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Differential embeddings into algebras of topological stable rank 1

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DIFFERENTIAL EMBEDDINGS INTO ALGEBRAS OF TOPOLOGICAL STABLE RANK 1

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In memoriam: H. Garth Dales (1944–2022)

Abstract. We identify a class of *smooth* Banach *-algebras that are differential subalgebras of commutative C*-algebras whose openness of multiplication is completely determined by the topological stable rank of the target C*-algebra. We then show that group algebras of Abelian groups of unbounded exponent fail to have uniformly open convolution. Finally, we completely characterise in the complex case (uniform) openness of multiplication in algebras of continuous functions in terms of the covering dimension.

1. Introduction

Gelfand's proof of Wiener's lemma [20], which asserts that the reciprocal of a function with absolutely convergent Fourier series that does not vanish anywhere has absolutely convergent Fourier series too, was central to the development of Banach-algebraic ramifications of Harmonic Analysis. Wiener's lemma may be rephrased as follows: the algebra of absolutely convergent Fourier series is inverse-closed when embedded into the algebra of all continuous functions on the unit circle. In the present paper we shall be concerned with algebras that have even a stronger property, namely that the norm of an invertible element is a function of the norm of the element and its supremum norm of its Gelfand transform (see Lemma 1.1); a property that the algebra of absolutely convergent series actually lacks.

When A is a Banach algebra and $i\colon A \to B$ is a continuous injective homomorphism, we say that A *admits norm-controlled inversion* in B, whenever there exists a function $h\colon (0,\infty)^2 \to (0,\infty)$ so that for every element $a \in A$, which is invertible in B, we have

$$\|a^{-1}\|_A \leqslant h(\|a\|_A, \|i(a^{-1})\|_B).$$

Since the embedding i is injective an invertible element a in algebra A remains invertible in B (strcitly speaking, $i(a)$ is invertible), however in this case norm-controlled inversion of A in B implies that the inverses are actually in A (*i.e.*, $i(A)$ is inverse-closed in B).

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Following Nikolskii [29], for $\delta > 1$, we say that a Banach algebra A is *δ-visible in B*, whenever

$$\psi\big(\delta^{-1}\big) = \sup\{\|a^{-1}\|_A \colon a \in A, \|a\|_A \leqslant 1, \|i(a^{-1})\|_B \leqslant \delta\} < \infty. \tag{1}$$

Then A admits norm-controlled inversion in B if and only if it is δ-visible in B for all $\delta > 1$. Should that be the case, the norm-control function h can be arranged to be

$$h(\|a\|_A, \|i(a^{-1})\|_B) = \frac{1}{\|a\|_A}\psi\big(\|a\|_A\|i(a^{-1})\|_B\big). \tag{2}$$

For a commutative (*-)semi-simple Banach (*-)algebra A we say, for short, that A *admits norm-controlled inversion*, whenever it admits norm-controlled inversion in $C(\Phi_A)$, the space of continuous functions on the maximal (*-)ideal space Φ_A of A, when embedded by the Gelfand transform. (For a commutative (*-)semi-simple Banach (*-)algebra the Gelfand transform is injective; see also [15, Proposition 30.2(ii)].)

The Wiener (convolution) algebra $\ell_1(\mathbb{Z})$ is a primary example of a commutative Banach *-algebra without the norm-controlled inversion in $C(\mathbb{T})$, the algebra of continuous functions on the unit circle. Indeed, in [29] Nikolskii showed that for $\delta \geqslant 2$ we have $\psi(\delta^{-1}) = \infty$, where ψ is given in (1). The same conclusion extends to convolution algebras $\ell_1(G)$ for any infinite Abelian group G that lack norm-controlled inversion in $C(\widehat{G})$, the algebra of continuous functions on the Pontryagin-dual group to G, but this behaviour appears rather exceptional. On the positive side, various weighted algebras of Fourier series (see [17]) as well as algebras of Lipschitz functions on compact subsets of Euclidean spaces enjoy the norm-controlled inversion.

Norm-controlled inversion is a consequence of a smoothness of the embedding as observed by Blackadar and Cuntz [9]. More specifically, let $i\colon A \to B$ be an injective homomorphism of Banach algebras. Then A is a *differential subalgebra* of B whenever there exists $D > 0$ such that for all $a, b \in A$ we have

$$\|ab\|_A \leqslant D(\|a\|_A\|i(b)\|_B + \|i(a)\|_B\|b\|_A). \tag{3}$$

When A and B are Banach *-algebras, we additionally require that i is *-preserving (hence it preserves the modulai of elements); we omit the symbol i, when the map i is clear from the context (for example, when it is the formal inclusion of algebras). Differential subalgebras (especially of C*-algebras) have been extensively studied; see, *e.g.*, [24, 21, 22, 31].

In the sequel, we shall make use of [21, Theorem 1.1(i)] that we record below:

Lemma 1.1. *Differential *-subalgebras of C*-algebras have norm-controlled inversion.*

Note that the condition of being a differential norm is *extremely weak* assumption, and norms satisfying (3) meet the *weak* form of smoothness (see [21, Theorem 1.1(v)]).

In the present paper we investigate the possible connections between smoothness of an embedding of Banach algebras and topological stable rank 1 (which for unital Banach algebras this is equivalent to having dense invertible group) with openness of multiplication of a given Banach algebra A, *i.e.*, the question of for which Banach algebras the map $m\colon A \times A \to A$ given by $m(a,b) = ab$ $(a, b \in A)$ is open, that is, it maps open sets to open sets. The problem of which Banach algebras have open multiplication was systematically investigated by Draga and the first-named author in [16], where it was observed that unital Banach algebras with open multiplication have topological stable rank 1 but not *vice versa*. For example, matrix algebras M_n have topological stable rank 1 but multiplication therein is not open unless $n = 1$ ([7]). On the other hand, the problem of openness of convolution in $\ell_1(\mathbb{Z})$ is persistently *open*.

Various function algebras have been observed to have open multiplication (even uniformly, where a map $f\colon X \to Y$ is uniformly open whenever for every $\varepsilon > 0$ there is $\delta > 0$ such that for all $x \in X$ one has $f(B(x,\varepsilon)) \supseteq B(f(x),\delta)$): spaces of continuous/bounded functions: [2, 3, 4, 6, 5, 10, 25, 30] and spaces of functions of bounded variation: [11, 26]. The first main result of the paper unifies various approaches to openness of multiplication. (All unexplained terminology may be found in the subsequent section.)

Theorem A. *Suppose that A is a unital symmetric dual Banach *-algebra that is a dense differential subalgebra of $C(X)$ with X zero-dimensional. If A shares with X densely many points, then A has open multiplication.*

Theorem A applies, in particular, to $A = C(X)$, which may be interpreted as complex counterpart of the main result of [8].

The proofs of the main results of [11, 26] centre around showing that the algebras of functions of p-bounded variation (for $p = 1$ and $p \in (1,\infty)$, respectively) are approximable by jointly non-degenerate products. Our theorem appears to be the first general providing sufficient conditions for openness in a given commutative Banach *-algebra (*i.e.*, a self-adjoint function algebra).

In [16, Corollary 4.13], Draga and the first-named author proved that $\ell_1(\mathbb{Z})$ does not have uniformly open convolution (whether it is open or not remains an open problem). We strengthen this result by showing that having bounded exponent (that is, $\sup_{g\in G} o(g) < \infty$, where $o(g)$ denotes the rank of an element $g \in G$)) is a sufficient condition for not having uniformly open convolution.

