PARABOLIC BOUNDARY-VALUE PROBLEMS IN GENERALIZED SOBOLEV SPACES

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2. Contents

The talk is devoted to

a new direction in the theory of PDEs.
It investigates their properties in generalized Sobolev spaces.

The feature of these spaces is that their regularity order (smoothness index) is a function parameter depending on frequency variables, rather than a number used for classical Sobolev spaces.

The structure of the talk

a) Background
b) Main results

9. Background. Parabolic problems

Let $2 \leq n \in \mathbb{Z}$, $T > 0$, and G be a bounded domain in $\mathbb{R}^n$ with a boundary $\Gamma := \partial G \in C^\infty$. Then $\Omega := G \times (0,T)$ is an open cylinder in $\mathbb{R}^{n+1}$, and $S := \Gamma \times (0,T)$ is its lateral boundary.

Parabolic initial-boundary-value problem for 2b-parabolic PDE in Ω:

$$Lu(x,t) \equiv \sum_{|\alpha|+2b\beta \leq 2m} a^{\alpha,\beta}(x,t)\, D_x^\alpha \partial_t^\beta u(x,t) = f(x,t), \quad (x,t) \in \Omega; \quad (1)$$

$$B_j u(x,t)\big|_S \equiv \sum_{|\alpha|+2b\beta \leq m_j} b_j^{\alpha,\beta}(x,t)\, D_x^\alpha \partial_t^\beta u(x,t)\big|_S = g_j(x,t),$$
$$x \in \Gamma,\ \ t \in (0,T),\ \ j = 1,\ldots,m; \quad (2)$$

$$\partial_t^k u(x,t)\big|_{t=0} = h_k(x),\ \ x \in G,\ \ k = 0,\ldots,m/b-1. \quad (3)$$

Here the integers b, m, and m_j satisfy $m \geq b \geq 1$, $m/b \in \mathbb{Z}$ and $0 \leq m_j \leq 2m-1$. All $a^{\alpha,\beta} \in C^\infty(\overline{\Omega},\mathbb{C})$ and $b_j^{\alpha,\beta} \in C^\infty(\overline{S},\mathbb{C})$.
We use the notation: $\alpha = (\alpha_1,\ldots,\alpha_n)$ is a multi-index, $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $D_x^\alpha := D_1^{\alpha_1}\ldots D_n^{\alpha_n}$, with $D_k := i\,\partial/\partial x_k$, $\partial_t := \partial/\partial t$.

10. Background. Parabolicity condition

Consider the polynomials in $\xi \in \mathbb{R}^n$ and $p \in \mathbb{C}$ (principal symbols):

$$L^{(0)}(x,t,\xi,p) := \sum_{|\alpha|+2b\beta=2m} a^{\alpha,\beta}(x,t)\,\xi^{\alpha}p^{\beta}, \quad x \in \overline{G},\ t \in [0,T]; \tag{4}$$

$$B_j^{(0)}(x,t,\xi,p) := \sum_{|\alpha|+2b\beta=m_j} b_j^{\alpha,\beta}(x,t)\,\xi^{\alpha}p^{\beta}, \quad x \in \Gamma,\ t \in [0,T]. \tag{5}$$

Problem (1)–(3) is called parabolic in Ω if:

1. $L^{(0)}(x,t,\xi,p) \neq 0$ for all $x \in \overline{G}$, $t \in [0,T]$, $\xi \in \mathbb{R}^n$, and $p \in \mathbb{C}$ such that $\mathrm{Re}\,p \geq 0$ and $|\xi| + |p| \neq 0$;
2. For each choice $x \in \Gamma$, $t \in [0,T]$, a tangent vector $\xi \in \mathbb{R}^n$ to the boundary Γ at x, and a number $p \in \mathbb{C}$ with $\mathrm{Re}\,p \geq 0$ such that $|\xi| + |p| \neq 0$, the polynomials $B_j^{(0)}(x,t,\xi+\zeta\nu(x),p)$, $j=1,\ldots,m$, in $\zeta \in \mathbb{C}$ are linearly independent modulo $\prod_{j=1}^{m}(\zeta - \zeta_j^{+}(x,t,\xi,p))$. Here, $\nu(x)$ is the unit vector of the inward normal to Γ at x, and $\zeta_j^{+}(x,t,\xi,p)$ are ζ-roots of $L^{(0)}(x,t,\xi+\zeta\nu(x),p)$ with $\mathrm{Im}\,\zeta > 0$.

