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# Essays in Macroeconomics with Heterogeneous Agents and Portfolio Choice

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# Abstract

This thesis studies how the asset portfolio heterogeneity of households influences wealth inequality and macroeconomic outcomes in macroeconomic models. Specifically, it analyses the implications of a change in firm leverage and differential asset taxation on inequality and other macroeconomic variables, and how to compute the macroeconomic models used to study these implications more efficiently.

*Chapter 1* studies the effects of a change in firm leverage on wealth inequality and macroeconomic aggregates. The effects are studied in a general equilibrium model with a continuum of heterogeneous agents, life-cycle, incomplete markets, and idiosyncratic and aggregate risk. In the benchmark model, an increase in firm leverage leads to an increase in capital accumulation, and a decrease in wealth inequality and government revenue. Furthermore, I show that if the model abstracts from capital income taxation, the change in leverage has only minor effects on macro aggregates and inequality, despite having significant implications for asset prices.

*Chapter 2* analyzes the redistributive and macroeconomic effects of differential taxation of financial assets with a different risk. Poor households in the US primarily hold their savings in safe financial assets, while wealthy households invest a substantially higher share of their wealth in (risky) equity. However, in many tax codes equity and safe assets are often taxed at different rates. The main reason for this is that investments in equity (which are relatively riskier) are subject to corporate and personal income tax, unlike debt, which is tax deductible for companies. This chapter first builds a simple theoretical two-period model showing that the optimal tax wedge between risky and safe assets is increasing in the underlying wealth inequality. The chapter then analyzes a quantitative model with a continuum of heterogeneous agents, parsimonious life-cycle, borrowing constraint, aggregate shocks, and uninsurable idiosyncratic shocks. The simulations of quantitative models show that elimination of the differential asset taxation leads to a welfare loss and that the optimal tax wedge between taxes on equity and debt is higher than that in the US tax code.

*Chapter 3* proposes a novel method to compute the simulation part of the Krusell-Smith algorithm when agents can trade in more than one asset (for example, capital and bonds). The Krusell-Smith algorithm is used to solve the general equilibrium models with both aggregate and uninsurable idiosyncratic risk, and can be used to both to solve the bounded rationality equilibria, and to approximate the rational expectations equilibria. When applied to solve a model with more than one financial asset, in the simulation part, the standard algorithm has to impose equilibria for each additional asset (find the market-clearing price), for each simulated period. This procedure entails root-finding for each simulated period, which is computationally very expensive. In this chapter, I show that it is possible to avoid root-finding by not imposing the equilibria in each period, and instead, simulate the model without market clearing. The proposed method updates the laws of motion for asset prices by using Newton-like methods (Broyden's method) on the simulated excess demand, instead of imposing equilibrium for each period and running a regression on the clearing prices. In the example model, the proposed version of the algorithm leads to a decrease in the computational time, even when measured conservatively.

*In Czech:*

Tato práce studuje vliv heterogeneity portfolia domácností na ekonomickou nerovnost a spojené makroekonomické dopady. Za využití makroekonomických modelů tato práce odpovídá na otázky jaké jsou dopady víceúrovňového zdanění finančních aktiv na ekonomickou nerovnost a ostatní makroekonomické proměnné, a jak zefektivnit numerické výpočty vybraných makroekonomických modelů.

*Kapitola 1* se zabývá vlivem změn finanční páky firem na ekonomickou nerovnost a makroekonomické agregáty, za pomoci modelu všeobecné rovnováhy s kontinuem heterogenních aktérů, životním cyklem, neúplným trhem a s idiosynkratickým i s agregátním rizikem. Výsledky naznačují, že navýšení finanční páky vede k vyšší akumulaci kapitálu, poklesu ekonomické nerovnosti a poklesu státních výnosů. Dále je ukázáno, že změna ve finanční páce má pouze malé dopady na makroekonomické agregáty a ekonomickou nerovnost, pokud je abstrahováno od pozitivního efektu skrze daňové zatížení firem. Avšak dopady na ceny finančních aktiv zůstávají nadále zásadní.

*Kapitola 2* analyzuje redistribuční a makroekonomické vlivy víceúrovňového zdanění finančních aktiv s rozdílnými úrovněmi rizika. Zatímco chudé americké domácnosti spoří spíše skrze bezpečnější finanční aktiva, bohatší domácnosti investují větší podíl svého bohatství do těch rizikovějších. V mnoha případech se úroveň zdanění jednotlivých finančních aktiv liší. To je často důsledkem dvojího zdanění vlastního kapitálu, kdy je daň placena jak na firemní úrovni, tak úrovni individuální. To však neplatí pro firemní dluh, jenž je možné zanechat do nákladů firem.

Za pomoci “two-period” teoretického makroekonomického modelu je ukázáno, že optimální poměr zdanění mezi rizikovými a bezpečnými finančními aktivy je přímo závislý na míře ekonomické nerovnosti. Toto zjištění je podpořeno výsledky z kvantitativního modelu s kontinuem heterogenních agentů, životním cyklem, dluhovým omezením, agregátními šoky a nepojistitelnými idiosynkratickými šoky. Simulace ukazují, že odstranění víceúrovňového zdanění vede k celkovému snížení blahobytu ve společnosti a optimální rozdíl mezi zdaněním vlastního kapitálu a dluhopisů je větší, než současný rozdíl ve Spojených státech.

*Třetí kapitola* navrhuje novou metodu výpočtu simulací na základě Krusell-Smithova algoritmu, kdy ekonomičtí aktéři mohou obchodovat s více než jedním aktivem (například vlastní kapitál a dluhopisy). Krusell-Smithův algoritmus se používá na řešení modelů všeobecné rovnováhy, které mají agregátní i nepojistitelné idiosynkratické riziko a může být použit jak na řešení modelu s omezenou racionalitou tak na aproximaci modelu s racionálními očekávanými. Při aplikaci algoritmu na model s více než jedním finančním aktivem, je nutné najít rovnovážnou cenu pro každé simulované období. Hledání rovnovážné ceny je zajištěno numerickým hledáním kořenů rovnic, což je výpočetně náročné. V této kapitole je



ukázáno, že je možné se hledání kořenů vyhnout tím, že se neřeší rovnovážná cena pro každé období ale místo toho se nasimuluje model bez rovnovážné ceny. Pohybová rovnice pro ceny aktiv je následně upravena pomocí Broydenovy metody (metoda podobná Newtonově metodě) na základě simulovaných převisů poptávky, namísto řešení rovnovážných cen pro každé období a následné lineární regrese na základě rovnovážných cen. Je demonstrováno, že navrhovaná úprava algoritmu vede k poklesu času potřebného k dokončení výpočtu i když je měřen konzervativně.

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# Introduction

The thesis studies the implications of portfolio heterogeneity on macroeconomic models and tax policy recommendations. The portfolio heterogeneity stems from the fact that poorer households tend to invest in safer assets, which have lower returns compared to riskier assets, such as equity. Riskier assets are disproportionately held by wealthier households.

In a broad sense, this thesis contributes to the developing field of macroeconomic models with aggregate risk and a continuum of heterogeneous agents. Since these models can be relatively challenging from the technical perspective, topics involving these types of economic setup may not have been studied as much as they warrant. This thesis involves models with both aggregate and uninsurable idiosyncratic risk and endogenous portfolio choice, in which the differences in households' portfolio are highlighted.

Almost half of US households do not own any equity or stocks. There is a wide range of possible reasons such a large portion of the population does not save in assets that, in the long-run, yield higher returns on average. Some reasons may be prohibitive participation costs, higher exposure to other economic risks (labor market, for example), and lack of financial literacy, among others. This thesis aims to incorporate households' different portfolio choices into the standard macroeconomic models with heterogeneous households. Since the composition of

savings is heterogeneous among different groups of households, changes in asset prices, or the way they are taxed, are bound to have redistributory consequences on the household side of the economy. To study the implications of such financial market changes on different households, this thesis aims to model the portfolio choices of the households as endogenous and to have a continuous distribution of wealth, rather than the rule-of-thumb type of modeling that exogenously prevents specific types of households from investing in the equity market and assumes only a limited number of household types. Endogeneity of households' portfolio is important, as some households could change their behavior if the economic environment changes.

The first chapter studies how a change in firm leverage affects wealth inequality and other macroeconomic aggregates. A change in firm leverage affects asset prices because it changes the supply of relatively safe debt, but also changes the riskiness of firm equity. As mentioned earlier, these changes can impact households very differently, as they hold starkly different portfolios. This question is particularly interesting to study, as there were significant changes in both wealth inequality and firm leverage that coincided in the US during the 1980s and 1990s. The chapter aims to quantify how much of the increase in wealth inequality can be explained by the decrease in firm leverage. The chapter also revisits the Modigliani-Miller result of firm leverage irrelevance in the context of a heterogeneous-agent model with a high equity premium.

The second chapter examines differential asset taxation when households have endogenously heterogeneous portfolios. The study is motivated by the fact that (risky) equity is often taxed differently than other (safer) financial assets. A famous example is “double dividend taxation,” which means that firm profits are taxed twice: first, the company pays corporate income tax, and then the owner has to pay personal income tax on the received dividend. However, if the firm finances its investment by debt instead of issuing equity, it can deduct the debt payments, and pay less in corporate income taxes. As a consequence, this makes financing new investments by debt more desirable than by equity. This feature of the US tax code prompted a debate and pressure on governments to reduce

this tax wedge, as it leads to excessive leveraging by firms. However, the debate surrounding this issue has not addressed the fact that the owners of these two types of assets are different. It is more wealthy households that primarily save in equity, while less affluent households mostly hold safer debt (often indirectly through banks). Therefore, the tax wedge has a redistributionary effect. The chapter examines what the consequences of abolishing the tax wedge are, and what the optimal tax wedge is between these two assets, from the perspective of the utilitarian social planner.

The third chapter proposes a novel version of the Krusell-Smith method of solving macroeconomic models with both aggregate and uninsurable idiosyncratic shocks and a borrowing constraint when households have an endogenous portfolio choice. This type of model is used for analysis in the first two chapters. The novelty is in the proposed way to update the perceived law of motions for the bond price (equity premium). Instead of clearing the bond market in every period in the preliminary simulations of the model, the proposed method lets the markets proceed with excess (or insufficient) demand, and uses the information about the excess demand to update the perceived law of motions. This procedure avoids root-finding, which is computationally expensive, and thus leads to a significantly shorter run-time of the algorithm.

## Chapter 1

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# Firm Leverage and Wealth Inequality

## 1.1 Introduction

The data shows that households with different amounts of wealth hold starkly different portfolios (Survey of Consumer Finance, 1998, 2016). Considering only financial assets, poorer households invest mostly in safe financial assets, with 46% of US households not holding any risky financial assets (equity) (Chang et al., 2018). At the same time, the richest households invest most of their financial savings in equity.<sup>1</sup> These facts motivate the study in this chapter, which analyses how the change in the corporate structure affects inequality and macroeconomic aggregates. In particular, Graham et al. (2015) documents that firm leverage has significantly risen in the US since the end of the Second World War, and one of the sharp increases occurred during the late 1980s.<sup>2</sup> Given the patterns observed in the data, one could speculate that the change in corporate structure (i.e. leverage) will have heterogeneous effects on the population, as it changes the supply and the riskiness of financial assets, which are disproportionately distributed among households.

The Modigliani-Miller theorem (see Modigliani and Miller (1958) and Modigliani and Miller (1963)) on the irrelevance of firms' financial (leverage) policy has been

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<sup>1</sup>Chang et al. (2018) and Carroll (2000)

<sup>2</sup>See the graph in the Appendix B.

shown to be valid in a wide range of environments. Algan et al. (2009) have shown that, although the theorem does not hold exactly in an environment with a borrowing constraint, the effects of a change in the representative firms' financial policy is sufficiently small that the theorem holds approximately. This is because, similar to the original models of Krusell and Smith (1997) and Krusell and Smith (1998), most agents are well insured and can offset the effects of the firm's financing decision by adjusting their portfolio. The households that are not sufficiently insured are very poor households, close to the borrowing constraint. However, there are not many of such households, and moreover, they hold a very small amount of wealth, so a change in their behavior does not influence the aggregate dynamics. Furthermore, in Algan et al. (2009), the equity risk premium is not sufficiently high, as it is common in standard macroeconomic models (Mehra and Prescott, 1985), making the two types of assets similar from the household's point of view.

This chapter studies the effects of a change in firm leverage on macroaggregates and wealth inequality in a model with a continuum of households, imperfect markets, borrowing and portfolio choice constraints, and idiosyncratic and aggregate risk. The model augments the Krusell and Smith (1997, 1998) and Algan et al. (2009) models by adding the parsimonious life cycle in the style of Krueger et al. (2016), capital depreciation shocks, capital taxation, and accounting for the beneficial tax treatment of debt. These additional features help generate moments from the data in the model, which are essential to match in order to examine the question at hand. In particular, life cycle helps the model to generate a more realistic mass of households close to the borrowing constraint,<sup>3</sup> depreciation shocks help to generate a realistic equity premium,<sup>4</sup> and the beneficial tax treatment of debt (debt tax shield) is another feature of the model that breaks the Modigliani-Miller theorem on the neutrality of debt.<sup>5</sup>

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<sup>3</sup>This is because, in the infinite horizon models, majority of households live "long enough" to eventually accumulate a relatively large capital buffer to insure themselves against a series of bad shocks that would move them close to the borrowing constraints. However, if the households die periodically, many of them will not be very far from the borrowing constraints.

<sup>4</sup>Depreciation shocks increase the equity premium, as they increase the stock returns volatility.

<sup>5</sup>Modigliani and Miller (1963)

This chapter studies how firm leverage influences inequality and macroaggregates in a dynamic general equilibrium model. In particular, it performs an exercise that compares the long-run equilibrium in a model with the level of leverage from the early 1980s (which is very similar to the value in 2008), with the one with the level of leverage in the early 1990s. This is a particularly interesting time period because, at the same time that the changes in leverage were happening, the wealth inequality has been steadily increasing (Saez and Zucman, 2016). Firm leverage in the US economy (measured as total debt over total capital) during the 1980s increased from roughly 35 to 48 percent in the early 1990s (see Graham et al. (2015)), who use Moody’s Industrial Manual to compute it), and subsequently fell back to approximately 35 – 37% percent in 2010 (see Graham et al. (2015), who use Moody’s Industrial Manual data).<sup>6</sup> The study of the causes of the increase of firm leverage is beyond the scope of this thesis. Instead, the analysis is limited to the consequences of such an increase. The potential reasons for the change in leverage in the mentioned period is briefly discussed in the Appendix G. The results of the model suggest that the sudden increase in firm leverage that occurred in the US during the 1980s did not contribute to the increase in wealth inequality, but on the contrary, it is the steady reduction in firm leverage since the early 1990s that has contributed to the subsequent increases in wealth inequality.

The benchmark model finds that the increase of leverage of the magnitude observed in the late 80s leads to an increase in capital accumulation, and a decrease in both wealth inequality and government revenue. In the version of the model without capital taxation and beneficial tax treatment of debt, the changes in leverage lead to significant changes in asset prices but do not have quantitatively significant effects on inequality and macroaggregates. This result is consistent with the results of Algan et al. (2009), who find that in the Krusell-Smith (1997,1998) setting, Modigliani-Miller theorem holds approximately.

In terms of the model complexity, this paper includes idiosyncratic risk, aggre-

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<sup>6</sup>To see the historical changes, see Figure 5. in Appendix B.



gate risk, portfolio choice with discrete participation decision, and parsimonious life cycle. In a general context, the model is built on the tradition of the models of Bewley (1977), Huggett (1993) and Aiyagari (1994) with incomplete markets. Huggett (1993) studies a model where infinitely lived agents face idiosyncratic risk and borrowing constraint, and can only trade bonds which are in zero supply. Aiyagari (1994) adds an endogenous supply of bonds (capital) from firms, and constructs a general equilibrium model. Krusell and Smith (1998) add aggregate risk to the Aiyagari (1994) model, which means that the aggregate capital and prices in the economy are not constant, as in Aiyagari (1994). Furthermore, Krusell and Smith (1997) add portfolio choice to the model with both idiosyncratic and aggregate risk and allow the households to save in both capital and zero supply bonds. Papers such as Harenberg and Ludwig (2018), Krueger and Kubler (2006) and Gomes et al. (2013) add an overlapping-generations (OLG) structure on top of the complexity of Krusell and Smith (1997). One of the more similar models is in Algan et al. (2009), which does not have OLG structure, but adds a positive supply of both bonds and capital, which are issued by a leveraged firm. The model in this dissertation adds a parsimonious OLG structure, risk in capital depreciation, and a discrete choice of households whether to participate in the stock market or not. This means that the agents do not only choose how much they went to invest in stocks and bonds, but also whether to pay stock market participation and participate in this market in the first place. Therefore, my model differentiates from Algan et al. (2009) in that it contains OLG structure and an additional discrete decision. Furthermore, it is different from Harenberg and Ludwig (2018), Gomes et al. (2013), and Krueger and Kubler (2006), because it has a more parsimonious OLG structure, and at the same time it has additional discrete decision and positive supply of bonds from a leveraged firm.

The remainder of the chapter describes the benchmark model and calibration, presents and discusses the results, studies several decompositions and extensions, and finally concludes.

## 1.2 Model

I construct a model based on Algan et al. (2009), and in the tradition of Krusell and Smith (1997). The model consists of a continuum of heterogeneous agents facing aggregate risk, uninsurable idiosyncratic labor risk and a borrowing constraint, and who save in two assets: risky equity and safe bonds. Unlike Algan et al. (2009), the model parsimoniously captures the life cycle dynamics of the households in the fashion of Krueger et al. (2016), where working-age agents face the retirement shock and retired households face the risk of dying. Furthermore, this model captures the beneficial tax treatment of debt.

### 1.2.1 Production technology

In each period  $t$ , the representative firm uses aggregate capital  $K_t$ , and aggregate labor  $L_t$ , to produce  $y$  units of a final good with the aggregate technology  $y_t = f(z_t, K_t, L_t)$ , where  $z_t$  is an aggregate total factor productivity (TFP) shock. I assume that  $z_t$  follows a stationary Markov process with transition function  $\Pi_t(z, z') = Pr(z_{t+1} = z' | z_t = z)$ . The production function is continuously differentiable, strictly increasing, strictly concave and homogeneous of degree one in  $K$  and  $L$ . Capital depreciates at the stochastic rate  $\delta_t \in (0, 1)$  and it accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1 - \delta_t)K_t$$

where  $I_t$  is aggregate investment. Particular aggregate production technology is:

$$Y_t = z_t A K_t^\Delta L_t^{1-\Delta}$$

### 1.2.2 Preferences

Households are indexed by  $i$ , and they have Epstein-Zin preferences (Epstein and Zin, 1989).

They are maximizing their lifetime utility, expressed recursively for the retired agents:

$$V_{R,i,t} = \{c_t^{1-\rho} + v\beta[E_t V_{R,i,t+1}^{(1-\alpha)}]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where  $V_{R,i,t}$  is the recursively defined value function of a retired household  $i$ , at time period  $t$ .

Working-age agents maximize:

$$V_{W,i,t} = \{c_t^{1-\rho} + \beta[(1-\theta)E_t V_{W,i,t+1}^{1-\alpha} + \theta E_t V_{R,i,t+1}^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where  $V_{W,i,t}$  is the recursively defined value function of a working-age household  $i$ , at time period  $t$ . Furthermore,  $\beta$  denotes the subjective discount factor,  $E_t$  denotes expectations conditional on information at time  $t$ ,  $\alpha$  is the risk aversion,  $\frac{1}{\rho}$  is the intertemporal elasticity of substitution,  $\theta$  is the probability of retiring, and  $v$  is the probability of retired agents dying.

### 1.2.3 Life cycle structure

In each period, working-age households have a chance of retiring  $1 - \theta$ , and retired households have a chance of dying  $1 - v$ , as in Castaneda et al. (2003) and Krueger et al. (2016). Therefore, the share of working age households in the total population is:

$$\Pi_W = \frac{1 - v}{(1 - \theta) + (1 - v)}$$

and the share of the retired households in the total population is:

$$\Pi_R = \frac{1 - \theta}{(1 - \theta) + (1 - v)}$$

The retired households who die in period  $t$  are replaced by new-born agents who start at a working age without any assets. For simplicity, the retired households have perfect annuity markets, which make their returns larger by a fraction of  $\frac{1}{v}$ , as in Krueger et al. (2016).

It is common that models with idiosyncratic risk, aggregate risk, and imperfect markets generate fewer households that hold zero or almost zero wealth (or

are near the borrowing constraint) than is observed in the data. Consequently, these models often do not generate aggregate dynamics, which are very different from the representative agent model. However, Krueger et al. (2016) show that it is possible to generate a realistic mass of households with zero wealth, or near the borrowing constraint, by adding parsimonious life-cycle to the model. More importantly, they also demonstrate that once the model generates a realistic lower tail of wealth distribution, the heterogeneity matters for the aggregate dynamics. Simultaneously, the approximate aggregation result from Krusell and Smith (1997, 1998) still holds if the change in wealth distribution is captured well enough by the changes of aggregate capital and aggregate productivity shock (or any other state variable that directly enters the aggregate capital perceived laws of motion).

Furthermore, Chang et al. (2018) study the Survey of Consumer Finance and show that the financial portfolio composition changes with the age of the households. They find that the share of savings invested in risky assets initially increases in the age of the household (controlling for the wealth of the household), but as the households retire, the households begin to reduce their risky share as they become older. As the model has only two life cycle stages; working-age and retired, it is not possible to replicate this inverse U-shaped curve, but the model will replicate the fact that the retired households save disproportionately more in the risky asset compared to the working-age households.<sup>7</sup>

#### 1.2.4 Idiosyncratic uncertainty

In each period, working-age households are subject to an idiosyncratic labor income risk that can be decomposed into two parts. The first part is the employment probability that depends on aggregate risk and is denoted by  $e_t \in (0, 1)$ .  $e = 1$  denotes that the agent is employed, and  $e = 0$  that the agent is unemployed. Conditional on  $z_t$  and  $z_{t+1}$ , I assume that the period  $t + 1$  realization of the employment shock follows the Markov process.

$$\Pi_e(z, z', e, e') = Pr(e_{t+1} = e' | e_t = e, z_t = z, z_{t+1} = z')$$

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<sup>7</sup>See also Fagereng et al. (2017)

This labor risk structure allows idiosyncratic shocks to be correlated with the aggregate productivity shocks, which is consistent with the data and generates the portfolio choice profile such that the share of wealth invested in risky assets is increasing in wealth. The condition imposed on the transition matrix and the law of large numbers implies that aggregate employment is only a function of the aggregate productivity shock.

If  $e = 1$  and the agent is employed, one can assume that the agent is endowed with  $l_t \in L \equiv \{l_1, l_2, l_3, \dots, l_m\}$  efficiency labor units, which she can supply to the firm. Labor efficiency is independent of the aggregate productivity shock, and it is governed by a stationary Markov process with transition function  $\Pi_l(l, l') = Pr(l_{t+1} = l' | l_t = l)$ . If the agent is unemployed, (s)he receives unemployment benefits  $g_{u,t}$ , which are financed by the government.

Comparing the calibration to the cases in Davila et al. (2012), it is not a priori clear whether the model will generate excessive or insufficient aggregate capital accumulation (in a constrained efficiency sense). This is because the Markov process capturing labor income uncertainty is not made to replicate wealth inequality (as in the insufficient capital accumulation case), nor the unemployment economy (excessive capital accumulation case). Instead, the income process is supposed to capture the actual labor income risks that workers face.

### 1.2.5 The representative firm

As in Algan et al. (2009), firm leverage in this model is given exogenously. The leverage of the firm is determined exogenously, by the parameter  $\lambda$ . The Modigliani-Miller theorem (1958, 1963) does not hold, as some of the agents are borrowing-constrained, and some are portfolio-constrained. Therefore, theoretically, the leverage of the firm has some macroeconomic relevance. Additionally, debt is taxed differently than equity returns, and this additionally breaks the Modigliani-Miller theorem.

In the economy, the representative firm can finance its investment with two

types of contracts. The first is a one-period risk-free bond that promises to pay a fixed return to the owner. The second is a risky equity that entitles the owner to claim the residual profits of the firm after the firm pays out wages and debt from the previous period. Both of these assets are freely traded in competitive financial markets. By construction, there is no default in the equilibrium.

The return on the bond  $r_{t+1}^b$  is determined by the clearing of the bond market:

$$\int_S g^{b,j,e} d\mu = \lambda K'$$

where  $g^{b,j,e}$  are the individual policy functions for bonds.

In each period  $t$ , the firm redistributes all the residual value of the firm, after production and depreciation have taken place, and wages and debt have been paid. Therefore, the return on the risky equity depends on the realizations of the aggregate shocks and is given by the equation:

$$(1 + r_{t+1}^s) = \frac{f(z_{t+1}, K_{t+1}, L_{t+1}) - f_L(z_{t+1}, K_{t+1}, L_{t+1})L_{t+1} - \lambda K_{t+1}(1 + r_{t+1}^b) + (1 - \delta_{t+1})K_{t+1}}{(1 - \lambda)K_{t+1}}$$

Since the return to stocks subject to corporate income tax, the post-tax gross return to equity  $1 + r_{t+1}^{s,p}$  is:

$$1 + r_{t+1}^{s,p} = 1 + r_{t+1}^s(1 - \tau_s)$$

An important caveat in having heterogeneous households who own the firm is that they do not necessarily have the same stochastic discount factor  $m_{t+1}^j$ , and therefore the definition of the objective function of the firm is not straightforward. I follow Algan et al. (2009), who assume that the firm is maximizing the welfare of the agents who have an interior portfolio choice, and consequently, the firm has the same stochastic discount factor  $m_{t+1}$  as the agents with the interior portfolio choice.

As in Algan et al. (2009), it is possible to use the fact that, for a given stochastic discount factor,  $V_t = K_{t+1}$ , which enables the elimination of the capital

Euler equation from the equilibrium conditions.

### 1.2.6 Financial markets

As stated earlier, households can save in two assets: risky equity and safe bonds (firm debt). There are borrowing constraints for both assets, and thus the lowest amounts of equity and debt that households can hold in period  $t$  are:  $\kappa^s$  and  $\kappa^b$ , respectively. Markets are assumed to be incomplete, in the sense that there are no markets for the assets contingent on the realization of individual idiosyncratic shocks. Furthermore, if the household wants to save a positive amount of resources in equity in period  $t$ , it has to pay  $\phi$  as a per-period cost of participating in the stock market.

### 1.2.7 Government

The government runs two social programs: social security (retirement benefits), and unemployment insurance, and are modeled as in Krueger et al. (2016). Both are financed by separate labor taxes. Social security is financed with a constant labor tax rate:  $\tau^{lss}$ , and the revenues  $T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$  are equally distributed in period  $t$  to all retired households, irrespective of their past contributions. Unemployment benefits are financed with a labor tax rate  $\tau_t^u$ . The amount of unemployment benefits  $g_{u,t}$  is determined by a constant  $\eta$ , which represents the fraction of the average wage in each period.

To satisfy the budget constraint, the government has to tax labor with the tax rate:

$$\tau_t^u = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\eta}}$$

where  $\Pi_u$  is the share of unemployed people in the total working-age population.