Theorem B. *Let G be an Abelian group of unbounded exponent, i.e., $\sup_{g\in G} o(g) = \infty$. Then convolution in $\ell_1(G)$ is not uniformly open.*

By Prüfer's first theorem (see [27, p. 173]), every Abelian group of bounded exponent is isomorphic to a direct sum of a finite number of finite cyclic groups and a direct sum of possibly infinitely many copies of a fixed finite cyclic group, so if one seeks examples of group convolution algebras with uniform multiplication, the only candidates to be found are groups that are effectively direct sums of any number of copies of a fixed cyclic group.

Finally, we establish a complex counterpart of Komisarski's result [25] linking openness of multiplication in the real algebra $C(X)$ of continuous functions on a compact space X with the covering dimension of X. In the complex case $C(X)$ has open multiplication if and only if X is zero-dimensional in which case multiplication is actually uniformly open with $\delta(\varepsilon) = \varepsilon^2/4$ ($\varepsilon > 0$). (See also [16, Proposition 4.16] for an alternative proof using direct limits that does not depend on the scalar field; we refer to [18] for a modern exposition of dimension theory and standard facts thereof.)

Theorem C. *Let X be a compact space. Then the following conditions are equivalent for the algebra $C(X)$ of continuous complex-valued functions on X:*

(i) *X has open multiplication,*
(ii) *X has uniformly open multiplication,*
(iii) *the covering dimension of X is at most* 1.

Moreover, $C(X)$ have equi-uniformly open multiplications for all compact spaces of dimension at most 1.

A necessary condition for a unital Banach algebra to have open multiplication is topological stable rank 1, that is, having dense group of invertibles. For a compact space X of dimension at least 2, this is not the case, so $C(X)$ does not have open multiplication ([16, Proposition 4.4]). The proof of Theorem C is split into three cases.

- The first one uses a reduction to spaces being topological (planar) realisations of graphs. Here we rely on certain ideas from an unpublished manuscript of Behrends for which we have a permission to include them in the present note. We kindly acknowledge this crucial contribution from Professor Behrends establishing the case of $X = [0, 1]$.
- Then we proceed via an inverse limit argument to conclude the result for all compact metric spaces of dimension at most 1.
- Finally, we apply a result of Madrešić [28] to conclude the general non-metrisable case from equi-uniform openness of multiplication of $C(X)$ for all 1-dimensional compact metric spaces X.

2. Preliminaries

2.1. **Banach algebras.** Compact spaces are assumed to be Hausdorff. All Banach algebras considered in this paper are over $\mathbb{C}$, the field of complex scalars, unless otherwise specified. We denote by $\mathbb{T}$ the unit circle in the complex plane.

A Banach algebra A has *topological stable rank 1*, whenever invertible elements are dense in A if A is unital or in the unitisation of A otherwise. Algebras whose elements have zero-dimensional spectra have topological stable rank 1 and include biduals of $C(X)$ for a compact space X, the algebra of functions of bounded variation, or the algebra of compact operators on a Banach space; we refer to [16, Section 2] for more details.

2.1.1. *Arens regularity, dual Banach algebras.* As observed by Arens [1], the biudal of a Banach algebra may be naturally endowed with two, rather than single one, multiplications (the left and right Arens products, denoted $\square$, $\diamond$, respectively). Even though these multiplications may be explicitly defined, the following 'computation' rule is perhaps easier to comprehend: for $f, g \in A^{**}$, where A is a Banach algebra, by Goldstine's theorem, one may choose bounded nets (f_j), (g_i) from A that are weak* convergent to f and g, respectively. Then

- $f \,\square\, g = \lim_j \lim_i f_j g_i$,
- $f \diamond g = \lim_i \lim_j f_j g_i$

are well-defined and do not depend on the choice of the approximating nets. A Banach algebra is *Arens-regular* when the two multiplications coincide. For a locally compact space X, the algebra $C_0(X)$ is Arens-regular, but a group G, the group algebra $\ell_1(G)$ (see Section 2.5) is Arens-regular if and only if G is finite ([33]).

A *dual Banach algebra* is a Banach algebra A that is a dual space to some Banach space E whose multiplication is separately $\sigma(A, E)$-continuous. Notable examples of dual Banach algebras include von Neumann algebras, Banach algebras that are reflexive as Banach spaces, or biduals of Arens-regular Banach algebras; see [13, Section 5] for more details.

Suppose that A is a dual Banach algebra and let $i\colon A \to C(X)$ be an injective homomorphism for some compact space X. We say that A *shares with* X *densely points* whenever there exists a dense set $Q \subset X$ such that $i^*(\delta_x) \in E$ $(x \in Q)$, *i.e.*, the functionals $i^*(\delta_x)$ $(x \in Q)$ are $\sigma(A, E)$-continuous (here $\delta_x \in C(X)^*$ is the Dirac delta evaluation functional at $x \in X$). Since for an Arens-regular Banach algebra the bidual endowed with the unique Arens product is a dual Banach algebra, we may record the following lemma.

Lemma 2.1. *Let A be a unital Arens-regular Banach algebra and let $i\colon A \to C(X)$ be an injective algebra homomorphism with dense range. Then A^{**} shares with the maximal ideal space of $C(X)^{**}$ densely many points.*

Proof. Since A is Arens-regular, A^{**} is naturally a dual Banach algebra with the unique Arens product. Since i^{***} extends i^*, for every $x \in X$, we have $i^{***}(\delta_x) = i^*(\delta_x) \in A^*$, so that $i^*(\delta_x)$ is $\sigma(A^{**}, A^*)$-continuous. It remains to invoke the fact that X can be identified with an open dense subset of the maximal ideal space of $C(X)^{**}$ via $x \mapsto (\delta_x)^{**} = \delta_{\iota(x)}$ for some point $\iota(x)$ in the maximal ideal space of $C(X)^{**}$ (see the discussion after [12, Definition 3.3]); the map ι is necessarily discontinuous unless X is finite). □

Let us record two permanence properties of differential embeddings; even though we shall not utilise (ii) in the present paper, we keep it for possible future reference.

Lemma 2.2. *Let A be a Banach algebra continuously embedded into another Banach algebra B by a homomorphism $i\colon A \to B$ as a differential subalgebra.*

(i) *Consider both in A^{**} and B^{**} either left or right Arens products. Then in either setting $i^{**}\colon A^{**} \to B^{**}$ is a differential embedding.*
(ii) *Let $\mathcal{U}$ be an ultrafilter. Then $i^{\mathcal{U}}\colon A^{\mathcal{U}} \to B^{\mathcal{U}}$ is a differential embedding between the respective ultrapowers.*

Proof. Case 1. Let $\{a_\alpha\}, \{b_\beta\} \subset A$ be bounded nets $\sigma(A^{**}, A^*)$-convergent to $a, b \in A^{**}$ respectively, satisfying for any α, β conditions $\|a_\alpha\|_A \leqslant \|a\|_{A^{**}}$ and $\|b_\beta\|_B \leqslant \|b\|_{B^{**}}$ (it is possible by the Krein–Šmulyan theorem). Then

$$\begin{aligned}\|ab\|_{A^{**}} &\leqslant \liminf_{\alpha,\beta} \|a_\alpha b_\beta\| \\ &\leqslant D\cdot \liminf_{\alpha,\beta} \big(\|a_\alpha\|_A\|i(b_\beta)\|_B + \|i(a_\alpha)\|_B\|b_\beta\|_A\big) \\ &\leqslant D\big(\|a\|_{A^{**}}\|i^{**}(b)\|_{B^{**}} + \|i^{**}(a)\|_{B^{**}}\|b\|_{A^{**}}\big).\end{aligned}$$

