$

$$+ \int_0^\tau \int \int_{\Gamma_+} \int_0^\infty I(t, x, \vec{\omega}, \nu)\, \vec{\omega} \cdot \vec{n} \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho\vec{u})_0|^2 + (\varrho e)_0 + \frac{1}{2\zeta} |\vec{B}_0|^2 - \frac{1}{2} \varrho_0 \Psi_0 - \frac{1}{2} \varrho_0 |\vec{\chi} \times \vec{x}|^2 + E_0^R \right) \, \mathrm{d}x,$$

where E_0^R is given by

$$E_0^R(x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(0, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu.$$

The transport equation (4.4), can be extended to the whole physical space $\mathbb{R}^3$ provided we set $\sigma_a(x, \nu, \vartheta) = \mathbb{I}_\Omega \sigma_a(\nu, \vartheta)$ and $\sigma_s(x, \nu, \vartheta) = \mathbb{I}_\Omega \sigma_s(\nu, \vartheta)$, where $\mathbb{I}_A$ is the characteristic function of a set A and take the initial distribution $I_0(x, \vec{\omega}, \nu)$ to be zero for $x \in \mathbb{R}^3 \setminus \Omega$. Accordingly, for any fixed $\vec{\omega} \in \mathcal{S}^2$, equation (4.4) can be viewed as a linear transport equation defined in $(0,T) \times \mathbb{R}^3$, with a right-hand side S. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$.

Definition 4.2.1 *We say that $\varrho, \vec{u}, \vartheta, \vec{B}, I$ is a weak solution of problem (4.1) – (4.6) iff*

$$\varrho \geq 0, \ \vartheta > 0 \ \textit{for a.a.}\ (t,x) \times \Omega, \ I \geq 0 \ \textit{a.a. in}\ (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),$$

$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \vartheta \in L^\infty(0,T; L^4(\Omega)),$$
$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)),$$
$$\vartheta \in L^2(0,T; W^{1,2}(\Omega)),$$
$$\vec{B} \in L^2(0,T; W^{1,2}(\Omega; \mathbb{R}^3)), \quad \vec{B} \cdot \vec{n} \Big|_{\partial\Omega} = 0$$
$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty))),$$

and if ϱ, $\vec{u}$, ϑ, $\vec{B}$, I satisfy the integral identities (4.26), (4.27), (4.41), (4.29), (4.42), together with the transport equation (4.4).

The stability result of (Ducomet, Kobera, Nečasová , 2014) reads now

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (4.14) – (4.19), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (4.20) – (4.25).*

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (4.1) – (4.13) in the sense of Definition 4.2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \text{ in } L^{5/3}(\Omega), \tag{4.43}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + \frac{1}{2\zeta}|\vec{B}_\varepsilon|^2 - \frac{1}{2}\varrho_\varepsilon \Psi_\varepsilon - \frac{1}{2}\varrho_\varepsilon |\vec{\chi} \times \vec{x}|^2 + E^R_\varepsilon \right)(0,\cdot)\, \mathrm{d}x \equiv \tag{4.44}$$

$$\int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E^R_{0,\varepsilon} + \frac{1}{2\zeta}|\vec{B}_{0,\varepsilon}|^2 - \frac{1}{2}\varrho_{\varepsilon,0}|\vec{\chi} \times \vec{x}|^2 - \frac{1}{2}\varrho_{\varepsilon,0}\Psi_{\varepsilon,0} \right)\mathrm{d}x \le E_0,$$

$$\int_\Omega [\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) + s^R(I_\varepsilon)](0,\cdot)\, \mathrm{d}x \equiv \int_\Omega (\varrho s + s^R)_{0,\varepsilon}\, \mathrm{d}x \ge S_0,$$

and

$$0 \le I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \le I_0, \ |I_{0,\varepsilon}(\cdot,\nu)| \le h(\nu) \text{ for a certain } h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W^{1,2}_0(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$
$$\vec{B}_\varepsilon \to \vec{B} \text{ weakly in } L^2(0,T; W^{1,2}(\Omega;\mathbb{R}^3)),\ \vec{B}\cdot\vec{n}\Big|_{\partial\Omega} = 0$$

and

$$I_\varepsilon \to I \text{ weakly-(*) in } L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, \vec{B}, I\}$ is a weak solution of problem (4.1) – (4.13).

4.3 Formal scaling analysis

In order to identify the appropriate limit regime we perform a general scaling, denoting by $L_{ref}, T_{ref}, U_{ref}, \rho_{ref}$, ϑ_{ref}, p_{ref}, e_{ref}, μ_{ref}, λ_{ref}, κ_{ref}, the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity), by I_{ref}, ν_{ref}, $\sigma_{a,ref}$, $\sigma_{s,ref}$, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients), by χ_{ref} the reference rotation velocity, and by ζ_{ref}, B_{ref} the reference electrodynamic quantities (permeability and magnetic induction).

We also assume the compatibility conditions $p_{ref} \equiv \rho_{ref} e_{ref}$, $\nu_{ref} = \frac{k\vartheta_{ref}}{h}$, $I_{ref} = \frac{2h\nu^3_{ref}}{c^2}$, $\tilde{\lambda} = \frac{\lambda_{ref}}{L_{ref}U_{ref}}$ and we denote by $Sr := \frac{L_{ref}}{T_{ref}U_{ref}}$, $Ma := \frac{U_{ref}}{\sqrt{p_{ref}/\rho_{ref}}}$, $Re := \frac{U_{ref}\rho_{ref}L_{ref}}{\mu_{ref}}$, $Pe := \frac{U_{ref}p_{ref}L_{ref}}{\vartheta_{ref}\kappa_{ref}}$, $Fr := \frac{U_{ref}}{\sqrt{G\rho_{ref}L^2_{ref}}}$, $\mathcal{C} := \frac{c}{U_{ref}}$, the Strouhal, Mach, Reynolds, Péclet, Froude and "infrarelativistic" dimensionless numbers corresponding to hydrodynamics, by $Ro := \frac{U_{ref}}{\chi_{ref}L_{ref}}$ the Rossby number, by

$Al := \frac{U_{ref}\rho_{ref}^{1/2}\varsigma_{ref}^{1/2}}{B_{ref}}$ the Alfven number and by $\mathcal{L} := L_{ref}\sigma_{a,ref}$, $\mathcal{L}_s := \frac{\sigma_{s,ref}}{\sigma_{a,ref}}$, $\mathcal{P} := \frac{2k^4\vartheta_{ref}^4}{h^3c^3\ \rho_{ref}e_{ref}}$, various dimensionless numbers corresponding to radiation.

Using these scalings and using carets to symbolize renormalized variables we get

$$S = \frac{I_{ref}}{L_{ref}}\ \hat{S},$$

where

$$\hat{S} = \mathcal{L}\hat{\sigma}_a\left(B(\hat{\nu},\hat{\vartheta}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2}\hat{I}(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} - \hat{I}\right).$$

Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\ \partial_t I + \vec{\omega}\cdot\nabla_x I = s = \mathcal{L}\sigma_a\left(B - I\right) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{4.45}$$

where we used the same notation $\mathfrak{B}$ for the dimensionless Planck function $\mathfrak{B}(\nu,\vartheta) = \dfrac{\nu^3}{e^{\frac{\nu}{\vartheta}} - 1}$.

Denoting also by $E^R = \int_{\mathcal{S}^2}\int_0^\infty I\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the (renormalized) radiative energy, by $\vec{F}^R = \int_{\mathcal{S}^2}\int_0^\infty I\vec{\omega}\,\mathrm{d}\nu\,\mathrm{d}\vec{\omega}$, the renormalized radiative momentum, by $s_E = \int_{\mathcal{S}^2}\int_0^\infty s\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative energy source, by $\vec{s}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\ d\vec{\omega}d\nu$, the renormalized radiative entropy with $n = n(I) = \frac{I}{\nu^3}$, by $\vec{q}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}\,d\nu$, the renormalized radiative entropy flux, and taking the first moment of (4.45) with respect to $\vec{\omega}$, we get first an equation for E^R

$$\frac{Sr}{\mathcal{C}}\ \partial_t E^R + \nabla_x\cdot\vec{F}^R = s_E. \tag{4.46}$$

The continuity equation is now

$$Sr\ \partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{4.47}$$

and the momentum equation reads

$$Sr\ \partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{Ma^2}\ \nabla_x p(\varrho,\vartheta) + \frac{2}{Ro}\varrho\vec{\chi}\times\vec{u} =$$

$$\frac{1}{Re}\ \mathrm{div}_x\mathbb{S} + \frac{1}{Fr^2}\varrho\nabla\Psi + \frac{1}{2Ro^2}\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2 + \frac{1}{Al^2}\vec{j}\times\vec{B}. \tag{4.48}$$

The balance of internal energy rewrites

$$Sr\ \partial_t\left(\varrho e + \frac{1}{\mathcal{C}}E^R\right) + \mathrm{div}_x\left(\varrho e\vec{u} + \vec{F}^R\right) + \frac{1}{Pe}\ \mathrm{div}_x\vec{q} = \frac{Ma^2}{Re}\ \mathbb{S}:\nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} +$$

$$Sr\frac{Ma^2}{Al^2}\vec{j}\cdot\vec{E},$$

and we get the balance of matter (fluid) entropy

$$Sr\partial_t\left(\varrho s\right) + \mathrm{div}_x\left(\varrho s\vec{u}\right) + \frac{1}{Pe}\ \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \varsigma, \tag{4.49}$$

with

$$\varsigma = \frac{1}{\vartheta}\left(\frac{Ma^2}{Re}\mathbb{S} : \nabla_x \vec{u} - \frac{1}{Pe}\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{Ma^2}{Al^2}\frac{\lambda}{\zeta}|\mathrm{curl}_x\vec{B}|^2\right) - \frac{S_E}{\vartheta},$$

and the balance of radiative entropy

$$\frac{Sr}{\mathcal{C}}\,\partial_t s^R + \mathrm{div}_x\vec{q}^R = \varsigma^R, \tag{4.50}$$

with

$$\varsigma^R = \mathcal{P}\mathcal{L}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\mathfrak{B})}{n(\mathfrak{B})+1}\right]\sigma_a(I-\mathfrak{B})\,d\vec{\omega}\,d\nu$$

$$+\mathcal{P}\mathcal{L}\mathcal{L}_s\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\,d\vec{\omega}\,d\nu + \frac{S_E}{\vartheta}.$$

The scaled equation for the electromagnetic field is

$$Sr\,\partial_t\vec{B} + \mathrm{curl}_x(\vec{B}\times\vec{u}) + \mathrm{curl}_x(\tilde{\lambda}\;\mathrm{curl}_x\vec{B}) = 0. \tag{4.51}$$

The scaled equation for total energy gives finally the total energy balance

$$Sr\,\frac{d}{dt}\int_\Omega\left(\frac{Ma^2}{2}\;\varrho|\vec{u}|^2 + \varrho e + \frac{1}{\mathcal{C}}\;E^R + \frac{Ma^2}{2Al^2}\frac{1}{\zeta}|\vec{B}|^2 - \frac{1}{2}\frac{Ma^2}{Fr^2}\varrho\Psi\right.$$

$$\left. - \frac{1}{2}\frac{Ma^2}{Ro^2}\varrho|\vec{\chi}\times\vec{x}|^2\right)\mathrm{d}x + \int_0^\infty\int_{\Gamma_+} I\,\vec{\omega}\cdot\vec{n}\;d\Gamma_+ d\nu = 0. \tag{4.52}$$