Furthermore, the government taxes the net profit of the firm by a corporate income tax with the rate  $\tau_s$ . Therefore, the net return to the investment in

stock is  $r_{t+1}^s(1 - \tau_s)$ . The revenue collected by corporate income tax is spent on wasteful government consumption. This is a simplifying assumption, since studying government expenditure is not a central question for this thesis chapter.

### 1.2.8 Household problem

Retired household  $i$  maximizes its lifetime utility subject to the following constraints:

$$\begin{aligned} c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} &\leq \omega_{i,t} \\ \omega_{i,t+1} &= T_{ss,t+1} + [(1 + r_{t+1}^s)s_{i,t+1} + (1 + r_{t+1}^b)b_{i,t+1}] \frac{1}{v} \\ (c_{i,t}, b_{i,t+1}, s_{i,t+1}) &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

Working age household  $i$  maximizes its expected lifetime utility subject to the constraints below:

$$\begin{aligned} c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} &\leq \omega_{i,t} \\ \omega_{i,t+1} &= \begin{cases} w_{t+1}l_{i,t+1}(1 - \tau_{t+1}^l) + (1 + r_{t+1}^s(1 - \tau_s))s_{i,t+1} + (1 + r_{t+1}^b)b_{i,t+1} & \text{if } e = 1 \\ g_{u,t+1}(1 - \tau_{t+1}^l) + (1 + r_{t+1}^s(1 - \tau_s))s_{i,t+1} + (1 + r_{t+1}^b)b_{i,t+1} & \text{if } e = 0 \end{cases} \\ (c_{i,t}, b_{i,t+1}, s_{i,t+1}) &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

### 1.2.9 Recursive household problem

The idiosyncratic state variables of the household problem are: current wealth  $\omega$ , households age, and if the household is not retired: current employment and productivity state  $e, l$ . By  $\Theta$ , I denote the vector of all discrete individual states (all except the current wealth).<sup>8</sup>

The aggregate state variables of the household problem are: state of the TFP shock  $z$ , state of the capital depreciation shock  $\delta$ , and distribution captured by the probability measure  $\mu$ .  $\mu$  is a probability measure on  $(S, \beta_s)$ , where  $S = [\underline{\omega}, \bar{\omega}] \times \Theta$ ,

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<sup>8</sup>In the benchmark model, there will be five elements of  $\Theta$ : retired, unemployed, and three levels of productivity for the employed households.



and  $\beta_s$  is the Borel  $\sigma$ -algebra.  $\underline{\omega}$  and  $\bar{\omega}$  denote the minimal and maximal allowed amount of wealth the household can hold.<sup>9</sup> Therefore, for  $B \in \beta_s$ ,  $\mu(B)$  indicates the mass of households whose individual states fall in  $B$ . Intuitively, one can think of  $\mu$  as a distribution variable that measures the mass of agents in a certain interval of wealth, for each possible combination of other idiosyncratic variables.

The recursive household problem for the retired households:

$$v_R(\omega; z, \mu, \delta) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + v \beta E_{z', \mu', \delta' | z, \mu, \delta} [v_R(\omega'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$\begin{aligned} c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} &= \omega \\ \omega' &= T'_{ss} + [s'(1 + r'^s) + b'(1 + r'^b)] \frac{1}{v} \\ \mu' &= \Gamma(\mu, z, z', d, d') \\ (c, b', s') &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

Working-age households:

$$v_W(\omega, e, l; z, \mu, \delta) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{e', l', z', \mu', \delta' | e, l, z, \mu, \delta} [(1 - \theta)v_W(\omega', e', l'; z', \mu', \delta')^{1-\alpha} + \theta v_R(\omega', e', l'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$\begin{aligned} c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} &= \omega \\ \omega' &= \begin{cases} w'l'(1 - \tau^l) + s'(1 + r'^s(1 - \tau_s)) + b'(1 + r'^b) & \text{if } e = 1 \\ g'_u w'(1 - \tau^l) + s'(1 + r'^s(1 - \tau_s)) + b'(1 + r'^b) & \text{if } e = 0 \end{cases} \\ \mu' &= \Gamma(\mu, z, z', d, d') \\ (c, b', s') &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

where  $\omega$  is the vector of individual wealth of all agents,  $\mu$  is the probability measure generated by the set  $\Omega x ExL$ ,  $\mu' = \Gamma(\mu, z, z', d, d')$  is a transition function

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<sup>9</sup> $\underline{\omega}$  is determined by the borrowing constraint, and  $\bar{\omega}$  is chosen such that there are always no agents with that amount of wealth in equilibrium.

and  $'$  denotes the next period.

### 1.2.10 General equilibrium

The economy-wide state is described by  $(\omega, e; z, \mu, d)$ . Therefore, the individual household policy functions are:  $c^j = g^{c,j}(\omega, e, l; z, \mu, d)$ ,  $b^j = g^{b,j}(\omega, e, l; z, \mu, d)$  and  $s^j = g^{s,j}(\omega, e, l; z, \mu, d)$ , and the law of motion for the aggregate capital is  $K' = \int_S g^{b,j}(\omega, e, l; z, \mu, d) + g^{s,j}(\omega, e, l; z, \mu, d)$ .

A recursive competitive equilibrium is defined by the set of individual policy and value functions  $\{v_R, g^{c,R}, g^{s,R}, g^{b,R}, v_W, g^{c,W}, g^{s,W}, g^{b,W}\}$ , the law of motion for the aggregate capital  $g^K$ , a set of pricing functions  $\{w, R^b, R^s\}$ , government policies in period  $t$ :  $\{\tau^{lb}, \tau^u, \tau^s, \tau^b\}$  and tax rates contingent on the aggregate states in period  $t + 1$ :  $\{\tau'^{lb}, \tau'^u, \tau'^s, \tau'^b\}$ , and forecasting equations  $g^L$ , such that:

1. The law of motion for the aggregate capital  $g^K$  and the aggregate “wage function”  $w$ , given the taxes satisfy the optimality conditions of the firm.
2. Given  $\{w, R^b, R^s\}$ , the law of motion  $\Gamma$ , the exogenous transition matrices  $\{\Pi_z, Pi_e, Pi_l\}$ , the forecasting equation  $g^L$ , the law of motion for the aggregate capital  $g^K$ , and the tax rates, the policy functions  $\{g^{c,j}, g^{b,j}, g^{s,j}\}$  solve the household problem.
3. Labor, shares and the bond markets clear (the goods market clears by the Walras’ law):

•

$$L = \int_S e l d\mu$$

•

$$\int_S g^{s,j}(\omega, e, l; z, \mu, \delta) d\mu = (1 - \lambda)K'$$

•

$$\int_S g^{b,j}(\omega, e, l; z, \mu, \delta) d\mu = \lambda K'$$

4. The law of motion  $\Gamma(\mu, z, z', \delta, \delta')$  for  $\mu$  is generated by the optimal policy functions  $\{g^c, g^b, g^s\}$ , which are endogenous, and by the transition matrices

for the aggregate shocks ( $z$  and  $\delta$ ).<sup>10</sup> Additionally, the forecasting equation for aggregate labor is consistent with the labor market clearing:  $g^L(z', \delta') = \int_S \epsilon l d\mu$ .

5. Government budget constraints are satisfied:

$$T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$$

$$\tau_t^u = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\phi}}$$

### 1.3 Parametrization

Parametrization and calibration mainly follow Algan et al. (2009) and Krueger et al. (2016). The model is calibrated to quarterly frequency. There are two possible realizations for TFP shocks: good and bad state. In addition, capital depreciation shock can also take two possible values. Therefore, there are four possible aggregate states overall. The probability of remaining in the same state is 0.875. A discount factor is calibrated to match the capital-output ratio and interest rate.<sup>11</sup>

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<sup>10</sup>  $\mu'$  is given by a function  $\Gamma$ , i.e.  $\mu' = \Gamma(\mu, z, z', \delta, \delta')$

<sup>11</sup>  $\beta$  is relatively low because the agents face high idiosyncratic risk, while having substantial risk aversion, which makes them have high precautionary savings. The variance of the depreciation shocks may not seem large, but depreciation equals, on average, 6.5% in the high state and 0.2% in the low state.

**Table 1: Internally-calibrated parameters**

Parameter	Symbol	Value	Target
Discount factor	$\beta$	0.8703	Capital-Output ratio
Subsistence constraint	$\gamma$	0.03	Portfolio choice pattern
Quarterly stock market participation costs	$\phi$	0.0044	Share of households with no equity

**Table 2: Externally-calibrated parameters**

Parameter	Symbol	Value	Target
Intertemporal elasticity of substitution	$\frac{1}{\rho}$	0.5	Capital-Output ratio
Expected depreciation rate	$E(\delta)$	0.033	Equity premium
Chance of not retiring	$\theta$	.994	Average working duration
Chance of not dying	$v$	0.983	Average retirement duration
Tax advantage of debt	$\tau^s$	0.3	Hennessy and Whited (2005)
Capital share	$\Delta$	0.4	Algan et al. (2009)

**Table 3: Parameters to generate a sizable equity premium**

Parameter	Symbol	Value
Risk aversion	$\alpha$	12
Variance of depreciation rate	$\sigma^2(\delta)$	0.001

**Table 4: Other parameters**

Parameter	Symbol	Value
Social security tax	$\tau^{lss}$	0.06
Unemployment replacement rate	$\eta$	0.042
Borrowing constraint: bonds	$\kappa^b$	-0.19
Borrowing constraint: stocks	$\kappa^s$	0.0

For the idiosyncratic labor income shocks transition matrix, I use the same values as Pijoan-Mas (2007).

$$\Pi_l = \begin{bmatrix} 0.9850 & 0.0100 & 0.0050 \\ 0.0025 & 0.9850 & 0.0125 \\ 0.0050 & 0.0100 & 0.9850 \end{bmatrix}$$

For the individual labor productivity levels, the following values are used:  $l \in \{36.5, 9.5, 1.2\}$  (they differ slightly from those used by Pijoan-Mas (2007) to account for the fact that, in the model presented here, the household has to pay taxes for social programs). This type of modeling the labor productivity process allows the generation of realistic earnings inequality while keeping the possible number of states relatively low. As in Algan et al. (2009), the expected unemployment duration is set to 1.5 quarters in a good TFP state, and 2.5 quarters in bad TFP state. The unemployment benefits  $g_u$  are set to 4.2% of the average wage in period  $t$ .

Following Krueger et al. (2016), I set  $\theta$  and  $v$  to match the expected work-life length to 40 years, and retirement to 15 years.

Table 5 shows the performance of the model concerning asset pricing and compares it to the values from Algan et al. (2009), which can be considered as a benchmark economy, where the Modigliani-Miller theorem holds approximately. The model asset pricing performance moves much closer to the data, but the classic asset pricing puzzles are still present (excessive bond interest rate, and insufficiently high equity premium).

Figure 1.



Data for portfolio choice are taken from Chang et al. (2018), who use Survey of Consumer Finance.

## 1.4 Solution method

Following Krusell and Smith (1997, 1998), households consider that the aggregate amount of capital and equity premium in the economy move according to the perceived laws of motion, which depend on the TFP aggregate state and aggregate capital. Instead of using a perceived laws of motion for the bond interest rate, like Krusell and Smith (1997) and Algan et al. (2009), I use the perceived law of motion for the equity premium. This method facilitates computation, because during the course of solving the model, predicting negative equity premium is eliminated if logarithmic rules are used (see Harenberg and Ludwig (2018)). For details about the solution algorithm, see the Appendix. The perceived laws of motion are:

$$\ln K' = a_0(z, \delta) + a_1(z, \delta) \ln K$$

$$\ln P^e = b_0(z, \delta) + b_1(z, \delta) \ln K'$$

For the benchmark economy ( $\lambda = 0.35$ ) the perceived aggregate laws of motion are: In a high TFP and high  $\delta$  state:

$$\ln K' = 0.085 + 0.941 \ln K$$

$$\ln P^e = -4.547 - 0.107 \ln K'$$

In a low TFP and high  $\delta$  state:

$$\ln K' = 0.075 + 0.945 \ln K$$

$$\ln P^e = -4.568 - 0.090 \ln K'$$

In a high TFP and low  $\delta$  state:

$$\ln K' = 0.112 + 0.948 \ln K$$

$$\ln P^e = -4.311 + 0.115 \ln K'$$

In a low TFP and low  $\delta$  state:

$$\ln K' = 0.111 + 0.947 \ln K$$

$$\ln P^e = -4.305 - 0.086 \ln K'$$

The perceived laws of motion predicts the actual movements of capital and equity premium with  $R^2 = 0.99991$  for capital and  $R^2 = 0.99900$  for equity premium.

The average error for the aggregate capital laws of motion is 0.020% percent of the capital stock, while the maximum error is 0.062% of the capital stock.

## 1.5 Results

The model is calibrated to match the share of households who do not participate in the equity markets in a low leverage economy, and generates 45.9% of non-participating households, which is only slightly less than 46.7%, which is what Chang et al. (2018) observe in the data. In a high leverage economy, this share is only slightly lower, with 45.0% of agents participating in the stock market.<sup>12</sup> Furthermore, the model roughly generates the portfolio choice pattern along the wealth distribution dimension (as shown in Figure 1.) Furthermore, the model does not generate the realistic right tail of the wealth distribution. This is a common issue in standard models in the field.<sup>13</sup> Furthermore, it is questionable how relevant the returns on bonds and market indices are for the saving behavior of the very richest households, who mostly own very diversified equity. Finally, the model generates a Sharpe ratio of 0.33.

I perform an exercise in which the leverage of the economy rises from  $\lambda = 0.35$ , which was the leverage in the US economy in the 1990s, to  $\lambda = 0.48$ , which was the value in 1984.

**Table 5. Quarterly statistics for the benchmark economy**

	$K/Y$	Wealth Gini	Households not owning any equity %	$r^b$	$E(r_{pt}^s) - r_{pt}^b$
Data	7	0.780	46.7	0.23	1.0 - 2.0
Economy $\lambda = 0.35$	7.00	0.5738	45.9	0.36	1.02
Economy $\lambda = 0.48$	7.21	0.5631	45.0	0.24	1.31
Algan et al. (2009) $\lambda = 0.37$	7.01	0.480	41.0	2.4	0.007

<sup>12</sup>Billas et al. (2017) discuss how an increase in stock market participation can influence inequality. On the one hand, it makes poorer households obtain equity premium, but at the same time, poorer households might make bad investment decisions, if they are not sophisticated enough in their stock investment behavior. The authors find that the increased stock market participation has not significantly changed inequality in the US during the 1990s. However, in my model, the latter channel is not existent, since only the representative firm issues equity (which is only subject only to the aggregate risk).

<sup>13</sup>It would be possible, for example, to generate it by introducing a stochastic discount factor in the fashion of Krusell and Smith (1998), but that would complicate the analysis and would be computationally expensive.



**Table 6: Results: Change in asset prices and inequality**

Unconditional moments	$\lambda = 35$ Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	1.02%	1.31%
$E(r^{b,p})$	0.36%	0.23%
$E(r^{s,p})$	1.39%	1.54%
Wealth Gini	0.5738	0.5631
$E(K)$	5.76	6.07
Share of wealth by quintiles (%)		
1st	0.92%	0.96%
2st	4.20%	4.65%
3st	11.40%	11.85%
4st	22.14%	22.26%
5st	61.57%	60.48%

**Table 7: Changes in the Wealth shares:**

Quintile	Wealth change				
	1	2	3	4	5
% change	+4.35	+10.71	+3.95	+0.54	-1.77

The change in leverage has important implications for asset prices. Increased leverage means increased variance of equity returns, and consequently, increased equity premium. However, this does not necessarily mean that the portfolios of the households holding equity are riskier; now the bonds are in higher demand, and the households will hold less risky equity and more safe bonds. Overall, the Sharpe ratio in the economy remains approximately the same: 0.33.<sup>14</sup>

The average long-run aggregate capital depends on the leverage. The economy with high leverage  $\lambda = 0.48$  has 5.38% more capital in the long run than the economy with relatively low leverage  $\lambda = 0.35$ . To see the intuition behind this, one can see that the increase in leverage effectively functions as a decrease in the capital tax rate. With the higher leverage, more of the firms' surplus (not

<sup>14</sup>Not reported in tables. There is a slight change from 0.3258 to 0.3327

profits, but a surplus as defined by  $\text{surplus} = \text{profits} + \text{debt interest payment}$ ) is distributed as debt interest payments, and less as profits. Since the debt has a beneficial tax treatment (debt repayment is tax-deductible), this effectively serves as the reduction in the capital tax rate. Consequently, the effective decrease in the overall capital tax rate leads to an increase in capital accumulation.

Intuitively, the effective reduction in the tax rate, or the effective reduction in the tax base (decrease in profits and increase in debt repayment), leads to a decrease in government revenue. More precisely, the government revenue from capital taxes in the low leverage economy is 1.85%, and in the high leverage economy, it is reduced in the low leverage economy to 1.74%. Furthermore, as a consequence of the capital increase, wages rise by 1.03% in the high leverage economy.

Wealth inequality decreases by 1.07 Gini points. This amounts to 12% of the change in wealth Gini in the observed period. Given that there were many policies and global economic changes during the studied period, this can be considered a significant decrease. The reason that drives the wealth inequality reduction is the different response to the change in the effective tax rate of the poor and rich. Increased leverage implies less corporate tax paid, which in turn implies increased net returns on savings. How will the households' savings respond to the increased interest rate depends on two counteracting effects: substitution and income effects. The substitution effect means a household wants to save more, since the reward for saving in the next period is higher, i.e., households want to substitute the consumption today for consumption in the future. On the other hand, there is an income effect, which means that, since the households are now richer (its savings are now more valuable since they give a higher return), it wants to consume more and save less. The change in wealth inequality will be determined by the size of the two effects for the poor and the rich. The results of the model are the following: first, the substitution effect is stronger than the income effect for all households, meaning that all households save more. Second, the income effect is relatively stronger for the rich, compared to poor households. This occurs because the more wealth the households already have, the stronger the income

effect is. This follows from the fact that the income effect of the increased interest rate is multiplicative; the more wealth the households already have, proportionately the more income it will receive if their savings behavior does not change.<sup>1516</sup>

Additionally, the return on bonds (safe asset) declines, both directly because of the change in leverage: increased supply of the bond would imply lower returns, and (mainly) indirectly through the increased aggregate capital and decreased marginal productivity of capital. The equity premium does not increase as much with the increased leverage as it would in the economy without equity taxation. This is because, absent of taxes, when leverage increases, the variance of equity returns also increases (compared to the low-leverage economy, the returns are even better in the good state, and even worse in the bad state because the firms need to repay the same amount to debt owners, regardless of the performance in the current period). However, when equity returns are taxed, this effect is dampened, because the tax bill will be higher in the good state, and lower in the bad state.

From the utilitarian standpoint, the change of leverage (including the transition path of the economy) is welfare improving from the utilitarian standpoint, and equivalent to the 5.7% permanent increase in consumption. This is hardly unexpected, as the increase in leverage implies a smaller tax base for corporate income tax, and the government revenue collected by corporate income tax is assumed to be wasteful in this model. In addition, the eventual capital accumulation raises wages, which are an important source of income for agents close to the borrowing constraint. Furthermore, as in line with Davila et al. (2012), there is a severe capital under-accumulation in this model. The welfare gains across

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<sup>15</sup>In other words: the intuition behinds result is: the rising returns on savings have substitution effect (consumption today is more expensive) which increases savings, and income effect (asset owners are now getting higher interest/dividends and are effectively richer) which decreases savings (increases consumption today). Since poor households do not own many assets, their income effect is much weaker than for wealthy households, and consequently, they increase their saving rates more than the wealthy households do. In the long run, this results in lower wealth inequality.

<sup>16</sup>The size of this effect, among other factors, depends on the intertemporal elasticity of substitution, which is partially governed by the parameter  $\rho$ . The magnitude of the change of the wealth inequality should increase with the increase of intertemporal elasticity of substitution (decrease in  $\rho$ , for example).

the household type and wealth can be seen in Appendix C.

To demonstrate the importance of modeling capital taxation and debt tax benefits in extension 1, I perform the same exercise of increasing leverage, but without capital income taxation. As can be seen in tables 8 and 9, the results on the aggregates and inequality are quite small. These results show that the “Modigliani-Miller” result of the irrelevance of leverage holds even in a Krusell and Smith (1997) type of economy, as stated in Algan et al. (2009). However, this chapter extends the result to the economy in which the equity premium is realistically high, and equity market participation constraints are present. It is important to generate a high equity premium since the Modigliani-Miller result is less surprising in a model where the two assets are not very different from a households’ perspective. In the absence of capital taxation, there are no real effects of the change in leverage because households are able to rebalance their asset holdings such that they can almost perfectly replicate their old portfolios. For example, when the leverage increases, the equity becomes more risky. But at the same time, there is less equity in the economy and there is more of safer bonds. Consequently, households will hold less equity and more bonds, and these two effects will approximately negate each-other. An important feature of this rebalancing is that the bond return remains approximately the same. This means that even the constrained agents, who hold only bonds and cannot rebalance their portfolios, will not be affected by the change in leverage.

### **Decomposition: Holding prices consistent with the levels of aggregate capital from $\lambda = 0.35$ economy**

The decomposition is performed to isolate the effect of higher capital accumulation from the other effects of the increase in leverage. To achieve this, leverage is set to  $\lambda = 0.48$ , but the set of prices is fixed such that they are consistent with the levels of capital in the  $\lambda = 0.35$  economy. Therefore, wages and output will be the same as in the  $\lambda = 0.35$  economy, but the asset prices (returns) will be different as a consequence of the changed leverage (but not as a consequence of the increased capital, as the output, and marginal capital productivity are held to be the same

as in  $\lambda = 0.35$  economy). The increase in the aggregate capital (or rather the changes in the interest rate and wages caused by the rise in the aggregate capital) decreases inequality. Disregarding the general equilibrium effects of the increase in aggregate capital, the change of leverage itself decreases inequality by 0.85 Gini points (of total of 1.07). The reason is that an increase in the leverage acts similarly to a decrease in the tax rate on capital income, which ultimately leads to the increased net return on savings. The key is that the poor households respond more strongly to the increased savings returns, meaning that they increase their savings more than the rich households. This is because the income effect of the increased savings returns is rather weak for the poor households, and relatively strong for the wealthy households since they have a considerable amount of accumulated savings. Furthermore, it increases the supply of the safe asset, in which the equity-constrained households exclusively save.

An increase in the aggregate capital further decreases wealth inequality (for the remaining 0.22 Gini points), as it decreases the interest rate (the primary source of income for the rich), and increases wages (the primary source of income for the poor). It is interesting to observe that the households in the lowest quintile benefit significantly from the increase in aggregate capital, with their wealth share increasing from 0.91% to 0.96%, as they are the ones who mostly rely on wages, which increase with the increase in aggregate capital.

To check that the changes in inequality are indeed driven by a change in the behavior of the rich and poor, and not young (which are poorer on average) and old (which are richer on average) households, I include the Gini indicator excluding retired households and focusing only on working-age households. The same inequality pattern is observed as when the whole population is included. Moreover, when excluding the retired households, the wealth inequality is reduced by 1.34 Gini points, and by 1.13 Gini points when ignoring the change in aggregate capital.

**Table 8: Results: Change in asset prices and inequality**

Unconditional moments	$\lambda = 35$ Economy	Decomposition Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	1.02%	1.31%	1.31%
$E(r^{b,p})$	0.36%	0.36%	0.23%
$E(r^{s,p})$	1.37%	1.67%	1.54%
Wealth Gini	0.5738	0.5653	0.5631
Wealth Gini excluding retired households	0.4727	0.4614	0.4593
$E(K)$	5.76	/	6.07
Share of wealth by quintiles (%)			
1st	0.92%	0.91%	0.96%
2nd	4.20%	4.57%	4.65%
3rd	11.40%	11.80%	11.85%
4th	22.14%	22.24%	22.26%
5th	61.57%	60.69%	60.48%

## 1.6 Extensions

In this section, I analyze four extensions. All four models are recalibrated to match the crucial moments from the data.<sup>17</sup>

First, I consider an economy without capital taxation. The effects of the change in leverage on the wealth inequality are rather small, and even smaller on the aggregate capital. This extension contributes to the literature by showing that the Modigliani-Miller theorem holds “approximately” (Algan et al., 2009) in the borrowing constraint general equilibrium models, even when inequality and equity premium are comparable to those that we observe in the data. Unlike in the case without corporate tax, in the benchmark model, the households cannot approximately replicate their portfolios after the change in equity. This is because the presence of corporate income tax not only changes the value of the firm (and payoffs), but it also changes the riskiness of equity. This means that in the aggregate states with higher equity payoff, the tax bill will be higher than in the states with lower equity payoff, and this reduces the equity payoff variance.

Second, I consider an economy in which there is no capital income taxation, but also assumes that 56% of households are exogenously equity-constrained,<sup>18</sup> i.e., they can only save in bonds. For this economy, I find that the effects of the changes in leverage are quite small on the macroaggregates and wealth distribution, even though a majority of households are portfolio constrained. The intuition behind this result is that, in the absence of the debt tax shield, as long as there is a sufficient number of rich, well-insured households that can adjust their portfolio after the leverage change, the safe interest rate will not change significantly (thus the constrained households will not be significantly affected), and the non-constrained households can almost exactly replicate their portfolios from the low-leverage economy.

Third, I consider an economy in which 4% of households are exogenously al-

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<sup>17</sup>Therefore, some differences between variables in different extensions are simply a result of the different calibration.

<sup>18</sup>There are additional 3.3% of households who endogenously chose not to invest in equity.

lowed to save only in equity. This number is consistent with the share of such households in the Survey of Consumer Finance in 2016. For a household to be considered to invest only in equity, I choose the threshold value that no more than 10% of their savings should be in safe assets.<sup>19</sup> Now, the exogenously constrained households are directly impacted by the change in the leverage, since the variance of equity returns is changed, and they cannot rebalance their portfolios. However, the amount of this type of household is too small to change the wealth inequality or macroaggregates significantly.

The fourth extension considers a benchmark economy, but in addition, the government revenues from capital income taxes are rebated equally to the households in a lump-sum fashion. The goal is to perform a welfare comparison between the economies with different leverage since in the benchmark economy the additional tax revenue in the low-leverage economy is wasteful. However, there are many ways in which the additional revenue might be used, so naturally the welfare comparison results would change depending on the way in which this revenue is used. In this extension, the budget constraint of the households is as follows:

$$c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} \leq \omega_{i,t}$$

$$\omega_{i,t+1} = \begin{cases} w_{t+1} l_{i,t+1} (1 - \tau_{t+1}^l) + (1 + r_{t+1}^s (1 - \tau_s)) s_{i,t+1} + (1 + r_{t+1}^b) b_{i,t+1} + T_{ls,t+1} & \text{if } e = 1 \\ g_{u,t+1} (1 - \tau_{t+1}^l) + (1 + r_{t+1}^s (1 - \tau_s)) s_{i,t+1} + (1 + r_{t+1}^b) b_{i,t+1} + T_{ls,t+1} & \text{if } e = 0 \end{cases}$$

$$(c_{i,t}, b_{i,t+1}, s_{i,t+1}) \geq (0, \kappa^b, \kappa^s)$$

$T_{ls,t+1}$  is a lump sum subsidy. This subsidy equally redistributes the revenue from the capital income taxes:

$$T_{ls,t+1} = \tau_s r_{t+1}^s K_t (1 - \lambda)$$

As reported in table 12., the increase in leverage leads to a decrease in wealth inequality, exactly as in the benchmark model. Furthermore, when examining the

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<sup>19</sup>These are mostly the assets held in checking accounts, which are often held for the purpose of liquidity, and not necessarily risk aversion.



long-run results, welfare still increases in the high-leverage economy. This means that these two results persist even though lump-sum subsidies are reduced as a consequence of the decreased capital tax revenue. Lump-sum subsidies should generally decrease wealth inequality and increase welfare, as they are financed by capital taxes (which are mostly paid by the wealthy households), and rebated to everyone equally, making the wealth-poor households net recipients and wealthy households net-payers. However, when taking transition into consideration, the welfare gains are neutralized by the transition path, during which the households increase savings and reduce consumption, and the leverage change is roughly welfare neutral. In particular, when the transition is taken into account, an increase in leverage leads to a fall of welfare equivalent to a 0.006% permanent decrease of consumption.