11. Background. Classical theory of parabolic problems

The parabolic problem (1)–(3) induces the mapping

$$\Lambda : u \mapsto \left(Lu, B_1 u, \ldots, B_m u, u|_{t=0}, \ldots, (\partial_t^{m/b-1} u)|_{t=0}\right) \tag{6}$$

defined on the Hilbert anisotropic Sobolev space $H^{s,s/(2b)}(\Omega)$ of order $s \geq 2m$ with respect to x and of order $s/(2b)$ with respect to t.

The classical theorem for parabolic problems.

Let $s \geq 2m$ and $s - 1/2 \notin \mathbb{Z}$. Then mapping (6) realizes an isomorphism between $H^{s,s/(2b)}(\Omega)$ and a subspace of the space

$$H^{s-2m,(s-2m)/(2b)}(\Omega) \oplus \bigoplus_{j=1}^{m} H^{s-m_j-1/2,(s-m_j-1/2)/(2b)}(S) \oplus \bigoplus_{k=0}^{m/b-1} H^{s-2bk-b}(G) \tag{7}$$

of right-hand sides

$$(f, g_1, \ldots, g_m, h_1, \ldots, h_{m/b-1}) \tag{8}$$

of the parabolic problem that satisfy the compatibility relations.

12. Background. Classical theory of parabolic problems

This result was proved by

- M.S. Agranovich, M.I. Vishik. Elliptic problems with parameter and parabolic problems of general form. Russian Math. Surveys 19 (1964), 53–157.
  (The case where $s, s/(2b) \in \mathbb{Z}$.)
- J.-L. Lions, E. Magenes. Problèmes aux limites non homogèenes et applications. Vol. 2. Dunod, Paris, 1968.
  (The case of $b = m$ and normal boundary conditions.)
- N.V. Zhitarashu. Theorems on complete collection of isomorphisms in the L_2-theory of generalized solutions for one equation parabolic in Petrovskii's sense. Mat. Sb. 128 (1985), 451–473.
  (The general case is treated.)

An analogous result is also true for Hölder spaces of fractional order:

- O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'tzeva. Linear and Quasilinear Equations of Parabolic Type. AMS, Providence, 1968.

13. Background. Examples of parabolic problems

Example 1. A general 2-nd order parabolic equation.

$$\partial_t u(x,t) + \sum_{|\alpha|\le 2} a_\alpha(x,t)\, D_x^\alpha u(x,t) = f(x,t), \quad (x,t)\in\Omega; \tag{9}$$

$$u(x,t) = g(x,t), \quad x\in\Gamma,\ t\in(0,T); \tag{10}$$

$$u(x,0) = h(x),\ x\in G. \tag{11}$$

Here, $b=2$.

Example 2. A parabolic equation with iterated Laplacian.

$$\partial_t u(x,t) + (-1)^m \Delta^m u(x,t) = f(x,t), \quad (x,t)\in\Omega; \tag{12}$$

$$\partial_\nu^{j-1} u(x,t)\big|_S = g_j(x,t), \quad x\in\Gamma,\ t\in(0,T),\ j=1,\dots,m; \tag{13}$$

$$\partial_t^k u(x,t)\big|_{t=0} = h_k(x),\ x\in G,\ k=0,\dots,m/b-1. \tag{14}$$

Here, $1\le m\in\mathbb{Z}$, $b=m$, and ν is the inner normal to S.

14. Main results. Necessary function spaces

Consider first the case of the zero Cauchy data. Let $s \in \mathbb{R}$ and $\varphi \in \mathscr{M}$.

The anisotropic Hörmander space $H_{+}^{s,s/(2b);\varphi}(\Omega) := \mathscr{B}_{2,\mu}^{+}(\Omega)$ where

$$\mu(\xi,\eta) := \left(1+|\xi|^2+|\eta|^{1/b}\right)^{s/2}\varphi\left((1+|\xi|^2+|\eta|^{1/b})^{1/2}\right) \qquad (15)$$

is a function of $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}$, and

$$\mathscr{B}_{2,\mu}^{+}(\Omega) := \{w \restriction \Omega : w \in \mathscr{B}_{2,\mu}(\mathbb{R}^{n+1}),\ \operatorname{supp} w \subseteq \mathbb{R}^n \times [0,\infty)\}. \qquad (16)$$

The Hilbert space $H_{+}^{s,s/(2b);\varphi}(S)$, where $S = \Gamma \times (0,T)$,

consists of all distributions $v \in \mathscr{D}'(S)$ such that

$$v_j(x,t) := \chi_j(\alpha_j(x)) \cdot v(\alpha_j(x),t) \text{ belongs to } \mathscr{B}_{2,\mu}^{+}(\mathbb{R}^{n-1} \times (0,T)) \qquad (17)$$

for each $j \in \{1,\dots,r\}$, and is endowed with the norm

$$\|v\|_{H_{+}^{s,s/(2b);\varphi}(S)} := \left(\sum_{j=1}^{r} \|v_j\|^2_{\mathscr{B}_{2,\mu}^{+}(\mathbb{R}^{n-1}\times(0,T))}\right)^{1/2}. \qquad (18)$$

Here, μ is given by (15), $\xi \in \mathbb{R}^{n-1}$; $\{\alpha_j : \mathbb{R}^{n-1} \leftrightarrow \Gamma_j\}$ and $\{\chi_j \in C_0^\infty(\Gamma_j)\}$ are an atlas and corresponding partition of unit on Γ. This space does not depend (up to equivalence of norms) on $\{\alpha_j\}$ and $\{\chi_j\}$.

