On the solvability of inhomogeneous boundary-value problems in Sobolev spaces

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Aim

The aim of the talk is

to present the results about the character of solvability of systems of linear differential equations of arbitrary order on a finite interval with the most general inhomogeneous boundary conditions. Boundary conditions can be both overdetermined and underdetermined. These boundary-value problems have essential features and require new research methods.

Contents

Generic boundary conditions

Statement of the problem

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m,\, n,\, r,\, l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here, $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, $f(\cdot) \in (W_p^n)^m$, $c \in \mathbb{C}^l$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen; $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

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are arbitrarily chosen; $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation.

Statement of the problem

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) of an order $\geq r$.

Thus, boundary conditions can contain derivatives whose order is greater than the order of the equation.

If $l < r$, then the boundary conditions are underdetermined.
If $l > r$, then the boundary conditions are overdetermined.

Example 1 (one-point problem)

Linear one-point boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b),$$

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) = c.$$

Here, $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, $f(\cdot) \in (W_p^n)^m$, $c \in \mathbb{C}^l$, $\alpha_k \in \mathbb{C}^{l\times m}$ are arbitrarily chosen; $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

Here, m is the number of differential equations of order r,

l is the number of scalar boundary conditions.

General case

In case $1 \leq p < \infty$, the linear continuous operator $B\colon (W_p^{n+r})^m \to \mathbb{C}^l$ admits the unique analytic representation

$$By = \sum_{k=0}^{n+r-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n+r)}(t)\mathrm{d}t, \quad y(\cdot) \in (W_p^{n+r})^m. \qquad (4)$$

Here, the matrices $\alpha_k \in \mathbb{C}^{l\times m}$, and the matrix-valued function $\Phi(\cdot) \in L_{p'}\big([a,b];\mathbb{C}^{l\times m}\big)$, $1/p + 1/p^{'} = 1$.

For $p = \infty$ this formula also defines an operator $B\colon (W_\infty^{n+r})^m \to \mathbb{C}^l$. However, there exist other operators from this class generated by the integrals over finitely additive measures.

Index of problem

With problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{5}$$

Theorem 1.

The linear operator (5) *is a bounded Fredholm operator with index* $mr - l$.

Characteristic matrix

Family of matrix Cauchy problems with the initial conditions

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1, \dots, r\}.$$

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By $[BY_k]$, we denote the numerical $m \times l$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

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Definition 1.

A block numerical matrix

$$\mathrm{M(L,B)} := \left([BY_0], \ldots, [BY_{r-1}] \right) \in \mathbb{C}^{rm \times l} \tag{6}$$

is characteristic matrix to problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m \times l}$.

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If $B = 0$, then $M(L,B) = O_{rm \times l}$ for all L.

Solvability of problem

Theorem 2.

The dimensions of kernel and cokernel of the operator (5) *are equal to the dimensions of kernel and cokernel of matrix* (6), *respectively:*

$$\dim\ker(L,B) = \dim\ker\big(M(L,B)\big),$$
$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\big(M(L,B)\big).$$

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Corollary 1.

The operator (5) *is invertible if and only if* $l=mr$ *and the square matrix* $M(L,B)$ *is nondegenerate.*

Example 2 (one-point problem)

Consider a linear *one-point* boundary-value problem

$$Ly(t) := y'(t) + Ay(t) = f(t), \quad t \in (a,b),$$

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) = c. \tag{7}$$

Here, A is a constant $m \times m$ – matrix, $f(\cdot) \in (W_p^{n-1})^m$, $\alpha_k \in \mathbb{C}^{l\times m}$, $c \in \mathbb{C}^l$,

$$B\colon (W_p^n)^m \to \mathbb{C}^l, \quad (L,B)\colon (W_p^n)^m \to (W_p^{n-1})^m \times \mathbb{C}^l,$$

$y(\cdot) \in (W_p^n)^m$.
We denote by $Y(\cdot) \in (W_p^n)^{m\times m}$ the unique solution of the Cauchy matrix problem

$$Y'(t) + AY(t) = O_m, \quad t \in (a,b), \quad Y(a) = I_m.$$

Example 2 (one-point problem)

Then the matrix-valued function $Y(\cdot)$ and its k-th derivative will have the following form:

$$Y(t) = \exp\big(-A(t-a)\big), \quad Y(a) = I_m;$$
$$Y^{(k)}(t) = (-A)^k \exp\big(-A(t-a)\big), \quad Y^{(k)}(a) = (-A)^k, \quad k \in \mathbb{N}.$$

Substituting these values into the equality (7), we have

$$M(L,B) = \sum_{k=0}^{n-1} \alpha_k (-A)^k.$$

It follows from Theorem 1 that $ind(L,B) = ind(M(L,B)) = m - l$. Therefore, by Theorem 2, we obtain

$$\dim\ker(L,B) = \dim\ker\left(\sum_{k=0}^{n-1} \alpha_k (-A)^k\right) = m - \operatorname{rank}\left(\sum_{k=0}^{n-1} \alpha_k (-A)^k\right),$$

$$\dim\operatorname{coker}(L,B) = -m + l + \dim\ker\left(\sum_{k=0}^{n-1} \alpha_k (-A)^k\right) = l - \operatorname{rank}\left(\sum_{k=0}^{n-1} \alpha_k (-A)^k\right)$$