In the sequel we analyze the asymptotic regime defined by

$$Ma = \varepsilon,\;\; Al = \varepsilon,\;\; Fr = \varepsilon^{1/2},\;\; \mathcal{C} = \varepsilon^{-1},\;\; Pe = \varepsilon^2$$

where $\varepsilon > 0$ is small and we put $Sr = 1$, $Re = 1$, $Ro = 1$, $\mathcal{P} = 1$, $\mathcal{L} = \mathcal{L}_s = 1$ in the previous system. Plugging this scaling into the previous system gives

$$\varepsilon\partial_t I + \vec{\omega}\cdot\nabla_x I = \sigma_a\,(B - I) + \sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\;\mathrm{d}\vec{\omega} - I\right), \tag{4.53}$$

$$\partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{4.54}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{\varepsilon^2}\;\nabla_x p(\varrho,\vartheta) + 2\varrho\vec{\chi}\times\vec{u} = \;\mathrm{div}_x\mathbb{S} + \frac{1}{\varepsilon}\varrho\nabla\Psi + \tag{4.55}$$

$$\frac{1}{2}\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2 + \frac{1}{\varepsilon^2}\vec{j}\times\vec{B},$$

$$\partial_t\left(\varrho e + \varepsilon E^R\right) + \mathrm{div}_x\left(\varrho e\vec{u} + \vec{F}^R\right) + \frac{1}{\varepsilon^2}\mathrm{div}_x\vec{q} = \varepsilon^2\mathbb{S} : \nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} + \vec{j}\cdot\vec{E}, \tag{4.56}$$

$$\partial_t\left(\varrho s + \varepsilon s^R\right) + \mathrm{div}_x\left(\varrho s\vec{u} + \vec{q}^R\right) + \;\frac{1}{\varepsilon^2}\mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) \geq \varsigma_\varepsilon, \tag{4.57}$$

with

$$\varsigma_\varepsilon = \frac{1}{\vartheta}\left(\varepsilon^2\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\varepsilon^2\vartheta} + \frac{\lambda}{\zeta}|\mathrm{curl}_x\vec{B}|^2\right)$$

$$+\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\mathfrak{B})}{n(\mathfrak{B})+1} \right] \sigma_a (I - \mathfrak{B}) \, d\vec{\omega} \, d\nu$$

$$+\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (I - \tilde{I}) \, d\vec{\omega} \, d\nu,$$

$$\partial_t \vec{B} + \mathrm{curl}_x (\vec{B} \times \vec{u}) + \mathrm{curl}_x (\tilde{\lambda} \, \mathrm{curl}_x \vec{B}) = 0, \tag{4.58}$$

and finally

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \varepsilon^2 \varrho |\vec{u}|^2 + \varrho e + \varepsilon E^R + \frac{1}{2\zeta} |\vec{B}|^2 - \frac{1}{2} \varepsilon \varrho \Psi - \frac{1}{2} \varrho \varepsilon^2 |\vec{\chi} \times \vec{x}|^2 \right) \mathrm{d}x$$

$$+\int_0^\infty \int_{\Gamma_+} I \, \vec{\omega} \cdot \vec{n} \, d\Gamma_+ d\nu = 0. \tag{4.59}$$

To compute the limit system, we consider now the formal expansions

$$(I, \varrho, \vec{u}, \vartheta, p, \vec{B}) = (I_0, \varrho_0, \vec{u}_0, \vartheta_0, p_0, \vec{B}_0) + \varepsilon (I_1, \varrho_1, \vec{u}_1, \vartheta_1, p_1, \vec{B}_1) + O(\varepsilon^2). \tag{4.60}$$

- We first observe from (4.55) that $\varrho_0 = const := \overline{\varrho}$ and $\vartheta_0 = const := \overline{\vartheta}$, moreover

$$\nabla_x p_1 = \overline{\varrho} \nabla_x \Psi(\overline{\varrho}). \tag{4.61}$$

Let us require to fix the constants in the Neumann problem for perturbations of the temperature

$$\int_\Omega \vartheta_i \, dx = 0 \quad \text{for any } i \geq 1. \tag{4.62}$$

From (4.54) we derive the incompressibility condition

$$\mathrm{div}_x \vec{u}_0 = 0, \tag{4.63}$$

and

$$\partial_t \varrho_1 + \mathrm{div}_x \left(\overline{\varrho} \vec{u}_1 + \varrho_1 \vec{u}_0 \right) = 0. \tag{4.64}$$

- From (4.53) we get now two stationary linear transport equations for the two moments I_0 and I_1

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_{a,0} \left(\mathfrak{B}_0 - I_0 \right) + \sigma_{s,0} \left(\tilde{I}_0 - I_0 \right), \tag{4.65}$$

$$\vec{\omega} \cdot \nabla_x I_1 = \sigma_{a,0} \left(\partial_\vartheta \mathfrak{B}_0 \vartheta_1 - I_1 \right) + \partial_\vartheta \sigma_{a,0} \left(\mathfrak{B}_0 - I_0 \right) \vartheta_1 + \partial_\vartheta \sigma_{s,0} \left(\tilde{I}_0 - I_0 \right) \vartheta_1 \tag{4.66}$$

$$+\sigma_{s,0} \left(\tilde{I}_1 - I_1 \right),$$

where $\tilde{I} := \frac{1}{4\pi} \int_{\mathcal{S}^2} I \, \mathrm{d}\vec{\omega}$, $\sigma_{a,0} = \sigma_a(\nu, \vartheta_0)$, $\sigma_{s,0} = \sigma_s(\nu, \vartheta_0)$ and $\mathfrak{B}_0 = \mathfrak{B}(\nu, \vartheta_0)$.

- The limit momentum equation reads

$$\overline{\varrho} \left(\partial_t \vec{u}_0 + \mathrm{div}_x (\vec{u}_0 \otimes \vec{u}_0) \right) + \nabla_x \Pi + 2 \overline{\varrho} \vec{\chi} \times \vec{u}_0 = \mathrm{div}_x \mathbb{S}(\vec{u}_0) + \frac{1}{\zeta} \mathrm{curl}_x \vec{B}_1 \times \vec{B}_1 + \vec{F}, \tag{4.67}$$

where $\mu_0 = \mu(\vartheta_0)$ is used in $\mathbb{S}(\vec{u}_0)$, $\vec{F} = \varrho_1 \nabla_x \Psi(\overline{\varrho})$ and Π is an effective pressure for which it holds $\nabla_x \Pi = \frac{1}{2} \overline{\varrho} \nabla_x |\vec{\chi} \times \vec{x}|^2 + \overline{\varrho} \nabla_x \Pi(\varrho_1) + p_{\varrho,\varrho}(\overline{\varrho}, \overline{\vartheta}) \varrho_1 \nabla_x \varrho_1$. Here we set $\vartheta_1 = 0$ which is consistent with the $O(\varepsilon^{-1})-$order of the internal energy equation (4.56) and the additional zero mean of $\vartheta - \vartheta_0$ requirement.

- The limit magnetic field $\vec{B}_1$ solves

$$\partial_t \vec{B}_1 + \mathrm{curl}_x(\vec{B}_1 \times \vec{u}_0) + \mathrm{curl}_x(\overline{\lambda}\ \mathrm{curl}_x \vec{B}_1) = 0, \tag{4.68}$$

for $\overline{\lambda} = \tilde{\lambda}(\vartheta_0)$.

- At the lowest order ($O(\varepsilon^0)$) the energy equation (4.56) gives

$$\overline{\kappa}\Delta\vartheta_2 = s_{E0} \tag{4.69}$$

where $-s_{E0} = \int_0^\infty \int_{\mathcal{S}^2} \sigma_{a,0}\,(I_0 - \mathfrak{B}_0)\ d\vec{\omega}\,d\nu$ and $\overline{\kappa} = \kappa(\overline{\vartheta})$.

- At the order ($O(\varepsilon)$) we simplify the energy equation (4.56). Observing that from (4.61) we have

$$\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) D\varrho_1 + \overline{\varrho}\vec{u}_0 \cdot \nabla_x \Psi(\overline{\varrho}) = 0, \tag{4.70}$$

where $D := \partial_t + \vec{u}_0 \cdot \nabla_x$, and from (4.64)

$$\overline{\varrho}\mathrm{div}_x \vec{u}_1 = -D\varrho_1,$$

and after (4.66)

$$S_{E1} = -\int_0^\infty \int_{\mathcal{S}^2} \sigma_{a,0} I_1\, d\vec{\omega}\, d\nu,$$

and simplifying by (4.7) we end up with

$$\partial_t \varrho_1 + \mathrm{div}_x(\varrho_1 \vec{u}_0) = -\overline{\alpha}\left(\overline{\kappa}\,\triangle\vartheta_3 + \int_0^\infty \int_{\mathcal{S}^2} \sigma_{a,0} I_1\, d\vec{\omega}\, d\nu\right),$$

where $\overline{\alpha} := \frac{\overline{\varrho}}{\overline{\vartheta}} \partial_\theta p(\overline{\varrho}, \overline{\vartheta})$.

Putting

$$\vec{U} = \vec{u}_0,\ \Theta = \vartheta_3,\ \vec{B} = \vec{B}_1,\ \overline{\varrho} = \varrho_0,\ \overline{\vartheta} = \vartheta_0,\ \overline{\mu} = \mu(\vartheta_0),\quad \sigma_a = \sigma_{a,0},\ \sigma_s = \sigma_{s,0},$$

$$\mathfrak{B} = \mathfrak{B}_0,\ \mathbb{D}(\vec{U}) = \frac{1}{2}\left(\nabla \vec{u}_0 + \nabla^T \vec{u}_0\right),$$

and

$$G = \frac{\int_0^\infty \int_{\mathcal{S}^2} \sigma_{a,0}\,(I_0 - \mathfrak{B}_0)\ d\vec{\omega}\,d\nu}{\overline{\kappa}}$$

we observe that the solution of the equation (4.65) is up to the boundary condition $(4.12)_2$ $I_0 = \mathfrak{B}_0$ which in turn entails that the equation for ϑ_2 turns in Ω into the Laplace homogeneous equation ($G = 0$) and therefore $\vartheta_2 = 0$ and we obtain the limit system in $(0, T) \times \Omega$

$$\mathrm{div}_x \vec{U} = 0, \tag{4.71}$$

$$\overline{\varrho}(\partial_t \vec{U} + \mathrm{div}_x(\vec{U} \otimes \vec{U})) +\ \nabla_x \Pi = \mathrm{div}_x(2\overline{\mu}\ \mathbb{D}(\vec{U})) + \frac{1}{\zeta}\mathrm{curl}_x \vec{B} \times \vec{B} + \vec{F} \tag{4.72}$$

$$\partial_t \vec{B} + \mathrm{curl}_x(\vec{B} \times \vec{U}) + \mathrm{curl}_x(\overline{\lambda}\ \mathrm{curl}_x \vec{B}) = 0, \tag{4.73}$$

$$\mathrm{div}_x \vec{B} = 0, \tag{4.74}$$

$$- \Delta\Theta = \frac{1}{\overline{\alpha\kappa}} \vec{U} \cdot \nabla_x \tilde{r} - \frac{1}{\overline{\kappa}} \int_0^\infty \sigma_a \int_{\mathcal{S}^2} I_1 \, d\vec{\omega} \, d\nu + \tilde{h}(t) \tag{4.75}$$

$$\vec{\omega} \cdot \nabla_x I_1 = -\sigma_a I_1 + \sigma_s \left(\tilde{I}_1 - I_1 \right), \tag{4.76}$$

together with the *Boussinesq relation* (4.61)

$$\nabla_x \tilde{r} = \frac{\overline{\varrho} \nabla_x \Psi(\overline{\varrho})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})}, \tag{4.77}$$

where $\tilde{r} := \varrho_1 - \overline{\varrho}$ and $\tilde{h}$ is an undetermined function which allows satisfaction of $(4.81)_2$.

We finally consider the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0, \;\; \nabla\Theta \cdot \vec{n}|_{\partial\Omega} = 0, \;\; \vec{B} \cdot \vec{n}|_{\partial\Omega} = 0, \;\; \mathrm{curl}_x \vec{B} \times \vec{n}|_{\partial\Omega} = 0 \tag{4.78}$$

for (4.71)-(4.75) and

$$I_1(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \;\; \vec{\omega} \cdot \vec{n} \leq 0 \tag{4.79}$$

for (4.76), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \vec{B}|_{t=0} = \vec{B}_0. \tag{4.80}$$

Moreover, we endow the system (4.71) – (4.77) with the additional conditions

$$\mathrm{div}_x \vec{B}_0 = 0, \quad \int_\Omega \Theta \;\mathrm{d}x = 0. \tag{4.81}$$

For this system we have the following existence result (see the Appendix for a short proof).