15. Main results. Zero Cauchy data. Isomorphism theorem

The parabolic problem with zero Cauchy data induces the mapping

$$\Lambda_0 : u \mapsto (Au, B_1 u, \ldots, B_m u), \quad \text{where} \tag{19}$$

$$u \in C^\infty(\overline{\Omega}) \text{ such that } \partial_t^k u(x,0) \equiv 0 \text{ whenever } 0 \leq k \in \mathbb{Z}. \tag{20}$$

Theorem 1 (Isomorphism Theorem).

Mapping (19) extends uniquely (by continuity) to an isomorphism

$$\Lambda_0 : H_+^{s,s/(2b);\varphi}(\Omega) \leftrightarrow$$

$$\leftrightarrow H_+^{s-2m,(s-2m)/(2b);\varphi}(\Omega) \oplus \bigoplus_{j=1}^{m} H_+^{s-m_j-1/2,(s-m_j-1/2)/(2b);\varphi}(S) \tag{21}$$

for all $s > 2m$ and $\varphi \in \mathscr{M}$.

If $\varphi(\cdot) \equiv 1$, isomorphism (21) acts between anisotropic Sobolev spaces.

16. Main results. Regularity Theorem

Let U be an open subset of $\mathbb{R}^{n+1}$ such that $\omega := U \cap \Omega \neq \varnothing$. Set $\pi_1 := U \cap \partial\Omega$, $\pi_2 := U \cap S$. Put

$$H^{s,s/(2b);\varphi}_{+,\mathrm{loc}}(\omega,\pi_1) := \{u \in \mathscr{D}'(\Omega) : \chi u \in H^{s,s/(2b);\varphi}_{+}(\Omega) : \text{ for every } \chi \in C^\infty(\overline{\Omega}) \text{ such that } \operatorname{supp}\chi \subset \omega \cup \pi_1\} \tag{22}$$

and

$$H^{s,s/(2b);\varphi}_{+,\mathrm{loc}}(\pi_2) := \{v \in \mathscr{D}'(S) : \chi v \in H^{s,s/(2b);\varphi}_{+}(S) \text{ for every } \chi \in C^\infty(\overline{S}) \text{ such that } \operatorname{supp}\chi \subset \pi_2\}. \tag{23}$$

Theorem 2 (Regularity Theorem).

Let $u \in H^{2m,m/b}_{+}(\Omega)$ be a solution to the parabolic problem where

$$f \in H^{s-2m,(s-2m)/(2b);\varphi}_{+,\mathrm{loc}}(\omega,\pi_1), \tag{24}$$

$$g_j \in H^{s-m_j-1/2,(s-m_j-1/2)/(2b);\varphi}_{+,\mathrm{loc}}(\pi_2), \quad j = 1,\ldots,m, \tag{25}$$

for some $s > 2m$ and $\varphi \in \mathscr{M}$. Then $u \in H^{s,s/(2b);\varphi}_{+,\mathrm{loc}}(\omega,\pi_1)$.

17. Main results. Classical smoothness of solutions

Theorem 3 (condition for solutions to be smooth).

Let an integer $p \geq 0$ satisfies $p+b+n/2>2m$. Assume that a solution $u \in H_{+}^{2m,m/b}(\Omega)$ to the parabolic problem satisfies the hypotheses of Theorem 2 for $s:=p+b+n/2$ and some $\varphi \in \mathscr{M}$ subject to

$$\int_{1}^{\infty} \frac{d\theta}{\theta\varphi^{2}(\theta)} < \infty. \tag{26}$$

Then

$$D_x^{\alpha}\partial_t^{\beta}u(x,t) \in C(\omega \cup \pi_1) \quad \text{whenever} \quad |\alpha|+2b\beta \leq p. \tag{27}$$

Condition (26) is exact (cannot be weakened).

If we use anisotropic Sobolev spaces only (the case of $\varphi(\cdot) \equiv 1$), we will demand in the hypotheses of Theorem 3 that $s > p+b+n/2$. This makes the result rougher.