Case 2. Let $a = [(a_\gamma)_{\gamma\in\Gamma}]$, $b = [(b_\gamma)_{\gamma\in\Gamma}] \in A^{\mathcal{U}}$. Then

$$\begin{aligned}\|ab\|_{A^{\mathcal{U}}} &= \lim_{\gamma,\mathcal{U}} \big\|a_\gamma b_\gamma\big\|_A \\ &\leqslant \lim_{\gamma,\mathcal{U}} D\Big(\big\|a_\gamma\big\|_A\big\|i(b_\gamma)\big\|_B + \big\|i(a_\gamma)\big\|_B\big\|b_\gamma\big\|_A\Big) \\ &\leqslant D\Big(\lim_{\gamma,\mathcal{U}}\big\|a_\gamma\big\|_A\cdot\lim_{\gamma,\mathcal{U}}\big\|i(b_\gamma)\big\|_B + \lim_{\gamma,\mathcal{U}}\big\|i(a_\gamma)\big\|_B\cdot\lim_{\gamma,\mathcal{U}}\big\|b_\gamma\big\|_A\Big) \\ &= D\Big(\big\|a\big\|_{A^{\mathcal{U}}}\big\|i^{\mathcal{U}}(b)\big\|_{B^{\mathcal{U}}} + \big\|i^{\mathcal{U}}(a)\big\|_{B^{\mathcal{U}}}\big\|b\big\|_{A^{\mathcal{U}}}\Big).\end{aligned}$$

□

2.2. **Banach *-algebras.** Let A be a unital Banach *-algebra. In this setting, for $a \in A$ we interpret $|a|^2$ as a^*a. We say that elements a, b in A are *jointly non-degenerate*, when $|a|^2 + |b|^2$ is invertible. When X is a compact space and $a, b \in C(X)$, we sometimes say that elements with $|a|^2 + |b|^2 \geqslant \eta$ (for some $\eta > 0$) are *jointly η-non-degenerate*. Let us introduce the following definition.

Definition 2.3. A unital Banach *-algebra A is *approximable by jointly non-degenerate products* whenever for all $a, b \in A$ and $\varepsilon > 0$ there exist jointly non-degenerate elements $a', b' \in A$ with $\|a - a'\|, \|b - b'\| < \varepsilon$ such that $ab = a'b'$.

Remark 1. It is readily seen that $C(X)$ for a zero-dimensional compact space X has this property. Indeed, let $f, g \in C(X)$ and $\varepsilon > 0$. Consider the sets

- $D_1 = \{x \in X : |f(x)| \geqslant \varepsilon/3\}$
- $D_2 = \{x \in X : |g(x)| \geqslant \varepsilon/3\}$
- $D_3 = \{x \in X : |f(x)|, |g(x)| \leqslant \varepsilon/2\}$.

Certainly, the sets D_1, D_2, D_3 are closed and cover the space X. As X is zero-dimensional, there exist pairwise clopen sets $D_1' \subseteq D_1$, $D_2' \subseteq D_2$, and $D_3' \subseteq D_3$ that still cover X, *i.e.*, $X = D_1' \cup D_2' \cup D_3'$. Let $f' = f \cdot \mathbb{1}_{D_1'\cup D_2'} + \frac{\varepsilon}{2}\mathbb{1}_{D_3'}$ and $g' = g \cdot \mathbb{1}_{D_1'\cup D_2'} + \frac{2}{\varepsilon} fg\mathbb{1}_{D_3'}$. Then f', g' are the sought jointly non-degenerate approximants. On the other hand, as $C(X)$ for $X = [0, 1]$ and compact spaces alike readily not approximable by jointly non-degenerate issues due to connectedness.

Kowalczyk and Turowska [26] showed that the algebra $BV[0, 1]$ of functions of bounded variation on the unit interval is approximable by jointly non-degenerate products and Canarias, Karlovich, and Shargorodsky [11] extended this result to alebras of bounded p-variation on the interval as well as certain further function algebras.

2.3. **Ultraproducts.** Ultraproducts of mathematical structures usually come in two main guises: the algebraic one (first-order) and the analytic one (second-order). Let us briefly summarise the link between these in the context of groups and their group algebras. This has been essentially developed by Daws in [14, Section 5.4] and further explained in [16, Section 2.3.2].

Let $(S_\gamma)_{\gamma\in\Gamma}$ be an infinite collection of semigroups and let $\mathcal{U}$ be an ultrafilter on Γ. The (algebraic) *ultraproduct* $\prod^{\mathcal{U}}_{\gamma\in\Gamma} S_\gamma$ with respect to $\mathcal{U}$ (denoted $S^{\mathcal{U}}$ when $S_\gamma = S$ for all $\gamma\in\Gamma$ and then termed the *ultrapower* of S with respect to $\mathcal{U}$) is the quotient of the direct product $\prod_{\gamma\in\Gamma} S_\gamma$ by the congruence

$$(g_\gamma)_{\gamma\in\Gamma} \sim (h_\gamma)_{\gamma\in\Gamma} \quad \text{if and only if} \quad \{\gamma\in\Gamma\colon g_\gamma = h_\gamma\}\in\mathcal{U}.$$

Then the just-defined ultraproduct is naturally a semigroup/group/Abelian group if S_γ are semigroups/groups/Abelian groups for $\gamma\in\Gamma$.

Let $(A_\gamma)_{\gamma\in\Gamma}$ be an infinite collection of Banach spaces. Then the $\ell_\infty(\Gamma)$-direct sum $A = (\bigoplus_{\gamma\in\Gamma} A_\gamma)_{\ell_\infty(\Gamma)}$, that is, the space of all tuples $(x_\gamma)_{\gamma\in\Gamma}$ with $x_\gamma\in A_\gamma$ $(\gamma\in\Gamma)$ and $\sup_{\gamma\in\Gamma}\|x_\gamma\| < \infty$ is a Banach space under the supremum norm. Moreover, the subspace $J = c_0^{\mathcal{U}}(A_\gamma)_{\gamma\in\Gamma}$ comprising all tuples $(x_\gamma)_{\gamma\in\Gamma}$ such that $\lim_{\gamma\to\mathcal{U}}\|x_\gamma\| = 0$ is closed. The (Banach-space) *ultraproduct* $\prod^{\mathcal{U}}_{\gamma\in\Gamma} A_\gamma$ of $(A_\gamma)_{\gamma\in\Gamma}$ with respect to $\mathcal{U}$ is the quotient space A/J. If A_γ $(\gamma\in\Gamma)$ are Banach algebras, then naturally so is A and J is then a closed ideal therein. Consequently, the ultraproduct is a Banach algebra. Let us record formally a link between these two constructions.

Lemma 2.4. *Let $(S_\gamma)_{\gamma\in\Gamma}$ be an infinite collection of semigroups and let $\mathcal{U}$ be a countably incomplete ultrafilter on Γ. Then there exists a unique contractive homomorphism*

$$\iota\colon \prod_{\gamma\in\Gamma}^{\mathcal{U}} \ell_1(S_\gamma) \to \ell_1\Big(\prod_{\gamma\in\Gamma}^{\mathcal{U}} S_\gamma\Big) \tag{4}$$

that satisfies

$$\iota\Big(\big[(e_{g_\gamma})_{\gamma\in\Gamma}\big]\Big) = e_{[(g_\gamma)_{\gamma\in\Gamma}]} \qquad \Big(\big[(e_{g_\gamma})_{\gamma\in\Gamma}\big] \in \prod_{\gamma\in\Gamma}^{\mathcal{U}} \ell_1(S_\gamma)\Big).$$

2.4. **Abelian groups.** Let G be a group. For $g\in G$ we denote by $o(g)$ the order of the element g. For a (locally compact) Abelian group G we denote by $\widehat{G}$ the Pontryagin dual group of G; for details and basic properties concerning this duality we refer to [23, Chapter 6].