**Table 9: Extension 1: Model without capital taxation: Outcomes in economies with different leverage  $\lambda$**

Unconditional moments	$\lambda = 35$ Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	1.82%	2.27%
$E(r^{b,p})$	0.18%	0.17%
Wealth Gini	0.5420	0.5414
$E(K)$	5.79	5.81
Share of wealth by quintile (%)		
1st	1.09%	1.09%
2nd	5.40%	5.43%
3rd	12.78%	12.81%
4th	22.01%	22.03%
5th	58.94%	58.86%

**Table 10: Extension 2: Majority of households (56%) can ex-ante only save in safe assets + no capital taxation**

Unconditional moments	$\lambda = 35$ Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	0.69%	0.86%
$E(r^{b,p})$	0.87%	0.87%
Wealth Gini	0.4386	0.4386
$E(K)$	5.86	5.86
Share of wealth by quintile (%)		
1st	2.57%	2.57%
2nd	8.59%	8.59%
3rd	15.96%	15.97%
4th	25.91%	25.92%
5th	49.46%	49.46%

**Table 11: Extension 3: 4% of households can only save in equity + no debt tax shield**

Unconditional moments	$\lambda = 35$ Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	1.87%	2.33%
$E(r^{b,p})$	0.10%	0.11%
Wealth Gini	0.5439	0.5474
$E(K)$	5.78	5.75
Share of wealth by quintile (%)		
1st	1.33%	1.11%
2nd	5.14%	5.12%
3rd	12.64%	12.64%
4th	22.02%	22.08%
5th	59.20%	59.37%

**Table 12: Extension 4: Tax revenue from capital taxation if rebated in a lump-sum fashion**

Unconditional moments	$\lambda = 35$ Economy	$\lambda = 48$ Economy
$E(r^{s,p} - r^{b,p})$	1.04%	1.31%
$E(r^{b,p})$	0.40%	0.35%
Wealth Gini	0.5671	0.5585
$E(K)$	5.62	5.77
Share of wealth by quintile (%)		
1st	2.14%	2.44%
2nd	4.37%	4.60%
3rd	11.13%	11.30%
4th	20.60%	20.54%
5th	62.10%	61.45%

## 1.7 Conclusion

This thesis chapter studies the effects of a change in firm leverage on wealth inequality and macroaggregates. The effects are examined in a model with heterogeneous agents, life-cycle, incomplete markets, and aggregate risk. The analysis

focuses on the particular change in the financial policy that occurred in recent US history, when firm leverage increased significantly during the 1980s, which was followed by a steady fall beginning in the early 1990s. I consider a general equilibrium model, that generates a sizable equity premium and a realistic amount of households that are portfolio constrained. Increasing the leverage leads to a decrease in the amount of corporate taxes paid since the debt repayment is tax-deductible. Analysis of the benchmark model shows that the increase in firm leverage leads to an increase in capital accumulation, a decrease in wealth inequality, and a decline in government revenues. The reduction in the inequality amounts to 1.07 Gini percentage points, aggregate capital increases by 5.38%, and government revenue decreases by 0.11% of output. The difference in wealth inequality amounts to the 12% of the increase of the wealth inequality in the US during the relevant period. The reduction in inequality is mainly driven by the fact that poor households increase their savings more than wealthy households as a response to the effective reduction in capital tax, caused by the increase in firm leverage, and indirectly by the increase in capital accumulation which increases wages (the main source of income of poor households) and decreases the returns to capital (the main source of income of the rich). The results of the model imply that increase in firm leverage that occurred in the US during the 1980s did not contribute to the increase in wealth inequality, but on the contrary, it is the reduction in firm leverage since the early 1990s that has contributed to the subsequent increases in the wealth inequality.

Furthermore, the chapter shows that, if the model abstracts from capital taxation, an increase in leverage has only minor effects on macroaggregates and inequality, despite having significant implications for asset prices. This is consistent with previous literature that shows that Modigliani-Miller theorem holds either exactly or approximately. Unlike in the past literature, this chapter shows that the Modigliani-Miller theorem holds approximately in imperfect market models with borrowing constraint, even if the equity premium is comparable to that observed in the data, wealth inequality is sizable, and a majority of households are portfolio constrained.

## Appendix A: Numerical algorithm

This appendix briefly describes the solution algorithm used to obtain the solution. The algorithm broadly follows the method used by Krusell and Smith (1997,1998), which replaces the infinite-dimensional wealth distribution with a finite set of moments of wealth distribution, as a state variable.

Furthermore, to solve the individual problem, given the laws of motion for aggregate variables, I use the “endogenous grid method” proposed by Carroll (2006). The method is augmented to allow for two choice variables (amount of savings and the composition of savings between stocks and bonds). Rather than using the perceived aggregate law of motion bond interest rate, I follow Harenberg and Ludwig (2018) and use a perceived law of motion for equity premium. This eliminates the possibility of guessing negative equity premium and facilitates the computation. I use a FORTRAN programming language for the numeric computation, since the computation is intensive, and requires a compile programming language for the run-time of the program to be feasibly short. The symbol  $m$  denotes whether the household is ex-ante constrained in portfolio choice or not. For the benchmark model (unlike the extensions),  $m$  can take only one value, since none of the agents are ex-ante constrained.

1. Guess the law of motion for aggregate capital  $K_{t+1}$  and equity premium  $P_t^e$ . This means guessing the starting 16 coefficients in the following equations (since there are two possible realizations of  $z$  and two for  $\delta$ ):

$$\ln K' = a_0(z, \delta) + a_1(z, \delta) \ln K$$

$$\ln P^e = b_0(z, \delta) + b_1(z, \delta) \ln K'$$

2. Given the perceived laws of motion, solve the individual problem described earlier. In this step, the endogenous grid method (Carroll, 2006) is used. Instead of constructing the grid on the state variable  $\omega$  and searching for the optimal decision for savings  $\tilde{\omega}$ , this method creates a grid on the "optimal savings amounts  $\tilde{\omega}$ , and evaluates the individual optimality conditions to obtain the level of wealth  $\omega$  at which it is optimal to save  $\tilde{\omega}$ . This way,

the root-finding process is avoided, since finding optimal  $\omega$ , given  $\tilde{\omega}$ , involves only the evaluation of a function (households optimality condition). However, the root-finding process is necessary to find the optimal portfolio choice of the household, which is performed after finding the optimal pairs  $\omega$  and  $\tilde{\omega}$ .

3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth, instead of a Monte Carlo simulation, I use the method proposed by Young (2010). For each realized value of  $\omega$ , the method distributes the mass of agents between two grid points:  $\omega_i$  and  $\omega_{i+1}$ , where  $\omega_i < \omega < \omega_{i+1}$ , based on the distance of  $\omega$ , based on Euclidean distance between  $\omega_i$ ,  $\omega$  and  $\omega_{i+1}$ . Do this in the following steps:

- (a) Set up an initial distribution in period 1:  $\mu$  over a simulation grid  $i = 1, 2, \dots, N_{sgrid}$ , for each pair of efficiency and employment status, where  $N_{sgrid}$  is the number of wealth grid points. Set up an initial value for aggregate states  $z$  and  $d$ .
- (b) Find the bond interest rate (expected equity premium  $P^e$ ) in the given period, which clears the market for bonds. This is performed by iterating on  $P^e$  (or bond price), until the following equation is satisfied (bond market clears):

$$\sum_j g^{m,b,j}(\omega, e, l; z, d, K, P^e) d\mu = \lambda \sum_j \{g^{m,b,j}(\omega, e, l; z, d, K, P^e) d\mu + g^{m,s,j}(\omega, e, l; z, d, K, P^e) d\mu\}$$

where  $g^{m,b,j}(\omega, e, l; z, d, K, P^e)$  and  $g^{m,s,j}(\omega, e, l; z, d, K, P^e)$  are the policy functions for bonds and shares, where  $j$  denotes the age of the household (working age or retired), that solve the following recursive household maximization problems: Retired households:

$$v_R^m(\omega; z, \mu, \delta, P^e) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + v\beta E_{z', \mu', \delta', P^{e'} | z, \mu, \delta, P^e} [v_R^m(\omega'; z', \mu', \delta', P^{e'})^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

Working-age households:

$$v_W^m(\omega, e, l; z, \mu, \delta, P^e) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{e', l', z', \mu' P^{e'}, \delta' | e, l, z, \mu, \delta, P^e} \left[ (1 - \theta) v_W(\omega', e', l'; z', \mu', \delta', P^{e'})^{1-\alpha} + \theta v_R(\omega', e', l'; z', \mu', \delta', P^{e'})^{1-\alpha} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

where  $v_j$  are the value functions, obtained in step 2. In this step, an additional state variable is included explicitly: expected equity premium  $P^e$ .

- (c) Depending on the realization for  $z'$  and  $d'$ , compute the joint distribution of wealth, labor efficiency and employment status.
  - (d) To generate a long time series of the movement of the economy, repeat substeps b) and c).
4. Use the time series from step 2 and perform a regression of  $\ln K'$  and  $P^e$  on constants and  $\ln K$ , for all possible values of  $z$  and  $d$ . This way, the new aggregate laws of motion are obtained.
  5. Compare the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently accurate, the solution algorithm is completed. If not, make a new guess for laws of motion, based on a linear combination of laws from steps 1. and 4. Then, proceed to step 2.

## Details of endogenous grid method implementation

The introduction of the fixed participation cost to the endogenous grid method solution is as follows: First, the grid is fixed over the saving decision choices, rather than over current wealth. Then, using the first-order-conditions (inverted Euler-equations), the current levels of wealth, for which it is optimal to choose a pre-selected amount of savings, are computed.<sup>20</sup> This is conducted for two cases: In the first case, the households are not allowed to save in equity ( $s = 0$ ), and therefore do not pay the fixed participation cost  $\phi$ , and the second case where households are allowed to choose their portfolio freely but must pay the participation cost  $\phi$ . This way, two choice-specific endogenous grids (on current wealth) are obtained, with corresponding value function values. The true value function is the upper envelope of the two choice-specific value functions (for every level of current wealth, the greater of the two choice-specific value functions is chosen). To obtain it, both of the choice-specific value functions defined over choice-specific endogenous grids are interpolated on the exogenous grid on current wealth so that they can be directly compared. Then, for each grid point on the exogenous grid, the maximum value of the two choice-specific value functions is chosen to obtain the actual value function over the exogenous grid of current wealth. Then, the algorithm proceeds to calculate the next iteration in the value function iteration.

A potential complication in the next step algorithm is that the value function might be non-concave in the neighborhood of the threshold where households start to invest in stocks and choose to pay the fixed participation cost. The reason is that the actual value function is an upper envelope of the two choice-specific value functions. In the non-concave part of the value function, the first-order conditions are no longer sufficient for the optimal solution, but only necessary. In other words, there might be multiple local optima in that region. This means

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<sup>20</sup>For some values of savings, the obtained current wealth values can be lower than the borrowing constraint. In this case, these grid points are discarded. If the lowest obtained point for current wealth, obtained when the minimum amount of savings is used, is higher than the minimum one implied by the borrowing constraint, then additional grid points are inserted in the region between the borrowing constraint and the lowest obtained point. In this section, the policy function is obtained by using budget constraints with the lowest possible savings. This is in line with the previous papers which use the endogenous grid method (Carroll (2006), Barillas and Fernandez-Villaverde (2007) and Iskhakov et al. (2017)).



that the two different grid points over the savings can produce the same point on the endogenous grid over current wealth. Therefore, it is possible that in the neighborhood of the threshold where households start to invest in stocks, a segment could arise where, instead of the value function, a value correspondence is obtained. This segment implies that for the given value of current wealth, two savings choices satisfy the first order condition. In these segments, similar as in the algorithm of Iskhakov et al. (2017), the savings value which obtains the higher overall utility would be chosen. However, in practice, the non-concavity does not arise. The numerical implementation of the algorithm (cubic and cubic spline interpolated value function defined over a grid on current wealth) cannot register the non-concavity. This is the case even when the grid points in the neighborhood of the threshold are as close as 0.0087% of the average wealth to each other. The reason is that in the neighborhood of the threshold, both choice-specific value functions have very similar curvature, and the fixed participation costs are very small. The concavity of the value function is always checked, if it would arise, in the segments with multiple satisfied first order conditions, the solution with the overall higher resulting value is selected in the style of Iskhakov et al. (2017). However, throughout the solutions, this case does not arise. A similar situation is the case in Bayer et al. (2019), who use liquid/illiquid assets and adjustment costs, and find that there are no non-concavities, even though theoretically, they can arise when using the Fella (2014) algorithm.

## Appendix B: Historical corporate leverage in the US

Figure 3.: Firm leverage in the US



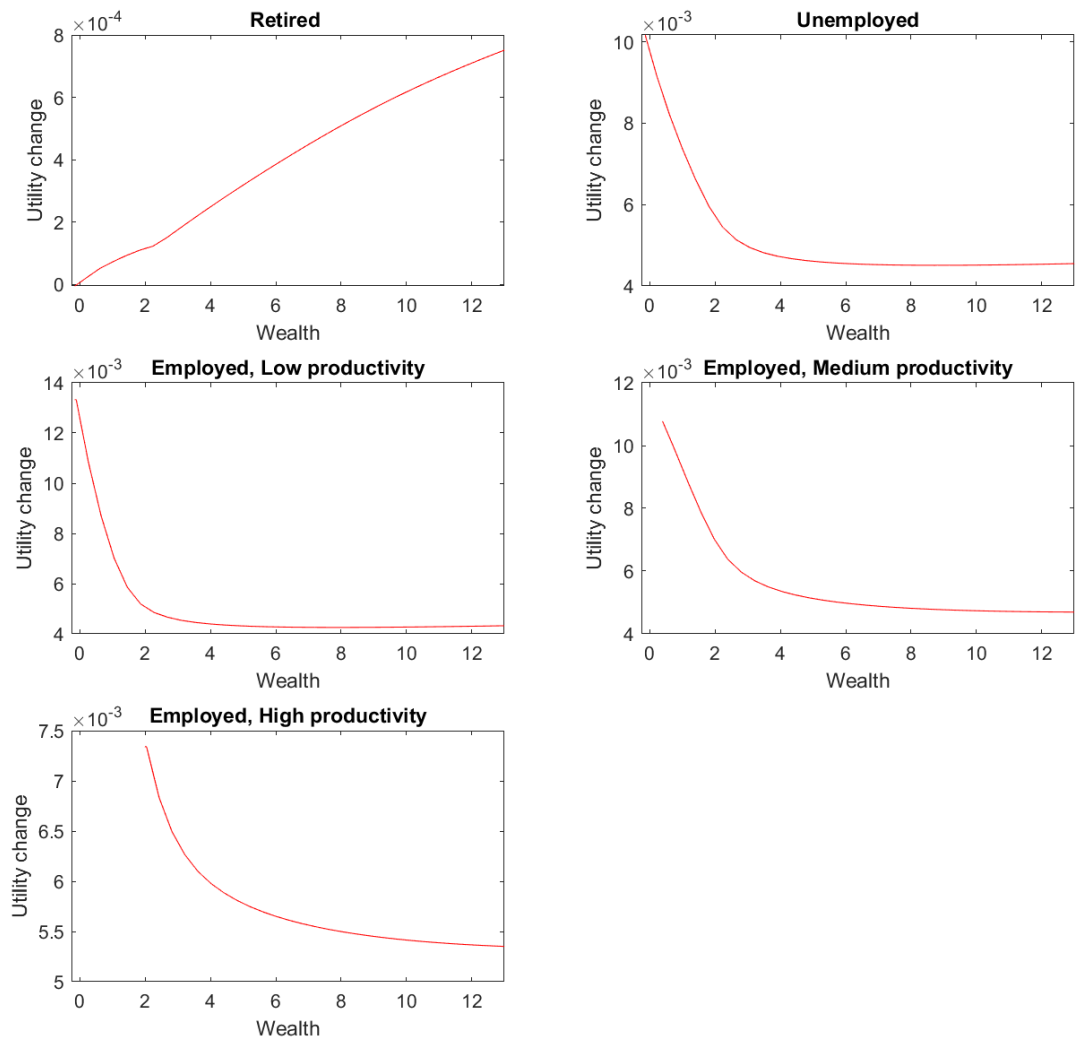
Taken from: Graham et al. (2015)<sup>21</sup>

The historical movement of corporate leverage in the US, taken from Graham et al. (2015), who use data from Moody's Industrial Manuals.

<sup>21</sup><https://corpgov.law.harvard.edu/wp-content/uploads/2015/04/graham-leary-roberts.jpg>

## Appendix C: Welfare changes

Figure 4.: Utility change in the increase in leverage; the change occurs at a high TFP state and a low capital depreciation state



## Appendix D: Policy functions

Figure 5.: Policy functions  $\lambda = 0.35$ , high TFP state, high capital depreciation state

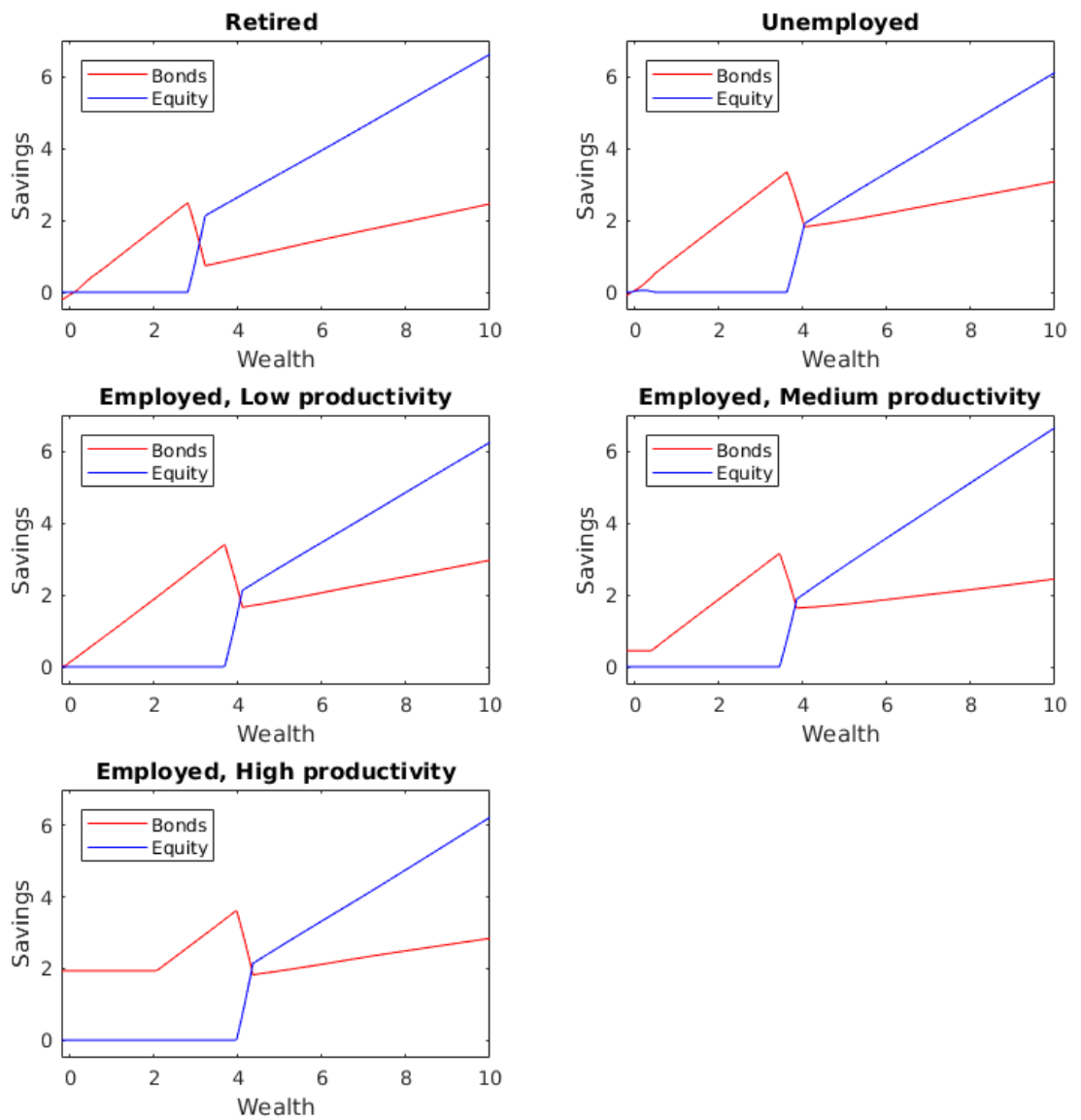
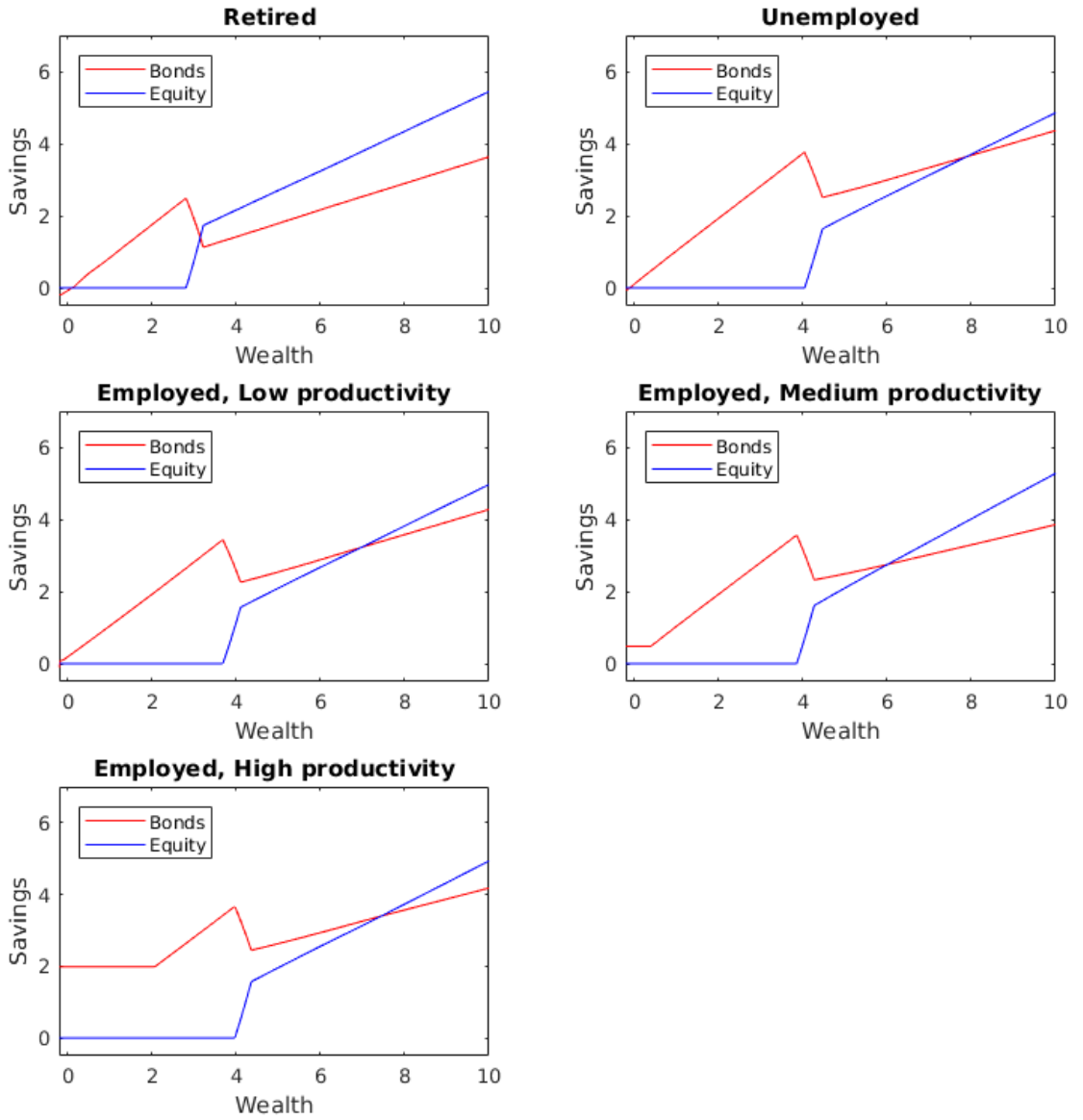
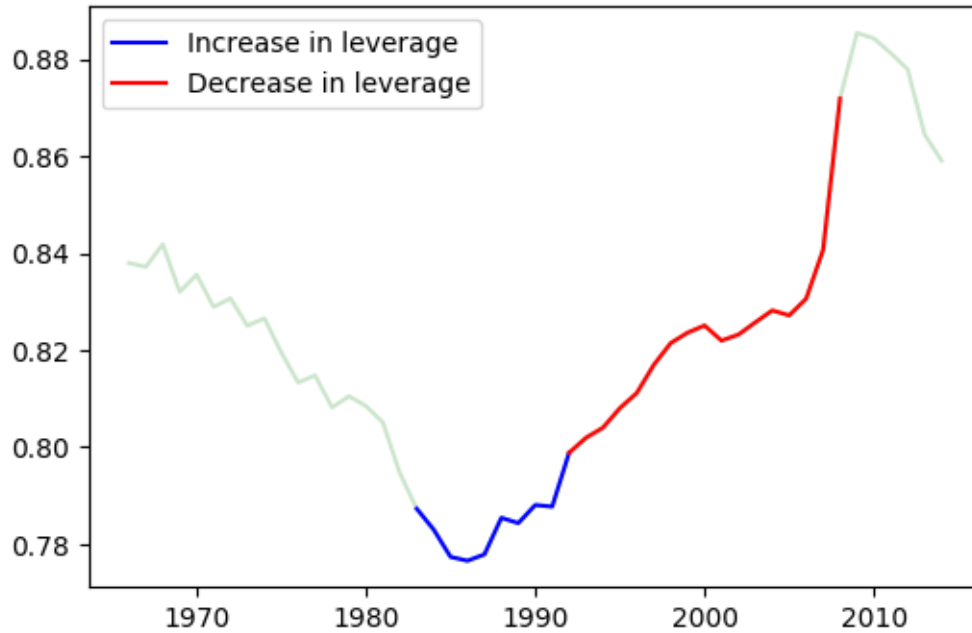


Figure 6.: Policy functions  $\lambda = 0.48$ , high TFP state, high capital depreciation state



## Appendix E: Leverage and wealth inequality

Figure 7.: Net Wealth Inequality and Corporate Leverage in the US



The inequality data is taken from World Inequality Database, which builds on Piketty et al. (2018).<sup>22</sup>

<sup>22</sup>URL=<https://wid.world/data/>

## Appendix F: Supply and demand for bonds

In the model without capital income taxation, the interest rate stays (approximately) the same with the change in leverage. This is because the increase in the supply of bonds is offset by the increase in the demand for bonds resulting from the decreased supply of equity and increase of the equity's riskiness. In the model with capital income taxation, the return on bond falls because in the long run, the aggregate capital rises, and consequently return on both assets (stocks and bonds) falls. This is because, with the Cobb-Douglas production function, increase of aggregate capital decreases its marginal productivity. The effects can be seen on the graph. However, when the leverage suddenly increases, in the short-run, the net return on both bonds and stocks increase.

