Limit theorems for solutions to one-dimensional boundary-value problems in Sobolev spaces

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Aim

The aim of the talk is

to present the results about the character of solvability and continuity in the parameters of solutions to systems of linear differential equations of arbitrary order on a finite interval with the most general inhomogeneous boundary conditions. These boundary-value problems have essential features and require new research methods.

Contents

Background

General boundary conditions

General boundary-value problems are a classic object of research in the theory of ordinary differential equations.
In the paper of I. Kiguradze (1987), for the first time sufficient conditions were obtained for uniform convergence of solutions to boundary-value problems with general inhomogeneous boundary conditions:

$$y'(t;n) = A(t;n)y(t;n) + f(t;n), \quad t \in [a,b],$$
$$B_n y(t;n) = c_n,$$

where $A(\cdot;n) \in (L_1)^{m\times m}$, $f(\cdot;n) \in (L_1)^m$, $c_n \in \mathbb{R}^m$ and linear continuous operators

$$B_n \colon C\big([a,b];\mathbb{R}^m\big) \to \mathbb{R}^m, \quad n \in \mathbb{N} \cup \{0\}.$$

These boundary conditions include classical boundary conditions, but cannot contain derivatives of the unknown function.

Kiguradze theorem

Suppose that a homogeneous boundary-value problem, where $n = 0$, has only a trivial solution and the following conditions are satisfied:

1) $\sup\limits_{n} \|A(\cdot; n)\|_1 < \infty$;
2) $\sup\limits_{n} \|f(\cdot; n)\|_1 < \infty$;
3) $\sup\limits_{n} \|B_n\| < \infty$;
4) $\max\limits_{t\in[a,b]} \Big|\int\limits_a^t A(s; n)ds - \int\limits_a^t A(s; 0)ds\Big| \to 0$;
5) $\max\limits_{t\in[a,b]} \Big|\int\limits_a^t f(s; n)ds - \int\limits_a^t f(s; 0)ds\Big| \to 0$;
6) $c_n \to c_0$;
7) $B_n y \to B_0 y, \quad y \in (W_1^1)^m$.

Then for sufficiently large n solutions to boundary-value problems exist, are unique, and

$$\big\|y(\cdot, 0) - y(\cdot, n)\big\|_\infty \to 0, \quad n \to \infty.$$

Distributions in coefficients

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Famous monographs:

1. S. Albeverio, F. Gesctezy, R. Hoegh–Krohn, and H. Holden, *Solvable Models in Quantum Mechanics.* Springer, New York (1988).
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For ordinary differential operators, the idea is to define such operators as quasi-differential with properly selected quasi–derivatives according to Shin-Zettl. In this connection, we can interpret some classes of quasi-differential operators as limits in the sense of norm or strong resolvent convergence of differential operators with smooth coefficients.

One-dimensional Schrödinger operator

Let the formal differential expression

$$l(y) = -y''(t) + q'(t)y(t), \quad q(\cdot) \in L_2\big([a,b],\mathbb{C}\big) = L_2 \tag{1}$$

be given on a compact interval, where the derivative of a function q is understood in the sense of distributions.
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This expression can be defined as the Shin–Zettl quasi-differential expression with following quasi-derivatives:

$$D^{[0]}y := y, \quad D^{[1]}y := y' - qy,$$
$$l(y) = D^{[2]}y := -\big(D^{[1]}y\big)' - qD^{[1]}y - q^2y.$$

If the function q is smooth, then this definition is equivalent to the classical one.

One-dimensional Schrödinger operator

Let us consider the set of quasi-differential expressions $l_\varepsilon(\cdot)$ of the form (1) with functions $q_\varepsilon(\cdot) \in L_2$, $\varepsilon \in [0, \varepsilon_0]$. In the Hilbert space with norm $\left\|\cdot\right\|_2$ each of these expressions generates a dense defined closed quasi-differential operator $L_\varepsilon y := l_\varepsilon(y)$.

$$Dom\big(L_\varepsilon\big) = \{y \in L_2 \colon D^{[2]}y \in L_2; \quad \alpha(\varepsilon)Y_a(\varepsilon) + \beta(\varepsilon)Y_b(\varepsilon) = 0\} \subset W_1^1,$$

where matrices $\alpha(\varepsilon)$, $\beta(\varepsilon) \in \mathbb{C}^{2\times 2}$, and vectors

$$Y_a(\varepsilon) := \{y(a), \quad D_\varepsilon^{[1]}(a)\}, Y_b(\varepsilon) := \{y(b), \quad D_\varepsilon^{[1]}(b)\} \in \mathbb{C}^2.$$

Note that the set $Dom\big(L_\varepsilon\big)$ may not contain any nontrivial function from C^1.