18. Main results. The classical smoothness. Comparison

Consider the following problem for parabolic 2-nd order constant-coefficients PDE in the cylinder $\Omega \subset \mathbb{R}^{n+1}$:

$$u_t' = \sum_{|\alpha|\leq 2} a_\alpha D_x^\alpha u + f, \qquad u\big|_S = 0, \qquad u\big|_{t=0} = 0. \tag{28}$$

It is known that there is a function $f \in C(\overline{\Omega})$ with $\operatorname{supp} f \subset \Omega$ that this problem has a generalized solution

$$u \in C^1(\Omega) \setminus C^{2,0}_{x,t}(\Omega) \ \text{ with } \operatorname{supp} u \subset \Omega. \tag{29}$$

Thus, "the fine" right-hand sides $f \in C(\overline{\Omega})$, $g \equiv 0$ and $h \equiv 0$ do not insure the inclusion $u \in C^{2,0}_{x,t}(\Omega)$.

$$\text{The classical result:} \quad f \in \bigcap_{\varepsilon>0} C^\varepsilon(\overline{\Omega}) \Longrightarrow u \in C^{2,1}_{x,t}(\overline{\Omega}). \tag{30}$$

Theorem 3 supplements this result giving another class of functions f that imply $u \in C^{2,1}_{x,t}(\overline{\Omega})$.

19. Main results. General Cauchy data

The Hilbert spaces $H^{s,s/(2b);\varphi}(\Omega \text{ or } S)$, $s \in \mathbb{R}$ and $\varphi \in \mathscr{M}$, are defined in the same way as $H_{+}^{s,s/(2b);\varphi}(\Omega \text{ or } S)$ omitting $\operatorname{supp} w \subseteq \mathbb{R}^n \times [0,\infty)$.

If $s > 2m$ and $s - 1/2 \notin \mathbb{Z}$, we let $\mathbb{Q}^{s-2m,(s-2m)/(2b);\varphi}$ denote the subspace of the space

$$H^{s-2m,(s-2m)/(2b);\varphi}(\Omega) \oplus \bigoplus_{j=1}^{m} H^{s-m_j-1/2,(s-m_j-1/2)/(2b);\varphi}(S) \oplus \bigoplus_{k=0}^{m/b-1} H^{s-2bk-b;\varphi}(G) \qquad (31)$$

of right-hand sides of the parabolic problem that satisfy the natural compatibility relations.

If $s > 2m$ and $s - 1/2 \in \mathbb{Z}$, we use the complex interpolation

$$\mathbb{Q}^{s-2m,(s-2m)/(2b);\varphi} := \left[\mathbb{Q}^{s-\varepsilon-2m,(s-\varepsilon-2m)/(2b);\varphi}, \mathbb{Q}^{s+\varepsilon-2m,(s+\varepsilon-2m)/(2b);\varphi}\right]_{1/2}. \qquad (32)$$

Here, $0 < \varepsilon < 1/2$.

20. Main results. General Cauchy data

The parabolic problem induces the mapping

$$\Lambda : u \mapsto \big(Lu, B_1 u, \ldots, B_m u, u|_{t=0}, \ldots, (\partial_t^{m/b-1} u)|_{t=0}\big), \qquad (33)$$

where $H^{2m,m/b}(\Omega)$.

Theorem 4 (General Isomorphism Theorem.)

The mapping (33) realizes an isomorphism

$$\Lambda : H^{s,s/(2b);\varphi}(\Omega) \leftrightarrow \mathbb{Q}^{s-2m,(s-2m)/(2b);\varphi} \qquad (34)$$

for all $s > 2m$ and $\varphi \in \mathscr{M}$.

If $\varphi(\cdot) \equiv 1$, this isomorphism acts between anisotropic Sobolev spaces.

Some versions of Theorems 2 and 3 are also valid for general (i.e., inhomogeneous) Cauchy data.

21. Main results. References

These results are established in the articles

- V. Los, V. Mikhailets, A. Murach. Parabolic problems in generalized Sobolev spaces. Comm. Pure Appl. Anal. 20 (2021), 3605–3636.
  (Inhomogeneous Cauchy data)
- V. Los, V. Mikhailets, A. Murach. An isomorphism theorem for parabolic problems in Hörmander spaces and its applications. Comm. Pure Appl. Anal. 16 (2017), 69–97.
  (Zero Cauchy data)

and are expounded in the monograph

- V. Los, V. Mikhailets, A. Murach. Parabolic Problems and Generalized Sobolev Spaces. Naukova Dumka, Kyiv, 2021, 164 pp. (Ukrainian.)
  (The preprint is available at arXiv:2109.03566.)