In the graphs below one can see the change in supply (by the firms) and demand (from the households) of bonds when the leverage changes. The supply of the bond is defined as the sum of savings in stocks multiplied by a fraction  $\frac{\lambda}{1-\lambda}$ .

Figure 8.: Change in Supply and Demand for Bonds in a model without corporate income taxation

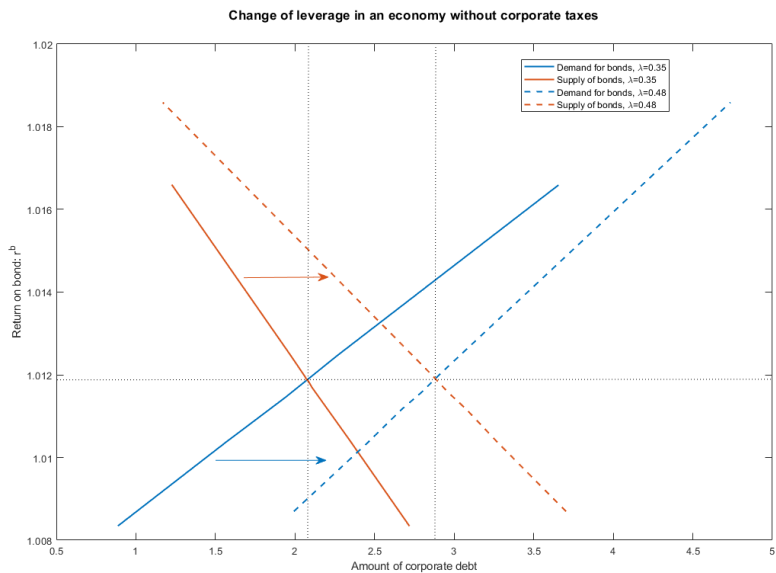
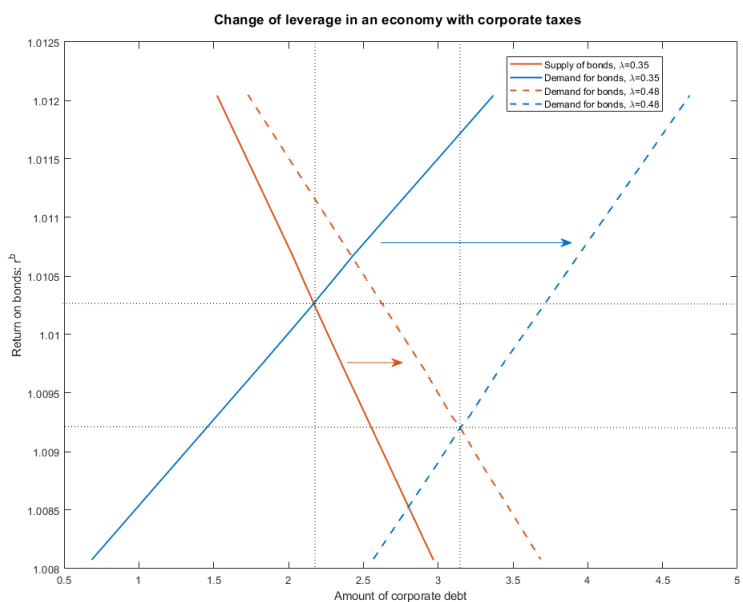


Figure 9.: Change in Supply and Demand for Bonds in a model with corporate income taxation





## Appendix G: Potential reasons for a change in corporate leverage

The scope of this chapter is to study the effects of a change in leverage, no matter the underlying reasons that generated it. Therefore, the focus of the paper is on understanding the channel of corporate leverage rather than explaining the overall change in the wealth inequality in the observed period. The increase in wealth inequality in the late 1980s and 1990s in the US is a complex topic, as many factors have contributed to the increase, either directly or indirectly. Determining all the underlying causes of the changes in inequality and leverage would significantly broaden the scope of this paper.

One potential reason is the change in government debt. Government debt and corporate debt are partial substitutes from the point of view of investors. Therefore, an increase in government debt could cause a decrease in corporate debt. However, during the 1980s and early 1990s, both government and corporate debt move in the same direction.<sup>23</sup> Furthermore, Graham et al. (2015) find that the newly created firms in the late 1980s and early 1990s tended to have lower leverage than their incumbent counterparts. Another potential cause for the change in corporate leverage, which would also influence wealth inequality is the decrease in corporate income taxes. However, the corporate tax rate and the tax advantage of debt do not seem to play a role in the change of corporate leverage in the US in the 20th-century (Graham et al., 2015), and the tax advantage of debt in the late 1980s decreased and then increased slightly in the beginning of 1990s. This is the opposite of what one might expect from the theoretical stand-point, as the leverage increased in the late 1980s and decreased in the early 1990s. Furthermore, Graham et al. (2015) state that financialization in the US leads to an increase in issuing both debt and equity.

Another possible reason for the change in leverage during the observed period is financial deregulation. One important act is the 1980 “Depository Institutions Deregulation and Monetary Control Act,” which deregulated the banking sector,

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<sup>23</sup>According to the data from Graham et al. (2015).

which could have contributed to the subsequent increase in leverage.

## Chapter 2

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# Capital Taxation with Portfolio Choice

## 2.1 Introduction

Traditionally, the optimal taxation literature has focused on the optimal taxation of capital, including studies such as Chamley (1986), Judd (1985), Straub and Werning (2020), Chari et al. (2018), and many others. The literature has so far usually assumed that the capital is homogeneous. However, financial capital has two main tranches: debt and equity. Since debt is the highest priority tranche, it is usually less risky than equity.<sup>1</sup> These tranches are usually taxed at different rates and subject to different types of taxes (for example, debt and interest are not subject to corporate income tax). Furthermore, empirically, different types of households allocate their wealth between these two tranches in starkly different proportions. Consequently, non-uniform taxation of these two tranches of capital leads to redistribution between different types of households. This thesis chapter analyzes, normatively and positively, the effects of such differential taxation, focusing on its redistributive dimension.

Survey data reveal the heterogeneity of the portfolio structure for households of different wealth. One of the most relevant surveys that shows this is the Survey of Consumer Finances (SCF), which collects data on US. households. Rich households are found to save disproportionately more in risky assets compared

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<sup>1</sup>Abstracting from the risk of inflation.

to poor households. The non-participation in financial markets by some poor households cannot alone explain this feature, because the share of wealth held in risky assets, conditional that a household participates in the financial market, is also increasing. The pattern that rich households invest more in risky assets is present even when controlling for the age of the households.

Significant portfolio differences between rich and poor households potentially has important policy implications for a government that maximizes utilitarian social welfare. More precisely, a utilitarian government might want to tax the assets in which the rich households save (risky asset) relatively heavily, in order to be able to tax the asset in which poor households save (safe assets) if the usual concave utility function is assumed. This feature of the tax code is widespread in developed countries in the form of double taxation of returns from equity (dividends).

At the same time, there is an ongoing debate about the double taxation of returns from equity. This feature is often interpreted as a higher tax on risky capital income (equity), compared to a safe asset (debt) (see Scheuer, 2013). It is often argued that this feature of the tax code has a negative impact on welfare, as it distorts the financing sources of firms, and therefore induces excessively large leverage. Consequently, in recent decades, some countries have attempted to reduce the difference between the two effective tax rates (for example, the “Bush tax reform” in 2003, with the introduction of qualified dividend, which was extended by the Obama administration in 2013). This is achieved both by reducing capital income tax, and by taxing the dividend payouts at the reduced rate in the personal income tax. However, the reduced tax rate for dividends in the personal income tax code is relevant only if the households hold the equity directly and not through pension funds (which is the way through which the majority of poor and middle-class people hold equity, see Rios-Rull and Kuhn (2016)).

This research takes a different, complementary perspective on the question of optimal taxation of debt and equity. Instead of focusing on the issuers of the assets (firms), it focuses on the holders of the assets (households). More precisely, it

analyzes the redistributionary effects of a distortionary policy which can shift the tax burden from poor to rich households. In other words, differential taxation can make insurance (precautionary savings in the safe asset) cheaper for poor households who need (and use) it the most. Therefore, given the heterogeneity of portfolios of rich and poor households, the social planner might find differential taxation of capital income optimal, even if it is associated with certain efficiency costs.

This chapter does not examine the normative question of whether capital income tax is optimal as such. Instead, it seeks to contribute to the literature by attempting to answer the following questions: given that capital is taxed, can the differential tax treatment of income from assets of a different risk be optimal, and what are the redistributional consequences of such a policy? Note that this question can be asked even if the average tax on the overall capital is zero. To the best of my knowledge, the redistributional effects of the differential financial capital taxation have not yet been analyzed. In addition to the redistributive consequences, I examine the effects on the aggregate savings rate. Therefore, this chapter has two dimensions: first, the normative one is to examine whether it is optimal to tax different types of capital differently. Second, the positive dimension attempts to quantify the effects of differential taxation of capital in the US.

The chapter analyzes the importance of differential taxation in a dynamic quantitative macroeconomic model. The model features endogenous portfolio choice, a continuum of heterogeneous agents, uninsurable idiosyncratic risk, and aggregate risk. Furthermore, the model is calibrated in attempt to match the wealth and earnings distribution in the US, as well as the generating substantial equity premium and the portfolio choice patterns. The model includes many features that break the uniform taxation result, such as uninsurable idiosyncratic labor income risk, which is correlated with the returns of the risky asset, borrowing and subsistence constraints, and parsimonious life-cycle.

The logic from the Ramsey taxation framework is that the utilitarian social planner can find it optimal to tax a risky asset at a higher rate for redistribution-

ary motives, even if the differential taxation is associated with efficiency costs. The two-period Ramsey taxation model is described in Section 3. Numerical simulations of the full-blown model show that poor households indeed prefer the relatively low taxation of safe assets, and wealthy households prefer relatively lower taxation of risky assets. This is because the households' portfolios, greatly governed by their exposure to labor income risk, are relatively skewed towards risky in rich and safe in poor households. However, taxing the risky asset at a relatively lower rate can promote capital accumulation, which consequently raises wages and decreases interest rates. As poorer households rely more on labor income, and rich households on capital income, these general equilibrium effects tend to reduce the inequality. Finally, it can also be interesting to consider the examined mechanisms in the context of Davila et al. (2012), who find that in these types of models, there is a severe under-accumulation of capital, compared to the constrained efficient outcome. A higher wedge between taxes on equity and debt slightly decreases capital accumulation and decreases the insurance of the relatively poor households. On the other hand however, it increases their possibilities of insurance, as the returns on safe assets, in which they primarily save, increase. Furthermore, heavy taxation of risky equity can be beneficial even for wealthy households, as it reduces the variance of the returns of the risky asset (the taxes are high when the returns are high, but are low when the returns are low).

The remainder of this chapter is organized as follows: Section 2 reviews the existing literature, Section 3 builds an analytical model and solves for the optimal tax formula. Section 4 describes the benchmark quantitative model. Section 5 describes the performed numerical exercises and presents the results. Finally, the chapter concludes.

## 2.2 Literature review

The topics of redistributive and capital income taxation have attracted much interest from economists. The famous Chamley-Judd result of zero long-run capital taxation (Chamley, 1986; Judd, 1985) was shown by Straub and Werning (2020)

not to hold generally even in the models from which it was derived. Many other papers have shown that capital taxation can be optimal in a life cycle model if the government (as is usually the case in the majority of countries), can not condition taxes on the age of a household: Erosa and Gervais (2002), Conesa et al. (2009). Furthermore, Panousi and Reis (2012) show that capital income tax, combined with other policy instruments, can increase the aggregate capital accumulation because it reduces the variance of the investment returns for entrepreneurs. In addition, Saez (2013) studies optimal progressive capital income taxes in an infinite horizon model where agents differ only in their initial wealth. Chari et al. (2018) revisit the results of Straub and Werning (2020), arguing that when one abstracts from the expropriation of the initial capital, zero capital taxation result reemerges in the basic models.

This thesis does not examine the normative question of whether capital income tax is optimal as such, but it poses the question: can the differential tax treatment of income from assets of different risk be optimal, and what are the redistributive consequences of such policy? To the best of the author's knowledge, the redistributive effects of the differential financial capital taxation have not yet been explored. In addition to the main question, other effects of differential capital income taxation on the economy, such as effects on saving rates and asset prices, will be examined as well.

The second related strand of literature examines the differential financial asset taxation. Notable papers that have recently examined the desirability of the differential financial asset taxation from the efficiency perspective are Ferris (2018), which looks at its effect on stock volatility, and Chetty and Saez (2010), which attempts to develop an empirically implementable formula for the efficiency cost of dividend taxation. They both find a significant efficiency cost of differential asset taxation. However, they do not consider the setup with heterogeneous households. There are important results regarding taxation and entrepreneur portfolio choices in the Mirrleesian, private information framework (Mirrlees, 1971), e.g. papers like Shourideh (2012) and Albanesi (2011) consider the optimal Mirrleesian taxation problem of entrepreneurial income. The results

are therefore relevant, but they crucially differ from this thesis chapter because they consider the problem of taxing the entrepreneurs, who have private information about their businesses, while my analysis focuses on portfolio choices of agents who seek to allocate their savings, and are not necessarily entrepreneurs. Shourideh (2012) studies the optimal taxation of wealthy individuals in the Mirrlessian environment in which the sources of inequality are capital income shocks and financial frictions. He finds that the optimal tax schedule is characterized by progressive savings tax and negative bequest tax. Albanesi (2011) studies optimal taxation of entrepreneurial capital in which entrepreneurs have private information, and entrepreneurial activity is thus subject to moral hazard. The main results of the model are that the differential asset taxation and double taxation of capital income are found to be optimal. Another paper with relevant results in the Mirrlesian tradition is Scheuer (2013). He constructs a highly abstract model with aggregate uncertainty, where heterogeneous agents trade consumption claims contingent on aggregate shocks in financial markets. The main feature of the optimal tax code is that optimal asset taxes are higher for the securities that payout in aggregate states where consumption is more volatile. This result is compatible with the result of the quantitative part of this chapter. He argues that this can provide a theoretical efficiency justification for the differential tax treatment of different asset classes (for example, debt and equity). However, he does not consider the redistributive effects of such differential taxation, which will be of crucial interest in my research.

Unlike in the above models in the Mirrlessian tradition, differential taxation of assets is often found not to be the “first best”, meaning that it would be optimal to transfer across different agents, without distorting the incentives in the economy. However, such taxes would be highly complex and almost certainly unimplementable by the government (for example, tax evasion, or simple unconstitutionality of laws that condition taxes on the age of a person). Therefore, governments often have to rely on the more straightforward (for example, linear) taxes, and in this context differential asset taxation can be found to be optimal, often referred to as the second best.



Analysis by Slavík and Yazici (2014) is an example of an arguably desirable differential capital taxation because of its heterogeneous effects in the population, although in a different context. They find that differential taxation of different types of physical capital can be beneficial because it can promote investments in types of physical capital that are complementary with a low skilled type of labor, which benefits the poor agents.

In this chapter, the redistribution channel comes from the stylized fact that poor households save mainly in a safe asset, while wealthy households invest mostly in risky assets. This has been well documented in US micro survey data (Survey of Consumer Finances) by Guiso et al. (2000), Poterba and Samwick (2003) and Chang et al. (2018). For example, Chang et al. (2018) document that, according to Survey of Consumer Finances, in the US in 1998, the poorest households (1st quintile of the wealth distribution), invested 5.3% of their wealth in risky assets, while the richest households (5th quintile of the wealth distribution) held 64.9% of their wealth in risky assets. They also show that the positive correlation of wealth and the share of wealth invested in the risky asset cannot be solely explained by the non-participation of poor households in the equity markets. The reasoning behind the fact that a wealthy household invests a larger share of their wealth in a risky asset is illustrated in the following way: wealthy agents do so because they have big capital stock to act as a buffer against the idiosyncratic risk of their future labor income, which allows them to bear more risk in the financial markets.<sup>2</sup> Importantly, it is not easily observed in the data what portion of the assets is invested in risky as opposed to safe assets. This is because most households (except perhaps the wealthy), hold a considerable amount of their wealth in mutual funds and retirement accounts (Rios-Rull and Kuhn, 2016). However, SCF data provide additional information showing where these funds are eventually invested (equity or safer assets). Chang et al. (2018) use this information to calculate the risky/safe split of the household's wealth.

Guiso et al. (2000), Chang et al. (2018) also notice that, in the data, the share of wealth invested in risky assets is increasing with the age of the household. This

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<sup>2</sup>See Algan et al. (2009) and Chang et al. (2018).

stylized fact can also motivate the inclusion of the life-cycle dimension into the question (disproportionately taxing risky asset means disproportionately taxing older households). This can as well provide a reason for the optimality of differential asset taxation, because (in the absence of age-dependent taxes) it provides another policy instrument that can distinguish between young and old agents. This would enable a government to differentiate between agents of various ages much more finely, compared to the case in which it relies only on having positive (uniform) capital income tax (which is found to be optimal by Erosa and Gervais (2002), in the context of non-age dependent labor taxes).<sup>3</sup>

Studies such as Poterba and Samwick (2003) show that the effect of the tax structure (a particular implementation of tax systems, for example, corporate and personal income tax) impacts the portfolio choice. They find that households with a high marginal personal income tax rate prefer to hold their wealth in stocks as opposed to interest-bearing assets because, in the US, most equity is taxed at the reduced *personal* income tax rate (though it is taxed by corporate tax as well). Unlike their study, this chapter abstracts from the issues of different tax instruments for the sake of simplicity and computational feasibility.

In addition to examining the distributional effects of differential asset taxation, this project will also study its effects on the aggregate savings rate in the economy. In particular, the presumed progressive nature of differential asset taxation can potentially increase the precautionary savings motive of the wealthy agents, increasing their savings rate.

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<sup>3</sup>See also Conesa et al. (2009).

## 2.3 Two-period Ramsey model

A simple two-period model is presented to develop the intuition and obtain insights from a minimal working example. Two agents differ only in their initial wealth  $m^h$ .

The setup is a standard Ramsey problem in which the agents invest in two types of capital (instead of traditionally in consumption goods). Let us say that the agent has the following preferences

$$u(c_h) = \frac{(c_h + L)^{1-\alpha}}{1-\alpha}$$

where  $c_h$  is the consumption of agent  $h$ ,  $L$  is the exogenous labor income (if it is negative, it can be interpreted as a subsistence level of consumption), and  $\alpha$  is a parameter (coefficient of risk aversion). The preferences are standard constant relative risk aversion (CRRA) preferences, with the addition of  $L$  (exogenous labor income or subsistence level).

Furthermore, assume that there are two possible states of the world: “Good” and “Bad”, and that prior to the realization of these states, agents have to decide how to allocate their wealth  $m_h$  between two assets: 1) a safe asset that pays off 1 unit of consumption good in both good and bad states, and 2) a risky asset that pays of  $r_g$  in a good state and  $r_b$  in a bad state of the world ( $r_g > 1$ ,  $r_b < 1$ , and  $\frac{r_g+r_b}{2} > 1$ , and technologies are linear). Thus, the consumer maximizes the expected utility:

$$E(u(c_h)) = \frac{1}{2} \frac{(c_g^h + L_g)^{1-\alpha}}{1-\alpha} + \frac{1}{2} \frac{(c_b^h + L_b)^{1-\alpha}}{1-\alpha}$$

where

$$c_g^h = R^h r_g + S^h$$

$$c_b^h = R^h r_b + S^h$$

subject to the constraint  $R^h q_r + S^h q_s = m^h$ , where  $q_r$  and  $q_s$  are (after-tax) prices.

Considering this setup, it is possible to rewrite the consumer utility function as a function of two goods: risky asset  $R$ , and safe asset  $S$ :

$$U(R^h, S^h) = \frac{1}{2} \left\{ \frac{(R^h r_g + S^h + L_g)^{1-\alpha}}{1-\alpha} + \frac{(R^h r_b + S^h + L_b)^{1-\alpha}}{1-\alpha} \right\}$$

For simplicity, it is assumed that  $L_b = 0$ . Now, we can set up a Ramsey problem in which the government has to tax the two goods ( $R$  and  $S$ ) to raise some exogenously given revenue  $G$ .

The drawback of this approach is that the government is taxing the agents when they are buying the assets and is not taxing the returns of the assets (which is more realistic). However, the main results extend to the case in which the government is taxing the returns on the two assets. This is verified numerically in Appendix B. The advantage of this approach is that we can use the existing results of taxation theory and compare our results to other standard Ramsey models. In addition, this approach removes the questions of debt and state-dependent taxes for the sake of simplicity.

Now consider the Ramsey problem by the government:

$$\max_{\tau_r, \tau_s} W = \sum^h U(c^h)$$

subject to:

$$\tau_r \sum^h R^h + \tau_s \sum^h S^h \geq G$$

and agents' maximizing decisions.

The questions being asked are: should there be differential taxation of the two assets, in which cases, and in what direction?

## Ramsey problem

The government sets  $\tau_r$  and  $\tau_s$  to raise the revenue  $G$ , and achieve the highest welfare possible for the agents, anticipating their optimal decisions.

**Notation:**

- let  $x_j^h(p, m^h)$  be the Marshallian demand for good  $j$  of agent  $h$
- let  $V^h(p, m^h)$  be the agent's indirect utility function
- let  $\hat{x}_j^h(p, m^h)$  be the compensated (Hicksian) demand for good  $j$
- let  $\chi(p, m)$  be the utilitarian social welfare function equal to  $\sum^h V^h(p, m^h)$
- let  $p_j = (1 + \tau_j)$  be the after tax price of asset  $j$
- let  $\lambda$  be a lagrangian multiplier on a government constraint

Using this notation, the Ramsey problem is:

$$\max_{\tau_r, \tau_s} \chi(p, m)$$

s.t.

$$\tau_r \left( \sum_{h=1}^2 x_r^h(p, m^h) \right) + \tau_s \left( \sum_{h=1}^2 x_s^h(p, m^h) \right) \geq G$$

Furthermore, following notation is used:

$$X_k = \sum_{h=1}^H x_k^h(p, m)$$

where  $X_k$  is the total demand for good  $k$

$$\beta^h = \frac{\frac{\partial \chi}{\partial V^h} \alpha^h}{\lambda} + \sum_{j=1}^J \tau_j \frac{\partial x_j^h(p, m^h)}{\partial m}$$

where  $\beta^h$  is the so called "net social marginal utility of income for agent  $h$ "

$$\bar{\beta} = \sum_{h=1}^H \frac{\beta^h}{H}$$

Now, denote by  $\theta_k$  the empirical covariance between  $\beta^h$  and  $h$ 's consumption of good  $k$ :

$$\theta_k = cov\left(\frac{\beta^h}{\bar{\beta}}, \frac{x_k^h}{\bar{X}_k}\right) = \frac{1}{H} \sum_{h=1}^H \left(\frac{\beta^h}{\bar{\beta}} - 1\right) \left(\frac{x_k^h}{\bar{X}_k} - 1\right)$$

Positive  $\theta_k$  means that good  $k$  is mostly consumed by agents with higher  $\beta^h$  (typically, poor agents).

*Lemma 1.* The optimal linear tax policy from the social planner, which maximizes the expected utility of the two agents, satisfies the following equations:

$$-\frac{\sum_{j=1}^J \tau_j p_j \left( \sum_{h=1}^H \frac{\partial x_k^h}{\partial p_j} \right)}{X_k} = 1 - \bar{\beta} - \bar{\beta} \theta_k$$

Where  $J = 2$  is the number of assets, and  $H = 2$  is the number of agents.

*Proof.* Appendix A.1 □

This is almost identical to the so-called “a many-person Ramsey tax rule” (Diamond, 1975).

Interpretation of the formula:

The left-hand side is called the “discouragement index for good  $k$  under a tax system”, and is a measure of a percentage reduction in demand on good  $k$  as a consequence of taxation.

The right-hand side is called the “redistributive factor of good  $k$ ”. In a representative agent economy  $\theta_k = 0$ .

Rewriting the formulas in terms of elasticities of Hicksian demand ( $\hat{\epsilon}_{i,j}^h = \frac{p_j}{x_i} \frac{dx_k^h}{dp_j}$ ):

$$\sum_{j=1}^J \left( \frac{\tau_j}{1 + \tau_j} \sum_{h=1}^H \hat{\epsilon}_{k,j}^h \right) = 1 - \bar{\beta} - \bar{\beta} \theta_k \quad (2.1)$$

### 2.3.1 The case with no labor income

Now, let’s consider a case where there is an inequality in the initial wealth, but there is no exogenous labor income in the second period.

*Lemma 2.* With no exogenous labor income in the second period, both agents invest the same share of their wealth in the risky asset, and consequently  $\theta_r = \theta_s = 0$ .

*Proof.* Marshalian demand for the risky asset is:

$$x_r^h = \frac{\frac{m^h}{p_s} (A - 1)}{r_g - r_b A + \frac{p_r}{p_s} (A - 1)}$$

where

$$A = \left( \frac{r_g - \frac{p_r}{p_s}}{\frac{p_r}{p_s} - r_b} \right)^{\frac{1}{\alpha}} > 1$$

Marshallian demand for the safe asset is:

$$x_s^h = \frac{\frac{m^h}{p_r}(Ar_b - r_g)}{1 - A + \frac{p_s}{p_r}(Ar_b - r_g)}$$

Both are linear in  $m^h$ .

Computing the income elasticities:

$$\epsilon_{r,m} = \frac{m^h}{x_r^h} \frac{dx_r^h}{dm^h} = 1 = \epsilon_{s,m}$$

□

This is the standard property of CRRA preferences.

**Proposition 1.** *If there is no exogenous labor income in the second period, the optimal tax schedule taxes both assets at the same rate.*

*Proof.* Appendix A.2

□

In this case, the social planner cannot use differential asset taxation to shift the tax burden from one agent to another because all agents will spend an equal share of their wealth on investing in the two assets. In this particular setup, the mentioned investment pattern is a consequence of constant relative risk aversion preferences. In addition, since there are no externalities, in the absence of taxes, the agents choose the optimal investment allocation. Therefore, the social planner should decrease the investment in assets proportionally. This is the standard equal taxation result in the Ramsey taxation literature. Finally, one can see the result in the following way: since the initial wealth is exogenous, the social planner can mimic the (non-distortive) lump-sum tax by taxing all the assets at the same rate.