Theorem 4.3.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.*

For any $T > 0$ the initial-bounday value problem (4.71) - (4.81) has at least one weak solution
$(\vec{U}, \Theta, \vec{B}, I_1)$ such that

1.

$$\vec{U} \in L^\infty(0, T; \mathcal{H}(\Omega)) \cap L^2(0, T; \mathcal{U}(\Omega)),$$
$$\vec{B} \in L^\infty(0, T; \mathcal{V}(\Omega)) \cap L^2(0, T; \mathcal{W}(\Omega)),$$

2.

$$\Theta \in L^\infty((0, T; W^{2,2}(\Omega)) \cap L^2((0, T; W^{q,2}(\Omega)) \qquad \textit{for any } q < \frac{5}{2},$$

3.

$$I_1 \in L^\infty((0, T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+),$$

with

$$\vec{\omega} \cdot \nabla_x I_1 \in L^p((0, T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+),$$

for any $p > 1$ and any $\vec{\omega} \in \mathcal{S}^2$.

The remaining part of the paper is devoted to the proof of the convergence of the primitive system (4.1)–(4.13) to the target system (4.71)–(4.81).

4.4 Global existence for the primitive system and uniform estimates

For the system (4.1)–(4.13) we prepare the initial data as follows

$$\begin{cases} \varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon}, \\ \vec{u}(0,\cdot) = \vec{u}_{0,\varepsilon}, \\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon^3 \vartheta^{(3)}_{0,\varepsilon}, \\ I(0,\cdot,\cdot,\cdot) = I_{0,\varepsilon} = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon}, \\ \vec{B}(0,\cdot) = B_{0,\varepsilon} = \varepsilon \vec{B}^{(1)}_{0,\varepsilon}, \end{cases} \tag{4.82}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ are spacetime constants and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\,\mathrm{d}x = 0 = \int_\Omega \vartheta^{(3)}_{0,\varepsilon}\,\mathrm{d}x$ for any $\varepsilon > 0$.

From Theorem 4.2.1 we get immediately (by combining the approximating schemes introduced in (Ducomet, Feireisl, Nečasová , 2011) and (Ducomet, Feireisl , 2006)) the existence of a weak solution to the radiative MHD system (4.1) – (4.13) $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon, \vec{B}_\varepsilon)$.

Theorem 4.4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (4.14) – (4.19), and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function $\mathfrak{B}$ comply with (4.20) – (4.25). Let the initial data $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon}, \vec{B}_{0,\varepsilon})$ be given by (4.82), where $(\varrho^{(1)}_{0,\varepsilon}, \vartheta^{(3)}_{0,\varepsilon}, I^{(1)}_{0,\varepsilon}, \vec{B}^{(1)}_{0,\varepsilon})$ are uniformly bounded measurable functions.*

Then for any $\varepsilon > 0$ small enough (in order to maintain positivity of $\varrho_{0,\varepsilon}$ and $\vartheta_{0,\varepsilon}$), there exists a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon, \vec{B}_\varepsilon)$ to the radiative Navier-Stokes system (4.1) – (4.11) for $(t, x, \vec{\omega}, \nu) \in (0,T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (4.12) – (4.13) and the initial conditions (4.82).

More precisely we have

- $$\int_0^T \int_\Omega \varrho_\varepsilon b(\varrho_\varepsilon) \left(\partial_t \phi + \vec{u}_\varepsilon \cdot \nabla_x \phi\right) \mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \beta(\varrho_\varepsilon) \mathrm{div}_x u_\varepsilon \ \phi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_{0,\varepsilon} b(\varrho_{0,\varepsilon})\ \phi(0,\cdot)\,\mathrm{d}x, \tag{4.83}$$
for any β such that $\beta \in (L^\infty \cap C)([0,\infty))$, $b(\varrho) = b(1) + \int_1^\varrho \frac{\beta(z)}{z^2}\,\mathrm{d}z$ and any $\phi \in C^\infty_c([0,T) \times \overline{\Omega})$,

- $$\int_0^T \int_\Omega \Big(\varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t \vec{\varphi} + \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla_x \vec{\varphi} + \frac{p_\varepsilon}{\varepsilon^2}\,\mathrm{div}_x \vec{\varphi} - 2\varrho_\varepsilon \vec{\chi} \times \vec{u}_\varepsilon \cdot \vec{\varphi} \Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x \vec{\varphi} - \frac{1}{\varepsilon} \varrho_\varepsilon \nabla_x \Psi_\varepsilon \cdot \vec{\varphi} - \frac{1}{\varepsilon^2} (\vec{j}_\varepsilon \times \vec{B}_\varepsilon) \cdot \vec{\varphi} - \frac{1}{2} \varepsilon^2 \varrho_\varepsilon \nabla_x |\vec{\chi} \times \vec{x}|^2 \cdot \vec{\varphi} \Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_\Omega \varrho_{0,\varepsilon} \vec{u}_{0,\varepsilon} \cdot \vec{\varphi}(0,\cdot)\,\mathrm{d}x, \tag{4.84}$$
for any $\vec{\varphi} \in C^\infty_c([0,T) \times \overline{\Omega}; \mathbb{R}^3)$ with $p_\varepsilon = p(\varrho_\varepsilon, \vartheta_\varepsilon)$, $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon, \vartheta_\varepsilon)$, and $\vec{j}_\varepsilon = \frac{1}{\zeta} \mathrm{curl}_x \vec{B}_\varepsilon$,

- $$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E^R_\varepsilon + \frac{1}{2\zeta}|\vec{B}_\varepsilon|^2 - \frac{1}{2}\varepsilon \varrho_\varepsilon \Psi_\varepsilon - \frac{1}{2}\varrho_\varepsilon \varepsilon^2 |\vec{\chi}\times\vec{x}|^2 \right) \mathrm{d}x\, \mathrm{d}t$$
$$+ \int_0^T \int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma_+\ d\nu\, \mathrm{d}t$$
$$= \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} + \frac{1}{2\zeta}|\vec{B}_{0,\varepsilon}|^2 - \frac{1}{2}\varepsilon \varrho_{0,\varepsilon}\Psi_{0,\varepsilon} - \right. \tag{4.85}$$
$$\left. -\frac{1}{2}\varepsilon^2 \varrho_{0,\varepsilon}|\vec{\chi}\times\vec{x}|^2 \right) \mathrm{d}x,$$
for a. a. $t \in (0,T)$ *with* $e_\varepsilon = e(\varrho_\varepsilon, \vartheta_\varepsilon)$, $\Psi_\varepsilon = \Psi(\varrho_\varepsilon)$, $\Psi_{0,\varepsilon} = \Psi(\varrho_{0,\varepsilon})$ *and* $E^R_\varepsilon(t,x) = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\ d\vec{\omega}\ d\nu$

- $$\int_0^T \int_\Omega \Big(\vec{B}_\varepsilon \cdot \partial_t \vec{\varphi} - (\vec{B}_\varepsilon \times \vec{u}_\varepsilon + \lambda_\varepsilon \mathrm{curl}_x \vec{B}_\varepsilon)\cdot \mathrm{curl}_x \vec{\varphi} \Big) \mathrm{d}x\, \mathrm{d}t + \int_\Omega \vec{B}_{0,\varepsilon}\cdot\vec{\varphi}(0,\cdot)\,\mathrm{d}x = 0, \tag{4.86}$$
for any vector field $\vec{\varphi} \in \mathcal{D}([0,T)\times\mathbb{R}^3, \mathbb{R}^3)$, *with* $\lambda_\varepsilon = \tilde{\lambda}(\vartheta_\varepsilon)$.

- $$\int_0^T \int_\Omega \Big(\big(\varrho_\varepsilon s_\varepsilon + \varepsilon s^R_\varepsilon\big)\partial_t \varphi + \big(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}^R_\varepsilon\big)\cdot\nabla_x\varphi \Big)\, \mathrm{d}x\, \mathrm{d}t +$$
$$\int_0^T \int_\Omega \frac{\vec{q}_\varepsilon}{\varepsilon^2 \vartheta_\varepsilon}\cdot\nabla_x\varphi\, \mathrm{d}x\, \mathrm{d}t + \big\langle \varsigma^m_\varepsilon + \varsigma^R_\varepsilon; \varphi \big\rangle_{[\mathcal{M};C]([0,T)\times\overline{\Omega})} =$$
$$- \int_\Omega \Big(((\varrho s)_{0,\varepsilon} + \varepsilon s^R_{0,\varepsilon})\varphi(0,\cdot) \Big)\, \mathrm{d}x, \tag{4.87}$$
where
$$\varsigma^m_\varepsilon \geq \frac{1}{\vartheta_\varepsilon}\left(\varepsilon^2 \mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\varepsilon^2\vartheta_\varepsilon} + \frac{\lambda_\varepsilon}{\zeta}\left|\mathrm{curl}_x \vec{B}_\varepsilon\right|^2 \right),$$
and
$$\varsigma^R_\varepsilon \geq \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\mathfrak{B}_\varepsilon)}{n(\mathfrak{B}_\varepsilon)+1} \right] \sigma_{a\varepsilon}(\mathfrak{B}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\, d\nu$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega}\, d\nu,$$
for any $\varphi \in C^\infty_c([0,T)\times\overline{\Omega})$ *with* $\varsigma^m_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$ *and* $\varsigma^R_\varepsilon \in \mathcal{M}^+([0,T)\times\overline{\Omega})$, *and with* $\sigma_{a\varepsilon} = \sigma_a(\nu,\vartheta_\varepsilon)$, $\sigma_{s\varepsilon} = \sigma_s(\nu,\vartheta_\varepsilon)$, $\mathfrak{B}_\varepsilon = \mathfrak{B}(\nu,\vartheta_\varepsilon)$, $\vec{q}_\varepsilon = \kappa(\vartheta_\varepsilon)$ $\nabla_x\vartheta_\varepsilon$, $s_\varepsilon = s(\varrho_\varepsilon,\vartheta_\varepsilon)$, $s^R_\varepsilon = s^R(I_\varepsilon)$, $\vec{q}^R_\varepsilon = \vec{q}^R(I_\varepsilon)$ *and* $\tilde{I}_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}$,

- $$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon\partial_t\psi + \vec{\omega}\cdot\nabla_x\psi)\, I_\varepsilon\ d\vec{\omega}\ d\nu\, \mathrm{d}x\, \mathrm{d}t$$

$$+\int_0^T\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\left[\sigma_{a\varepsilon}\left(\mathfrak{B}_\varepsilon - I_\varepsilon\right)+\sigma_{s\varepsilon}\left(\tilde{I}_\varepsilon - I_\varepsilon\right)\right]\psi\ d\vec{\omega}\ d\nu\,\mathrm{d}x\,\mathrm{d}t,$$

$$=-\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\varepsilon I_{0,\varepsilon}\psi(0,x,\vec{\omega},\nu)\ d\vec{\omega}\ d\nu\,\mathrm{d}x+\int_0^T\int_{\Gamma_+}\int_0^\infty I_\varepsilon\vec{\omega}\cdot\vec{n}_x\psi\ d\Gamma_+\ d\nu\,\mathrm{d}t, \tag{4.88}$$

for any $\psi\in C_c^\infty([0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$.

4.4.1 Uniform estimates

We recall from (Feireisl, Novotný , 2009) the necessary definitions in the formalism of essential and residual sets ((see Ducomet, Nečasová , 2014)).