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Theorem (Mikhailets, Goriunov 2010)

Suppose that the resolvent set of the operator L_0 is not empty and

i) $\|q_\varepsilon - q_0\|_2 \to 0, \quad \varepsilon \to 0+;$

ii) $\alpha(\varepsilon) \to \alpha(0), \quad \beta(\varepsilon) \to \beta(0).$

Then $L_\varepsilon \to L_0$ in the sense of norm resolvent convergence.

Thus, each of the introduced operators is the limit of similar operators with smooth coefficients.

Generic boundary conditions

Statement of the problem

Let $(a, b) \subset \mathbb{R}$ and $\{m, n, r\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a, b), \tag{2}$$

$$By = c. \tag{3}$$

Here, $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, $f(\cdot) \in (W_p^n)^m$, $c \in \mathbb{C}^{rm}$, linear continuous operator

$$B \colon (W_p^{n+r})^m \to \mathbb{C}^{rm} \tag{4}$$

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The solutions of equation (2) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (3) with operator (4) is generic condition for this equation.

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It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order $\geq r$.

Invertibility of the operator

With problem (2), (3), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^{rm}. \tag{5}$$

By $[BY_k]$, we denote the numerical $m \times m$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1.

A block numerical matrix

$$\text{M(L,B)} := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{rm\times rm} \tag{6}$$

is characteristic matrix to problem (2), (3). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m\times m}$.

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Theorem 1.

The operator (5) is invertible if and only if the matrix $M(L, B)$ is nondegenerate.

Continuity in a parameter

Parameterized boundary-value problem

Boundary-value problem depending in a parameter $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \ \ t \in (a,b), \tag{7}$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon). \tag{8}$$

Problem (7), (8) is a Fredholm one with zero index for every ε.

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Definition 2.

The solution to the problem (7), (8) depends continuously in a parameter ε at $\varepsilon = 0$ if the conditions are satisfied:

($*$) there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen $f(\cdot;\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;

($**$) the convergence of right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ and $c(\varepsilon) \to c(0)$ implies the convergence of solutions

$$y(\cdot;\varepsilon) \to y(\cdot;0) \quad \text{in} \quad (W_p^{n+r})^m \quad \text{as} \quad \varepsilon \to 0+.$$

The continuous dependence in a parameter

Consider the following conditions:

(0) the homogeneous boundary-value problem

$$L(0)y(t,0) = 0, \quad t \in (a,b), \quad B(0)y(\cdot,0) = 0$$

has only the trivial solution;

(I) $A_{r-j}(\cdot;\varepsilon) \to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$ for every $j \in \{1,\dots,r\}$;

(II) $B(\varepsilon)y \to B(0)y$ in $\mathbb{C}^{rm}$ for every $y \in (W_p^{n+r})^m$.

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Theorem 2.

The solution to the problem (7), (8) depends continuously in the parameter ε at $\varepsilon=0$ if and only if this problem satisfies Conditions (0), (I), and (II).

Remark

Gnyp, Mikhailets, and Murach (2016) gave a constructive criterion in Sobolev spaces W_p^{n+r}, where $1 \leq p < \infty$. The proof based on the unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m. \tag{9}$$

Here, $\alpha_k \in \mathbb{C}^{rm\times m}$, $\Phi(\cdot) \in L_{p'}\left([a,b];\mathbb{C}^{rm\times m}\right)$, $1/p + 1/p^{'} = 1$.

Our method of proof allows to investigate such problems in Sobolev spaces W_p^{n+r}, where $1 \leq p \leq \infty$, and some others function spaces.