### 2.3.2 The case with exogenous labor income in the second period

The feature that both rich and poor people invest the same share in risky assets is counter-factual. A stylized fact is that the rich invest disproportionately more in risky assets. To generate this pattern, the exogenous labor income in the second period is assumed;  $L_g > 0$  in the good period and  $L_b = 0$  in the bad period (where  $L_b$  is set to 0 for simplicity). Alternatively, instead of adding labor, it is possible to assume a subsistence constraint also known as The Stone-Geary utility function, which would be equivalent to setting  $L_g = L_b < 0$ . This means that the agents have exogenous labor income, which is perfectly positively correlated with the returns to risky assets. This feature breaks the homotheticity.<sup>4</sup>

Marshallian demands are:<sup>5</sup>

$$x_r^h = \frac{\frac{m^h}{p_s} - L_g(A - 1)}{r_g - r_b A + \frac{p_r}{p_s}(A - 1)}$$

$$x_s^h = \frac{\frac{m^h}{p_r}(Br_b - r_g) - L_g}{1 - B + \frac{p_s}{p_r}(Br_b - r_g)} = \frac{\frac{m^h}{p_s} - p_r m^h(A - 1) + \frac{p_r}{p_s} L_g}{r_g - r_b A + \frac{p_r}{p_s}(A - 1)}$$

The introduction of labor income  $L_g$  makes the demand for the safe asset less sensitive to changes in income (prices). The reason is that since the agent wants to smooth the consumption across the two states, and in the good state the agent receives  $L_g$ , independently of the portfolio choice, the agent primarily wants to buy the asset that has a higher return in the bad state: the safe asset.

**Proposition 2.** *In the case with exogenous labor income in the second period, if there is no initial inequality:  $\forall h : m^h = m$ , it is optimal for the social planner to tax both assets at the same rate.*<sup>6</sup>

<sup>4</sup>In the quantitative model, both features: subsistence constraint and risky labor income are present, which reinforces the pattern.

<sup>5</sup> $B = \left(\frac{1 - \frac{p_s}{p_r} r_g}{\frac{p_s}{p_r} r_b - 1}\right)^{\frac{1}{\alpha}}$

<sup>6</sup>If there are more than two assets, or if the labor supply is endogenous, the government



*Proof.* Appendix A.3. □

Agents choose the first best allocation in the absence of taxes. Furthermore, since there is no initial inequality, they will choose exactly the same portfolio, and the social planner does not have any incentive or the tools to shift the tax burden between different agents. A social planner can mimic the lump-sum tax (which is non-distortive) by having equal taxes on both assets. However, if the wealth in the first period was endogenous (for example, if the agents were choosing their labor income), or there were additional investments on which the agents would spend their initial wealth, it would be optimal to tax the safe asset more. This is because the income elasticity of demand for the risky asset is higher than the income elasticity of demand for the safe asset. (see Appendices A.3. and A 5.)

Let us now assume that two agents differ in their initial unearned wealth  $m^h$ . In this case, we have an effect from Proposition 2, with an additional, redistributive effect. Since with the introduction of risky labor income the share of wealth invested in risky assets is increasing in wealth, the rich agent will invest disproportionately more in the risky asset compared to the poor agent. Therefore, the government can tax the poor agent less by taxing the safe asset by a lower rate than the risky asset.

**Proposition 3.** *In the case with exogenous labor income in the second period, a mean preserving spread in  $M$  increases the ratio of optimal taxes  $\frac{\tau_r}{\tau_s}$ .*

*Proof.* Appendix A.4. □

This model gives the intuition as to why the government may want to tax the risky assets (for example equity) at a higher rate in order to shift the tax burden from the poor to the relatively rich households. The result that the optimal tax rate  $\frac{\tau_r}{\tau_s}$  is increasing in the initial inequality extends to the, more realistic, case in which the government taxes the returns on assets. The example is considered in Appendix B.

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can find it optimal to tax the safe asset at a higher rate.

This result may seem contradictory to the intuition of the “production efficiency theorem” from Diamond and Mirrlees (1971), which states that the intermediary production goods should not be taxed. However, Diamond and Mirrlees (1971) assume that the government can tax the final goods at different rates, while in the presented model this is not possible because the two assets (intermediary goods) are used to produce the same final good. Therefore, the government cannot distort the after-tax prices of the final goods for the redistribution purposes, and it is therefore forced to use distortionary “intermediate goods” prices. Furthermore, the outcome shows that the result (taxing the safe industry more if the household has decreasing relative risk aversion) from Atkinson and Stiglitz (1972) can be overturned in the presence of heterogeneous agents.

## 2.4 Quantitative Model

I construct the model based on (Algan et al., 2009), and in the tradition of Krusell and Smith (1997). The model consists of a continuum of heterogeneous agents facing aggregate risk, uninsurable idiosyncratic labor risk and a borrowing constraint, and who save in two assets: risky equity and safe bonds. Unlike the above-mentioned models, the model parsimoniously captures the life cycle of the households, in the fashion of Castaneda et al. (2003), in which working-age agents face the retirement shock and retired households face the risk of dying. In this model, I introduce the fiscal policy, in which the government uses capital income taxes to finance exogenous government spending  $G$ , and labor taxes to finance unemployment benefits, social security (benefits to the retired households), and to balance the budget. The tax rate on the income from bonds (safe asset) is exogenously set to be lower than the tax rate on equity (risky asset) by  $C^f$ . To evaluate the effects of differential taxation, the coefficient  $C^f$  is varied, and the implications of this variation on the economy are studied. Unlike in the analytical, two-period model, the general equilibrium effects will be present. This is important, firstly because the analysis would be incomplete without considering the (potentially important) welfare implication of such effects. Moreover, the portfolio choice problem is notoriously sensitive, and changing the interest rate

(or taxes) in the partial equilibrium even slightly, can cause drastic changes to the optimal portfolio choice. In the general equilibrium model, such responses are mitigated by the price adjustments stemming from the general equilibrium effects.

### 2.4.1 Production technology

In each period  $t$ , the representative firm uses aggregate capital  $K_t$ , and aggregate labor  $L_t$ , to produce  $y$  units of final good with the aggregate technology  $y_t = f(z_t, K_t, L_t)$ , where  $z_t$  is an aggregate productivity shock. I assume that  $z_t$  can take only two values, and it follows a stationary Markov process with transition function  $\Pi_t(z, z') = Pr(z_{t+1} = z' | z_t = z)$ . The production function is continuously differentiable, strictly increasing, strictly concave and homogeneous of degree one in  $K$  and  $L$ . Capital depreciates at the stochastic rate  $\delta_t \in (0, 1)$  and it accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1 - \delta_t)K_t$$

where  $I_t$  is aggregate investment. The particular aggregate production technology is:

$$Y_t = z_t A K_t^\gamma L_t^{1-\gamma}$$

### 2.4.2 Preferences

Households are indexed by  $i$  and they have identical, recursive preferences, for the retired agents:

$$V_{R,i,t} = \left[ c_t^{1-\rho} + v\beta \left[ E_t V_{R,i,t+1}^{1-\alpha} \right]^{\frac{1-\rho}{1-\alpha}} \right]^{\frac{1}{1-\rho}}$$

where  $V_{R,i,t}$  is the recursively defined value function of a retired household  $i$ , at time period  $t$ .

Working-age agents maximize:

$$V_{W,i,t} = \left\{ c_t^{1-\rho} + \beta \left[ (1 - \theta) E_t V_{W,i,t+1}^{1-\alpha} + \theta E_t V_{R,i,t+1}^{1-\alpha} \right]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

### 2.4.3 Life cycle structure

In each period, working-age households have a chance of retiring  $\theta$ , and retired households have a chance of dying  $v$ , similarly as in Castaneda et al. (2003) and Krueger et al. (2016). Therefore, the share of working age households in the total population is:

$$\Pi_W = \frac{1 - v}{(1 - \theta) + (1 - v)}$$

and the share of the retired households in the total population is:

$$\Pi_R = \frac{1 - \theta}{(1 - \theta) + (1 - v)}$$

The retired households who die in period  $t$  are replaced by new-born agents who start at a working age without any assets. For simplicity, the retired households have perfect annuity markets, which make their returns larger by a fraction of  $\frac{1}{v}$ , as in Krueger et al. (2016).

### 2.4.4 Idiosyncratic uncertainty

In each period, working-age households are subject to an idiosyncratic labor income risk that can be decomposed into two parts. The first part is the employment probability that depends on aggregate risk and is denoted by  $e_t \in (0, 1)$ .  $e = 1$  denotes that the agent is employed, and  $e = 0$  that the agent is unemployed. Conditional on  $z_t, z_{t+1}$  I assume that the period  $t + 1$  realization of the employment shock follows the Markov process.

$$\Pi_e(z, z', e, e') = Pr(e_{t+1} = e' | e_t = e, z_t = z, z_{t+1} = z')$$

This labor risk structure allows idiosyncratic shocks to be correlated with the aggregate productivity shocks, which is consistent with the data and generates the portfolio choice profile such that the share of wealth invested in risky assets is increasing in wealth. A condition imposed on the transition matrix and the law of large numbers implies that the aggregate employment is only the function of the aggregate productivity shock.

In case that  $e = 1$  and the agent is employed, one can assume that the agent

is endowed with  $l_t \in L \equiv \{l_1, l_2, l_3, \dots, l_m\}$  efficiency labor units, which she can supply to the firm. Labor efficiency is independent of the aggregate productivity shock, and is governed by the stationary Markov process with transition function  $\Pi_l(l, l') = Pr(l_{t+1} = l' | l_t = l)$ . If the agent is unemployed, (s)he receives an exogenous amount of final good  $g_u$ , which can be interpreted as home production or social insurance.

### 2.4.5 The representative firm

As in Algan et al. (2009), firm leverage in my model is given exogenously. The leverage of the firm is determined exogenously by the parameter  $\lambda$ . The Modigliani-Miller theorem (1958, 1963) is rendered invalid by the fact that some of the agents are borrowing constrained. Therefore, the leverage of the firm has macroeconomic relevance.

In the economy, the representative firm can finance its investment with two types of contracts. The first is a one-period risk-free bond that promises to pay a fixed return to the owner. The second is risky equity that entitles the owner to claim the residual profits of the firm after the firm pays out wages and debt from the previous period. Both of these assets are freely traded in competitive financial markets. By construction, there is no default in the equilibrium.

The return on the bond  $r_{t+1}^b$  is determined by the clearing of the bond market:

$$\int g^{b,j} d\mu = \lambda K'$$

where  $g^{b,j}$  are the individual policy functions for bonds.

Next, the return on the risky equity depends on the realizations of the aggregate shocks and is given by the equation:

$$(1 + r_{t+1}^s) = \frac{f(z_{t+1}, K_{t+1}, L_{t+1}) - f_L(z_{t+1}, K_{t+1}, L_{t+1})L_{t+1} - \lambda K_{t+1}(1 + r_{t+1}^b) + (1 - \delta_{t+1})K_{t+1}}{(1 - \lambda)K_{t+1}}$$

An important caveat in having heterogeneous households that own the firm is

that they do not necessarily have the same stochastic discount factor  $m_{t+1}^j$ , and therefore the definition of the objective function of the firm is not straightforward. I follow Algan et al. (2009), who assume that the firm is maximizing the welfare of the agents who have interior portfolio choice, and consequently the firm has the same stochastic discount factor  $m_{t+1}$ , as the agents with the interior portfolio choice.

**Proposition 4** *In equilibrium, the aggregate capital stock  $K_{t+1}$  is equal and ex-dividend firm value  $V_t$  are equal to the present discounted value of the firm's net cash flows:*

$$K_{t+1} = V_t = E_t \left\{ \sum_{j=1}^{\infty} m_{t,t+j}^f [f_K(z_{t+j}, K_{t+j}, L_{t+j})K_{t+j} - I_{t+j}] \right\}$$

where  $m_{t,t+j}^f$  is the stochastic discount factor of the firm.

*Proof.* See Algan et al. (2009) □

This proposition is used to eliminate the capital Euler equation from the equilibrium conditions, and instead use  $V_t = K_{t+1}$ .

## 2.4.6 Financial markets

As stated earlier, households can save in two assets; risky equity and safe bonds (firm debt). There are borrowing constraints for both assets, so the lowest amounts of equity and debt that households can hold in period  $t$  are respectively:  $\kappa^s$  and  $\kappa^b$ . Markets are assumed to be incomplete, in the sense that there are no markets for the assets contingent on the realization of individual idiosyncratic shocks.

## 2.4.7 Government

The government has a twofold function in the model. First, in each period  $t$ , the government has to collect enough tax revenue to finance  $G_t = \eta Y_t$ , which

is equal to a fraction  $\eta$  of the overall production of the economy in the period  $t$ . The government balances the budget by a labor tax  $\tau_t^{lb}$ , and capital income taxes:  $\tau_t^s, \tau_t^b$ , which are the taxes on income from shares and bonds, respectively. An important simplifying assumption is that the government is forced to have a tax rate on bonds as a set difference between a tax rate on equity return and a constant  $C^f$ :  $\tau_t^b = \tau_t^s - C^f$ . In each period  $t$ , tax rates  $\tau_{t+1}^s$  and  $\tau_{t+1}^b$  have to be known. Therefore, to accommodate the variation in tax revenue collected by the capital tax rates (which depend on the realization of the aggregate shocks  $z_{t+1}$  and  $\delta_{t+1}$ , the government balances the budget with a special tax on labor:  $\tau_{t+1}^{lb}$ . A more realistic setting would have been to allow the government to run a budget deficit, but this is not computationally feasible since it would require the introduction of the additional state variable; the government debt (Gomes et al., 2013). However, running a balanced budget every period by adjusting a labor tax  $\tau_t^{lb}$ , should not significantly influence the results since the labor supply in this model is exogenous (therefore, this tax is not distortive, but is redistributive).

The government budget constraint in period  $t$  is:

$$S_t R_t^s \tau_t^s + B_t R_t^b \tau_t^b + w_t L_t \tau_t^{lb} = G_t$$

or equivalently

$$((1 - \lambda)K_t r_t^s + \lambda K_t r_t^b) \tau_t^s - (\lambda K_t r_t^b) C^f + w_t L_t \tau_t^{lb} = \eta Y_t$$

Concerning capital income taxes, the government follows a simple fiscal rule in period  $t$ , such that the expected revenue from capital income taxes in period  $t + 1$  is equal to the expected wasteful government expenditure:

$$E_t \left[ ((1 - \lambda)K_{t+1} r_{t+1}^s + \lambda K_{t+1} r_{t+1}^b) \tau_{t+1}^s - (\lambda K_{t+1} r_{t+1}^b) C^f + w_{t+1} L_{t+1} \tau_{t+1}^{lb} \right] = E_t [\eta Y_{t+1}]$$

As a rule, after the realization of aggregate shocks in period  $t + 1$ , revenues

from capital income taxes will not be equal to  $\eta Y_{t+1}$ , and the difference will be collected (or returned to the taxpayers as a tax break) with the special labor income tax  $\tau^{lb}$ . Therefore, contingent on the realization of the aggregate shocks in the period  $t + 1$ ,  $\tau_{t+1}^{lb}$  is known in the period  $t$ .

Second, the government runs two social programs: social security (retirement benefits), and unemployment insurance, and are modeled as in Krueger et al. (2016). Both are financed by separate labor taxes. Social security is financed with a constant labor tax rate:  $\tau^{lss}$ , and the revenues  $T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$  are equally distributed in period  $t$  to all retired households, irrespective of their past contributions. Unemployment benefits are financed with a labor tax rate  $\tau_t^u$ . The amount of the unemployment benefits  $g_{u,t}$  is determined by a constant  $\phi$ , which represents the fraction of the average labor earnings that are paid to the unemployed agent. Therefore,  $g_{u,t} = \phi w_t L_t$ .

To satisfy the budget constraint the government has to tax labor with the tax rate:

$$\tau_t^u = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\phi}}$$

where  $\Pi_u$  is the share of the unemployed people in the total working-age population.

## 2.4.8 Household problem

Retired household  $i$  maximizes its lifetime utility subject to the following constraints:

$$\begin{aligned} c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} &\leq \omega_{i,t} \\ \omega_{i,t+1} &= T_{ss,t+1} + \left[ (1 + r_{t+1}^s (1 - \tau_{t+1}^s)) s_{i,t+1} + (1 + r_{t+1}^b (1 - \tau_{t+1}^b)) b_{i,t+1} \right] \frac{1}{v} \\ (c_{i,t}, b_{i,t+1}, s_{i,t+1}) &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

Working age household  $i$  maximizes its expected lifetime utility subject to the constraints below:



$$\begin{aligned}
& c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} \leq \omega_{i,t} \\
\omega_{i,t+1} = & \begin{cases} w_{t+1} l_{i,t+1} (1 - \tau_{t+1}^l) + (1 + r_{t+1}^s (1 - \tau_{t+1}^s)) s_{i,t+1} + (1 + r_{t+1}^b (1 - \tau_{t+1}^b)) b_{i,t+1} & \text{if } e = 1 \\ g_{u,t+1} + (1 + r_{t+1}^s (1 - \tau_{t+1}^s)) s_{i,t+1} + (1 + r_{t+1}^b (1 - \tau_{t+1}^b)) b_{i,t+1} & \text{if } e = 0 \end{cases} \\
& (c_{i,t}, b_{i,t+1}, s_{i,t+1}) \geq (0, \kappa^b, \kappa^s)
\end{aligned}$$

First order conditions imply that the following equations are satisfied:

$$\begin{aligned}
1 & \geq E_t \{ m_{i,t,t+1}^j (1 + r_{t+1}^s (1 - \tau_{t+1}^s)) \} \quad \text{and} \quad s_{i,t+1}^j \geq \kappa^s \\
1 & \geq E_t \{ m_{i,t,t+1}^j (1 + r_{t+1}^b (1 - \tau_{t+1}^b)) \} \quad \text{and} \quad b_{i,t+1}^j \geq \kappa^b
\end{aligned}$$

where pricing kernel is  $m_{i,t,t+1}^j = \beta \left[ \frac{c_{i,t+1}^j}{c_{i,t}^j} \right]^{\frac{(1-\alpha)(\rho-1)}{\rho}} (R_{t+1}^a)^{\frac{1-\alpha}{\rho}}$  for household  $i$  between periods  $t$  and  $t+1$ , where  $R_{t+1}^a$  is the gross post tax return on asset  $a$ , and  $j$  denotes if the household is of working age  $W$ , or retired  $R$ . Furthermore, define  $q_{t,t+1}^s$  and  $q_{t,t+1}^b$  as the stochastic marginal rates of substitution of households that are unconstrained at period  $t$  in their choices of shares and bonds, respectively (they have an interior solution to their portfolio choice problem).

Asset pricing equations are:

$$\begin{aligned}
1 & = E_t \{ q_{t,t+1}^s (1 + r_{t+1}^s (1 - \tau_{t+1}^s)) \} \\
1 & = E_t \{ q_{t,t+1}^b (1 + r_{t+1}^b (1 - \tau_{t+1}^b)) \}
\end{aligned}$$

or more concisely:

$$1 = E_t \{ q_{t,t+1}^a R_{t+1}^a \}$$

## 2.4.9 Recursive household problem

It is possible to define the household's problem recursively: Retired households:

$$v_R(\omega; z, \mu, \delta) = \max_{c, b', s'} \left\{ c^{1-\rho} + v \beta E_{z', \mu', \delta' | z, \mu, \delta} [v_R(\omega'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} = \omega$$

$$\omega' = T'_{ss} + [s'(1 + r'^s(1 - \tau'^s)) + b'(1 + r'^b(1 - \tau'^b))] \frac{1}{v}$$

Working age households:

$$v_W(\omega, e, l; z, \mu, \delta) =$$

$$\max_{c, b', s'} \left\{ c^{1-\rho} + \beta E_{e', l', z', \mu', \delta' | e, l, z, \mu, \delta} [(1 - \theta)v_W(\omega', e', l'; z', \mu', \delta')^{1-\alpha} + \theta v_R(\omega', e', l'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} = \omega$$

$$\omega' = \begin{cases} w'l'(1 - \tau'^l) + s'(1 + r'^s(1 - \tau'^s)) + b'(1 + r'^b(1 - \tau'^b)) & \text{if } e = 1 \\ g'_u + s'(1 + r'^s(1 - \tau'^s)) + b'(1 + r'^b(1 - \tau'^b)) & \text{if } e = 0 \end{cases}$$

$$(c, b', s') \geq (0, \kappa^b, \kappa^s)$$

where  $\omega$  is the vector of individual wealth of all agents,  $\mu$  is the probability measure generated by set  $\Omega x E x L$ ,  $\mu' = \Gamma(\mu, z, z', d, d')$  is a transition function and  $'$  denotes the next period.

## 2.4.10 General equilibrium

The economy-wide state is described by  $(\omega, e; z, \mu, d)$ . Therefore, the individual household policy functions are:  $c^j = g^{c,j}(\omega, e, l; z, \mu, d)$ ,  $b'^j = g^{b,j}(\omega, e, l; z, \mu, d)$  and  $s'^j = g^{s,j}(\omega, e, l; z, \mu, d)$ , and laws of motion for the aggregate capital is  $K' = g^K(\omega, e, l; z, \mu, d)$ .

A recursive competitive equilibrium is defined by the set of individual policy and value functions  $\{v_R, g^{c,R}, g^{s,R}, g^{b,R}, v_W, g^{c,W}, g^{s,W}, g^{b,W}\}$ , laws of motion for the aggregate capital  $g^K$ , a set of pricing functions  $\{w, R^b, R^s\}$ , government policies in period  $t$ :  $\{\tau^{lb}, \tau^u, \tau^s, \tau^b\}$  and tax rates contingent on the aggregate states in period  $t + 1$ :  $\{\tau'^{lb}, \tau'^u, \tau'^s, \tau'^b\}$ , and a forecasting equation  $g^L$ , such that:

1. The laws of motion for the aggregate capital  $g^K$  and the aggregate "wage function"  $w$ , given the taxes satisfy the optimality conditions of the firm.

2. Given  $\{w, R^b, R^s\}$ , the laws of motion  $\Gamma$ , the exogenous transition matrices  $\{\Pi_z, Pi_e, Pi_l\}$ , the forecasting equation  $g^L$ , the laws of motion for the aggregate capital  $g^K$ , and the tax rates, the policy functions  $\{g^{c,j}, g^{b,j}, g^{s,j}\}$  solve the household problem.

3. Labor, shares and the bond markets clear:

- $L = \int e l d\mu$
- $\int g^{s,j}(\omega, e, l; z, \mu, \delta) d\mu = (1 - \lambda)K'$
- $\int g^{b,j}(\omega, e, l; z, \mu, \delta) d\mu = \lambda K'$

4. The laws of motion  $\Gamma(\mu, z, z', \delta, \delta')$  for  $\mu$  is generated by the optimal policy functions  $\{g^c, g^b, g^s\}$ , the laws of motion for aggregate capital  $g^K$  and by the transition matrices for the shocks. Additionally, the forecasting equation for aggregate labor is consistent with the labor market clearing:  $g^L(z', \delta') = \int e l d\mu$

5. Government budget constraints are satisfied:

$$E_t \left[ ((1 - \lambda)K_{t+1}r_{t+1}^s + \lambda K_{t+1}r_{t+1}^b) \tau_{t+1}^s - (\lambda K_t r_t^b) C^f + w_{t+1} L_{t+1} \tau_{t+1}^{lb} \right] = E_t [\eta Y_{t+1}]$$

$$g_{u,t} = \phi w_t L_t$$

$$T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$$

$$\tau_t^u = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\phi}}$$

### 2.4.11 Parametrization and calibration

The model is calibrated to a quarterly frequency. The goal of the calibration is to match the important patterns of wealth inequality and portfolio choice in the US economy. Moreover, it is calibrated to match the capital-output ratio excluding housing, following Algan et al. (2009). The capital does not include housing, even

though housing is a substantial part of assets owned by poor households (Kuhn and Rios-Rull, 2016). Housing is not included for the reason of simplicity and the fact that not all of the value of the housing should be considered as savings, because housing has a use-value of providing accommodation and can be partly considered consumption. However, excluding (mostly risky, since it is financed by mortgages) housing wealth, can cause the financial capital portfolio allocation to be misspecified because it implies excessively low background risk. Parameter  $\eta$ , which governs how much asset tax revenue the government has to collect in each period as the share of the output. The value is set as an approximation of how much the US federal government collected in 2016 from taxes from savings, as a percentage of GDP.<sup>7</sup> Overall, there are four possible aggregate states of the economy, since both TFP and capital depreciation shocks can take two possible values.

### Preferences, firm and households constraints

**Table 1: Internally-calibrated parameters**

Parameter	Symbol	Value	Target
Discount factor	$\beta$	0.877	Capital-Output ratio: 7
Subsistence constraint	$\gamma$	0.036	Portfolio choice pattern
Quarterly stock market participation costs	$\phi$	0.002	Share of households with no equity 46%

**Table 2: Externally-calibrated parameters**

Parameter	Symbol	Value	Target
Intertemporal elasticity of substitution	$\frac{1}{\rho}$	0.5	Capital-Output ratio: 7
Expected depreciation rate	$E(\delta)$	0.033	Equity premium 1 – 2%
Chance of not retiring	$\theta$	.994	Average working duration 40 years
Chance of not dying	$v$	0.983	Average retirement duration 15 years
Tax advantage of debt	$\tau^s$	0.3	Hennessy and Whited (2005)
Capital share	$\Delta$	0.4	Algan et al. (2009)

<sup>7</sup>The revenues taken into account are from taxes of corporate income and gains, and one third (capital share) of personal income, profits, and gains.

**Table 3: Parameters to generate sizable equity premium**

Parameter	Symbol	Value
Risk aversion	$\alpha$	10
Variance of depreciation rate	$\sigma^2(\delta)$	0.0001

**Table 4: Other parameters**

Parameter	Symbol	Value
Social security tax	$\tau^{lss}$	0.06
Unemployment replacement rate	$\eta$	0.042
Borrowing constraint: bonds	$\kappa^b$	0.00
Borrowing constraint: stocks	$\kappa^s$	0.0

**Idiosyncratic labor income shocks**

For the idiosyncratic labor income shocks transition matrix, I use the same values as Pijoan-Mas (2007) and Algan et al. (2009).

$$\Pi_l = \begin{bmatrix} 0.9850 & 0.0100 & 0.0050 \\ 0.0025 & 0.9850 & 0.0125 \\ 0.0050 & 0.0100 & 0.9850 \end{bmatrix}$$

For the individual labor productivity levels, the following values are used:  $l \in \{44, 8, 1.5\}$  (they differ slightly from the ones used by Algan et al. (2009)). This type of modeling the labor productivity process allows the generation of a realistic size of earnings and wealth inequality, while keeping the possible number of states relatively low.

The average unemployment duration during booms is set to 1.6 quarters, while for the recession, it is set to 2.8 quarters.

**Table 5: Quarterly statistics**

	$K/Y$	Wealth GINI	$R^b$	$E(R_{pt}^s) - R_{pt}^b$
Data	7.01	0.78	0.23	1.0 - 2.0
Economy	7.03	0.59	1.83	0.09

The data figures in Table 5. are taken from Algan et al. (2009). The reported returns from the model are post-tax.