Given three numbers $\overline{\varrho}\in\mathbb{R}_+$, $\overline{\vartheta}\in\mathbb{R}_+$ and $\overline{E}\in\mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess}:=\left\{(\varrho,\vartheta)\in\mathbb{R}^2\ :\ \frac{\overline{\varrho}}{2}<\varrho<2\overline{\varrho},\ \frac{\overline{\vartheta}}{2}<\vartheta<2\overline{\vartheta}\right\}, \tag{4.89}$$

and $\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess}:=\left\{E^R\in\mathbb{R}\ :\ \frac{\overline{E}}{2}<E^R<2\overline{E}\right\}, \tag{4.90}$$

with $\mathcal{O}_{ess}:=\mathcal{O}^H_{ess}\cup\mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res}:=(\mathbb{R}_+)^2\backslash\mathcal{O}^H_{ess},\quad \mathcal{O}^R_{res}:=\mathbb{R}_+\backslash\mathcal{O}^R_{ess},\quad \mathcal{O}_{res}:=(\mathbb{R}_+)^3\backslash\mathcal{O}_{ess}. \tag{4.91}$$

Let $\left\{\varrho_\varepsilon,\vec{u}_\varepsilon,\vartheta_\varepsilon,\vec{B}_\varepsilon,I_\varepsilon\right\}_{\varepsilon>0}$ be a family of solutions of the scaled radiative Navier-Stokes system given in Theorem 4.4.1. We call $\mathcal{M}^\varepsilon_{ess}\subset(0,T)\times\Omega$ the set

$$\mathcal{M}^\varepsilon_{ess}=\left\{(t,x)\in(0,T)\times\Omega\ :\ \left(\varrho_\varepsilon(t,x),\vartheta_\varepsilon(t,x),E^R_\varepsilon(t,x)\right)\in\mathcal{O}_{ess}\right\},$$

and $\mathcal{M}^\varepsilon_{res}=(0,T)\times\Omega\backslash\mathcal{M}^\varepsilon_{ess}$ the corresponding residual set.

To any measurable function h we associate its decomposition into essential and residual parts

$$h=[h]_{ess}+[h]_{res},$$

where $[h]_{ess}=h\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}$ and $[h]_{res}=h\cdot\mathbb{I}_{\mathcal{M}^\varepsilon_{res}}$.

Denoting by $H_{\overline{\vartheta}}$ the Helmholtz function for matter

$$H_{\overline{\vartheta}}(\varrho,\vartheta)=\varrho e-\overline{\vartheta}\varrho s,$$

and for radiation

$$H^R{}_{\overline{\vartheta}}(I)=E^R-\overline{\vartheta}s^R,$$

and using (4.87) we rewrite (4.85) as

$$\int_\Omega\left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon|\vec{u}_\varepsilon|^2+H_{\overline{\vartheta}}(\varrho_\varepsilon,\vartheta_\varepsilon)+\varepsilon H^R{}_{\overline{\vartheta}}(I_\varepsilon)+\frac{1}{2\zeta}|\vec{B}_\varepsilon|^2-\frac{1}{2}\varepsilon\varrho_\varepsilon\Psi_\varepsilon-\frac{1}{2}\varepsilon^2\varrho_\varepsilon|\vec{\chi}\times\vec{x}|^2\right)\mathrm{d}x$$

$$+\int_0^T\int_0^\infty\int_{\Gamma_+}I_\varepsilon(t,x,\vec{\omega},\nu)\,\vec{\omega}\cdot\vec{n}_x\ d\Gamma\ d\nu\,\mathrm{d}t+\overline{\vartheta}\left(\varsigma^m_\varepsilon+\varsigma^R_\varepsilon\right)\left[[0,T]\times\overline{\Omega}\right]=$$

$$\int_\Omega \left(\frac{\varepsilon^2}{2} \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} + \frac{1}{2\zeta} |\vec{B}_{0,\varepsilon}|^2 - \frac{1}{2} \varepsilon \varrho_{0,\varepsilon} \Psi_{0,\varepsilon} - \frac{1}{2} \varepsilon^2 \varrho_{0,\varepsilon} |\vec{\chi} \times \vec{x}|^2 \right) \mathrm{d}x.$$

Observing that the total mass is a constant of motion $M = \int_\Omega \varrho_\varepsilon \, dx = \overline{\varrho} |\Omega|$ and using Hardy-Littlewood-Sobolev inequality, we get $\frac{\varepsilon}{2} \int_\Omega \varrho_\varepsilon \Psi_\varepsilon \, \mathrm{d}x \leq \frac{G\varepsilon}{2} C M^{2/3} \|\varrho_\varepsilon\|^{4/3}_{L^{4/3}(\Omega)}$. By virtue of (4.14) and (4.18) we have also $\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \geq a \vartheta^4_\varepsilon + \frac{3p_\infty}{2} \varrho_\varepsilon^{5/3}$, so we have got the lower bound

$$\int_\Omega \left[H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{2} \varepsilon \varrho_\varepsilon \Psi_\varepsilon \right] \, dx \geq c \int_\Omega H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) \, dx,$$

for ε small and a $c(\varepsilon) < 1$ and we deduce finally the dissipation energy-entropy inequality

$$\int_\Omega \left(\frac{\varepsilon^2}{2} \, \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) + \frac{1}{2\zeta} |\vec{B}_\varepsilon|^2 - \right.$$
$$\left. \frac{\varepsilon^2}{2} \varrho_\varepsilon |\vec{\chi} \times \vec{x}|^2 + \varepsilon {H^R}_{\overline{\vartheta}}(I_\varepsilon) \right) \mathrm{d}x$$
$$+ \int_0^T \int_0^\infty \int_{\Gamma_+} I_\varepsilon(t, x, \vec{\omega}, \nu) \, \vec{\omega} \cdot \vec{n}_x \, d\Gamma \, d\nu \, \mathrm{d}t + \overline{\vartheta} \left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon \right) \left[[0, T] \times \overline{\Omega} \right] \leq$$
$$C \int_\Omega \left(\frac{\varepsilon^2}{2} \, \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) + \right.$$
$$\left. \frac{1}{2\zeta} |\vec{B}_{0,\varepsilon}|^2 + \varepsilon {H^R}_{\overline{\vartheta}}(I_{0,\varepsilon}) \right) \mathrm{d}x. \tag{4.92}$$

Now, according to in (Ducomet, Nečasová , 2014; Lemma 4.1) (see Feireisl, Novotný , 2009) we have the following properties for material and radiative Helmholtz functions.

Lemma 4.4.1. Let $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$ two given constants and let

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e - \overline{\vartheta} \varrho s,$$

and

$${H^R}_{\overline{\vartheta}}(I) = E^R - \overline{\vartheta} s^R.$$

Let $\mathcal{O}_{ess}$ and $\mathcal{O}_{res}$ be the sets of essential and residual values introduced in (4.89) – (4.91).

There exist positive constants $C_j = C_j(\overline{\varrho}, \overline{\vartheta})$ for $j = 1, \cdots, 8$ such that

1.
$$C_1 \left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \right) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \leq$$
$$C_2 \left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2 \right), \tag{4.93}$$
for all $(\varrho, \vartheta) \in \mathcal{O}^H_{ess}$,

2.
$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \geq$$
$$\inf_{\tilde{\varrho}, \tilde{\vartheta} \in \mathcal{O}_{res}} \left\{ H_{\overline{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) - (\tilde{\varrho} - \overline{\varrho}) \partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right\} = C_3, \tag{4.94}$$
for all $(\varrho, \vartheta) \in \mathcal{O}^H_{res}$,

3.

$$H_{\overline{\vartheta}}(\varrho,\vartheta)-(\varrho-\overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})-H_{\overline{\vartheta}}(\overline{\varrho},\overline{\vartheta})\geq C_4\left(\varrho e(\varrho,\vartheta)+\varrho|s(\varrho,\vartheta)|\right), \quad (4.95)$$

for all $(\varrho,\vartheta)\in\mathcal{O}^H_{res}$,

4.

$$C_5|E^R-\overline{E}|^2\leq H^R{}_{\overline{\vartheta}}(I)\leq C_6|E^R-\overline{E}|^2, \quad (4.96)$$

for all $E\in\mathcal{O}^R_{ess}$,

5.

$$H^R{}_{\overline{\vartheta}}(I)\geq \inf_{\tilde{I}\in\mathcal{O}_{res}} H^R{}_{\overline{\vartheta}}(\tilde{I})=C_7, \quad (4.97)$$

for all $E\in\mathcal{O}^R_{res}$,

6.

$$H^R{}_{\overline{\vartheta}}(I)\geq C_8\left(E^R(I)+|s^R(I)|\right) \quad (4.98)$$

for all $E\in\mathcal{O}^R_{res}$.

Using (4.92) and Lemma 4.4.1, we get the following energy estimates

Lemma 4.4.2. Suppose that the initial data satisfy

$$\|[\varrho_{0,\varepsilon}-\overline{\varrho}]_{ess}\|^2_{L^2(\Omega)}\leq C\varepsilon^2,\ \|[\vartheta_{0,\varepsilon}-\overline{\vartheta}]_{ess}\|^2_{L^2(\Omega)}\leq C\varepsilon^2,\ \|E^R_{0,\varepsilon}-\overline{E}\|^2_{L^2(\Omega)}\leq C\varepsilon,$$

$$\|\vec{B}_{0,\varepsilon}\|^2_{L^2(\Omega;\mathbb{R}^3)}\leq C\varepsilon^2,$$

and

$$\|\sqrt{\varrho_{0,\varepsilon}}\ \vec{u}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)}\leq C.$$

Then the following estimates hold

$$\operatorname*{ess\ sup}_{t\in(0,T)}|\mathcal{M}^\varepsilon_{res}(t)|\leq C\varepsilon^2, \quad (4.99)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[\varrho_\varepsilon-\overline{\varrho}]_{ess}(t)\|^2_{L^2(\Omega)}\leq C\varepsilon^2, \quad (4.100)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[\vartheta_\varepsilon-\overline{\vartheta}]_{ess}(t)\|^2_{L^2(\Omega)}\leq C\varepsilon^2, \quad (4.101)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[E^R_\varepsilon-\overline{E}]_{ess}(t)\|^2_{L^2(\Omega)}\leq C\varepsilon, \quad (4.102)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}\leq C\varepsilon^2, \quad (4.103)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}\leq C\varepsilon^2, \quad (4.104)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}\leq C\varepsilon, \quad (4.105)$$

$$\operatorname*{ess\ sup}_{t\in(0,T)}\|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)}\leq C\varepsilon. \quad (4.106)$$

$$\left(\varsigma^m_\varepsilon+\varsigma^R_\varepsilon\right)\left[[0,T]\times\overline{\Omega}\right]\leq C\varepsilon^2, \quad (4.107)$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\| \frac{\vec{B}_\varepsilon(t)}{\varepsilon} \right\|_{L^2(\Omega;\mathbb{R}^3)} \le C, \tag{4.108}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\ \vec{u}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{R}^3)} \le C. \tag{4.109}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega \left([\varrho_\varepsilon]_{res}^{\frac{5}{3}} + [\vartheta_\varepsilon]_{res}^4 \right)(t)\ dx \le C\varepsilon^2, \tag{4.110}$$

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le C, \tag{4.111}$$

$$\left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \le C, \tag{4.112}$$

$$\left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon^2} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \le C, \tag{4.113}$$

$$\left\| \frac{\vec{B}_\varepsilon}{\varepsilon} \right\|_{L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))} \le C. \tag{4.114}$$

Proof: Estimate (4.99) follows from (4.94). Bounds (4.100), (4.101) and (4.102) follow from (4.93) and (4.96). Estimates (4.103) and (4.104) follow from (4.95). Bounds (4.105) and (4.106) follow from (4.98). Estimates (4.107), (4.108) and (4.109) follow from the dissipation energy-entropy inequality (4.92). Bound (4.110) follows from (4.103) and (4.18) (cf. a lower bound for ϱe before (4.92)).

From (4.107) we see that

$$\left\| \nabla_x \vec{u}_\varepsilon + \nabla_x^T \vec{u}_\varepsilon - \frac{2}{3}\operatorname{div}_x \vec{u}_\varepsilon \mathbb{I} \right\|_{L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^{3\times 3}\right)\right)} \le C. \tag{4.115}$$

From (4.20), (4.109) and (4.115) we get (4.111). Details can be found in (Ducomet, Kobera, Nečasová , 2014) and (Feireisl, Novotný , 2009). From (4.107) we get

$$\left\| \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon^2} \right) \right\|_{L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^3\right)\right)} + \left\| \nabla_x \left(\frac{\log \vartheta_\varepsilon}{\varepsilon^2} \right) \right\|_{L^2\left(0,T;L^2\left(\Omega;\mathbb{R}^3\right)\right)} \le C,$$

which, using Poincaré inequality, gives (4.112) and (4.113). Finally by (4.22), (4.36) and (4.107) one gets

$$\left\| \frac{\operatorname{curl}_x \vec{B}_\varepsilon}{\varepsilon} \right\|_{L^2\left(0,T;L^2(\Omega;\mathbb{R}^3)\right)} \le C,$$

and (4.114) follows by using (Duvaut, Lions , 1976; Theorem 6.1).