If G is an (Abelian) divisible group, that is, for any $g\in G$ and $n\in\mathbb{N}$ there is $h\in G$ such that $g = nh$, then G is an injective object in the category Abelian groups, which means that for any Abelian groups $H_1\subset H_2$, every homomorphism $\varphi\colon H_1\to G$ extends to a homomorphism $\overline{\varphi}\colon H_2\to G$. Direct sums of arbitrary many copies of $\mathbb{Q}$, the additive group of rationals, are divisible.

Let us record for the future reference the following observation, likely well known to algebraically-oriented model theorists.

Lemma 2.5. *Suppose that G is an Abelian group with $\sup_{g\in G} o(g)=\infty$. Then $\mathbb{Z}^{(\mathbb{R})}$ embeds into some ultrapower of G with respect to an ultrafilter on $\mathbb{N}$.*

Proof. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $(g_n)_{n=1}^\infty$ be a sequence in G such that $\sup_n o(g_n)=\infty$. Then $g=[(g_n)_{n=1}^\infty]$ has infinite order in $H=G^{\mathcal{U}}$. Let $\mathcal{A}$ be an almost disjoint family of infinite subsets of $\mathbb{N}$ that has cardinality continuum. Then all but at most one elements $\mathcal{A}$ are not in $\mathcal{U}$ (as $\mathcal{U}$ is non-principal and closed under finite intersections), so let us assume that $\mathcal{A}\subset\mathcal{U}'$. For each $A\in\mathcal{A}$ we set

$$g_A(i)=\begin{cases} g, & i\notin A,\\ 0, & i\in A.\end{cases}\quad (i\in\mathbb{N}).$$

Then $h_A=[(g_A(i))_{i=1}^\infty]\in H^{\mathcal{U}}$ and $o(h_A)=\infty$ $(A\in\mathcal{A})$. Moreover, $\{h_A\colon A\in\mathcal{A}\}$ is a $\mathbb{Z}$-linearly independent set of cardinality continuum. As such, the subgroup it generates is isomorphic to $\mathbb{Z}^{(\mathbb{R})}$. It remains to notice that canonically $(G^{\mathcal{U}})^{\mathcal{U}}\cong G^{\mathcal{U}\otimes\mathcal{U}}$, as required. □

2.5. **Semigroup algebras.** Let S be a semigroup written multiplicatively. In the Banach space $\ell_1(S)$ one can define a convolution product by

$$x*y=\sum_{t\in S}\Big(\sum_{r\cdot s=t}x_r y_s\Big)e_t\quad (x=(x_s)_{s\in S},\ y=(y_s)_{s\in S}\in\ell_1(S)),$$

where $(e_s)_{s\in S}$ is the canonical unit vector basis of $\ell_1(S)$, together with $\ell_1(S)$ becomes a Banach algebra. For the additive semigroup of natural numbers, the convolution in $\ell_1(\mathbb{N})$ renders the familiar Cauchy product.

Suppose that $T\subseteq S$ is a subsemigroup. Then $\ell_1(T)$ is naturally a closed subspace of $\ell_1(S)$, which is moreover a closed subalgebra. Every surjective semigroup homomorphism $\vartheta\colon T\to S$ implements a surjective homomorphism $\iota_\vartheta\colon \ell_1(T)\to\ell_1(S)$ on the Banach-algebra level by the action

$$\iota_\vartheta e_t=e_{\vartheta(t)}\quad (t\in T). \tag{5}$$

When G is an Abelian (discrete) group, then the (compact) dual group $\widehat{G}$ is the maximal ideal space of the convolution algebra $\ell_1(G)$. More information on semigroup algebras may be found in [12, Chapter 4].

3. Proofs of Theorems A and B

The crux of Theorem A lies at the subsequent lemma whose proof shares with the proofs of the main results of [26] and [11] the idea for the construction of the approximation scheme for sought elements; the result itself is however more general and so are the techniques applied along the way.

Lemma 3.1. *Suppose that A is a unital symmetric Banach *-algebra such that there exists an injective *-homomorphisim $i\colon A\to C(X)$ for some compact space X such that A has norm-controlled inversion in $C(X)$. Let us consider either case:*

- *$A=C(X)$,*
- *$A=E^*$ is a dual Banach algebra that shares with X densely many points.*

Then multiplication in A is open at all pairs of jointly non-degenerate elements.

Furthermore, suppose that i has dense range in $C(X)$. If A has open multiplication, then the maximal ideal space of A is of dimension at most 1.

Proof. Suppose that A has norm-controlled inversion implemented by a *-homomorphism $i\colon A \to C(X)$. We have

$$\|i(f)\|_\infty = \sup_{x\in X} |(i(f))(x)| \leqslant C\|f\|_A \quad (f \in A), \tag{6}$$

where $C = \|i\| \geqslant 1$. Suppose that $F, G \in A$ are jointly non-degenerate (in particular, $|F| + |G|$ is invertible in $C(X)$ being nowhere zero). Fix $\varepsilon \in (0, 1)$ and let

$$\gamma := \min\left\{1, \frac{1}{2} \inf_{x\in X} \big((|i(F))(x)| + |(i(G))(x)|\big)\right\} \tag{7}$$

Set

$$K := 2 \cdot \max\big\{\|F\|_A, \|G\|_A, 1\big\}, \tag{8}$$

$$\widehat{T} := \frac{2C}{\gamma^2} \cdot \psi\left(\frac{4K^2}{\gamma^2}\right) > 0, \tag{9}$$

where the function ψ satisfies (1). Moreover, let $T := \max\{\widehat{T}, 1\}$. Pick an arbitrary element $H \in A$ so that

$$\|H\|_A < \frac{\varepsilon \cdot \gamma}{CK^3T^2} \tag{10}$$

and consider

$$F_0 := F, \quad G_0 := G, \quad H_0 := H \tag{11}$$

We then define recursively the sequences $(F_n)_{n=0}^\infty$, $(G_n)_{n=0}^\infty$, and $(H_n)_{n=0}^\infty$ by

$$F_{n+1} := F_n + \frac{H_n\overline{G_n}}{|F_n|^2 + |G_n|^2}, G_{n+1} := G_n + \frac{H_n\overline{F_n}}{|F_n|^2 + |G_n|^2}, H_{n+1} := -\frac{H_n^2\overline{F_nG_n}}{(|F_n|^2 + |G_n|^2)^2}. \tag{12}$$

We *claim* that

(i)
$$F_nG_n + H_n = FG + H \quad (n = 0, 1, 2, \ldots),$$

(ii)
$$\|F_n\|_A, \|G_n\|_A \leqslant \tfrac{1}{2}K + 1 - 2^{-n} < K,$$

(iii)
$$\inf_{x\in X} \big(|(i(F_n))(x)| + |(i(G_n))(x)|\big) \geqslant \gamma + \gamma \cdot 2^{-n} > 0,$$

(iv)
$$\|H_n\|_A \leqslant \frac{1}{2^n} \cdot \frac{\varepsilon \cdot \gamma}{CK^3T^2}.$$

Note that (iii) implies that sequences (12) are well defined. We will prove these claims by induction.