**Table 6: Wealth distribution: Owned share of overall wealth %**

Quintile	Data	Model
Q1	-0.2	1.4
Q2	1.2	3.67
Q3	4.6	11.6
Q4	11.9	21.0
Q5	82.5	62.4
T1 %	33.5	4.7

The data in Table 5. is taken from Krueger et al. (2016), who use the Panel Study of Income Dynamics (PSID). The model replicates the bottom tail of the wealth distribution fairly closely, but it does not generate a thick enough top tail of the distribution, which is common in similar types of models (without discount rate heterogeneity and without shocks to the asset returns).

Figure 1)

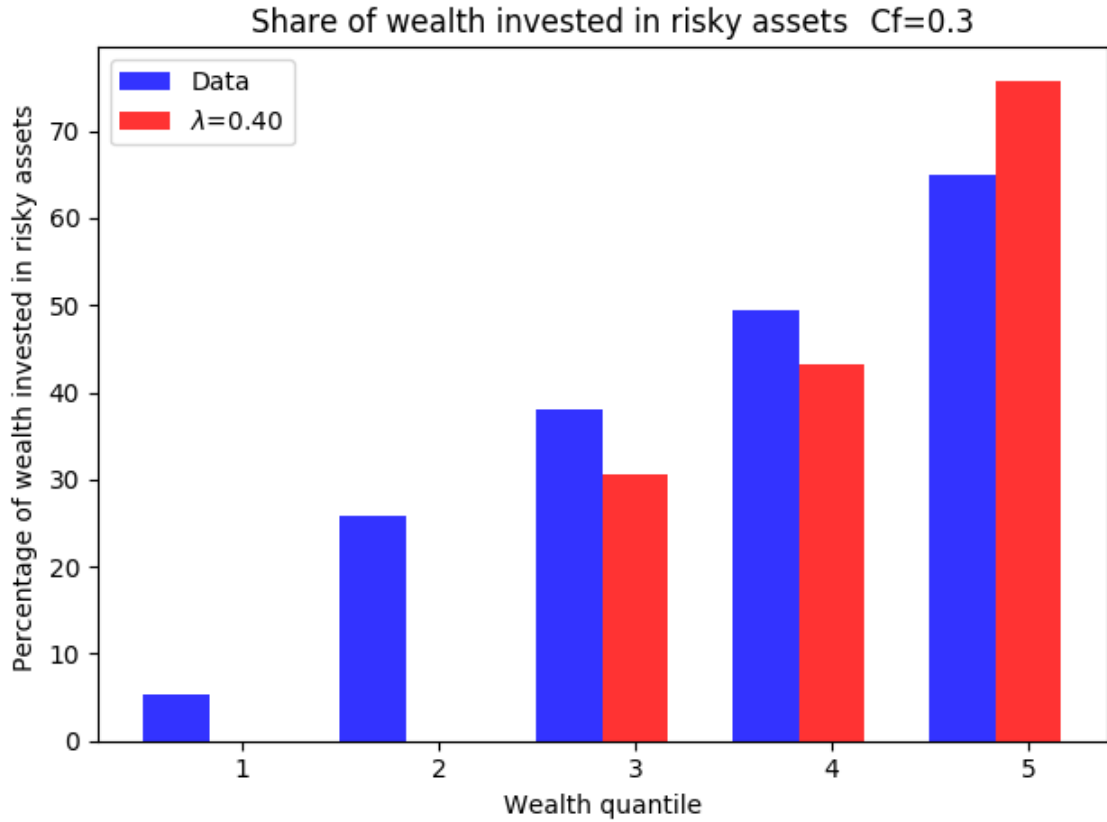


Figure 1 reports the data from Chang et al. (2018), who use SCF. The model matches the portfolio choices along the wealth distribution reasonably well. The only major deviations seem to be the not risky enough portfolio in the second quintile of the distribution, and excessively risky portfolio of the richest quintile.

#### 2.4.12 Solution method

It is well known that solving these types of models is difficult since the state variables include the cross-sectional distribution of agents over the wealth for each (un)employment status. When the model features aggregate risk, the cross-sectional distribution of agents over the wealth is a time-varying infinite-dimensional object. In this model, I follow the approach of Krusell and Smith (1997, 1998), who reduce the state-space to include only a finite set of cross-sectional distribution moments. In the simulation of the model, I use a non-stochastic simulation routine described by Young (2010) to keep track of the household wealth distribu-

tion. The procedure is the alternative to the Monte Carlo simulation procedure, and is used to speed up the convergence of wealth distribution while using the fine grid for household wealth.

The approximate equilibrium laws of motion for capital and bond interest rate are the following:

$$\ln K' = a_0(z, d) + a_1(z, d)\ln K$$

and

$$\ln P^e = b_0(z, d) + b_1(z, d)\ln K'$$

While solving the individual household problem, given the aggregate laws of motion, I use the “endogenous grid method” proposed by Carroll (2006), augmented to allow for two choice variables by the agents. This method reduces the computational time, as it avoids the root-finding process. In the benchmark model, the  $R^2$  of the laws of motion are 99.8% for capital and 99.9% for equity premium.

The numerical implementation of the solution algorithm is discussed in more detail in Appendix D.

## 2.5 Exercises and results

The main computational exercises are in changing the ratio of the taxes of safe (bonds) and risky (stocks) assets  $C^f = \tau_s - \tau_b$ .  $C^f$  can be thought of as a measure for the debt tax shield. This exercise keeps the government revenue from the capital taxes the same, but it changes the composition of the raised revenues (between revenues from stocks and bonds). The benchmark value for  $C^f$  is set at  $C^f = 0.3$ , which is taken from Hennessy and Whited (2005). The welfare is measured from the perspective of the utilitarian social planner.



### 2.5.1 Exercise 1: Eliminating the tax wedge

In the first exercise, the tax wedge (or debt tax shield) is eliminated, which means that the  $C^f$  is set to 0. All other parameters are held exactly the same.

**Table 7: Results: Eliminating the tax wedge**

Unconditional moments	$C^f = 0.0$ Economy	$C^f = 0.3$ Economy
Post-tax $E(r^{sP} - r^{bP})$	0.448%	0.401%
Pretax $E(r^b)$	1.93%	1.55%
Pretax $E(r^s)$	2.53%	2.80%
Post-tax $E(r^{bP})$	1.55%	1.59%
Post-tax $E(r^{sP})$	2.00%	1.99%
Wealth GINI	0.5833	0.5831
$E(K)$	5.766	5.747
Share of wealth by quintile (%)		
1st	1.85%	1.80%
2st	3.19%	3.26%
3st	11.40%	11.46%
4st	20.91%	20.84%
5st	62.64%	62.62%

**Table 8:  $C_f = 0$ : Welfare gains from abolishing the beneficial tax treatment of debt**

	Wealth change				
Quintile	1	2	3	4	5
% change	-0.71	+0.00	-0.18	-0.25	-0.32

The elimination of the debt tax shield is found to be welfare reducing and is equivalent to a 0.3% permanent decrease in consumption (including the transition path of the economy). The decline in welfare is mainly due to the decreased return in the safe asset, in which the poor households (households with a high marginal utility of wealth) mostly save. On the other hand, the households that gain are the wealthy households (which have a lower marginal utility of wealth), as they save mostly in the risky equity. Therefore, the policy that taxes the risky asset more effectively functions as insurance. It also serves as insurance for the equity

owners, since it reduces the variance of the after-tax returns. This is useful for households, as the income from labor and asset returns are correlated. Therefore, the households that would benefit the most from this are households that have significant income from both labor and assets. Furthermore, wealth inequality increases slightly, as the poorer households save less in the economy without the debt tax shield. The welfare gains by quintile can be seen in Table 8.

### 2.5.2 Exercise 2: Finding the optimal tax wedge

In this section, the optimal  $C^f$  for the long-run is calculated. The optimal level of debt tax shield is found to be  $C^f = 0.60$ . This means that in the optimum, the returns on the equity are heavily taxed, and the returns to bonds are heavily subsidized. As discussed earlier, this helps all, and especially poor households, to insure themselves against the bad aggregate shocks. The reason subsidizing a return to safe assets is particularly useful for poor households, is that it enables them to receive high returns on savings without exposing themselves to the uncertainty of equity returns and without paying equity market participation costs. The share of households participating in the stock markets drops slightly from 41.75% to 40.23%, when the economy moves from  $C^f = 0.30$  to  $C^f = 0.60$ . The reported welfare changes are in terms on % of consumption equivalent variation. The aggregate capital is decreasing when the tax wedge  $C^f$  increases. This is partly due to the fact that the savings in a risky asset are more elastic to the change in the net return.<sup>8</sup>

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<sup>8</sup>The reasons for this are discussed in the two-period model section.

**Table 9: Results: Optimal long-run  $C^f$** 

Moments	$C^f = 0.0$	$C^f = 0.3$	$C^f = 0.60$	$C^f = 0.90$
Welfare change from the benchmark model: $C_f = 0.3$ <sup>9</sup>	-0.028%	0.0%	+0.105%	-0.001%
Wealth GINI	0.5833	0.5831	0.5826	0.5829
$E(K)$	5.766	5.749	5.740	5.726
Pretax $E(r^b)$	1.93	1.55	1.26	1.06
Pretax $E(r^s)$	2.53	2.80	2.99	3.15
Post-tax $E(r^{bP})$	1.55	1.59	1.61	1.63
Post-tax $E(r^{sP})$	2.00	1.99	1.98	1.98
Post-tax $E(r^{sP} - r^{bP})$	0.448	0.401	0.371	0.354

### 2.5.3 Exercise 3: Changing leverage with the changing tax wedge

An important caveat of the performed analysis is that the firm leverage was taken as exogenous. However, it is intuitive to expect that the firms would adjust their leverage after a change in taxes to optimize their financing policy in an attempt to avoid excessive taxation. To control for the possible change in firm financing policy, this exercise performs the change in the tax wedge, exactly as in Exercise 1, but at the same time, exogenously changes the leverage.

The effect of changing the leverage does not overturn or dampen the effects of simulated tax reforms on welfare and wealth inequality. On the contrary, the effect of eliminating the debt tax shield on welfare and inequality is underestimated. As shown in Table 10., when the change of leverage is taken into consideration, the welfare and inequality changes are even larger because the welfare gains from increasing the debt tax shield in the model mostly stem from the increased after-tax return on the safe asset (in which poor and constrained agents save). Therefore, when the debt tax shield is decreased, the firms would presumably decrease their debt financing. This decreases the issuance of bonds and consequently decreases their return.

**Table 10: Results: Economies with exogeneously different leverages**

Moments	$C^f = 0.0, \lambda = 0.40$	$C^f = 0.0, \lambda = 0.20$	$C^f = 0.6, \lambda = 0.40$	$C^f = 0.6, \lambda = 0.60$
Welfare change	-0.028%	-0.080%	+0.105%	+0.141%
Wealth GINI	0.5833	0.5838	0.5826	0.5819
$E(K)$	5.766	5.763	5.740	5.741
$E(r^{sP} - r^{bP})$	0.448	0.343	0.371	0.478

## 2.6 Conclusion

This chapter considers the macroeconomic consequences of differential taxation of risky and safe financial assets. The social planner may wish to tax safe assets at the lower rate in order to reduce the tax burden of poor households, that mainly save in safe assets. The theoretical part of the chapter finds that the optimal difference between taxes on risky and safe assets is increasing in wealth inequality. The quantitative part of the chapter finds that 1) the elimination of the debt tax shield is welfare reducing, and it is equivalent to a permanent consumption decrease of 0.3%, and 2) that the optimal tax shield is larger than in the current US tax code. In the general equilibrium model, the distortionary taxation that taxes the risky asset more is useful for the utilitarian social planner for multiple reasons: it shifts the tax burden from poor households owning safe assets to wealthy households holding equity, and it reduces the variance of the after-tax returns of the risky asset, which is beneficial for equity owners across the board.

## 2.7 Appendix A. Two-period Ramsey model and proofs

A.1. *Proof of Lemma 1.* The proof closely follows Diamond (1975).

Government maximizes:

$$\max_{\tau_r, \tau_s} \chi(p, m)$$

s.t.

$$\tau_r \left( \sum_{h=1}^2 x_r^h(p, m^h) \right) + \tau_s \left( \sum_{h=1}^2 x_s^h(p, m^h) \right) \geq G$$

The FOCs are the following, for the two tax rates for goods  $k$ :

$$\sum_{h=1}^H \frac{\partial \chi}{\partial V^h} \frac{\partial V^h}{\partial \tau_k} = -\lambda p_k \sum_{h=1}^H \left\{ x_k^h(p, m^h) + \sum_{j=1}^J \tau_j \frac{\partial x_j^h(p, m^h)}{\partial p_k} \right\}$$

Using the Roy's identity we get:

$$\begin{aligned} \frac{\partial V^h}{\partial q_k} &= -\frac{\partial V^h}{\partial m} x_k^h \\ \frac{\partial V^h}{\partial q_k} &= -\alpha^h x_k^h \end{aligned}$$

And  $\alpha$  is defined as agent  $h$ 's marginal utility of income.

From the Slutsky equation:

$$\frac{\partial x_j^h}{\partial p_k} = \frac{\partial \hat{x}_j^h}{\partial p_k} - x_k^h(p, m^h) \frac{\partial x_j^h}{\partial m}$$

Using the Slutsky equation and Roy's identity, we get:

$$\sum_{h=1}^H \frac{\chi}{\partial V^h} \alpha^h x_k^h = \lambda p_k \sum_{h=1}^H \left\{ x_k^h(p, m^h) + \sum_{j=1}^J \tau_j p_j \left[ \frac{\partial \hat{x}_j^h}{\partial p_k} - x_j^h(p, m^h) \frac{x_j^h(p, m^h)}{\partial m} \right] \right\}$$

rewriting:

$$\sum_{j=1}^J \tau_j p_j \left( \sum_{h=1}^H \frac{\partial x_j^h}{\partial p_k} \right) = \frac{\sum_{h=1}^H \frac{\partial \chi}{\partial V^h} \alpha^h x_k^h}{\lambda} - \sum_{h=1}^H x_k^h(p, m^h) + \sum_{h=1}^H x_k^h(p, m^h) \sum_{j=1}^J \tau_j \frac{\partial x_j^h(p, m^h)}{\partial m}$$

Using the symmetry of the substitution matrix

$$\frac{\partial \hat{x}_j^h}{\partial p_k} = \frac{\partial \hat{x}_k^h}{\partial p_j}$$

the following is obtained:

$$\sum_{j=1}^J p_j \tau_j \left( \sum_{h=1}^H \frac{\partial x_j^h}{\partial p_k} \right) = \frac{\sum_{h=1}^H \frac{\partial \chi}{\partial V^h} \alpha^h x_k^h}{\lambda} - \sum_{h=1}^H x_k^h(p, m^h) + \sum_{h=1}^H x_k^h(p, m^h) \sum_{j=1}^J \tau_j \frac{\partial x_j^h(p, m^h)}{\partial m}$$

To simplify the expression, the following notation is used:

$$X_k = \sum_{h=1}^H x_k^h(p, m)$$

where  $X_k$  is the total demand for good  $k$

$$\beta^h = \frac{\frac{\partial \chi}{\partial V^h} \alpha^h}{\lambda} + \sum_{j=1}^J \tau_j \frac{\partial x_j^h(p, m^h)}{\partial m}$$

where  $\beta^h$  is the so called “net social marginal utility of income for agent  $h$ ”

$$\bar{\beta} = \sum_{h=1}^H \frac{\beta^h}{H}$$

Now, it is possible to rewrite the formula:

$$\sum_{j=1}^J p_j \tau_j \left( \sum_{h=1}^H \frac{\partial \hat{x}_k^h}{\partial p_j} \right) = -X_k \left( 1 - \sum_{h=1}^H \beta^h \frac{x_k^h}{X_k} \right)$$

Now, denote by  $\theta_k$  the empirical covariance between  $\beta^h$  and  $h$ 's consumption of good  $k$ :

$$\theta_k = cov\left(\frac{\beta^h}{\bar{\beta}}, \frac{x_k^h}{\bar{X}_k}\right) = \frac{1}{H} \sum_{h=1}^H \left( \frac{\beta^h}{\bar{\beta}} - 1 \right) \left( \frac{x_k^h}{\bar{X}_k} - 1 \right)$$

Positive  $\theta_k$  indicates that good  $k$  is mostly consumed by agents with higher  $\beta^h$  (in the context of the model; poor agents).

$\theta_k$  can also be expressed as:

$$\theta_k = \sum_{h=1}^H \frac{\beta^h x_k^h}{X_k} - 1$$

Finally, the standard “many-person” Ramsey optimal rule formula can be written:

$$- \frac{\sum_{j=1}^J \tau_j p_j (\sum_{h=1}^H \frac{\partial \hat{x}_k^h}{\partial p_j})}{X_k} = 1 - \bar{\beta} - \bar{\beta} \theta_k \quad (2.2)$$

□

*A.2. Proof of Proposition 1.*

$$\frac{U_R(kR, kS)}{U_S(kR, kS)} = \frac{U_R(R, S)}{U_S(R, S)}$$

It is a well-established result in the Ramsey framework that in the case of homogeneous utility there should be uniform taxation of goods: risky and safe assets should be taxed at the same rate.

Marshallian demand for the risky asset is:

$$x_r^h = \frac{\frac{m^h}{p_s} (A - 1)}{r_g - r_b A + \frac{p_r}{p_s} (A - 1)}$$

where

$$A = \left( \frac{r_g - \frac{p_r}{p_s}}{\frac{p_r}{p_s} - r_b} \right)^{\frac{1}{\alpha}} > 1$$

To see this, first consider a case in which there is no initial inequality:  $m^h = \frac{\sum_h m^h}{H}$

$$\frac{\tau_r}{1 + \tau_r} \hat{\epsilon}_{r,r} + \frac{\tau_s}{1 + \tau_s} \hat{\epsilon}_{r,s} = 1 - \bar{\beta} - \bar{\beta} \theta_k$$

rearranging:

$$\frac{\frac{\tau_s}{1 + \tau_s}}{\frac{\tau_r}{1 + \tau_r}} = \frac{-\hat{\epsilon}_{r,r} + \hat{\epsilon}_{s,r}}{-\hat{\epsilon}_{s,s} + \hat{\epsilon}_{r,s}}$$

Rewriting in terms of uncompensated (Marshallian) elasticities, using the Slutsky equation:



$$\frac{-(\epsilon_{r,r} + \epsilon_{r,m}B_r^h) + \epsilon_{s,r} + \epsilon_{s,m}B_r^h}{-(\epsilon_{s,s} + \epsilon_{s,m}B_s^h) + \epsilon_{r,s} + \epsilon_{r,m}B_s^h}$$

Rewriting using the “symmetry” equation ( $\epsilon_{i,j} = \epsilon_{j,i} \frac{B_j}{B_i} + B_j(\epsilon_{j,m} - \epsilon_{i,m})$ ), where  $B_i$  is a budget share of a good  $i$ ):

$$\frac{-(\epsilon_{r,r} + \epsilon_{r,m}B_r^h) + \epsilon_{r,s} \frac{B_r^h}{B_s^h} + \epsilon_{r,m}B_r^h}{-(\epsilon_{s,s} + \epsilon_{s,m}B_s^h) + \epsilon_{s,r} \frac{B_s^h}{B_r^h} + \epsilon_{s,m}B_r^h} = \frac{-\epsilon_{r,r} + \epsilon_{r,s} \frac{B_r^h}{B_s^h}}{-\epsilon_{s,s} + \epsilon_{s,r} \frac{B_s^h}{B_r^h}}$$

Using the homogeneity condition ( $\epsilon_{r,s} + \epsilon_{r,r} + \epsilon_{r,m} = 0$ ):

$$\frac{\epsilon_{r,s} + \epsilon_{r,m} + \epsilon_{r,s} \frac{B_r^h}{B_s^h}}{\epsilon_{s,r} + \epsilon_{s,m} + \epsilon_{s,r} \frac{B_s^h}{B_r^h}} = \frac{\epsilon_{r,m} + \epsilon_{r,s} \frac{B_s^h + B_r^h}{B_s^h}}{\epsilon_{s,m} + \epsilon_{s,r} \frac{B_s^h + B_r^h}{B_r^h}}$$

Using the “symmetry” property again:

$$\frac{\frac{\tau_s}{1+\tau_s}}{\frac{\tau_r}{1+\tau_r}} = \frac{\frac{B_s^h + B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{r,m} + (B_s^h + B_r^h)(\epsilon_{s,m} - \epsilon_{r,m})}{\frac{B_s^h + B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,m}} \quad (2.3)$$

If there is no labor (leisure), budget shares sum up to 1 ( $B_s^h + B_r^h$ ), and there should be no tax distortion.

Marshallian demand for the risky asset is:

$$x_r^h = \frac{\frac{m^h}{p_s}(A-1)}{r_g - r_b A + \frac{p_r}{p_s}(A-1)}$$

where

$$A = \left( \frac{r_g - \frac{p_r}{p_s}}{\frac{p_r}{p_s} - r_b} \right)^{\frac{1}{\alpha}} > 1$$

Marshallian demand for the safe asset is:

$$x_s^h = \frac{\frac{m^h}{p_r}(Ar_b - r_g)}{1 - A + \frac{p_s}{p_r}(Ar_b - r_g)}$$

Both are linear in  $m^h$ .

Computing the income elasticities:

$$\epsilon_{r,m} = \frac{m^h}{x_r^h} \frac{dx_r^h}{dm^h} = 1 = \epsilon_{s,m}$$

Therefore, from (3), we see that there will be uniform taxation between two assets.

The result extends to the case with initial inequality, as the elasticities do not depend on  $m^h$ :

$$\frac{\partial^3 x_j^h}{\partial p_i \partial^2 m^h} = 0$$

□

*A.3. Proof of Proposition 2.* Consider the equation (2):

$$\frac{\frac{\tau_s}{1+\tau_s}}{\frac{\tau_r}{1+\tau_r}} = \frac{\frac{B_s^h+B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{r,m} + (B_s^h + B_r^h) (\epsilon_{s,m} - \epsilon_{r,m})}{\frac{B_s^h+B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,m}}$$

Adding and subtracting  $\epsilon_{s,m}$  in the numerator;

$$\frac{\frac{\tau_s}{1+\tau_s}}{\frac{\tau_r}{1+\tau_r}} = \frac{\frac{B_s^h+B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,m} + (1 - B_s^h + B_r^h) (\epsilon_{r,m} - \epsilon_{s,m})}{\frac{B_s^h+B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,m}}$$

Because  $B_s^h + B_r^h = 1$ , we have:

$$\frac{\frac{\tau_s}{1+\tau_s}}{\frac{\tau_r}{1+\tau_r}} = 1$$

□

However, if there are more than the two mentioned assets  $(B_s^h + B_r^h) \leq 1$ , and if  $\epsilon_{r,m} > \epsilon_{s,m}$ , the safe asset should be taxed more.

$$\epsilon_{r,m} = \frac{m^h}{x_r^h} \frac{dx_r^h}{dm^h} = \frac{\frac{m^h}{p_s}(A-1)}{\frac{m^h}{p_s}(A-1) - L_g} > 1$$

$$\epsilon_{s,m} = \frac{m^h}{x_s^h} \frac{dx_s^h}{dm^h} = \frac{\frac{m^h}{p_s}(r_g - r_b A + \frac{p_r}{p_s}(A-1)) - p_r(A-1)}{\frac{m^h}{p_s}(r_g - r_b A + \frac{p_r}{p_s}(A-1)) - p_r(A-1) + L_g} < 1$$

Therefore,  $\epsilon_{r,m} > \epsilon_{s,m}$ , and the safe asset would be taxed at the higher rate in the optimum.

*A.4. Proof of Proposition 3.* Formally, in examining equation 1:  $\theta_s$  is positive, and  $\theta_r$  is negative. For example, examine  $\theta_r$ :

$$\theta_r = \sum_{h=1}^H \frac{\beta^h x_k^h}{\bar{\beta} X_k} - 1$$

Let us consider a case with only two agents: rich and poor, and denote the rich agent with superscript  $C$ , and the poor agent with superscript  $P$ . They differ in the initial income  $m^C > m^P$ .

$\beta^P > \beta^C$  because  $\alpha^P > \alpha^C$ , and the term  $\frac{\partial x_j^h(p, m^h)}{\partial m}$  cannot compensate for this, because it would imply that the tax policy is not optimal. Therefore, the net marginal utility of one additional unit of income is greater for the poor agent. It immediately follows that  $\frac{\beta^P}{\beta^C}$  is increasing in  $\frac{m^C}{m^P}$ .

Furthermore,  $\frac{x_r^P}{x_r^P + x_r^C} < 0.5$ , because, as we have seen before  $\epsilon_{r,m} > 1$ .

Using these facts:

$$\theta_r = \frac{\overbrace{\beta^P}^{>\bar{\beta}} x_r^P + \overbrace{\beta^C}^{<\bar{\beta}} x_r^C}{\bar{\beta}(x_r^P + x_r^C)} - 1$$

Since the average of  $\beta^C$  and  $\beta^P$  is  $\bar{\beta}$ , this implies that  $\theta_r < 0$ . Since there are only two assets, and  $\theta_r$  is negative,  $\theta_s$  must be positive:  $\theta_s > 0$ .

Examining the optimal many-person Ramsey taxation formulas, this implies that the risky asset should be taxed more (compared to the safe asset), the larger the initial inequality (ceteris paribus) is. In other words, the mean-preserving spread in  $M$  results in a higher taxation of the risky asset.

The extension to the case with  $H$  agents is straightforward. Namely, the LHS of (3) is not changed by the mean-preserving spread in  $m$ , as the elasticities and Marshallian demands are linear in income  $m^h$ . Therefore, LHS of (3) is just a function of  $m$  ( $m = \sum_h^H m^h$ ). Again, the proof is obvious when the following property is used:

$$\frac{\partial^3 x_j^h}{\partial p_i \partial^2 m^h} = 0$$

□

## Appendix A5. Introduction of Non-taxable good (consumption, leisure)

Consider the case where now the agents do not have only fixed wealth  $m^h$ , but can also increase their investment budget by decreasing  $l$ , which can be interpreted as either leisure or consumption in the first period.

$$U(R^h, S^h) = \frac{1}{2} \left\{ \frac{(R^h r_g + S^h + L_g)^{1-\alpha}}{1-\alpha} + \frac{(R^h r_b + S^h + L_b)^{1-\alpha}}{1-\alpha} \right\} + v(l^h)$$

s.t.

$$R^h p_r + S^h p_s + l^h = m^h$$

$v$  is well behaved and increasing in  $l$ .