Our goal in the next Section will be to prove that the incompressible system (4.71)-(4.80) is the limit of the primitive system (4.83)-(4.88) in the following sense

Theorem 4.4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (4.14) – (4.19) with $P \in C^1([0,\infty)) \cap C^2(0,\infty)$, and that the transport coefficients μ, η, κ, λ, σ_a, σ_s and the equilibrium function $\mathfrak{B}$ comply with (4.20) – (4.25).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon)$ be a weak solution of the scaled system (4.1) – (4.11) for $(t, x, \vec{\omega}, \nu) \in [0, T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (4.12) – (4.13) and initial conditions
$(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \vec{B}_{0,\varepsilon}, I_{0,\varepsilon})$ given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\ \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon^3 \vartheta^{(3)}_{0,\varepsilon},\ I_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon},$$

$$\vec{B}_\varepsilon(0,\cdot) = \varepsilon \vec{B}^{(1)}_{0,\varepsilon},$$

where $\overline{\varrho} > 0,\ \overline{\vartheta} > 0,\ \overline{I} > 0$ are constants in $(0, T) \times \Omega$ and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon}\, dx = 0,\ \int_\Omega \vartheta^{(3)}_{0,\varepsilon}\, dx = 0,\ \int_\Omega I^{(1)}_{0,\varepsilon}\, dx = 0,\ \int_\Omega \vec{B}^{(1)}_{0,\varepsilon}\, dx = 0 \quad \text{for all } \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho^{(1)}_{0,\varepsilon} \to \varrho^{(1)}_0 \ \ \text{weakly} - (*) \text{ in } L^\infty(\Omega), \\ \vec{u}_{0,\varepsilon} \to \vec{U}_0 \ \ \text{weakly} - (*) \text{ in } L^\infty(\Omega; \mathbb{R}^3), \\ \vartheta^{(3)}_{0,\varepsilon} \to \vartheta^{(3)}_0 \ \ \text{weakly} - (*) \text{ in } L^\infty(\Omega), \\ I^{(1)}_{0,\varepsilon} \to I^{(1)}_0 \ \ \text{weakly} - (*) \text{ in } L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+), \\ \vec{B}^{(1)}_{0,\varepsilon} \to \vec{B}^{(1)}_0 \ \ \text{weakly} - (*) \text{ in } L^\infty(\Omega; \mathbb{R}^3). \end{cases}$$

Then

$$\operatorname{ess}\sup_{t \in (0,T)} \|\varrho_\varepsilon(t) - \overline{\varrho}\|_{L^{\frac{5}{3}}(\Omega)} \le C\varepsilon, \tag{4.116}$$

and up to subsequences

$$\vec{u}_\varepsilon \to \vec{U} \ \text{weakly in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \tag{4.117}$$

$$\frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^3} =: \vartheta^{(3)}_\varepsilon \to \Theta \ \text{weakly in } L^{\frac{4}{3}}(0, T; W^{1,\frac{4}{3}}(\Omega)) \tag{4.118}$$

$$I_\varepsilon \to \overline{I} = B_0 \ \text{weakly in } L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \tag{4.119}$$

$$\frac{\vec{B}_\varepsilon}{\varepsilon} = \vec{B}^{(1)}_\varepsilon \to \vec{B} \ \text{weakly in } L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^3)), \tag{4.120}$$

and

$$\frac{I_\varepsilon - \overline{I}}{\varepsilon} = I^{(1)}_\varepsilon \to I_1 \ \text{weakly in } L^2(0, T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)), \tag{4.121}$$

where $(\vec{U}, \Theta, \vec{B}, I_1)$ solves the system (4.71)-(4.76).

4.5 Proof of Theorem 4.4.2

Let us first quote the following result of (Ducomet, Nečasová , 2014), (see Feireisl, Novotný , 2009).

Proposition 4.5.1. Let $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}, \{I_\varepsilon\}_{\varepsilon>0}$ be three sequences of non-negative measurable functions such that

$$\left[\varrho_\varepsilon^{(1)}\right]_{ess} \to \varrho^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{ess} \to \vartheta^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[I_\varepsilon^{(1)}\right]_{ess} \to I^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)), \ a.e. \ in \ \mathcal{S}^2\times\mathbb{R}_+,$$

where

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}, \ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}, \ I_\varepsilon^{(1)} = \frac{I_\varepsilon-\overline{I}}{\varepsilon}.$$

Suppose that

$$\operatorname*{ess\ sup}_{t\in(0,T)} |\mathcal{M}_{res}^\varepsilon(t)| \le C\varepsilon^2. \tag{4.122}$$

Let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ be given functions. Then

$$\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess}-G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \to \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\ \varrho^{(1)} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\ \vartheta^{(1)},$$

$weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega))$, and if we denote

$$\left[G^R(I_\varepsilon)\right]_{ess} := \left[G^R(I_\varepsilon(\cdot,\cdot,\vec{\omega},\nu))\right]_{ess} = G^R(I_\varepsilon)\cdot\mathbb{I}_{\mathcal{M}_{ess}^\varepsilon}, \ for\ a.a.\ (\vec{\omega},\nu)\in\mathcal{S}^2\times\mathbb{R}_+,$$

we have got

$$\frac{\left[G^R(I_\varepsilon)\right]_{ess}-G^R(\overline{I})}{\varepsilon} \to \frac{\partial G(\overline{I})}{\partial I}\ I^{(1)},$$

$weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)), \ a.e. \ in \ \mathcal{S}^2\times\mathbb{R}_+$.

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ then

$$\left\|\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess}-G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\ \left[\varrho^{(1)}\right]_{ess} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\ \left[\vartheta^{(1)}\right]_{ess}\right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

and

$$\left\|\frac{\left[G^R(I_\varepsilon)\right]_{ess}-G^R(\overline{I})}{\varepsilon} - \frac{\partial G(\overline{I})}{\partial I}\ \left[I^{(1)}\right]_{ess}\right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

$for\ a.a.\ (\vec{\omega},\nu)\in\mathcal{S}^2\times\mathbb{R}_+$.

Clearly, this result provides us with the convergence properties (4.116) – (4.117), (4.120) – (4.121). The convergence of radiative intensity (4.119) follows from (4.105), (4.102), and the linearity of (4.53), cf. the section 4.5.2 Radiative transfer equation. The equilibrium Planck function $\mathfrak{B}_0$ does not satisfy the boundary condition $(4.13)_1$ however since it is isotropic; therefore has to be modified at the boundary $\partial\Omega$. The last convergence (4.118) is postponed to Section 4.5.3.

To conclude the proof of Theorem 4.4.2, let us prove that the limit quantities $(\vec{U},\Theta,\vec{B},I_1)$ solve the target system (4.71)-(4.76).

As a number of terms in the equations of our model are similar to those of the radiative Navier-Stokes-Fourier analyzed in (Ducomet, Nečasová , 2014) we only focus on the new contributions.

4.5.1 Continuity and Momentum equations

For the continuity equation, one expects that in the low Mach number limit, it reduces to the incompressibility constraint. In fact, from Lemma 4.4.2 we know that $\int_0^T \|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\,\mathrm{d}t \le C$ so passing to the limit after possible extraction of a subsequence, we deduce that

$$\vec{u}_\varepsilon \to \vec{U} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{4.123}$$

In the same stroke $\varrho_\varepsilon \to \overline{\varrho}$, weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (4.83) which gives $\int_0^T\int_\Omega \vec{U}\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t = 0$ for all $\phi \in \mathcal{D}((0,T)\times\overline{\Omega})$, which rewrites

$$\mathrm{div}_x\vec{U} = 0, \quad \text{a.e. in } (0,T)\times\Omega, \quad \vec{U}\Big|_{\partial\Omega} = 0,$$

provided $\partial\Omega$ is regular.

For the momentum equation one knows that due to possible strong time oscillations of the gradient component of velocity, one has only $\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon \to \overline{\varrho\vec{U}\otimes\vec{U}}$ weakly in $L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))$. However one can show by the analysis in (Feireisl, Novotný , 2009) that one can pass to the limit in the convective term and obtain

$$\int_0^T\int_\Omega \overline{\varrho\,\vec{U}\otimes\vec{U}} : \nabla_x\vec{\phi}\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \overline{\varrho}\,\vec{U}\otimes\vec{U} : \nabla_x\vec{\phi}\,\mathrm{d}x\,\mathrm{d}t.$$

According to the hypotheses on the pressure law, the temperature ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega))\cap L^2(0,T;L^6(\Omega))$, which implies that $\mathbb{S}_\varepsilon \to \mu(\overline{\vartheta})(\nabla_x\vec{U}+\nabla_x^T\vec{U})$ weakly in $L^{\frac{34}{23}}(0,T;L^{\frac{34}{23}}(\Omega;\mathbb{R}^3))$.

So taking a divergence free test vector field $\vec{\phi}$ in (4.84), we have

$$\int_0^T\int_\Omega \Big(\varrho_\varepsilon\vec{u}_\varepsilon\cdot\partial_t\vec{\phi} + \varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon : \nabla_x\vec{\phi} - 2\varrho_\varepsilon\vec{\chi}\times\vec{u}_\varepsilon\cdot\vec{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T\int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x\vec{\phi} - \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\nabla_x\Psi_\varepsilon\cdot\vec{\phi} - \frac{1}{\zeta}\frac{\mathrm{curl}_x\vec{B}_\varepsilon}{\varepsilon}\times\frac{\vec{B}_\varepsilon}{\varepsilon}\cdot\vec{\phi} - \frac{1}{2}\varepsilon^2\varrho_\varepsilon\nabla_x|\vec{\chi}\times\vec{x}|^2\cdot\vec{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$- \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon}\cdot\vec{\phi}(0,\cdot)\,\mathrm{d}x. \tag{4.124}$$

Moreover, using (4.29) together with estimates (4.108), (4.114) and Aubin-Lions lemma we get

$$\frac{\vec{B}_\varepsilon}{\varepsilon} \to \vec{B} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \text{ and strongly in } L^2(0,T;L^q(\Omega,\mathbb{R}^3)), \tag{4.125}$$
$$\frac{1}{\zeta}\frac{\mathrm{curl}_x\vec{B}_\varepsilon}{\varepsilon}\times\frac{\vec{B}_\varepsilon}{\varepsilon} \to \frac{1}{\zeta}\mathrm{curl}_x\vec{B}\times\vec{B} \text{ weakly in } L^{\frac{5}{4}}((0,T)\times\Omega;\mathbb{R}^3),$$

for any $1 \le q < 6$.

Then passing to the limit and using (4.117)-(4.121), we get

$$\int_0^T\int_\Omega \Big(\overline{\varrho}\vec{U}\cdot\partial_t\vec{\phi} + \overline{\varrho}\vec{U}\otimes\vec{U} : \nabla_x\vec{\phi} - 2\overline{\varrho}\vec{\chi}\times\vec{U}\cdot\vec{\phi}\Big)\,\mathrm{d}x\,\mathrm{d}t =$$

$$\int_0^T \int_\Omega \left(\mu(\overline{\vartheta}) \left(\nabla_x \vec{U} + \nabla_x^T \vec{U} \right) : \nabla_x \vec{\phi} - \varrho_1 \nabla_x \Psi(\overline{\varrho}) \cdot \vec{\phi} - \frac{1}{\zeta} \mathrm{curl}_x \vec{B} \times \vec{B} \cdot \vec{\phi} \right) \mathrm{d}x \, \mathrm{d}t -$$

$$\int_\Omega \overline{\varrho} \vec{U}_0 \cdot \vec{\phi} \, \mathrm{d}x,$$

provided that $\vec{u}_{0,\varepsilon} \to \vec{U}_0$ weakly-$*$ in $L^\infty(\Omega; \mathbb{R}^3)$.