Degree of convergence of the solutions

We supplement our result with a two-sided estimate of the error $\left\|y(\cdot;0)-y(\cdot;\varepsilon)\right\|_{n+r,p}$ of solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \left\|L(\varepsilon)y(\cdot;0)-f(\cdot;\varepsilon)\right\|_{n,p} + \left\|B(\varepsilon)y(\cdot;0)-c(\varepsilon)\right\|_{\mathbb{C}^{rm}}.$$

Here, $y(\cdot;0)$ is an approximate solution to problem (7), (8).

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Here,$y(\cdot;0)$ is an approximate solution to problem (7), (8).

Theorem 3.

Let the problem (7), (8) satisfies Conditions (0), (I), and (II). Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$, γ_1, and γ_2, such that

$$\gamma_1\, \widetilde{d}_{n,p}(\varepsilon) \leq \|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p} \leq \gamma_2\, \widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_2)$. Here, the numbers ε_2, γ_1, and γ_2 do not depend on $y(\cdot;0)$, and $y(\cdot;\varepsilon)$.

Thus, the error and discrepancy of the solution to problem (7), (8) are of the same degree of smallness.

Multipoint boundary-value problem

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (7)

multipoint Fredholm boundary condition

$$B(\varepsilon)y(\cdot,\varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon), \varepsilon) = q(\varepsilon), \qquad (10)$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m \times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a, b]$ are arbitrarily given.

The solution $y(\cdot, \varepsilon)$ to problem (7), (10) is continuous in the parameter ε if it exists, is unique, and satisfies the limit relation

$$\big\| y(\cdot,\varepsilon) - y(\cdot,0) \big\|_{n+r,p} \to 0 \quad \text{as} \quad \varepsilon \to 0+ . \qquad (11)$$

The limit theorem, $p = \infty$

Assumptions as $\varepsilon \to 0+$:

(α) $t_{j,k}(\varepsilon) \to t_j$ for all $j \in \{1, \ldots, N\}$, and $k \in \{1, \ldots, \omega_j(\varepsilon)\}$;

(β) $\sum\limits_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \to \beta_j^{(l)}$ for all $j \in \{1, \ldots, N\}$, and $l \in \{0, \ldots, n+r-1\}$;

(γ) $\sum\limits_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(l)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1, \ldots, N\}$, $k \in \{1, \ldots, \omega_j(\varepsilon)\}$, and $l \in \{0, \ldots, n+r-1\}$;

(δ) $\sum\limits_{k=1}^{\omega_0(\varepsilon)} \left\|\beta_{0,k}^{(l)}(\varepsilon)\right\| \to 0$ for all $k \in \{1, \ldots, \omega_0(\varepsilon)\}$, and $l \in \{0, \ldots, n+r-1\}$.

Theorem 4.

Let the boundary-value problem (7), (10) for $p = \infty$ satisfies the assumptions (α), (β), (γ), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (11).

The limit theorem, $1 \le p < \infty$

Assumptions as $\varepsilon \to 0+$:

(γ_p) $\sum_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(n+r-1)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right|^{1/p'} = O(1)$ for all $j \in \{1, \dots, N\}$, and $k \in \{1, \dots, \omega_j(\varepsilon)\}$;

(γ') $\sum_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(l)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1, \dots, N\}$, $k \in \{1, \dots, \omega_j(\varepsilon)\}$, and $l \in \{0, \dots, n+r-2\}$.

Theorem 5.

Let the boundary-value problem (7), (10) for $1 \le p < \infty$ satisfies the assumptions (α), (β), (γ_p), (γ'), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (11).

Applications

Approximation

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t)y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{12}$$

$$By = c, \tag{13}$$

where $1 \leq p < \infty$.

A sequence of multipoint boundary-value problems

$$(L_k y_k)(t) := y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t)y_k^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{14}$$

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c. \tag{15}$$

Approximation

Theorem 6.

For the boundary-value problem (12), (13) there is a sequence of multipoint boundary-value problems of the form (14), (15) such that they are well-posedness for sufficiently large k and the asymptotic property is fulfilled

$$y_k \to y \quad \text{in} \quad (W_p^{n+r})^m \quad \text{for} \quad k \to \infty.$$

The sequence can be chosen independently of f and c, and constructed explicitly.

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Thank you!