It follows from (11) that $F_0G_0 + H_0 = FG + H$. We obtain from (7)–(11) that

- $\|F_0\|_A = \|F\|_A \leqslant K/2$,
- $\|G_0\|_A = \|G\|_A \leqslant K/2$,
- $\|H_0\|_A = \|H\|_A < \frac{\varepsilon\cdot\gamma}{CK^3T^2}$,
- $\inf_{x\in X}\big(|F_0(x)|+|G_0(x)|\big) = \inf_{x\in X}\big(|F(x)|+|G(x)|\big) \geqslant 2\gamma > 0$.

That is, (i)–(iv) are satisfied for $n=0$.

Now we assume that (i)–(iv) are fulfilled for some $n = 0, 1, 2, \ldots$ Consequently, sequences (12) are well defined. Then, taking into account (8), we see that $K/2 \geqslant 1$ and

$$F_nG_n + H_n = FG + H, \tag{13}$$

$$\|F_n\|_A \leqslant \frac{K}{2} + 1 - 2^{-n} < K, \tag{14}$$

$$\|G_n\|_A \leqslant \frac{K}{2} + 1 - 2^{-n} < K, \tag{15}$$

$$\inf_{x\in X}\big(|(i(F_n))(x)| + |(i(G_n))(x)|\big) \geqslant \gamma + \gamma\cdot 2^{-n} > \gamma, \tag{16}$$

$$\|H_n\|_A \leqslant \varepsilon\cdot 2^{-n}\cdot\frac{\gamma}{CK^3T^2}. \tag{17}$$

Let us show that (i)–(iv) are fulfilled for $n+1$.

For (i), it follows from (12)–(13) that

$$\begin{aligned}
F_{n+1}G_{n+1} + H_{n+1} &= \left(F_n + \frac{H_n\cdot\overline{G_n}}{|F_n|^2+|G_n|^2}\right)\left(G_n + \frac{H_n\cdot\overline{F_n}}{|F_n|^2+|G_n|^2}\right) - \frac{H_n^2\cdot\overline{F_nG_n}}{(|F_n|^2+|G_n|^2)^2}\\
&= F_nG_n + H_n\frac{F_n\overline{F_n}+G_n\overline{G_n}}{|F_n|^2+|G_n|^2} + H_n^2\frac{\overline{F_nG_n}}{(|F_n|^2+|G_n|^2)^2} - H_n^2\frac{\overline{F_nG_n}}{(|F_n|^2+|G_n|^2)^2}\\
&= F_nG_n + H_n = FG + H.
\end{aligned}$$

Hence, (i) is satisfied for $n+1$.

As for (ii), using (14)–(15) we conclude that

$$\begin{aligned}
\||F_n|^2+|G_n|^2\|_A &\leqslant \|F_n\cdot\overline{F_n}\|_A + \|G_n\cdot\overline{G_n}\|_A\\
&\leqslant \|F_n\|_A\|\overline{F_n}\|_A + \|G_n\|_A\|\overline{G_n}\|_A\\
&= \|F_n\|_A^2 + \|G_n\|_A^2\\
&\leqslant 2K^2.
\end{aligned} \tag{18}$$

It follows from (16) that

$$\begin{aligned}
\gamma^2 &\leqslant \inf_{x\in X}\big(|(i(F_n))(x)| + |(i(G_n))(x)|\big)^2\\
&= \inf_{x\in X}(|(i(F_n))(x)|^2 + 2|(i(F_n))(x)|\cdot|(i(G_n))(x)| + |(i(G_n))(x)|^2)\\
&\leqslant 2\inf_{x\in X}\big(|(i(F_n))(x)|^2 + |(i(G_n))(x)|^2\big),
\end{aligned}$$

hence

$$\sup_{x\in X}\left(|(i(F_n))(x)|^2+|(i(G_n))(x)|^2\right)\geqslant \inf_{x\in X}\left(|(i(F_n))(x)|^2+|(i(G_n))(x)|^2\right)\geqslant \frac{\gamma^2}{2}>0. \tag{19}$$

By (6) and (19) we obtain

$$\||F_n|^2+|G_n|^2\|_A\geqslant \frac{1}{C}\cdot\frac{\gamma^2}{2}>0. \tag{20}$$

It then follows from (12), (14)–(15) and (19) that

$$\begin{aligned}\|F_{n+1}\|_A&\leqslant \|F_n\|_A+\|H_n\|_A\|G_n\|_A\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_A\\&\leqslant\left(\frac{K}{2}+1-2^{-n}\right)+\|H_n\|_A K\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_A.\end{aligned} \tag{21}$$

Since A admits norm-controlled inversion in $C(X)$, we follows from (18), (19), (20) that

$$\begin{aligned}\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_A&\leqslant \frac{1}{\||F_n|^2+|G_n|^2\|_A}\cdot\psi\Big(\||F_n|^2+|G_n|^2\|_A\cdot\left\|\left(|F_n|^2+|G_n|^2\right)^{-1}\right\|_\infty\Big)\\&\leqslant\frac{2C}{\gamma^2}\cdot\psi\left(2K^2\cdot\frac{2}{\gamma^2}\right)=\widehat{T}.\end{aligned} \tag{22}$$

Combining (21)–(22) with (17) and taking into account that $\varepsilon\in(0,1)$, $\gamma\in(0,1]$, $K\geqslant 2$, $C\geqslant 1$, and $T\geqslant 1$, we obtain

$$\begin{aligned}\|F_{n+1}\|_A,\|G_{n+1}\|_A&\leqslant \tfrac{K}{2}+1-2^{-n}+K\widehat{T}\cdot\varepsilon\cdot 2^{-n}\cdot\tfrac{\gamma}{CK^3T^2}\\&\leqslant \tfrac{K}{2}+1-2^{-n}+2^{-n}\cdot\tfrac{1}{2}\\&=\tfrac{K}{2}+1-2^{-n-1}.\end{aligned} \tag{23}$$

Thus, (ii) is fulfilled for $n+1$.

In order to verify (iii), since $\varepsilon\in(0,1)$, $\gamma\in(0,1]$, $K\geqslant 2$, $C\geqslant 1$, and $T\geqslant 1$, it follows from (12), (6), (15), (17) and (22) that for $x\in X$ we have

$$\begin{aligned}|(i(F_n))(x)|&\leqslant|(i(F_{n+1}))(x)|+|(i(H_n))(x)|\frac{|(i(G_n))(x)|}{|(i(F_n))(x)|^2+|(i(G_n))(x)|^2}\\&\leqslant|(i(F_{n+1}))(x)|+C\|H_n\|_A\|G_n\|_A\left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_A\\&\leqslant|(i(F_{n+1}))(x)|+C\cdot\varepsilon\cdot 2^{-n}\frac{\gamma}{CK^3T^2}\cdot K\widehat{T}\\&<|(i(F_{n+1}))(x)|+2^{-n}\cdot\frac{\gamma}{K^2}\\&<|(i(F_{n+1})(x)|+2^{-n}\cdot\frac{\gamma}{4}.\end{aligned}$$

Consequently,

$$|(i(F_{n+1}))(x)|>|(i(F_n))(x)|-2^{-n-2}\gamma\quad(x\in X). \tag{24}$$

In the same way we observe that

$$|(i(G_{n+1}))(x)| > |(i(G_n))(x)| - 2^{-n-2}\gamma \quad (x \in X). \tag{25}$$

We conclude from (16) and (24)–(25) that

$$\begin{aligned}\inf_{x\in X}\big(|(i(F_{n+1}))(x)| + |(i(G_{n+1}))(x)|\big) &\geqslant \inf_{x\in X}\big(|(i(F_n))(x)| + |(i(G_n))(x)|\big) - 2\cdot 2^{-n-2}\gamma\\ &\geqslant \gamma + \gamma\cdot 2^{-n} - \gamma\cdot 2^{-n-1}\\ &= \gamma + \gamma\cdot 2^{-n-1},\end{aligned}$$

so (iii) is fulfilled for $n+1$.