Moreover as in (Feireisl, Novotný , 2009), the formal relation between $\varrho^{(1)}$ and $\overline{\varrho}$ is recovered by multiplying the momentum equation by ε. One gets, using Proposition 4.5.1 and passing to the limit $\varepsilon \to 0$,

$$\int_0^T \int_\Omega \left(\nabla_x p^{(1)} - \overline{\varrho} \nabla_x \Psi(\overline{\varrho}) \right) \cdot \vec{\varphi} \, \mathrm{d}x \, \mathrm{d}t = 0, \tag{4.126}$$

which is the weak formulation of

$$\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \nabla_x \varrho^{(1)} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \nabla_x \vartheta^{(1)} - \overline{\varrho} \nabla_x \Psi(\overline{\varrho}) = 0. \tag{4.127}$$

This rewrites as

$$\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \nabla_x \varrho_1 - \overline{\varrho} \nabla_x \Psi(\overline{\varrho}) = 0, \tag{4.128}$$

once we establish that $\vartheta^{(1)} = \vartheta_1 = \vartheta_2 = 0$ in the section 4.5.3. That means we have got an explicit formula for ϱ_1

$$\varrho_1 = \frac{\overline{\varrho} \Psi(\overline{\varrho})}{\partial_\varrho p(\overline{\varrho}, \overline{\vartheta})} + h(t), \tag{4.129}$$

where h is an undetermined function.

4.5.2 Radiative transfer equation

Using the L^∞ bound shown in (Ducomet, Kobera, Nečasová , 2014) for I_ε, based on the initial data bound (4.82), it is clear that $I_\varepsilon \to I_0$ weakly in $L^2((0,T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$, and we have also by virtue of (4.112) $\vartheta_\varepsilon \rightharpoonup \overline{\vartheta}$ in $L^2(0,T; W^{1,2}(\Omega))$.

By using the cut-off hypotheses (4.23), (4.25) and the same notation for any time-independent test function $\psi \in C_c^\infty(\overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, we can pass to the limit in (4.88) and we get

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \cdot \nabla_x \psi \; I_0 \; d\vec{\omega} \; d\nu \, \mathrm{d}x + \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \Big[\sigma_a(\nu, \overline{\vartheta}) \left(\mathfrak{B}(\nu, \overline{\vartheta}) - I_0 \right) +$$

$$\sigma_s(\nu, \overline{\vartheta}) \left(\tilde{I}_0 - I_0 \right) \Big] \psi \; d\vec{\omega} \; d\nu \, \mathrm{d}x = \int_{\Gamma_+} \int_0^\infty I_0 \, \vec{\omega} \cdot \vec{n}_x \; \psi \; d\Gamma \; d\nu,$$

which is the weak formulation of the stationary problem

$$\vec{\omega} \cdot \nabla_x I_0 = S_0, \tag{4.130}$$

with the boundary condition

$$I_0 = 0 \quad \text{on } \Gamma_-, \tag{4.131}$$

where $S_0 = \sigma_a(\nu, \overline{\vartheta}) \left(\mathfrak{B}(\nu, \overline{\vartheta}) - I_0 \right) + \sigma_s(\nu, \overline{\vartheta}) \left(\tilde{I}_0 - I_0 \right)$. The solution of (4.130) – (4.131) is the function equal to $\mathfrak{B}(\nu, \overline{\vartheta}) = \mathfrak{B}_0$ in Ω and 0 on Γ_-. This solution is unique at least for a particular type of domains thanks to the linearity of (4.130).

Subtracting from (4.88) and dividing by ε gives

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon \partial_t \psi + \vec{\omega} \cdot \nabla_x \psi) \ \frac{I_\varepsilon - I_0}{\varepsilon} \ d\vec{\omega} \ d\nu \, \mathrm{d}x \, \mathrm{d}t +$$

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\frac{S_\varepsilon - S_0}{\varepsilon} \right] \psi \ d\vec{\omega} \ d\nu \, \mathrm{d}x \, \mathrm{d}t =$$

$$- \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon \frac{I_{0,\varepsilon} - I_0}{\varepsilon} \ \psi(0, x, \vec{\omega}, \nu) \ d\vec{\omega} \ d\nu \, \mathrm{d}x +$$

$$\int_0^T \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x \ \frac{I_\varepsilon - I_0}{\varepsilon} \psi \ d\Gamma \ d\nu \, \mathrm{d}t,$$

for any $\psi \in C_c^\infty([0,T) \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, with $S_\varepsilon - S_0 := S(I_\varepsilon) - S(I_0)$. From Proposition 4.5.1, we get

$$\frac{S_\varepsilon - S_0}{\varepsilon} \to S_1 := \partial_\vartheta(\sigma_a \mathfrak{B})(\nu, \overline{\vartheta})\vartheta^{(1)} - \partial_\vartheta \sigma_a(\nu, \overline{\vartheta})\vartheta^{(1)} I_0 - \sigma_a(\nu, \overline{\vartheta}) I_1 +$$

$$\partial_\vartheta \sigma_s(\nu, \overline{\vartheta})\vartheta^{(1)} \tilde{I}_0 + \sigma_s(\nu, \overline{\vartheta}) \tilde{I}_1 - \partial_\vartheta \sigma_s(\nu, \overline{\vartheta})\vartheta^{(1)} I_0 - \sigma_s(\nu, \overline{\vartheta}) I_1 =$$

$$-\sigma_a(\nu, \overline{\vartheta}) I_1 + \sigma_s(\nu, \overline{\vartheta}) \left(\tilde{I}_1 - I_1 \right)$$

weakly-* in $L^\infty(0,T; L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+))$ with $I_1 := I^{(1)}$.

Passing to the limit we find the limit equation based on the assumption $I_{0,\varepsilon}^{(1)} \to I_0^{(1)}$ weakly-* in $L^\infty(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \cdot \nabla_x \psi \ I_1 \ d\vec{\omega} \ d\nu \, \mathrm{d}x + \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} S_1 \psi \ d\vec{\omega} \ d\nu \, \mathrm{d}x =$$

$$\int_{\Gamma_+} \int_0^\infty I_1 \, \vec{\omega} \cdot \vec{n}_x \ \psi \ d\Gamma \ d\nu, \tag{4.132}$$

using the same notation for any time-independent test function $\psi \in C_c^\infty(\overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$ which is the weak formulation of the stationary problem

$$\vec{\omega} \cdot \nabla_x I_1 = S_1, \tag{4.133}$$

with the boundary condition

$$I_1 = 0 \ \text{ on } \Gamma_-. \tag{4.134}$$

4.5.3 Entropy balance

First of all we analyze the weak limit of (4.87), then we subtract it from (4.87) and divide by ε as in the last section. We follow the ideas of (Feireisl, Novotný , 2009) and (Kukučka , 2014).

The most obvious convergence in (4.87) is in the entropy production rate measures. By virtue of (4.107) it holds

$$\left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R ; \varphi \right\rangle_{[\mathcal{M};C]([0,T) \times \overline{\Omega})} \to 0 \ \text{ as } \varepsilon \to 0+, \tag{4.135}$$

and

$$\frac{1}{\varepsilon}\left\langle \varsigma^m_\varepsilon + \varsigma^R_\varepsilon ; \varphi \right\rangle_{[\mathcal{M};C]([0,T)\times\overline{\Omega})} \to 0 \quad \text{as } \varepsilon \to 0+\,. \tag{4.136}$$

We split the heat flux term into residual and essential parts as follows:

$$-\int_0^T \int_\Omega \frac{\vec{q}_\varepsilon}{\varepsilon^2 \vartheta_\varepsilon} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \frac{\kappa([\vartheta_\varepsilon]_{res})}{[\vartheta_\varepsilon]_{res}} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon^2} \cdot \nabla_x \varphi \, dx \, dt + \int_0^T \int_\Omega \frac{\kappa([\vartheta_\varepsilon]_{ess})}{[\vartheta_\varepsilon]_{ess}} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon^2} \cdot \nabla_x \varphi \, dx \, dt. \tag{4.137}$$

The first term on the rhs vanishes in the limit. The argument is as follows:

Firstly, from (4.107), (4.36) and (4.21) we get an exact estimate

$$\int_0^T \int_\Omega \vartheta_\varepsilon \left| \nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^2} \right|^2 dx \, dt \leq c. \tag{4.138}$$

From (4.112) we know that $\left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon} \right\|_{L^2(0,T;W^{1,2}(\Omega))} \leq c$, thus

$$\vartheta_\varepsilon \to \overline{\vartheta} \qquad \text{in } L^2(0,T;W^{1,2}(\Omega)) \tag{4.139}$$

strongly. On the residual set we now apply the Sobolev embedding and interpolate (4.139) with the information in (4.110). (Similarly we get $\vartheta^{(1)}_\varepsilon \to 0$ in $L^2(0,T;W^{1,2}(\Omega))$ as well.) This leads to the convergence

$$[\vartheta_\varepsilon]_{res} \to [\overline{\vartheta}]_{res} = 0 \qquad \text{in } L^{\frac{14}{3}}(0,T;L^{\frac{14}{3}}(\Omega)), \tag{4.140}$$

meaning that the first integral in (4.137) converges. With the intention that its limit is zero we apply (4.21) and split the integral into two parts. The second part, namely,

$$\int_0^T \int_\Omega \left([\vartheta_\varepsilon]^{\frac{3}{2}}_{res} - \overline{\vartheta}^{\frac{3}{2}} + \overline{\vartheta}^{\frac{3}{2}} \right) \sqrt{[\vartheta_\varepsilon]_{res}} \nabla_x \frac{\vartheta_\varepsilon}{\varepsilon^2} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{4.141}$$

converges to zero as $\varepsilon \to 0$ with the rate ε^2 by the Poincaré inequality

$$\left\| \vartheta^{\frac{3}{2}}_\varepsilon - \overline{\vartheta}^{\frac{3}{2}} \right\|_{L^2(0,T;L^2(\Omega))} \leq c \left\| \sqrt{\vartheta_\varepsilon} \nabla_x \vartheta_\varepsilon \right\|_{L^2(0,T;L^2(\Omega))} \leq C\varepsilon^2 \tag{4.142}$$

by (4.138), (4.99) and (4.110). The first part, namely,

$$\int_0^T \int_\Omega [1]_{res} \, \vartheta^{-1}_\varepsilon \nabla_x \frac{\vartheta_\varepsilon}{\varepsilon^2} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \tag{4.143}$$

converges to zero as $\varepsilon \to 0$ with the rate ε by Cauchy-Schwarz inequality, (4.113) and (4.99). The second term on the rhs of (4.137) converges by virtue of (4.138) and (4.139), at least for a subsequence, to

$$\int_0^T \int_\Omega \kappa(\overline{\vartheta}) \overline{\vartheta}^{-1} \nabla_x \vartheta_2 \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t. \tag{4.144}$$

For the convergence of the initial entropies in (4.87) we use Proposition 4.5.1 and we get

$$-\int_\Omega \left\{ \left(\frac{[(\varrho s)_{0,\varepsilon}]_{ess} - \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta})}{\varepsilon} + \varepsilon \frac{\left[s^R_{0,\varepsilon}\right]_{ess} - s^R(\overline{I})}{\varepsilon} \right) \varphi(0,\cdot) \right\} \mathrm{d}x$$

$$\to - \int_\Omega \overline{\varrho} \left(\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho_0^{(1)} \right) \phi(0,\cdot)\, \mathrm{d}x. \tag{4.145}$$

In particular

$$[(\varrho s)_{0,\varepsilon}]_{ess} \to \overline{\varrho} s(\overline{\varrho}, \overline{\vartheta}) \qquad \varepsilon \to 0+, \tag{4.146}$$

$$\left[s^R_{0,\varepsilon}\right]_{ess} \to s^R(\overline{I}) \qquad \varepsilon \to 0+, \tag{4.147}$$

weakly-(*) in $L^\infty\left(0,T;L^2(\Omega)\right)$.

Residual parts of the initial conditions disappear thanks to the L^∞ weak star convergences of the initial data in Theorem 4.4.2 for ε sufficiently small.