Example 3 (two-point problem)

Consider the *two-point* boundary-value problem with the coefficient $A(t) \equiv O_m$ and the boundary conditions at the points $\{t_0, t_1\} \subset [a, b]$ containing derivatives of integer and / or *fractional* orders. They are given by equality

$$Ly(t) := y'(t) + Ay(t) = f(t), \quad t \in (a, b),$$

$$By = \sum_j \alpha_{0j} y^{(\beta_{0j})}(t_0) + \sum_j \alpha_{1j} y^{(\beta_{1j})}(t_1).$$

Here, both sums are finite, numerical matrices $\alpha_{kj} \in \mathbb{C}^{l \times m}$. In this case, the matrix-valued function $Y(\cdot) = I_m$. Therefore, the characteristic matrix has the form

$$M(L, B) = \sum_j \alpha_{0j} I_m^{(\beta_{0j})} + \sum_j \alpha_{1j} I_m^{(\beta_{1j})} = \alpha_{0,0} + \alpha_{1,0},$$

because the derivatives $I_m^{(\beta_{kj})} = 0$.

Example 3 (two-point problem)

Therefore, by Theorem 2, we have

$$\dim\ker(L,B)=\dim\ker\left(\alpha_{0,0}+\alpha_{1,0}\right)=m-\operatorname{rank}\left(\alpha_{0,0}+\alpha_{1,0}\right),$$
$$\dim\operatorname{coker}(L,B)=\dim\ker\left(\alpha_{0,0}+\alpha_{1,0}\right)-m+l=l-\operatorname{rank}\left(\alpha_{0,0}+\alpha_{1,0}\right).$$

Example 4 (multipoint problem)

Consider problem (1), (2), where $r = 1$, putting $A(t) \equiv 0$ with the next boundary conditions:

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n)}(t) \mathrm{d}t, \quad y(\cdot) \in (W_p^n)^m.$$

Then we have

$$BY = \sum_{s=0}^{n-1} \alpha_s Y^{(s)}(a) + \int_a^b \Phi(t) Y^{(n)}(t) \mathrm{d}t, \quad Y(\cdot) = I_m,$$

$$M(L, B) = \alpha_0.$$

The numerical matrix α_0 does not depend on p, $\alpha_1, \ldots, \alpha_{n-1}$, and $\Phi(\cdot)$. Thus, the statement of Theorem 2 holds:

$$\dim \ker(M(L, B)) = \dim \ker(\alpha_0),$$
$$\dim \operatorname{coker}(M(L, B)) = \dim \operatorname{coker}(\alpha_0).$$

Application

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Boundary-value problems depending on $k \in \mathbb{N}$

$$L(k)y(t,k) := y^{(r)}(t,k) + \sum_{j=1}^{r} A_{r-j}(t,k)y^{(r-j)}(t,k) = f(t,k), \quad t \in (a,b), \tag{8}$$

$$B(k)y(\cdot,k) = c(k), \quad k \in \mathbb{N}. \tag{9}$$

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The sequence of linear continuous operators

$$(L(k), B(k))\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l,$$

and characteristic matrices

$$M\big(L(k), B(k)\big) := \big([B(k)Y_0(\cdot,k)], \ldots, [B(k)Y_{r-1}(\cdot,k)]\big) \subset \mathbb{C}^{mr\times l}.$$

Convergence of the characteristic matrices

Let's formulate a sufficient condition for convergence of the characteristic matrices.

Theorem 3.

If the sequence of operators $\big(L(k), B(k)\big)$ converges strongly to the operator $\big(L, B\big)$ then the sequence of characteristic matrices $M\big(L(k), B(k)\big)$ converges to the matrix $M\big(L, B\big)$ for $k \to \infty$.

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From Theorem 3 follows sufficient conditions of semicontinuity from above the dimensions of the kernel and cokernel of the operator (L, B).

Corollary 2.

Under assumptions in Theorem 3, the following inequalities hold starting with sufficiently large k:

$$\dim\ker\left(L(k), B(k)\right) \leq \dim\ker\left(L, B\right),$$

$$\dim\operatorname{coker}\left(L(k), B(k)\right) \leq \dim\operatorname{coker}\left(L, B\right).$$

Remark

The Corollary 2 implies the consequences of the stability of the invertibility of the sequence of operators $(L(k), B(k))$, the existence and uniqueness of the solution to problem (8), (9). In particular, for sufficiently large k, we have:

1) if $l = mr$ and operator (L, B) is invertible, then the operators $(L(k), B(k))$ are also invertible;
2) if problem (1), (2) has a solution, then problems (8), (9) also have a solution;
3) if problem (1), (2) has a unique solution, then problems (8), (9) also have a unique solution.