Finally, for (iv), by (14)–(15), (17) and (22), for $\varepsilon \in (0,1)$, $\gamma \in (0,1]$, $K \geqslant 2$, and $C \geqslant 1$, we then have

$$\begin{aligned}\|H_{n+1}\|_A &\leqslant \|H_n\|_A^2 \|\overline{F_n}\|_A \|\overline{G_n}\|_A \left\|\tfrac{1}{|F_n|^2+|G_n|^2}\right\|_A^2\\ &= \|H_n\|_A^2 \|F_n\|_A \|G_n\|_A \left\|\tfrac{1}{|F_n|^2+|G_n|^2}\right\|_A^2\\ &\leqslant \left(\varepsilon\cdot 2^{-n}\cdot \tfrac{\gamma}{CK^3T^2}\right)^2 \cdot K^2\cdot \widehat{T}^2\\ &\leqslant \varepsilon\cdot 2^{-n}\cdot \tfrac{\gamma}{CK^3T^2}\cdot \tfrac{\gamma}{CK^3T^2}\cdot K^2\cdot \widehat{T}^2\\ &\leqslant \varepsilon\cdot 2^{-n}\cdot \tfrac{\gamma}{CK^3T^2}\cdot \tfrac{1}{K}\\ &\leqslant \varepsilon\cdot 2^{-n-1}\cdot \tfrac{\gamma}{CK^3T^2},\end{aligned}$$

which verifies (iv) for $n+1$.

It follows from (6) and (iv) that

$$\lim_{n\to\infty} |(i(H_n))(x)| \leqslant C \lim_{n\to\infty} \|H_n\|_A \leqslant \varepsilon\cdot \frac{\gamma}{K^3T^2} \lim_{n\to\infty} 2^{-n} = 0 \quad (x \in X). \tag{26}$$

Suppose that $m, n \in \mathbb{N}$, $m > n$. For $\varepsilon \in (0,1)$, $\gamma \in (0,1]$, $K \geqslant 2$, $C \geqslant 1$, and $T \geqslant 1$, by (12), (15), (17), (22), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \|F_{n+1} - F_n\|_A &\leqslant \sum_{n=0}^{\infty} \|H_n\|_A \|G_n\|_A \left\|\frac{1}{|F_n|^2+|G_n|^2}\right\|_A\\ &\leqslant \sum_{n=0}^{\infty} \frac{1}{2^n}\cdot \frac{\varepsilon\gamma}{CK^3T^2}\cdot K\widehat{T}\\ &\leqslant \varepsilon\cdot \frac{1}{K^2}\sum_{n=0}^{\infty} 2^{-n}\\ &< \varepsilon\cdot \frac{1}{2}\cdot \sum_{n=0}^{\infty}\frac{1}{2^n} < \varepsilon.\end{aligned} \tag{27}$$

Case 1. From (27) for any $\varepsilon_1 > 0$ there exist N such that for $m, n \in \mathbb{N}, m > n > N$ holds

$$\|F_m - F_n\|_\infty \leqslant \sum_{j=n}^{m-1} \|F_{j+1} - F_j\|_\infty < \varepsilon_1, \tag{28}$$

which means that the sequence $(F_n)_{n=1}^{\infty}$ is uniformly Cauchy, so it converges uniformly to some continuous function f. Similarly, there exists a continuous function g that is the limit of the uniformly convergent sequence $(G_n)_{n=1}^{\infty}$.

In particular we obtain

$$\lim_{n\to\infty} F_n(x) = f(x) \quad \text{and} \quad \lim_{n\to\infty} G_n(x) = g(x). \tag{29}$$

Using (29), (i) and (26), we see that

$$\begin{aligned} f(x)\cdot g(x) &= \lim\nolimits_{n\to\infty} \big(F_n(x)\cdot G_n(x)\big) \\ &= \lim\nolimits_{n\to\infty} \big(F_n(x)\cdot G_n(x) + H_n(x)\big) \\ &= F(x)\cdot G(x) + H(x). \end{aligned} \tag{30}$$

Moreover, from (27) we have

$$\|f - F\|_\infty \leqslant \sum_{n=0}^{\infty} \|F_{n+1} - F_n\|_\infty < \varepsilon. \tag{31}$$

We show that $\|g - G\|_\infty < \varepsilon$ in the same way.

Case 2. A is a dual Banach algebra with $A = E^*$ that shares with X densely many points as witnessed by some dense set $Q \subset X$.

In view of (ii), the sequences $(F_n)_{n=0}^{\infty}$ and $(G_n)_{n=0}^{\infty}$ are uniformly bounded by constant K. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. By the Banach–Alaoglu theorem, $(F_n)_{n=0}^{\infty}$ and $(G_n)_{n=0}^{\infty}$ converge to some elements $f, g \in A$, $\|f\|, \|g\| \leqslant K$ with respect to $\sigma(A,E)$ along $\mathcal{U}$. Using (i) and (26), we see that for $x \in Q$

$$\begin{aligned} (i(f))(x)\cdot(i(g))(x) &= \langle \delta_x, i(fg)\rangle \\ &= \langle i^*(\delta_x), fg\rangle \\ &= \lim_{n\to\mathcal{U}} \langle i^*(\delta_x), F_nG_n\rangle \\ &= \lim_{n\to\mathcal{U}} \langle \delta_x, i(F_nG_n)\rangle \\ &= \lim_{n\to\mathcal{U}} \big((i(F_n))(x)\cdot(i(G_n))(x)\big) \\ &= \lim_{n\to\mathcal{U}} \big((i(F_n))(x)\cdot(i(G_n))(x) + (i(H_n))(x)\big) \end{aligned} \tag{32}$$

nonetheless, it follows from (i) that

$$\|i\big(F_nG_n + H_n - (FG+H)\big)\|_\infty \leqslant C\cdot\|F_nG_n + H_n - (FG+H)\|_A = 0,$$

hence for any $x \in Q$ we have

$$i\big(f(x)g(x)\big) = i\big(F(x)G(x) + H(x)\big). \tag{33}$$

Since Q is dense subset of X, by continuity of i and elements belonging to $C(X)$, there are equal everywhere. This means that

$$fg = FG + H, \tag{34}$$

because i is injective. Similarly, for $x \in Q$

$$\begin{aligned}(i(f))(x)-(i(F))(x) &= \langle \delta_x, i(f-F)\rangle \\ &= \langle i^*(\delta_x), f-F\rangle \\ &= \lim_{n\to\mathcal{U}} \langle i^*(\delta_x), F_n - F\rangle \\ &= \lim_{n\to\mathcal{U}} \langle \delta_x, i(F_n - F)\rangle \\ &= \lim_{n\to\mathcal{U}} \big((i(F_n))(x) - (i(F))(x)\big) \\ &= \lim_{n\to\mathcal{U}} \sum_{j=0}^{n} \big((i(F_{j+1}))(x) - (i(F_j(x))\big)\end{aligned} \tag{35}$$

but from (27) we know that

$$\sum_{n=0}^{\infty} \big\|i(F_{n+1}) - i(F_n)\big\|_\infty \leqslant C \cdot \sum_{n=0}^{\infty} \|F_{n+1} - F_n\|_A,$$

so for any $x \in Q$

$$(i(f))(x)-(i(F))(x) = \sum_{n=0}^{\infty} \big((i(F_{n+1}))(x) - (i(F_n))(x)\big),$$

hence, again by density of Q in X, continuity of i and elements belonging to $C(X)$, we have this equality everywhere. Moreover, since i is injective, we obtain

$$f - F = \sum_{n=0}^{\infty} \big(F_{n+1} - F_n\big),$$

so from (27) we have

$$\|f-F\|_A \leqslant \sum_{n=0}^{\infty} \|F_{n+1} - F_n\|_A < \varepsilon. \tag{36}$$

We show that $\|g-G\|_A < \varepsilon$ in the same way.