For the convergence in advective part of the entropy balance (4.87) we use (4.139) and the fact that

$$\varrho_\varepsilon \to \overline{\varrho} \qquad \varepsilon \to 0+ \tag{4.148}$$

in $L^\infty(0,T;L^{\frac{5}{3}}(\Omega))$. This allows to make the limit of entropy to a constant for a subsequence

$$s(\varrho_\varepsilon, \vartheta_\varepsilon) \to s(\overline{\varrho}, \overline{\vartheta}) \qquad \varepsilon \to 0+ \ \text{ a. e. in } (0,T)\times\Omega. \tag{4.149}$$

The convergence of entropy of a photon gas follows from Proposition 4.5.1 as

$$\varepsilon \frac{\left[s^R_\varepsilon\right]_{ess} - \overline{s^R}}{\varepsilon} \to 0, \tag{4.150}$$

$$s^R_\varepsilon \to \overline{s^R} \tag{4.151}$$

weakly-(*) in $L^\infty(0,T;L^2(\Omega))$ as $\varepsilon \to 0+$ according to (4.106) and (4.99) again. The convergence of the next term containing $\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon$ is again split into two terms, first one on the residual, second one on the essential set. For the second one we use again Proposition 4.5.1, the first one

$$\int_0^T \int_\Omega \left[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\right]_{res} \vec{u}_\varepsilon \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}t \tag{4.152}$$

converges to 0 just in $L^1((0,T)\times\Omega)$ as $\varepsilon \to 0+$ because of the estimates (4.99), (4.104), (4.110).

While the convergence of the equilibrial radiative entropy flux can be readily improved, e. g. to the space $L^{\frac{12}{11}}((0,T)\times\Omega)$ because of the Gibbs' relation between specific entropy and energy, cf. (4.18) and (4.19), the integral with the material entropy flux part does not seem to have a right regularity to be meaningful. However, we can use usual cut-off functions $T_K(z) := \min(K,z)$, choose K large enough, e. g. $K = \varepsilon^{-\frac{1}{6}}$ and split the integral into two parts

$$\int_0^T \int_\Omega \left|\left[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon, \vartheta_\varepsilon)\right]_{res}\right| |\vec{u}_\varepsilon| \, |\nabla_x \varphi| \, \mathrm{d}x\,\mathrm{d}t =$$

$$\int_0^T \int_\Omega |[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}|\, T_{\varepsilon^{-\frac{1}{6}}}(|\vec{u}_\varepsilon|)\, |\nabla_x\varphi| \,\mathrm{d}x\,\mathrm{d}t +$$
$$\int_0^T \int_\Omega |[\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{res}| \left[|\vec{u}_\varepsilon| - T_{\varepsilon^{-\frac{1}{6}}}(|\vec{u}_\varepsilon|)\right] |\nabla_x\varphi| \,\mathrm{d}x\,\mathrm{d}t.$$

The first part converges to 0 by (4.104), the second one is of order $O(\varepsilon)$ by Sobolev embedding, estimate (4.99) and Markov-Chebyshev inequality. The limiting part of this estimate is the first part, where the need to improve the regularity the material part of the entropy flux faces the problem that we have not got generally a better estimate than (4.104).

Previous works (Novotný, Růžička, Thäter , 2011), (Kukučka , 2014), (Feireisl, Novotný , 2009) rely on the closedness of the equation of state to the ideal gas law so that $\varrho_\varepsilon s_\varepsilon$ is estimated essentially by $\varrho_\varepsilon|\log\varrho_\varepsilon|$, ϑ_ε^3 and $\varrho_\varepsilon|\log\vartheta_\varepsilon|$, the last one being the most restrictive, leading to the convergence in (4.152) in $L^2(0,T;L^{\frac{30}{29}}(\Omega))$. Without such an assumption we would estimate the entropy by $\varrho_\varepsilon^2\vec{u}_\varepsilon$ which is not tractable in view of (4.110). Nevertheless, in our case of low stratification we do not need to identify the limit of the entropy flux on the essential set since it vanishes after an integration by parts.

After (4.139) and (4.148)

$$\int_0^T \int_\Omega [\varrho_\varepsilon s_\varepsilon(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess}\, \vec{u}_\varepsilon\cdot\nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \overline{\varrho} s(\overline{\varrho},\overline{\vartheta})\vec{U}\cdot\nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t = 0. \quad (4.153)$$

The last term contains the nonequlibirial radiative entropy flux $\vec{q}^R_\varepsilon$. Let us recall

$$\vec{q}^R_\varepsilon = \vec{q}^R(I_\varepsilon) = -\int_0^\infty \int_{\mathcal{S}^2} \nu^2 \left\{n_\varepsilon \log n_\varepsilon - (n_\varepsilon+1)\log(n_\varepsilon+1)\right\}\ \vec{\omega}\, d\vec{\omega}\, d\nu,$$

with $n_\varepsilon = n(I_\varepsilon) = \frac{I_\varepsilon}{\nu^3}$. We claim

$$J_\varepsilon := \int_0^T \int_\Omega \vec{q}^R_\varepsilon \cdot \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_\Omega \overline{\vec{q}}^R \cdot \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t =: J_0 \quad (4.154)$$

because of the convergence on the essential set $\mathcal{M}^\varepsilon_{ess}$ that follows from (4.161) and on the residual set $\mathcal{M}^\varepsilon_{res}$ we use (4.106). Collecting now all the aforementioned convergences in this section we readily get the weak formulation of (4.69). With (4.62) we see that $\vartheta_2 \equiv 0$ and search for $\nabla_x\Theta = \nabla_x\vartheta_3 := w - \lim_{L^{\frac{4}{3}}((0,T)\times\Omega),\varepsilon\to 0+} \nabla_x \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon^3}$.

Let us realize that we can extract from (4.107) the bound

$$\int_0^T \int_\Omega \vartheta_\varepsilon^{r-2} \frac{|\nabla_x(\vartheta_\varepsilon - \overline{\vartheta})|^2}{\varepsilon^4} \,\mathrm{d}x\,\mathrm{d}t < c \quad (4.155)$$

with a constant c independent of ε. Therefore $\left|\frac{\nabla_x(\vartheta_\varepsilon - \overline{\vartheta})}{\varepsilon^3}\right|$ is bounded in $L^{\frac{4}{3}}((0,T)\times\Omega)$ and Θ exists.

We subtract equation (4.87) from its limit and divide by ε

$$\int_0^T \int_\Omega \left\{ \varrho_\varepsilon\ \frac{s_\varepsilon - \overline{s}}{\varepsilon}\left(\partial_t\varphi + \vec{u}_\varepsilon\cdot\nabla_x\varphi\right) + \frac{s^R_\varepsilon - \overline{s^R}}{\varepsilon}\ \varepsilon\partial_t\varphi + \frac{\vec{q}^R_\varepsilon - \overline{\vec{q}^R}}{\varepsilon}\cdot\nabla_x\varphi \right\} \mathrm{d}x\,\mathrm{d}t +$$

$$\int_0^T\int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\frac{\vartheta_\varepsilon}{\varepsilon^3}\cdot\nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\left\langle\varsigma^m_\varepsilon+\varsigma^R_\varepsilon;\varphi\right\rangle_{[\mathcal{M};C]([0,T]\times\overline{\Omega})}=$$

$$-\int_\Omega\left\{\left(\varrho_{0,\varepsilon}\frac{s_{0,\varepsilon}-\overline{s}}{\varepsilon}+\varepsilon\frac{s^R_{0,\varepsilon}-\overline{s^R_0}}{\varepsilon}\right)\varphi(0,\cdot)\right\}\,\mathrm{d}x. \tag{4.156}$$

We claim that all the terms in (4.156) are uniformly bounded, especially

$$\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon^3}\to\frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\,\nabla_x\vartheta_3, \tag{4.157}$$

weakly in $L^{\frac{6r+16}{6r+15}}\left((0,T)\times\Omega;\mathbb{R}^3\right)$ which gives for $r=3$ the summability with the exponent of $\frac{34}{33}$. To show this we restrict ourselves to the residual set $\mathcal{M}^\varepsilon_{res}$, since on the essential set $\mathcal{M}^\varepsilon_{ess}$ the boundedness is easy. For the set $A_\varepsilon:=\{(t,x):|\nabla_x\vartheta_\varepsilon(t,x)|<1\}$ we use the estimates (4.99), (4.110) with Hölder's inequality and $r\in[3,5]$

$$K_0:=\int_{\mathcal{M}^\varepsilon_{res}\cap A_\varepsilon}\int\left|[\vartheta_\varepsilon]^{r-1}_{res}\,\nabla_x\frac{\vartheta_\varepsilon}{\varepsilon^3}\right|\,\mathrm{d}x\,\mathrm{d}t\le\varepsilon^{-3}\int\int_{\mathcal{M}^\varepsilon_{res}}\vartheta^{r-1}_\varepsilon\,\mathrm{d}x\,\mathrm{d}t\le$$

$$T\varepsilon^{-3}\|\,[\vartheta_\varepsilon]_{res}\,\|^{r-1}_{L^4(\Omega)}\operatorname*{ess\,sup}_{t\in(0,T)}\|\mathbb{I}_{\mathcal{M}^\varepsilon_{res}(t)}\|_{L^{\frac{4}{5-r}}(\Omega)}\le C\varepsilon^{-3}\varepsilon^{\frac{r-1}{2}}\varepsilon^2=C\varepsilon^{\frac{r-3}{2}}\le c \tag{4.158}$$

with c independent of ε. In the opposite case (the complement of this set in $\mathcal{M}^\varepsilon_{res}$) we estimate as follows

$$K_1:=\int_{\mathcal{M}^\varepsilon_{res}\setminus A_\varepsilon}\int\left|[\vartheta_\varepsilon]^{r-1}_{res}\,\nabla_x\frac{\vartheta_\varepsilon}{\varepsilon^3}\right|\,\mathrm{d}x\,\mathrm{d}t=$$

$$\int_{\mathcal{M}^\varepsilon_{res}\setminus A_\varepsilon}\int\left|[\vartheta_\varepsilon]^{\frac{r}{4}+\frac{1}{2}}_{res}\,|\nabla_x\vartheta_\varepsilon|^{-\frac{1}{2}}\underbrace{\vartheta_\varepsilon^{\frac{3}{4}r-\frac{3}{2}}\frac{|\nabla_x\vartheta_\varepsilon|^{\frac{3}{2}}}{\varepsilon^3}}_{\in L^{\frac{4}{3}}(\mathcal{M}^\varepsilon_{res}\setminus A_\varepsilon)}\right|\,\mathrm{d}x\,\mathrm{d}t\le$$

$$\int_0^T\int_\Omega\left|[\vartheta_\varepsilon]^{\frac{r}{4}+\frac{1}{2}}_{res}\,\vartheta_\varepsilon^{\frac{3}{4}r-\frac{3}{2}}\frac{|\nabla_x\vartheta_\varepsilon|^{\frac{3}{2}}}{\varepsilon^3}\right|\,\mathrm{d}x\,\mathrm{d}t<c \tag{4.159}$$

provided $[\vartheta_\varepsilon]^{\frac{r}{4}+\frac{1}{2}}_{res}$ is uniformly bounded in $L^4\left((0,T)\times\Omega\right)$, that is $[\vartheta_\varepsilon]^{r+2}_{res}$ is uniformly bounded in the space $L^1\left((0,T)\times\Omega\right)$. However we know that $[\vartheta_\varepsilon]_{res}$ is bounded in $L^\infty\left(0,T;L^4(\Omega)\right)\cap L^r\left(0,T;L^{3r}(\Omega)\right)$ as in (Feireisl, Novotný , 2009). By interpolation we get $[\vartheta_\varepsilon]_{res}$ is uniformly bounded in $L^{r+\frac{8}{3}}\left((0,T)\times\Omega\right)$ and that is why K_1 converges; moreover when we reiterate the same argument with a $s-$power of its integrand, we obtain the bound $s\le\frac{6r+16}{6r+15}$ for Hölder's inequality.

Similarly to (Feireisl, Novotný , 2009), using Proposition 4.5.1 and energy estimates, we see that

$$\varrho_\varepsilon\,\frac{s_\varepsilon-\overline{s}}{\varepsilon}\to\overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)}+\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\right)=\overline{\varrho}\varrho_1\partial_\varrho s(\overline{\varrho},\overline{\vartheta})$$

weakly-$*$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$, and remind (4.136), (4.150), (4.106), (4.99) and (4.145).