Application

For each $k \to \infty$, we write the operator $B(k)$ in the form (4), where $\alpha_s = \alpha_s(k)$, $\Phi(t) = \Phi(t, k)$.

In the case of $1 \leq p < \infty$, based on a unique analytic representation of the operator B in (4), we formulate necessary and sufficient conditions that guarantees a strong convergence of the sequence of operators $(L(k), B(k))$ to the operator (L, B).

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Theorem 4.

Condition $(L(k), B(k)) \xrightarrow{s} (L, B)$ is equivalent to conditions:

1. $\|L(k) - L\| \to 0$;
2. $L(k)y \to Ly$ for each $y \in (W_p^{n+r})^m$;
3. $\alpha_s(k) \to \alpha_s$ in $\mathbb{C}^{l \times m}$ for each $s \in \{0, \ldots, n-1\}$;
4. $\|\Phi(\cdot, k)\|_q = O(1)$;
5. $\int_a^t \Phi(\tau, k) d\tau \to \int_a^t \Phi(\tau) d\tau$ in $\mathbb{C}^{l \times m}$ for each $t \in (a, b]$.

Application

In the case of $1 \leq p < \infty$, we formulate necessary and sufficient conditions that guarantees the uniform convergence of the sequence of operators $(L(k), B(k))$ to the operator (L, B).

Theorem 5.

Condition $\left\| (L(k), B(k)) - (L, B) \right\| \to 0$ is equivalent to conditions:

1. $\|L(k) - L\| \to 0$;
6. $\|\Phi(\cdot, k) - \Phi(\cdot)\|_q \to 0$.

The condition 6 is stronger than conditions 4 and 5.

Example 5

$$L(k)y(t,k) := y'(t,k) + A(t,k)y(t,k) = f(t,k), \quad B(k)y(\cdot,k) = c(k). \tag{10}$$

Denote by $Y(\cdot,k) \in (W_p^n)^{m\times m}$, respectively, the solution of the sequence of matrix differential equations

$$Y'(t,k) + A(t,k)Y(t,k) = 0, \quad t \in (a,b), \quad k \in \mathbb{N}, \quad Y(a,k) = I_m. \tag{11}$$

Denote by $M(L(k), B(k)) := [B(k)Y(\cdot,k)] \in \mathbb{C}^{m\times l}$.

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Denote by $M(L(k),B(k)) := [B(k)Y(\cdot,k)] \in \mathbb{C}^{m\times l}$.
From (4), we have

$$B(k)Y = \sum_{s=0}^{n-1} \alpha_s(k)Y^{(s)}(a) + \int_a^b \Phi(t,k)Y^{(n)}(t)\mathrm{d}t. \tag{12}$$

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Suppose that for the problem (10) the conditions of the Theorem 4 are satisfied:

a) $\alpha_s(k) \to \alpha_s$ in $\mathbb{C}^{l\times m}$ for each $s \in \{0,\dots,n-1\}$;

b) $\|\Phi(\cdot,k)\|_q = O(1)$;

c) $\int\limits_a^t \Phi(\tau,k)d\tau \to \int\limits_a^t \Phi(\tau)d\tau$ in $\mathbb{C}^{l\times m}$ for each $t \in (a,b]$.

Example 5

Then we have a strong convergence of the sequence of operators $(L(k), B(k))$ to the operator (L, B).
Then by the Theorem 3 we have the convergence of the sequence of characteristic matrices.
In (11), put $A(t, k) \to 0$, then $Y(t, k) \to I_m$. Substituting this value into equality (12), we have

$$M(L(k), B(k)) \to \alpha_0.$$

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Therefore, starting with some number k

$$\dim\ker\big(M(L(k), B(k))\big) \leq \dim\ker(\alpha_0),$$
$$\dim\operatorname{coker}\big(M(L(k), B(k))\big) \leq \dim\operatorname{coker}(\alpha_0).$$

In particular, if the numerical matrix α_0 is square and nondegenerate, then starting from some number k_0 all boundary-value problems are well-posedness.

Approximation

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{13}$$

$$By = c, \tag{14}$$

where $1 \leq p < \infty$.

A sequence of multipoint boundary-value problems

$$(L_k y_k)(t) := y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y_k^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{15}$$

$$B_k y_k := \sum_{j=0}^{N} \sum_{l=0}^{n+r-1} \beta_k^{(l,j)} y^{(l)}(t_{k,j}) = c. \tag{16}$$

Approximation

Theorem 10.

For the boundary-value problem (13), (14) there is a sequence of multipoint boundary-value problems of the form (15), (16) such that they are well-posedness for sufficiently large k and the asymptotic property is fulfilled

$$y_k \to y \quad \text{in} \quad (W_p^{n+r})^m \quad \text{for} \quad k \to \infty.$$

The sequence can be chosen independently of f and c, and constructed explicitly.

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Thank you!