Here  $l$  is a non-taxed good, in the tradition of Ramsey literature. Define the wealth that the agent spends on investing as :  $W^h = m^h - l^h$ . Furthermore, for simplicity, assume that there is no inequality and  $m^h = 0 \forall h$ . The optimal taxation equation looks like this:

$$\frac{\frac{\tau_s}{1+\tau_s}}{\frac{\tau_r}{1+\tau_r}} = \frac{\frac{B_s^h + B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,W} + (1 - B_s^h + B_r^h) (\epsilon_{r,W} - \epsilon_{s,W})}{\frac{B_s^h + B_r^h}{B_r^h} \epsilon_{s,r} + \epsilon_{s,W}}$$

However, now there are more than two budget elements, so  $(B_s^h + B_r^h) \leq 1$ , and  $\epsilon_{r,W} > \epsilon_{s,W}$ , the safe asset should be taxed more.

$$\epsilon_{r,W} = \frac{W^h}{x_r^h} \frac{dx_r^h}{dW^h} = \frac{\frac{W^h}{p_s} (A-1)}{\frac{W^h}{p_s} (A-1) - L_g} > 1$$

$$\epsilon_{s,W} = \frac{W^h}{x_s^h} \frac{dx_s^h}{dW^h} = \frac{\frac{W^h}{p_s} (r_g - r_b A + \frac{p_r}{p_s} (A-1)) - p_r (A-1)}{\frac{W^h}{p_s} (r_g - r_b A + \frac{p_r}{p_s} (A-1)) - p_r (A-1) + L_g} < 1$$

Therefore,  $\epsilon_{r,W} > \epsilon_{s,W}$ , and the safe asset would be taxed at the higher rate in the absence of inequality. Still, increase in inequality increases the optimal ratio of tax on the risky and tax on the safe asset. The difference with the case without  $l$  is that now, the complete taxable wealth  $W^h = m^h - l^h$  is not fixed, since agents decide on  $l$ . Therefore, the standard result reemerges: assets with higher income elasticity should be taxed less.

## 2.8 Appendix B. Taxation of asset returns

In Appendix B, the case in which government taxes the returns on assets is considered.

The utilitarian social planner maximizes social welfare by choosing the taxes in the first period to collect the expected revenue  $G = E \left[ \sum_h^H R^h r_r t_r + \sum_h^H r_f S^h t_f \right]$ . Where  $r_r$  has two equally likely possible realizations:  $r_g$  in a good, and  $r_b$  in a bad state. Denote the tax rate on the risky asset as  $t_r$ , and the tax rate on safe asset as  $t_f$ . Furthermore, denote the net returns:  $R_g = r_g(1 - t_r)$ ,  $R_b = r_b(1 - t_r)$  and  $R_f = r_f(1 - t_f)$ .

Then, the problem of the households is:

$$\max_{R^h} \frac{1}{2} u(Y^h R_f + R^h (R_g - R_f) + L_g) + \frac{1}{2} u(Y^h R_f + R^h (R_b - R_f) + L_b)$$

Taking the first order conditions and rearranging, we get the optimal choices of the amount of wealth invested in the risky asset:

$$R^{h*} = \frac{Y^h R_f (1 - A) + L_b - L_g A}{A R_g - A R_f - R_b + R_f}$$

where  $A = \left( \frac{R_b - R_f}{R_f - R_g} \right)^{-\frac{1}{\alpha}}$ .

This appendix verifies numerically, in a two-agent economy, that, even when the asset returns are taxed, the ratio of optimal taxes  $\frac{t_r}{t_f}$  is increasing in the initial inequality  $\frac{Y^C}{Y^P}$ .

Consider the following example:

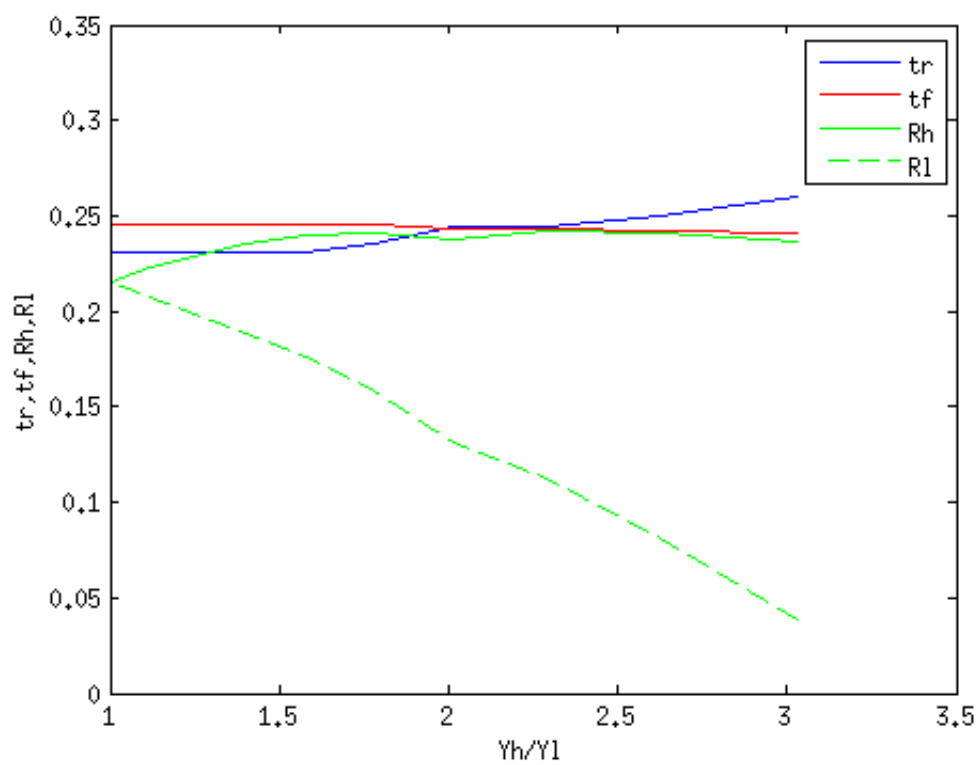
$$u(x) = \frac{x^\alpha}{1 - \alpha}$$

with the following parameter values:

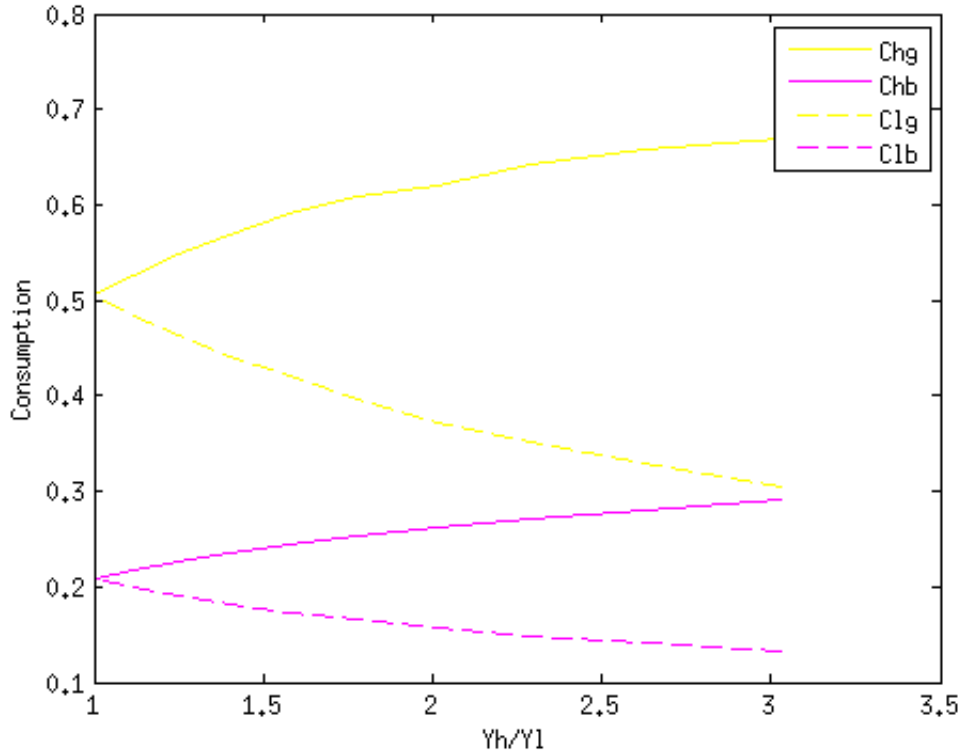
**Table 11: Parameter values**

$r_g$	$r_b$	$r_f$	$\alpha$	$G$	$\sum_h^2 Y^h$	$L_g$	$L_b$
0.076	0.01	0.02	2	0.15	25	0.20	0.04

Figure 6. Taxation of asset returns



**Figure 7.** Taxation of asset returns: consumption in the second period



## 2.9 Appendix C. Solution algorithm

This appendix briefly describes the solution algorithm used to obtain the solution of the quantitative model. The algorithm uses the method used by Krusell and Smith (1997, 1998), which replaces the infinite-dimensional wealth distribution with a finite set of moments of wealth distribution, as a state variable.

Furthermore, for solving the individual problem, given the laws of motion for aggregate variables, the “endogenous grid method” proposed by Carroll (2006) is used. The method is augmented to allow for two choice variables (amount of savings and the composition of savings between stocks and bonds). I use a FORTRAN programming language for the numeric computation, since the computation is intensive, and requires a “low-level” programming language for the runtime of the program to be feasibly short.

1. Guess the law of motion for aggregate capital  $K_{t+1}$  and interest rate  $P_t^e$ .

There will be for equations, as there are 4 possible aggregate state realizations (two realizations for TFP and two for depreciation shock).

2. Given the perceived laws of motion, solve the individual problem described in section 2.4.8. In this step endogenous grid method Carroll (2006) is used. Instead of constructing the grid on the state variable  $\omega$ , and searching for the optimal decision for savings  $\tilde{\omega}$ , this method creates a grid on the optimal savings amounts  $\tilde{\omega}$ , and evaluates the individual optimality conditions to obtain the level of wealth  $\omega$  at which it is optimal to save  $\tilde{\omega}$ . This way, the root-finding process is avoided, since finding optimal  $\omega$ , given  $\tilde{\omega}$ , involves only evaluation of a function (households optimality condition). However, root finding process is necessary to find the optimal portfolio choice of the household, which is performed after finding the optimal pairs  $\omega$  and  $\tilde{\omega}$ .
3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth distribution, instead of a Monte Carlo simulation, the method proposed by Young (2010) is used. For each realized value of  $\omega$ , the method distributes the mass of agents between two grid points:  $\omega_i$  and  $\omega_{i+1}$ , where  $\omega_i < \omega < \omega_{i+1}$ , based on the distance of  $\omega$ , based on Euclidean distance between  $\omega_i$ ,  $\omega$  and  $\omega_{i+1}$ . Do this in the following steps:
  - (a) Set up an initial distribution in period 1:  $\mu$  over a simulation grid  $i = 1, 2, \dots, N_{sgrid}$ , for each pair of efficiency and employment status, where  $N_{sgrid}$  is the number of wealth grid points. Set up an initial value for aggregate states  $z$  and  $d$ .
  - (b) Find the bond interest rate in the given period  $R^b$ , which clears the market for bonds. This is performed by iterating on  $P^e$ , until the following equation is satisfied (bond market clears):

$$\sum_j g^{b,j}(\omega, e, l; z, d, K, P^e) d\mu = \lambda \sum_j \{g^{b,j}(\omega, e, l; z, d, K, P^e) d\mu + g^{s,j}(\omega, e, l; z, d, K, P^e) d\mu\}$$

where  $g^{b,j}(\omega, e, l; z, d, K, P^e)$  and  $g^{s,j}(\omega, e, l; z, d, K, P^e)$  are the policy functions for bonds and shares, where  $j$  denotes the age of the household (working age or retired), that solve the following recursive household maximization problems: Retired households:

$$v_R(\omega; z, K, \delta, P^e) = \max_{c, b', s'} \left\{ c^{1-\rho} + v\beta E_{z', K', \delta', P^{e'} | z, K, \delta, P^e} [v_R(\omega'; z', K', \delta', P^{e'})^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$



Working age households:

$$v_W(\omega, e, l; z, K, \delta, P^e) =$$

$$\max_{c, b', s'} \left\{ c^{1-\rho} + \beta E_{e', l', z', K', \delta', P^{e'} | e, l, z, K, \delta, P^e} [(1-\theta)v_W(\omega', e', l'; z', K', \delta', P^{e'})^{1-\alpha} + \theta v_R(\omega', e', l'; z', K', P^{e'}, \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

where  $v_j$  are the value functions, obtained in step 2. In this step, an additional state variable is included explicitly:  $P^e$ .

- (c) Depending on the realization for  $z'$  and  $d'$ , compute the joint distribution of wealth, labor efficiency and employment status.
  - (d) To generate a long time series of the movement of the economy, repeat sub-steps b) and c).
4. Use the time series from step 2 and perform a regression of  $\ln K'$  and  $P^e$  on constants and  $\ln K$ , for all possible values of  $z$  and  $d$ . This way, the new aggregate laws of motion are obtained.
  5. Make a comparison of the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently good, the solution algorithm is completed. If not, make a new guess for the laws of motion, based on the linear combination of laws from steps 1 and 4. Then, proceed to step 2.

## 2.10 Appendix D. Quantitative model: $\tau^{lb} = 0$

In this simple extension, I consider the case in which  $\tau^{lb} = 0$ . This is conducted so that the effect of the changing  $\tau^{lb}$ , which inevitably comes as a consequence of changing  $C_f$ , does not confound the analysis of the changing leverage. For example, when  $C_f$  increases, the government will collect more tax revenue than before in the good aggregate states, and less revenue in bad aggregate states (compared with the case with low  $C_f$ ). This means that labor taxes  $\tau^{lb}$  will be higher in the good aggregate states and lower in the bad aggregate states (compared with the case with low  $C_f$ ). The poorer households lose more, with the caveat that the second quintile loses the most. This can be explained by the fact that the poorest households (in the first quintile) do not have much savings, while it is the households in the second quintile that save the most in the safe asset.

**Table 12:**  $\tau^{lb} = 0$ : Long-run results

Moments	$C^f = 0.0$	$C^f = 0.3$
Welfare	37.88	38.28
Wealth GINI	0.5959	0.5943
$E(K)$	5.765	5.762
Pretax $E(r^b)$	2.24	1.77
Pretax $E(r^s)$	2.36	2.61
Post-tax $E(r^{bP})$	1.70	1.84
Post-tax $E(r^{sP})$	1.79	1.93
Post-tax $E(r^{sP} - r^{bP})$	0.094	0.092

**Table 13:**  $\tau^{lb} = 0$ : Percent changes in welfare when moving from  $C_f = 0.3$  to  $C_f = 0$ 

Quintile	Welfare change				
	1	2	3	4	5
% change	-0.30	-0.40	-0.30	-0.27	-0.26

## Chapter 3

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# Avoiding root-finding in the Krusell-Smith algorithm simulation

### 3.1 Introduction

This chapter proposes a novel method to compute the simulation part of the Krusell-Smith algorithm when agents can trade with more than one asset. The classic example is the macroeconomic model with both idiosyncratic and aggregate risk, with a borrowing constraint, where agents can choose to save in both risky capital and safe bonds. The idea is to avoid root-finding in the simulation part of the algorithm, where it is necessary to find a market-clearing bond price. Instead, the proposed algorithm lets the economy proceed to the next period with the markets uncleared and updates the perceived law of motions for the bond price based on the simulated excess demand for bonds. The idea of finding a market-clearing price by having information on excess demand can be traced far back in the history of economics, not necessarily as a solution method, but as an actual process by which general equilibrium emerges in existing markets. The process was called *tâtonnement* (French for “trial and error” or “groping”) by Walras (1874) (translated to English: Walras (1954)). The proposed algorithm, however, does not imply anything about the process of reaching equilibrium, but uses the idea purely as a part of the solution algorithm.

The computational gain of using the proposed algorithm is a shorter time duration due to the avoidance of bond market clearing. Market clearing involves a root-finding process, which is computationally very expensive. The root-finding

consists of finding a bond interest rate (or equity premium), which will clear the bond market in each simulated period. In the general equilibrium, all the markets are supposed to clear, but in the process of finding the general equilibrium laws of motion, it can be computationally beneficial not to impose market clearing, and use the information on excess demand to make subsequent updates.

The proposed method could be especially useful in computing asset pricing models (for example models with risky and safe assets) with both aggregate and uninsurable idiosyncratic risk, since methods that use linearization in the neighborhood of the aggregate steady state are considered less accurate than global solution methods for these particular types of models. For example, Reiter (2009) proposes a solution using projection and perturbation instead of attempting to represent the cross-sectional distribution of wealth by a small number of statistics in order to reduce the dimensions in state space as in Krusell and Smith (1997), Den Haan (1997) and Reiter (2002). However, a solution method based on projection and perturbation most likely is not accurate enough for solving the models with asset pricing, as it assumes the linearity in the aggregate variables, which is not sufficient for the problems of portfolio choice and asset pricing (Reiter, 2009). In addition to this specific application, further use of this method could be useful to accelerate the Krusell-Smith algorithm where any type of market-clearing has to be imposed during the simulation of the model (for example clearing of the labor market in a model where labor supply is determined endogenously).

The rest of the thesis chapter is organized as follows: Section 2 describes the sample model, Section 3 describes the classical Krusell-Smith algorithm (Krusell and Smith, 1997) used to solve the models with both aggregate and idiosyncratic risk and a portfolio choice, Section 4 illustrates the proposed algorithm, Section 5 shows the computational performance comparisons between the classic and the proposed algorithm. Section 6 discusses the results and potential applications of the proposed algorithm, and Section 7 concludes.

## 3.2 Example model

The presented model is based on Algan et al. (2009), and in the tradition of Krusell and Smith (1997). The model consists of a continuum of heterogeneous

agents facing aggregate risk, uninsurable idiosyncratic labor risk and a borrowing constraint, and who save in two assets: risky equity and safe bonds. Unlike Algan et al. (2009), the model parsimoniously captures the life cycle dynamics of the households, in the fashion of Krueger et al. (2016), where working-age agents face the retirement shock and retired households face the risk of dying.

### 3.2.1 Production technology

In each period  $t$ , the representative firm uses aggregate capital  $K_t$ , and aggregate labor  $L_t$ , to produce  $y$  units of final good with the aggregate technology  $y_t = f(z_t, K_t, L_t)$ , where  $z_t$  is an aggregate total factor productivity (TFP) shock. I assume that  $z_t$  follows a stationary Markov process with transition function  $\Pi_t(z, z') = Pr(z_{t+1} = z' | z_t = z)$ . The production function is continuously differentiable, strictly increasing, strictly concave and homogeneous of degree one in  $K$  and  $L$ . Capital depreciates at the constant rate  $\delta \in (0, 1)$  and it accumulates according to the standard law of motion:

$$K_{t+1} = I_t + (1 - \delta)K_t$$

where  $I_t$  is aggregate investment. The particular aggregate production technology is:

$$Y_t = z_t A K_t^\Delta L_t^{1-\Delta}$$

### 3.2.2 Parsimonious life-cycle structure

In each period, working-age households have a chance of retiring  $\theta$ , and retired households have a chance of dying  $v$ , similarly as in Castaneda et al. (2003) and Krueger et al. (2016). Therefore the share of working age households in the total population is:

$$\Pi_W = \frac{1 - v}{(1 - \theta) + (1 - v)}$$

and the share of the retired households in the total population is:

$$\Pi_R = \frac{1 - \theta}{(1 - \theta) + (1 - v)}$$

The retired households who die in period  $t$  are replaced by new-born agents

who start at a working age without any assets. For simplicity, the retired households have perfect annuity markets, which make their returns larger by a fraction of  $\frac{1}{v}$ , as in Krueger et al. (2016). This life-cycle structure with stochastic aging and death helps capture important life-cycle aspects of the economy and risks that households face without adding an excessive computational burden.

### 3.2.3 Preferences

Households are indexed by  $i$ , and they have Epstein-Zin preferences (Epstein and Zin, 1989). These preferences are often used in asset-pricing models, since they allow one to separate the intertemporal elasticity of substitution and risk aversion.

Households are maximizing their lifetime utility, expressed recursively for the retired agents:

$$V_{R,i,t} = \{c_t^{1-\rho} + v\beta[E_t V_{R,i,t+1}^{(1-\alpha)}]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where  $V_{R,i,t}$  is the recursively defined value function of a retired household  $i$ , at time period  $t$ .

Working-age agents maximize:

$$V_{W,i,t} = \{c_t^{1-\rho} + \beta[(1-\theta)E_t V_{W,i,t+1}^{1-\alpha} + \theta E_t V_{R,i,t+1}^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}}\}^{\frac{1}{1-\rho}}$$

where  $V_{i,t}$  is recursively defined value function of household  $i$ , at time period  $t$ . Furthermore,  $\beta$  denotes the subjective discount factor,  $E_t$  denotes expectations conditional on information at time  $t$ ,  $\alpha$  is the risk aversion,  $\frac{1}{\rho}$  is the intertemporal elasticity of substitution.

### 3.2.4 Idiosyncratic uncertainty

In each period, working-age households are subject to idiosyncratic labor income risk that can be decomposed into two parts. The first part is the employment probability that depends on aggregate risk and is denoted by  $e_t \in (0, 1)$ .  $e = 1$  denotes that the agent is employed, and  $e = 0$  that the agent is unemployed. Conditional on  $z_t$  and  $z_{t+1}$ , I assume that the period  $t + 1$  realization of the employment shock follows the Markov process.

$$\Pi_e(z, z', e, e') = Pr(e_{t+1} = e' | e_t = e, z_t = z, z_{t+1} = z')$$

This labor risk structure allows idiosyncratic shocks to be correlated with the aggregate productivity shocks, which is consistent with the data and generates the portfolio choice profile such that the share of wealth invested in risky asset is increasing in wealth. The condition imposed on the transition matrix and the law of large numbers implies that the aggregate employment is only a function of the aggregate productivity shock.

If  $e = 1$  and the agent is employed, one can assume that the agent is endowed with  $l_t \in L \equiv \{l_1, l_2, l_3, \dots, l_m\}$  efficiency labor units, which she can supply to the firm. Labor efficiency is independent of the aggregate productivity shock, and is governed by a stationary Markov process with transition function  $\Pi_l(l, l') = Pr(l_{t+1} = l' | l_t = l)$ . If the agent is unemployed, (s)he receives unemployment benefits  $g_{u,t}$ , which are financed by the government.

### 3.2.5 The representative firm

As in Algan et al. (2009), firm leverage in this model is given exogenously. The leverage of the firm is determined exogenously, by the parameter  $\lambda$ . The Modigliani-Miller theorem (1958, 1963) does not hold, as some of the agents are borrowing constrained, and some are portfolio constrained. Therefore, theoretically, the leverage of the firm has some macroeconomic relevance. Additionally, debt is taxed differently than equity returns, and this additionally breaks the Modigliani-Miller theorem.

In the economy, the representative firm can finance its investment with two types of contracts. The first is a one-period risk-free bond, that promises to pay a fixed return to the owner. The second is risky equity that entitles the owner to claim the residual profits of the firm after the firm pays out wages and debt from the previous period. Both of these assets are freely traded in competitive financial markets. By construction, there is no default in the equilibrium.

The return on the bond  $r_{t+1}^b$  is determined by the clearing of the bond market:

$$\int_S g^{b,j,e} d\mu = \lambda K'$$

where  $g^{b,j,e}$  are the individual policy functions for bonds.

In each period  $t$ , the firm redistributes all the residual value of the firm, after production and depreciation have taken place, and wages and debt has been paid. Therefore, the return on the risky equity depends on the realizations of the aggregate shocks and is given by the following equation:

$$(1+r_{t+1}^s) = \frac{f(z_{t+1}, K_{t+1}, L_{t+1}) - f_L(z_{t+1}, K_{t+1}, L_{t+1})L_{t+1} - \lambda K_{t+1}(1 + r_{t+1}^b) + (1 - \delta)K_{t+1}}{(1 - \lambda)K_{t+1}}$$

An important caveat in having heterogeneous households that own the firm is that they do not necessarily have the same stochastic discount factor  $m_{t+1}^j$ , and therefore the definition of the objective function of the firm is not straightforward. I follow Algan et al. (2009), who assume that the firm is maximizing the welfare of the agents who have interior portfolio choice, and consequently the firm has the same stochastic discount factor  $m_{t+1}$  as the agents with the interior portfolio choice.

As in Algan et al. (2009), it is possible use the fact that for a given stochastic discount factor  $V_t = K_{t+1}$ , which enables the elimination of the capital Euler equation from the equilibrium conditions.

### 3.2.6 Financial markets

As stated earlier, households can save in two assets: risky equity and safe bonds (firm debt). There are borrowing constraints for both assets, the lowest amounts of equity and debt that households can hold in period  $t$  are:  $\kappa^s$  and  $\kappa^b$ , respectively. Markets are assumed to be incomplete in the sense that there are no markets for the assets contingent on the realization of individual idiosyncratic shocks. Furthermore, if the household wants to save a positive amount of resources in equity in the period  $t$ , it has to pay  $\phi$  as a per period cost of participating in the stock market.

### 3.2.7 Government

The government runs a unemployment insurance program, which is modeled as in Krueger et al. (2016) and is financed by special labor income taxes. Unemployment benefits are financed with a labor tax rate  $\tau_t^u$ . The amount of the



unemployment benefits  $g_{u,t}$  is determined by a constant  $\eta$ , which represents the fraction of the average wage in each period.

To satisfy the budget constraint, government has to tax labor at the tax rate:

$$\tau_t^u = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\eta}}$$

where  $\Pi_u$  is the share of unemployed people in the total working age population.

### 3.2.8 Household problem

Household  $i$  maximizes its expected lifetime utility subject to the constraints below:

$$\begin{aligned} c_{i,t} + s_{i,t+1} + b_{i,t+1} + \phi \mathbb{I}_{\{s_{i,t+1} \neq 0\}} &\leq \omega_{i,t} \\ \omega_{i,t+1} &= \begin{cases} w_{t+1} l_{i,t+1} (1 - \tau_{t+1}^l) + (1 + r_{t+1}^s) s_{i,t+1} + (1 + r_{t+1}^b) b_{i,t+1} & \text{if } e = 1 \\ g_{u,t+1} (1 - \tau_{t+1}^l) + (1 + r_{t+1}^s) s_{i,t+1} + (1 + r_{t+1}^b) b_{i,t+1} & \text{if } e = 0 \end{cases} \\ (c_{i,t}, b_{i,t+1}, s_{i,t+1}) &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

### 3.2.9 Recursive household problem

The idiosyncratic state variables of the household problem are: current wealth  $\omega$ , current employment and productivity state  $e, l$ .  $\Theta$  denotes the vector of all discrete individual states (all except the current wealth).<sup>1</sup>

The aggregate state variables of the household problem are: state of the TFP shock:  $z$ , and distribution which is captured by the probability measure  $\mu$ .  $\mu$  is a probability measure on  $(S, \beta_s)$ , where  $S = [\underline{\omega}, \bar{\omega}] \times \Theta$ , and  $\beta_s$  is the Borel  $\sigma$ -algebra.  $\underline{\omega}$  and  $\bar{\omega}$  denote the minimal and maximal allowed amount of wealth the household can hold.<sup>2</sup> Therefore, for  $B \in \beta_s$ ,  $\mu(B)$  indicates the mass of households whose individual states fall in  $B$ . Intuitively, one can think of  $\mu$  as a

<sup>1</sup>In the benchmark model, there will be 5 elements of  $\Theta$ : three levels of productivity for the employed households, unemployment, and retirement.