In each of the above cases, we have obtained the appropriate functions f and g, which, to simplify the notation, have been marked with the same symbols. So, for every $H \in A$ satisfying (10), there exist f and g in A such that

$$\|f-F\|_A < \varepsilon, \quad \|g-G\|_A < \varepsilon$$

(see respectively (31) or (36)) and $FG + H = fg$ (see respectively (30) or (34)). This means that

$$B_A(F\cdot G, \delta) \subset B_A(F,\varepsilon)\cdot B_A(G,\varepsilon)$$

with $\delta := \varepsilon \cdot \frac{\gamma}{CK^3T^2}$. Hence, the multiplication in A is locally open at the pair $(F,G) \in A^2$.

Suppose now that i has dense range in $C(X)$. By inverse-closedness of A, A has topological stable rank 1 if and only if $C(X)$ has topological stable rank 1. Consequently, if $C(X)$

fails to have dense invertibles (which happens exactly when $\dim X > 1$), then A does not have open multiplication. □

Applying Lemma 1.1 we obtain the following conclusion that proves Theorem A.

Lemma 3.2. *Suppose that A is a unital symmetric Banach *-algebra such that there exists an injective *-homomorphisim $i\colon A \to C(X)$ for some compact space X such that A is a differential subalgebra of $C(X)$. Let us consider either case:*

- *$A = C(X)$,*
- *$A = E^*$ is a dual Banach algebra that shares with X densely many points.*

Then multiplication in A is open at all pairs of jointly non-degenerate elements.

Corollary 3.3. *Let A be a (complex) reflexive Banach space with a K-unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ ($K \geqslant 1$). Then A is naturally a Banach *- algebra when endowed with multiplication*

$$a \cdot b = \sum_{\gamma\in\Gamma} a_\gamma b_\gamma e_\gamma \quad (a = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma, b = \sum_{\gamma\in\Gamma} b_\gamma e_\gamma \in A).$$

and coordinate-wise complex conjugation. Let $A^\#$ denote the unitisation of A. Then $A^\#$ has open multiplication.

Proof. It is clear any pair of elements of $A^\#$ is jointly non-degenerate. Since the basis $(e_\gamma)_{\gamma\in\Gamma}$ is K-unconditional, we have

$$\begin{aligned} \|ab\|_A &= \Big\| \sum_{\gamma\in\Gamma} a_\gamma b_\gamma e_\gamma \Big\|_A \\ &\leqslant K \Big\| \sum_{\gamma\in\Gamma} a_\gamma \cdot \|b\|_{\ell_\infty(\Gamma)} \cdot e_\gamma \Big\|_A \\ &= K\|a\|_A \|b\|_{\ell_\infty(\Gamma)} \\ &\leqslant K(\|a\|_A\|b\|_{\ell_\infty(\Gamma)} + \|a\|_{\ell_\infty(\Gamma)}\|b\|_A). \end{aligned}$$

This means that $A^\#$ is a differential subalgebra of $c(\Gamma)$, the unitisation of the algebra of functions that vanish at infinity on Γ. Since the formal inclusion from $A^\#$ to $c(\Gamma)$ has dense range, the conclusion follows. □

Since the bidual of $C(X)$ is isometric to $C(Z)$ for some compact, zero-dimensional space (in particular, $C(X)^{**}$ has uniformly open multiplication), using Lemmas 2.1 and 2.2 we may record the following corollary.

Corollary 3.4. *Suppose that A is an Arens-regular Banach *-algebra that is densely embedded as a differential subalgebra of $C(X)$ for some compact space X. Then A^{**} has open multiplication at all pairs of jointly non-degenerate elements.*

We now turn our attention to Theorem B.

Proof of Theorem B. By Lemma 2.5, there exists an ultrafilter $\mathcal{U}$ such that $\mathbb{Z}^{(\mathbb{R})}$ embeds into $G^{\mathcal{U}}$. As $\mathbb{Z}^{(\mathbb{R})}$ is a free Abelian group, it admits a surjective homomorphism φ onto $\mathbb{Q}^{(\mathbb{N})}$. Since $\mathbb{Q}^{(\mathbb{N})}$ is divisible, it is an injective object in the category of Abelian groups, so φ extends to a homomorphism $\overline{\varphi}\colon G^{\mathcal{U}} \to \mathbb{Q}^{(\mathbb{N})}$. In particular, the infinite-dimensional space $\widehat{\mathbb{Q}^{(\mathbb{N})}} \cong \mathbb{T}^{\mathbb{N}}$ embeds topologically into $\widehat{G^{\mathcal{U}}}$.

Consequently, $\dim \widehat{G^{\mathcal{U}}} = \infty > 1$. By [16, Corollary 4.10], multiplication in $\ell_1(G^{\mathcal{U}})$ is not open. However, $\ell_1(G^{\mathcal{U}})$ is a quotient of the Banach-algebra ultrapower $(\ell_1(G))^{\mathcal{U}}$ ([16, Section 2.3.2]), so by [16, Corollary 3.3], convolution in $\ell_1(G)$ is not uniformly open. □

4. Proof of Theorem C

The present section is devoted to the proof of Theorem C. We start by proving a special case of $X = [0, 1]$; the argument is a slightly improved version of a proof due to Behrends. We are indebted for his permission to include it here.

Theorem 4.1. *The (complex) algebra $C[0, 1]$ has uniformly open multiplication.*

In order to prove Theorem 4.1 we require a number of auxiliary results.

Anywhere below Δ will denote a set of all $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ such that $|\gamma| = 1$ and the polynomial $\gamma z^2 + \beta z + \alpha$ has two roots of different absolute value. In particular, in this situation, there is a uniquely determined root, so we can introduce the following definition

Definition 4.2. We denote by $Z\colon \Delta \to \mathbb{C}$ the map that assigns to (α, β, γ) the root of the quadratic polynomial $\gamma z^2 + \beta z + \alpha$ with the smaller absolute value.

Remark 2. The root function is locally analytic, so the function Z is continuous.

Let us now fix a non-degenerate interval $[a_0, b_0]$.

Lemma 4.3. *Let $f, g \in C[a_0, b_0]$. If $|f| \geqslant \eta$ for some $\eta > 0$ and $|g| = 1$ then for every $\varepsilon > 0$ there is $\delta > 0$ such that if $d \in C[a_0, b_0]$ and $\|d\| \leqslant \delta$ there is $\phi \in C[a_0, b_0]$ with $\|\phi\| \leqslant \varepsilon$ and*

$$f(t)\phi(t) + g(t)\phi^2(t) = d(t) \quad (t \in [a_0, b_0]).$$

Proof. Fix $(\alpha, \beta, \gamma) \in \mathbb{C}^3$ satisfying $|\beta| \geqslant \eta$, $|\gamma| = 1$ and arbitrary, strictly positive η, ε. It is enough to find $\delta > 0$ such that if $|\alpha| \leqslant \delta$ then $(\alpha, \beta, \gamma) \in \Delta$ and $|Z(\alpha, \beta, \gamma)| \leqslant \varepsilon$. Indeed, by Remark 2, this allows us to define function ϕ as $\phi(t) := Z(-d(t), f(t), g(t))$ for $t \in [a_0, b_0]$.