Moreover the advective part weakly converges according to Proposition 4.5.1 again

$$\varrho_\varepsilon \, \frac{s_\varepsilon - \overline{s}}{\varepsilon} \vec{u}_\varepsilon \to \overline{\varrho} \left(\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho^{(1)} + \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \vartheta^{(1)} \right) \vec{U} = \overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho_1 \vec{U},$$

weakly in $L^2(0,T; L^{3/2}(\Omega; \mathbb{R}^3))$. This allows to pass to the limit in all terms of (4.156) except the nonequilibrial radiative entropy flux term

$$\int_0^T \int_\Omega \frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t. \tag{4.160}$$

Let us compute the limit of $\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon}$.

Applying once more Proposition 4.5.1 with $G^R(I) = n(I) \log n(I) - (n(I)+1) \log(n(I)+1)$ and integrating on $\mathcal{S}^2 \times \mathbb{R}_+$, we find

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \, \log \frac{n(\overline{I})+1}{n(\overline{I})} \vec{\omega} \, I^{(1)} \, d\vec{\omega} \, d\nu,$$

weakly-$*$ in $L^\infty(0,T; L^2(\Omega; \mathbb{R}^3))$ on the essential set $\mathcal{M}^\varepsilon_{ess}$ and as $\log \left[\frac{n(\overline{I})+1}{n(\overline{I})} \right] = \frac{\nu}{\overline{\vartheta}}$, we have got

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \frac{1}{\overline{\vartheta}} \, \vec{F}^R(I^{(1)}),$$

with the radiative momentum $\vec{F}^R(I^{(1)}) = \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \, I^{(1)} \, d\vec{\omega} \, d\nu$. So

$$\int_0^T \int_\Omega \left(\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \right) \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t \to \int_0^T \int_\Omega \frac{\mathrm{div}_x \vec{F}^R(I^{(1)})}{\overline{\vartheta}} \, \varphi \, \mathrm{d}x \, \mathrm{d}t. \tag{4.161}$$

by the Proposition 4.5.1 , (4.106) and (4.99) on $\mathcal{M}^\varepsilon_{res}$. As we have got from (4.133)

$$\mathrm{div}_x \vec{F}^R = \int_0^\infty \int_{\mathcal{S}^2} \left[\partial_\vartheta \sigma_a(\nu, \overline{\vartheta}) \left(\mathfrak{B}(\nu, \overline{\vartheta}) - I_0 \right) \vartheta^{(1)} + \right.$$
$$\left. \sigma_a(\nu, \overline{\vartheta}) \left(\partial_\vartheta \mathfrak{B}(\nu, \overline{\vartheta}) \vartheta^{(1)} - I_1 \right) \right] \, d\vec{\omega} \, d\nu,$$

the limit contribution in (4.156) becomes

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \frac{-\sigma_a(\nu, \overline{\vartheta}) I_1(t, x, \vec{\omega}, \nu)}{\overline{\vartheta}} \varphi \;\; d\vec{\omega} \, d\nu \;\; \mathrm{d}x \, \mathrm{d}t.$$

Gathering all of these terms, we find the limit equation for entropy

$$\overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \int_0^T \int_\Omega \varrho_1 \left(\partial_t \phi + \vec{U} \cdot \nabla_x \phi \right) \, \mathrm{d}x \, \mathrm{d}t + \frac{\overline{\kappa}}{\overline{\vartheta}} \int_0^T \int_\Omega \nabla_x \Theta \cdot \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t -$$
$$\frac{1}{\overline{\vartheta}} \int_0^T \int_\Omega \int_0^\infty \sigma_a(\nu, \overline{\vartheta}) \int_{\mathcal{S}^2} I_1(t, x, \vec{\omega}, \nu) \varphi \, d\vec{\omega} \, d\nu \, \mathrm{d}x \, \mathrm{d}t =$$
$$-\overline{\varrho} \partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \int_\Omega \varrho_0^{(1)} \varphi(0, \cdot) \, \mathrm{d}x.$$

Using (4.128) we easily verify that we finally obtained the thermal equation (4.75) once we take the Maxwell relation $\partial_\vartheta p = \partial_\varrho s$ into account.

4.5.4 Maxwell equation

From (4.123) and (4.125) we get

$$\frac{\vec{B}_\varepsilon}{\varepsilon} \times \vec{u} \to \vec{B} \times \vec{U} \text{ weakly in } L^q(0,T;L^q(\Omega,\mathbb{R}^3)) \text{ for } q \in \left[1, \frac{5}{3}\right),$$

and

$$\tilde{\lambda}(\vartheta_\varepsilon)\,\mathrm{curl}_x \frac{\vec{B}_\varepsilon}{\varepsilon} \to \overline{\lambda}\,\mathrm{curl}_x \vec{B} \text{ weakly in } L^{\frac{34}{6p+17}}(0,T,L^{\frac{34}{6p+17}}(\Omega,\mathbb{R}^3)).$$

Then it is easy to pass to the limit in (4.86), realizing that $\frac{34}{6p+17} > 1$ for $1 \le p < \frac{17}{6}$.

This last step ends the proof of Theorem 4.4.2.

4.A Appendix: Proof of Theorem 4.3.1

1. Consider now the linearly coupled problem for the remaining equations

$$\mathrm{div}_x \vec{U} = 0, \tag{4.A.1}$$

$$\partial_t \vec{U} + (\vec{U}\cdot\nabla_x)\vec{U} + \nabla_x \Pi - \overline{\mu}\Delta\vec{U} + \frac{1}{\zeta}\nabla_x\left(\frac{\vec{B}^2}{2}\right) - \frac{1}{\zeta}(\vec{B}\cdot\nabla_x)\vec{B} = \vec{F}, \tag{4.A.2}$$

$$\partial_t \vec{B} + (\vec{U}\cdot\nabla_x)\vec{B} + (\vec{B}\cdot\nabla_x)\vec{U} - \overline{\lambda}\Delta\vec{B} = 0, \tag{4.A.3}$$

$$\mathrm{div}_x \vec{B} = 0, \tag{4.A.4}$$

$$-\triangle\Theta = \vec{U}\cdot\vec{\beta} - \frac{1}{\overline{\kappa}}\int_0^\infty \sigma_a \int_{\mathcal{S}^2} I_1\,d\vec{\omega}\,d\nu + \tilde{h} \tag{4.A.5}$$

$$\vec{\omega}\cdot\nabla_x I_1 + \sigma_a I_1 - \sigma_s\left(\tilde{I}_1 - I_1\right) = 0, \tag{4.A.6}$$

where $\vec{\beta} \in (L^\infty(\Omega))^3$, together with the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0,\ \nabla\Theta\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0,\ \mathrm{curl}_x\vec{B}\times\vec{n}|_{\partial\Omega} = 0 \tag{4.A.7}$$

for (4.A.1)-(4.A.5) and

$$I_1(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \le 0 \tag{4.A.8}$$

for (4.A.6), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \vec{B}|_{t=0} = \vec{B}_0. \tag{4.A.9}$$

We first consider the solution $(\vec{U},\vec{B},I_1)$ of the "radiative-MHD problem"

$$\mathrm{div}_x \vec{U} = 0, \tag{4.A.10}$$

$$\partial_t \vec{U} + (\vec{U}\cdot\nabla_x)\vec{U} + \nabla_x \Pi - \overline{\mu}\Delta\vec{U} = \frac{1}{\zeta}\mathrm{curl}_x\vec{B}\times\vec{B} + \vec{F}, \tag{4.A.11}$$

$$\partial_t \vec{B} + (\vec{U}\cdot\nabla_x)\vec{B} + \vec{B}\cdot\nabla_x\vec{U} - \overline{\lambda}\Delta\vec{B} = 0, \tag{4.A.12}$$

$$\mathrm{div}_x \vec{B} = 0, \tag{4.A.13}$$

$$\vec{\omega} \cdot \nabla_x I_1 + \sigma_a I_1 - \sigma_s \left(\tilde{I}_1 - I_1 \right) = 0, \tag{4.A.14}$$

with

$$\vec{U}|_{\partial\Omega} = 0,\ \vec{B} \cdot \vec{n}|_{\partial\Omega} = 0,\ \mathrm{curl}_x \vec{B} \times \vec{n}|_{\partial\Omega} = 0,$$

and

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \vec{B}|_{t=0} = \vec{B}_0,\ I_1(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega} \cdot \vec{n} \leq 0.$$

The MHD part has a weak solution $\vec{U} \in L^2(0, T; \mathcal{U}(\Omega))$, $\vec{B} \in L^2(0, T; \mathcal{W}(\Omega))$ thanks to an extension of the Leray-Hopf Theorem (see Sermange, Temam , 1983). Moreover the stationary radiative equation (4.A.14) also has got a weak solution $I_1 \in L^2((0, T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+)$ according to (Bardos et al. , 1988; Theorem 1 and Proposition 2).

Then we consider the solution Θ of the stationary diffusion equation

$$-\Delta\Theta = \vec{U} \cdot \vec{\beta} - \frac{1}{\bar{\kappa}} \int_0^\infty \sigma_a \int_{\mathcal{S}^2} I_1 \, d\vec{\omega} \, d\nu + \tilde{h} \tag{4.A.15}$$

with

$$\nabla\Theta \cdot \vec{n}|_{\partial\Omega} = 0.$$

subject to $\int_\Omega \Theta \, \mathrm{d}x = 0$ for all times. It admits a weak solution $\Theta \in L^\infty((0, T; W^{2,2}(\Omega)) \cap L^2((0, T; W^{q,2}(\Omega))\ \forall q < \frac{5}{2}$, thanks to classical elliptic regularity theory and due to regularity of the rhs due to (Golse, Lions, Perthame, Sentis , 1988).

Since the "radiative-MHD problem" does not depend on the temperature perturbation Θ the proof is complete.

Conclusion

In this thesis we study a model in radiative magnetohydrodynamics, which describes astrophysical plasmas. The model consists of the fundamental mass balance (2.10), momentum balance (2.11), the total energy balance (2.48) and an entropy equation/inequality (2.44), (2.45), the Maxwell system in the magnetohydrodynamical approximation (2.2)–(2.6) and the radiative transfer equation (2.13), (2.17) – (2.19).

We establish the existence of weak (renormalized) solutions to this system in the Theorem (2.2.1). Its proof is based on the theory developed in (Lions , 1996, 1998) and (Feireisl, Novotný , 2009) and on a velocity averaging lemma (Golse, Lions, Perthame, Sentis , 1988).

Then we investigate a particular limit of small Péclet, Mach, Froude and Alfvén numbers of the rescaled Navier-Stokes-Fourier system (4.47), (4.48), (4.49), (4.50), (4.51), (4.52). We identify the limit system (4.71), (4.72), (4.73), (4.74), (4.75), (4.76) with a Boussinesq-type relation (4.77). We establish the existence of a weak solution to the limit system in Theorem (4.3.1) and finally prove weak convergence of weak solutions to the primitive system towards weak solutions to the limit system in Theorem (4.4.2) for well-prepared data. All this is achieved for boundary conditions expressing mechanical and energetical isolation and a contact with a perfect conductor. The radiation may not enter the domain Ω. The essential role in the theory plays the entropy inequality. This time not all terms in the entropy production are non-negative, but thanks to some cut-off assumptions can be estimated at the right hand side.

The subject of (singular) limits in fluid mechanics is an active research field and with similar methods various other limit regimes can be tackled.

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List of Abbreviations

— BBGKY Bogoliubov–Born–Green–Kirkwood–Yvon hierarchy

— BGK Bhatnagar-Gross-Crook equation or operator

— CE compressible Euler system

— EOS equation of state

— IBVP initial boundary value problem

— IE incompressible Euler system

— lhs left hand side

— LTE local thermodynamical equilibrium

— MHD magnetohydrodynamics

— NSF Navier-Stokes-Fourier system

— rhs right hand side

— RTE radiative transfer equation

— SNSF scaled Navier-Stokes-Fourier system