<sup>2</sup> $\underline{\omega}$  is determined by the borrowing constraint, and  $\bar{\omega}$  is chosen such that there are always no agents with that amount of wealth in equilibrium.

distribution variable that measures the amount of agents in a certain interval of wealth, for each possible combination of other idiosyncratic variables.

The recursive household problem for the retired households is:

$$v_R(\omega; z, \mu, \delta) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + v\beta E_{z', \mu', \delta' | z, \mu, \delta} [v_R^m(\omega'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$\begin{aligned} c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} &= \omega \\ \omega' &= T'_{ss} + [s'(1 + r'^s) + b'(1 + r'^b)] \frac{1}{v} \\ \mu' &= \Gamma(\mu, z, z', d, d') \\ (c, b', s') &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

The recursive household problem for the working-age households is:

$$v_W(\omega, e, l; z, \mu, \delta) =$$

$$\max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{e', l', z', \mu', \delta' | e, l, z, \mu, \delta} [(1 - \theta)v_W(\omega', e', l'; z', \mu', \delta')^{1-\alpha} + \theta v_R(\omega', e', l'; z', \mu', \delta')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$\begin{aligned} c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} &= \omega \\ \omega' &= \begin{cases} w'l'(1 - \tau^l) + s'(1 + r'^s(1 - \tau_s)) + b'(1 + r'^b) & \text{if } e = 1 \\ g'_u w'l'(1 - \tau^l) + s'(1 + r'^s(1 - \tau_s)) + b'(1 + r'^b) & \text{if } e = 0 \end{cases} \\ \mu' &= \Gamma(\mu, z, z', d, d') \\ (c, b', s') &\geq (0, \kappa^b, \kappa^s) \end{aligned}$$

$$v(\omega; z, \mu) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{z', \mu' | z, \mu} [v(\omega'; z', \mu')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

subject to:

$$\begin{aligned} c + s' + b' + \phi \mathbb{I}_{\{s' \neq 0\}} &= \omega \\ \omega' &= T'_{ss} + [s'(1 + r'^s) + b'(1 + r'^b)] \frac{1}{v} \end{aligned}$$

where  $\omega$  is the vector of individual wealth of all agents,  $\mu$  is the probability measure generated by set  $\Omega xExL$ ,  $\mu' = \Gamma(\mu, z, z')$  is a transition function and '

denotes the next period.

### 3.2.10 General equilibrium

The economy-wide state is described by  $(\omega, e; z, \mu)$ . Therefore the individual household policy functions are:  $c^j = g^{c,j}(\omega, e, l; z, \mu)$ ,  $b^j = g^{b,j}(\omega, e, l; z, \mu)$  and  $s^j = g^{s,j}(\omega, e, l; z, \mu)$ , and law of motion for the aggregate capital is  $K' = g^K(\omega, e, l; z, \mu)$ .

A recursive competitive equilibrium is defined by the set of individual policy and value functions  $\{v_R, g^{c,R}, g^{s,R}, g^{b,R}, v_W, g^{c,W}, g^{s,W}, g^{b,W}\}$ , the laws of motion for the aggregate capital  $g^K$ , a set of pricing functions  $\{w, R^b, R^s\}$ , government policies in period  $t$ :  $\{\tau^l\}$ , and forecasting equations  $g^L$ , such that:

1. The law of motion for the aggregate capital  $g^K$  and the aggregate “wage function”  $w$ , given the taxes satisfy the optimality conditions of the firm.
2. Given  $\{w, R^b, R^s\}$ , the law of motion  $\Gamma$ , the exogenous transition matrices  $\{\Pi_z, Pi_e, Pi_l\}$ , the forecasting equation  $g^L$ , the law of motion for the aggregate capital  $g^K$ , and the tax rates, the policy functions  $\{g^{c,j}, g^{b,j}, g^{s,j}\}$  solve the household problem.
3. Labor, shares and the bond markets clear (goods market clears by Walras’ law):

•

$$L = \int_S \epsilon l d\mu$$

•

$$\int_S g^{s,j}(\omega, e, l; z, \mu) d\mu = (1 - \lambda)K'$$

•

$$\int_S g^{b,j}(\omega, e, l; z, \mu) d\mu = \lambda K'$$

4. The law of motion  $\Gamma(\mu, z, z')$  for  $\mu$  is generated by the optimal policy functions  $\{g^c, g^b, g^s\}$ , which are endogenous, and by the transition matrices for the aggregate shocks  $z$ .<sup>3</sup> Additionally, the forecasting equation for aggregate labor is consistent with the labor market clearing:  $g^L(z') = \int_S \epsilon l d\mu$ .

---

<sup>3</sup> $\mu'$  is given by a function  $\Gamma$ , i.e.  $\mu' = \Gamma(\mu, z, z', d, d')$

5. Government budget constraints are satisfied:

$$T_t^{ss} = \frac{L_t}{\Pi_R} w_t L_t \tau^{lss}$$

$$\tau_t^l = \frac{1}{1 + \frac{1 - \Pi_u(z)}{\Pi_u(z)\phi}}$$

### 3.3 Classical solution algorithm

1. Guess the law of motion for aggregate capital  $K_{t+1}$  and equity premium  $P_t^e$ . This means guessing the starting 8 coefficients following the equations (since there are two possible realizations of  $z$ ):

$$\ln K' = a_0(z) + a_1(z)\ln K$$

$$\ln P^e = b_0(z) + b_1(z)\ln K'$$

2. Given the perceived laws of motion, solve the individual problem described earlier. In this step, the endogenous grid method (Carroll, 2006) is used. Instead of constructing the grid on the state variable  $\omega$ , and searching for the optimal decision for savings  $\tilde{\omega}$ , this method creates a grid on the optimal savings amounts  $\tilde{\omega}$ , and evaluates the individual optimality conditions to obtain the level of wealth  $\omega$  at which it is optimal to save  $\tilde{\omega}$ . This way, the root-finding process is avoided, since finding optimal  $\omega$  given  $\tilde{\omega}$ , involves only the evaluation of a function (households optimality condition). However, root-finding process is necessary to find the optimal portfolio choice of the household, which is carried out after finding the optimal pairs  $\omega$  and  $\tilde{\omega}$ .
3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth distribution, instead of a Monte Carlo simulation, the method proposed by Young (2010) is used. For each realized value of  $\omega$ , the method distributes the mass of agents between two grid points:  $\omega_i$  and  $\omega_{i+1}$ , where  $\omega_i < \omega < \omega_{i+1}$ , based on the distance of  $\omega$ , based on Euclidean distance between  $\omega_i$ ,  $\omega$  and  $\omega_{i+1}$ . Do this in the following steps:
  - (a) Set up an initial distribution in period 1:  $\mu$  over a simulation grid  $i = 1, 2, \dots, N_{sgrid}$ , for each pair of efficiency and employment status, where  $N_{sgrid}$  is the number of wealth grid points. Set up an initial value for aggregate states  $z$ .
  - (b) Find the bond interest rate (expected equity premium  $P^e$ ) in the given period, which clears the market for bonds. This is performed by iterating on  $P^e$  (or on a bond return), until the following equation is

satisfied (bond market clears)<sup>4</sup>

$$\sum g^b(\omega, e, l; z, K, P^e)d\mu = \lambda \sum \{g^b(\omega, e, l; z, K, P^e)d\mu + g^s(\omega, e, l; z, K, P^e)d\mu\}$$

where  $g^b(\omega, e, l; z, K, P^e)$  and  $g^s(\omega, e, l; z, K, P^e)$  are the policy functions for bonds and shares that solve the following recursive household maximization problems: Retired households:

$$v(\omega; z, \mu, P^e) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{z', \mu', P^{e'} | z, \mu, P^e} [v(\omega'; z', \mu', P^{e'})^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

where  $v$  is the value function, obtained in step 2. In this step, an additional state variable is included explicitly: expected equity premium  $P^e$ .

- (c) Depending on the realization for  $z'$ , compute the joint distribution of wealth, labor efficiency and employment status.
  - (d) To generate a long time series of the movement of the economy, repeat substeps b) and c).
4. Use the time series from step 2 and perform a regression of  $\ln K'$  and  $P^e$  on constants and  $\ln K$ , for all possible values of  $z$  and  $d$ . This way, the new aggregate laws of motion are obtained.
  5. Make a comparison of the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently good, the solution algorithm is completed. If not, make a new guess for the laws of motion, based on a linear combination of laws from steps 1 and 4. Then, proceed to step 2.

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<sup>4</sup>Similar to Algan et al. (2009), the iteration is performed using the bisection until the excess demand is relatively close to zero, and then the updating is continued using the secant method.

### 3.4 Proposed solution algorithm

1. Guess the law of motion for aggregate capital  $K_{t+1}$  and equity premium  $P_t^e$ . This means guessing all initial coefficients. In this particular case, this would mean 8 coefficients overall, since both relationships are assumed to be linear, and there are two possible realizations of aggregate state  $z$  (2 equations  $\times$  2 coefficients  $\times$  2 aggregate states).

$$\ln K' = a_0(z) + a_1(z)\ln K$$

$$\ln P^e = b_0(z) + b_1(z)\ln K'$$

2. Given the perceived laws of motion, solve the individual problem described earlier. In this step, the endogenous grid method (Carroll, 2006) is used. Instead of constructing the grid on the state variable  $\omega$ , and searching for the optimal decision for savings  $\tilde{\omega}$ , this method creates a grid on the optimal savings amounts  $\tilde{\omega}$ , and evaluates the individual optimality conditions to obtain the level of wealth  $\omega$  at which it is optimal to save  $\tilde{\omega}$ . This way, the root-finding process is avoided, since finding optimal  $\omega$ , given  $\tilde{\omega}$ , involves only the evaluation of a function (households optimality condition). However, the root-finding process is necessary to find the optimal portfolio choice of the household, which is performed after finding the optimal pairs  $\omega$  and  $\tilde{\omega}$ .
3. Simulate the economy, given the perceived aggregate laws of motion. To keep track of wealth, instead of a Monte Carlo simulation, the method proposed by Young (2010) is used. For each realized value of  $\omega$ , the method distributes the mass of agents between two grid points:  $\omega_i$  and  $\omega_{i+1}$ , where  $\omega_i < \omega < \omega_{i+1}$ , based on the distance of  $\omega$ , based on Euclidean distance between  $\omega_i$ ,  $\omega$  and  $\omega_{i+1}$ . Do this in the following steps:
  - (a) Set up an initial distribution in period 1:  $\mu$  over a simulation grid  $i = 1, 2, \dots, N_{sgrid}$ , for each pair of efficiency and employment status, where  $N_{sgrid}$  is the number of wealth grid points. Set up an initial value for aggregate states  $z$ .
  - (b) Simulate the economy given the perceived laws of motion.

$$\sum g^b(\omega, e, l; z, K)d\mu = \lambda \sum \{g^b(\omega, e, l; z, K)d\mu + g^s(\omega, e, l; z, K)d\mu\}$$

where  $g^b(\omega, e, l; z, K)$  and  $g^s(\omega, e, l; z, K)$  are the policy functions for bonds and shares.

$$v(\omega; z, \mu) = \max_{c, b', s'} \left\{ u(c - \gamma)^{1-\rho} + \beta E_{z', \mu' | z, \mu} [v(\omega'; z', \mu')^{1-\alpha}]^{\frac{1-\rho}{1-\alpha}} \right\}^{\frac{1}{1-\rho}}$$

The market for bonds will not necessarily clear. Instead, in each period there will be an excess demand, which will be denoted by  $\chi_t$ .

where  $v_j$  are the value functions obtained in step 2. Unlike in the previous algorithm, expected equity premium is not included as the additional state variable.

- (c) Depending on the realization for  $z'$ , compute the joint distribution of wealth, labor efficiency and employment status.
  - (d) To generate a long time series of the movement of the economy, repeat substeps b) and c).
4. Use the time series from step 3 and perform a regression of  $\ln K'$  on constants and  $\ln K$ , for all possible values of  $z$ . This way, the new aggregate laws of motion for capital are obtained.

However, now for the law of motion for the equity premium, we cannot run a regression, since we do not have “true” market clearing bond prices (equity premium). Instead, we have excess demand in each time-period, given the perceived equity premium. We can use this information to update the perceived law of motion for equity premium. To do this, the Broyden method (Broyden, 1965) is used:

Consider a system of equations

$$f(x^*) = 0$$

, where  $x$  are the “true” coefficients of the perceived law of motion for equity premium

$$x^* = (b_0^*(z), b_1^*(z))$$

and

$$f(x) = (f_1(b_0^*(z), b_1^*(z)), f_2(b_0^*(z), b_1^*(z)))$$



$f_1$  and  $f_2$  denote the error measures that is chosen.<sup>5</sup> For this algorithm, I propose these two measures to be coefficients of a linear regression of excess demand on a constant and capital. The true solution to the model would have the coefficients of this regression equal to 0. This would mean that the mean value of excess demand is 0 and also that the excess demand do not depend on the amount of capital  $K$ . Therefore, to obtain  $f_1$  and  $f_2$ , one has to run the following regressions:<sup>6</sup>

$$\xi_t(z) = \varrho_1(z) + \varrho_2(z)K_t + \epsilon_t$$

One can also use a linear coefficient, and instead of a coefficient on a constant to use an average excess demand for a given aggregate state. In this particular example this provides a faster convergence. After this, step error measures are obtained:

$$f_1(b_0^*(z), b_1^*(z)) = \varphi \sum \xi_t(z)$$

where  $\varphi$  is arbitrary constant.<sup>7</sup>

$$f_2(b_0^*(z), b_1^*(z)) = \varrho_2(z)$$

Now, the goal is to find the true  $x^*$ . This is conducted in the following steps:

- (a) First, define  $\chi^n = f(x^n)$ . Where  $\chi^n$  and  $x^n$  denote the excess demand measure and the coefficients in the iteration  $n$ .

$$\chi^n = (f_1(b_0(z), b_1(z)), f_2(b_0(z), b_1(z)))$$

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<sup>5</sup>One particular error measure is proposed, but many others can be used, depending on the model and the convenience. For example, another one can be using a simple sum of excess demand in each period. Then, the sample would be partitioned into two, depending if the capital is higher or lower than a certain threshold. This would have to be done, as we need to determine two coefficients for each aggregate state. If, for example, the perceived law of motion would have a quadratic form, the sample would be partitioned into three partitions, depending on the level of capital, etc.

<sup>6</sup>In additional, if the perceived law of motion was quadratic, we would use a quadratic regression, since we would need to obtain three parameters for each realization of the aggregate state.

<sup>7</sup>Alternatively, it is possible to simply use  $f_1(b_0^*(z), b_1^*(z)) = \varphi \varrho_1(z)$ .  $\varphi$  is used only as a parameter that gives relative weight of the two error outputs.

Furthermore:  $\Delta x_n = x_n - x_{n-1}$ ,  $\Delta \chi_n = \chi_n - \chi_{n-1}$

- (b) For the initial iteration, we guess the Jacobian matrix. For each additional iteration, we update the Jacobian matrix by:

$$J_n = J_{n-1} + \frac{\Delta \chi - J_{n-1} \Delta x_n}{\|\Delta x_n\|^2} \Delta x_n^T$$

after updating the matrix, we update the guess of the perceived law of motion for equity premium:

$$x_{n+1} = x_n - J_n^{-1} f(x_n)$$

We do these steps two times, for  $z = \textit{good}$  and  $z = \textit{bad}$ .

5. Compare the laws of motion from step 4 and step 1. If they are almost identical and their predictive power is sufficiently good, the solution algorithm is completed. If not, make a new guess for the laws of motion, based on a linear combination of laws from steps 1 and 4. Then, proceed to step 2.

### 3.5 Performance comparison on an example model

To demonstrate the potential reduction in the computation speed of the discussed model, I solve the model described in section 2, both with the classical solution method (Krusell and Smith, 1997) and the proposed method from section 4. To compare the two algorithms, the parametrized model will be solved 20 times by the two algorithms, each time starting from the different initial perceived law of motions. The initial perceived law of motion is obtained as follows: Each parameter of the true laws of motion is randomly perturbed by a normally distributed shock with the standard deviation  $\sigma = 0.01$ . The size of the perturbation is large enough so that the initial guess is not too close to the solution, and not too large to cause all of the households to have a corner portfolio solution.<sup>8</sup> The stopping criterion for the perceived laws of motion for equity premium is that the excess demand of the bonds have to be on, average smaller, than 0.1% of the aggregate capital, without imposing the market-clearing.<sup>9</sup> When updating the

<sup>8</sup>This is important since taking the numerical derivative of excess demand may not behave properly. For details see the discussion in Section 6.

<sup>9</sup>If the market-clearing is imposed, at least in the last iteration, the excess demand will be orders of magnitudes smaller. For details, see the discussion in Section 6.

laws of motion parameters, the weight of the new guess is always 1. This is only because, for this specific model, it happens to minimize the time for convergence. In the value function iteration, 85 grid points are used in the individual wealth dimension, and 12 grid points are used in the aggregate capital dimension. Cubic splines are used to interpolate the values in between the grid points. The code is written in a FORTRAN 90 programming language and compiled using Intel Fortran Compiler. All the simulations are executed on a personal computer using Linux Mint 18 (64-bit) operating system, with Intel i7-67000 Central Processing Unit (4 cores and 8 threads), clocked at 2.60 GHz. I report both the number of iterations necessary obtain a solution (a convergence), and an overall run-time.

### 3.5.1 Parametrization

The model is parametrized to a quarterly frequency. The choice of the main parameters are reported in Table 1.

**Table 1: Parameters**

Parameter	Symbol	Value
Risk aversion	$\alpha$	10
Intertemporal elasticity of substitution	$\frac{1}{\rho}$	0.50
Discount factor	$\beta$	0.901
Expected depreciation rate	$E(\delta)$	0.033
Standard deviation of depreciation rate	$\sigma(\delta)$	$1.0E - 4$
Leverage	$\lambda$	0.35
Average tax rate for funding social security	$\tau^{lss}$	0.07
Borrowing constraint: bonds	$\kappa^b$	-0.23
Borrowing constraint: stocks	$\kappa^s$	0.00
Chance of not retiring	$\theta$	0.994
Chance of not dying	$v$	0.983

The TFP shocks and capital depreciation shocks are assumed to be perfectly correlated, and thus there are only two aggregate states<sup>10</sup> *good*, where TFP is high and depreciation is low, and *bad*, where TFP is low and depreciation is high.

<sup>10</sup>Since, unlike in the previous two chapters (which have four aggregate state realizations), the TFP and depreciation shocks are perfectly correlated.

### Idiosyncratic shocks

There are 5 possible idiosyncratic states in which the household can find itself (5 for each aggregate state). The labor productivity among the working-age employed households is governed by the transitional Markov matrix:

$$\Pi_l = \begin{bmatrix} 0.9850 & 0.0100 & 0.0050 \\ 0.0025 & 0.9850 & 0.0125 \\ 0.0050 & 0.0100 & 0.9850 \end{bmatrix}$$

and for the individual labor productivity levels, the following values are used:  $l \in \{36.5, 9.5, 1.2\}$ . In addition to this risk, the households face a risk of becoming unemployed, which is the same regardless of the labor productivity level. Finally, working-age households also face a risk of becoming retired  $1 - \theta$ . The average unemployment spell is set to 1.5 quarters in the good state (boom) and 2.5 quarters in the bad state (recession). The replacement rate for the unemployed is set to 4.2% of the average wage in the given period. The probabilities of becoming/remaining unemployed when the economy moves from a good to bad state and vice-versa is adjusted to match the movement of the overall employment, which is set to 95.9% in the good state and 92.8% in the bad state.

### Generated Moments Appendix

The selection of the moments in the model is presented in Table 2.

**Table 2: Moments in the model**

Moment	Symbol	Value
Capital-output ratio	$K/Y$	7.01
Average interest rate	$r^b$	1.43 %
Expected return to capital	$E\{r^s\}$	1.44 %
Average equity premium	$E\{r^s - r^b\}$	0.01 %

### 3.5.2 Solution for perceived laws of motions

$$\ln K' = a_0(z, \delta) + a_1(z, \delta) \ln K$$

$$\ln P^e = b_0(z, \delta) + b_1(z, \delta) \ln K'$$

For the example model, the perceived aggregate law of motions are:

In a good TFP and  $\delta$  state:

$$\ln K' = 0.113 + 0.936 \ln K$$

$$\ln P^e = -8.800 - 0.629 \ln K'$$

In a bad TFP and  $\delta$  state:

$$\ln K' = 0.111 + 0.934 \ln K$$

$$\ln P^e = -8.100 - 0.407 \ln K'$$

The perceived laws of motion predict the actual movements of capital and equity premium with  $R^2 = 0.99995$  for capital and  $R^2 = 0.99999$  for equity premium.

The average error for the aggregate capital law of motion is 0.0026% percent of the capital stock, while the maximum error is 0.0110% of the capital stock.

### 3.5.3 Comparison

**Table 3: Algorithm execution comparisons (including the obtaining of derivatives)**

Algorithm	Average Iterations	Average run-time
Krusell and Smith (1997)	3.2	26 min. 18 sec.
Proposed algorithm	9.8	17 min. 59 sec.

The use of the proposed algorithm leads to a reduction in the run-time of 32%. The execution performance of the proposed model is measured conservatively, since taking the numerical derivatives to construct the initial Jacobian matrix is considered. Alternatively, if one has a reasonably good guess for the Jacobian matrix (perhaps from the previous simulations of the model with similar parameters), it can be guessed directly, without taking the numeric derivative. If the initial Jacobian was guessed, instead of computed, then the proposed algorithm would take 2 iterations less, and lead to a 46% reduction in run-time.

**Table 4: Bond market errors: absolute average excess demand in terms of percentage of aggregate capital)**

Algorithm	Before imposing market clearing	After imposing market clearing
Krusell and Smith (1997)	0.0949%	0.0021%
Proposed algorithm	0.0950%	0.0019%

After obtaining final laws of motion, the simulation of the model is run with clearing of the bond market in each time period (like in the classical version of the algorithm). This is to compare and show that the obtained laws of motion are of approximately the same accuracy (they are basically approximately identical). In terms of  $R^2$ , the proposed algorithm generates  $R^2$  of 0.99995418 for capital and 0.99999712 for equity premium, while the classical version of the algorithm generates  $R^2$  of 0.99995418 for capital and 0.99999733. Both by looking at the  $R^2$  and Table 4, one can see that the laws of motion produce almost identical results.

### 3.6 Discussion

The main reason for the computational speed-up in the proposed algorithm is avoiding root-finding (finding the bond market-clearing price) for each simulated period  $t$ . However, the proposed algorithm takes more iterations to converge to the true solution. Therefore, the proposed algorithm is able to perform each iteration much faster (on average four and a half times faster), but takes more iterations to converge (on average three times more). However, the speed-up coming from a faster simulation of the economy outweighs the increased number of iterations, which leads to a reduction in total run-time.

The reported speed-up due to the proposed algorithm is conservative. The reason is twofold. First, the reported time and number of iterations includes numerically taking derivatives used to construct the initial guess for the Jacobian matrix  $J$ . If one would have a reasonably good guess for the Jacobian, which is often the case if the changes in parameters are small compared to the previously computed model, then it is possible to avoid the first two iterations of the proposed algorithm. For example, if the values of initial Jacobian were guessed, instead of computed, the proposed algorithm would take 2 iterations less and would lead to a 46% reduction in run-time. The second reason is that the initial guess for the Value function computation stage is always the same, and it is the value of consuming the entire wealth in one period. An alternative option would be to use the value function from the previous iteration as the initial guess for the value function for the current iteration. The choice is also biased towards the classical algorithm from Krusell and Smith (1997), since the proposed algorithm performs more iterations and Value function iterations to converge. Using better (circumstantial) initial value function guesses would decrease the speed-up from the proposed algorithm even more (for example: final guesses from previous iterations).

As mentioned in section 5, all the initial guesses for the laws of motion are such that at least some households have an internal portfolio choice. This is to ensure that the derivative of excess demand with respect to perceived equity premium would not be zero. This condition is important when constructing the initial Jacobian matrix in the proposed algorithm. If the condition is not satisfied, this does not mean that the proposed algorithm cannot be used. One can simply use the classical version of the algorithm until the condition is satisfied,

and then continue updating using the proposed version of the algorithm.<sup>11</sup>

Furthermore, the threshold for the excess demand caused by using the predicted equity premium is 0.1% (on average).<sup>12</sup> This is true for both the classical and the proposed versions of the algorithm. However, the actual excess demand are orders of magnitudes smaller in the classical algorithm, because the classical algorithm imposes the bond market-clearing each period, and the equity premium is then not restricted by the (linear) shape of the perceived laws of motion. However, this should not be perceived as a disadvantage of the proposed algorithm. One can see it only as a way to arrive at the true laws of motion, and then when the correct perceived laws of motion are computed, in the last iteration approximately exact market-clearing can be imposed.

The proposed version of the algorithm is particularly useful in asset pricing models with uninsurable idiosyncratic and aggregate risk. This is because the perturbation methods in the style of Reiter (2009) are not precise when applied to these types of models, as they assume linearity in the aggregate states (Reiter, 2009). To this date, the usual method for solving these types of models are variations of the algorithm described in Krusell and Smith (1997). The proposed algorithm can be used to improve on the classical Krusell-Smith algorithm whenever a market-clearing has to be imposed explicitly,<sup>13</sup> such as models with endogenous labor supply (although one might opt not to use Krusell-Smith algorithms at all).

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<sup>11</sup>For similar reasons, the proposed version of the algorithm tends to perform better when the guess for the equity premium laws of motion are relatively good, and perceived laws of motion for aggregate capital are relatively bad, the classical version of the algorithm tends to do better if the opposite is true.

<sup>12</sup>This may seem like a large value, but the changes in the equity premium producing such excess demand are very small, also by measuring how much they impact the welfare of the agents.

<sup>13</sup>The computation of the model without portfolio choice (Krusell and Smith, 1998) likely cannot be improved using the proposed algorithm, as in the case with only one good the market clears by Walras's law. Therefore, allowing non-clearing of the markets would be superfluous, as we can clear it directly from the budget constraint. One might use a Newton-like method to update the laws of motion for capital, instead of using the regression. However, this will probably require more iterations to arrive at the solution. One can see this in Table 3, where the proposed algorithm takes more iterations to arrive at the solution. The time savings come from not clearing the bond market in each time period  $t$ , and thus performing each iteration is shorter.



## 3.7 Conclusion

This chapter shows how to reduce the run-time of the Krusell-Smith algorithm (Krusell and Smith, 1997) by proposing an alternative version of the algorithm. The reduction in computation time is achieved by avoiding the computationally expensive root-finding procedure to clear the bond markets in every simulated period while finding the correct perceived laws of motion. Instead, the proposed algorithm lets the economy proceed with the uncleared bond markets, and uses the information on the excess demand to update the perceived laws of motion. The guesses on the perceived laws of motion are updated using the Newton-like method described in Broyden (1965).

Measured conservatively, the proposed algorithm leads to a decrease in computation time of 32% in the example model. By using better circumstantial initial guesses on the value function and initial Jacobian matrix, the computational improvement would be even higher.

The described algorithm is useful in reducing the computational time of asset pricing models with uninsurable idiosyncratic and aggregate risk, although it can be used in other models that require market-clearing to be explicitly imposed.

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