Denote by z_1, z_2 the roots of the polynomial $\gamma z^2 + \beta z + \alpha$. By Vieta's formulas

$$\gamma\,(z_1 + z_2) = -\beta,$$

hence either $|z_1| \geqslant \eta/2$ or $|z_2| \geqslant \eta/2$. Without loss of generality we may assume that $|z_1| \geqslant \eta/2$.

Again, by Vieta's formulae,

$$\gamma z_1 z_2 = \alpha,$$

so that $z_2 = \alpha/(\gamma z_1)$, hence $|z_2| \leqslant 2|\alpha|/\eta$. Thus, it suffices to choose $|\alpha| \leqslant \delta$ where $\delta > 0$ satisfies $2\delta/\eta \leqslant \varepsilon$ and $2\delta/\eta < \eta/2$. Then $|z_2| < |z_1|$ and $|z_2| \leqslant \varepsilon$, so conclusion follows by the definition of Z. □

Lemma 4.4. *For nonzero complex numbers z and w define $c := i\overline{w}z/|\overline{w}z|$. Then $|c| = 1$ and $|z + cw|^2 = |z|^2 + |w|^2$.*

Proof. The proof is trivial. □

We denote by $\overline{I}$ (respectively I^o) the closure (respectively, the interior) of an, interval.

Lemma 4.5. *Suppose that $I_1, \dots, I_k$ are open subintervals of $[a_0, b_0]$ with pairwise disjoint clousers. Then for any continous function $\phi\colon [a_0, b_0] \setminus \bigcup_j I_j \to \mathbb{T}$ there exists a continuous extension $\psi\colon [a_0, b_0] \to \mathbb{T}$.*

Proof. Since we know the values of ψ at the endpoints of the intervals $\overline{I_j}$ for $j = 1, 2, \dots, k$, we may connect them with any path in $\mathbb{T}$ to define ψ. □

We observe that is has been crucial to worth with complex numbers in the proof of Lemma 4.5 as there is no analogue of this lemma in the real case.

Lemma 4.6. *For any function $h \in C[a_0, b_0]$ and arbitrary $\eta_2 > \eta_1 > 0$ there are pairwise disjoint closed subintervals $J_1, \dots, J_k$ of $[a_0, b_0]$ such that*

$$\{t \in [a_0, b_0] \colon |h(t)| \leqslant \eta_1\} \subset \bigcup_{j=1}^{k} J_j \subset \{t \in [a_0, b_0] \colon |h(t)| < \eta_2\}.$$

Proof. Define set $K := \{t \colon |h(t)| \leqslant \eta_1\}$ and open set $O := \{t \colon |h(t)| < \eta_2\}$. Since $K \subset O$ we may find for any $t \in K$ an open subinterval I_t in such a way that $t \in I_t \subset \overline{I_t} \subset O$. Moreover, since K is compact, it is possible to cover K with finitely many of them. Hence their closures are desired intervals J_j (it might be necessary to pass to unions if they are not disjoint). □

Lemma 4.7. *Let $h_1, h_2 \in C[a_0, b_0]$. Suppose that h_1 and h_2 are jointly η^2-non-degenerate for some $\eta > 0$. Then there are continuous $\beta_1, \beta_2 \colon [a_0, b_0] \to \mathbb{T}$ such that*

$$|h_1(t)\beta_1(t) + h_2(t)\beta_2(t)| \geqslant \eta \quad (t \in [a_0, b_0]).$$

Proof. By compactness, we may find $\eta_0 > 0$ such that that h_1 and h_2 are jointly $(\eta^2 + \eta_0^2)$-non-degenerate; we also will assume that $2\eta_0^2 < \eta^2$. Next we choose, with a $\tau \in [0, 1]$ that will be fixed later, pairwise disjoint closed intervals $J_1, \dots, J_k$ and pairwise disjoint closed intervals $J_{k+1}, \dots, J_l$ such that

$$\left\{t \colon |h_1(t)| \leqslant \tau \cdot \frac{\eta_0}{2}\right\} \subset \bigcup_{j=1}^{k} J_j \subset \{t \colon |h_1(t)| < \tau \cdot \eta_0\}$$

and

$$\left\{t \colon |h_2(t)| \leqslant \tau \cdot \frac{\eta_0}{2}\right\} \subset \bigcup_{j=k+1}^{l} J_j \subset \{t \colon |h_2(t)| < \tau \cdot \eta_0\}$$

(see Lemma 4.6). As a consequence of $2\eta_0^2 < \eta^2$ no J_j with $j \leqslant k$ intersects $J_{j'}$ with $j' > k$: the family $(J_j)_{j=1}^{l}$ comprises disjoint intervals.

We now define β_1 and β_2. The function β_1 is the function constantly equal one, and β_2 is constructed as follows. On $[a_0,b_0]\setminus\bigcup_{j=1}^{l} J_j^o$ we put

$$\beta_2(t) := i\frac{h_1(t)\overline{h_2(t)}}{\left|h_1(t)\overline{h_2(t)}\right|}.$$

The values are in $\mathbb{T}$ so that, by Lemma 4.5, we may find a $\mathbb{T}$-valued continuous extension to all of $[a_0,b_0]$ that will be also denoted by β_2.

We *claim* that β_1 and β_2 have the desired properties. By construction both functions are continuous and they satisfy $|\beta_1(t)| = |\beta_2(t)| = 1$ for all t. For $t\in[a_0,b_0]\setminus\bigcup_{j=1}^{l} J_j^o$ Lemma 4.4 implies that

$$|h_1(t)\beta_1(t)+h_2(t)\beta_2(t)|^2 = |h_1(t)|^2+|h_2(t)|^2 > \eta^2.$$

Now let a t in one of the J_j with $j\leqslant k$ be given. Then $|h_1(t)|^2 < \tau^2\eta_0^2$ so that $|h_2(t)|^2 \geqslant \eta^2+(1-\tau^2)\,\eta_0^2$, and it follows that $|h_2(t)| \geqslant \sqrt{\eta^2+(1-\tau^2)\,\eta_0^2}$. We choose τ so small so that $\sqrt{\eta^2+(1-\tau^2)\,\eta_0^2}-\tau\eta_0 \geqslant \eta$. We may then continue our estimation as follows:

$$\begin{aligned}|h_1(t)\beta_1(t)+h_2(t)\beta_2(t)| &\geqslant |h_2(t)|-|h_1(t)|\\ &\geqslant \sqrt{\eta^2+(1-\tau^2)\,\eta_0^2}-\tau\eta_0\\ &\geqslant \eta.\end{aligned}$$

The argument for $\bigcup_{j=k+1}^{l} J_j$ is analogous. □

Lemma 4.8. *Let $h_1,h_2\in C\,[a_0,b_0]$. Suppose that h_1,h_2 are jointly non-degenerate. Then for every $\varepsilon>0$ there is a positive δ such that for every $d\in C\,[a_0,b_0]$ satisfying $\|d\|\leqslant\delta$ there are $z_1,z_2\in C\,[a_0,b_0]$ such that*

- *$|z_1(t)|\,,|z_2(t)|\leqslant\varepsilon$ and*
- *$h_1(t)z_1(t)+h_2(t)z_2(t)+z_1(t)z_2(t)=d(t)$ $(t\in[a_0,b_0])$.*

Proof. Choose β_1,β_2 as in the preceding lemma and put $f:=h_1\beta_1+h_2\beta_2$ and $g:=h_1h_2$. Choose $\delta>0$ according to Lemma 4.3 for β_1 and β_2: for every $d\in C[a_0,b_0]$ with $\|d\|\leqslant\delta$ we may find ϕ with $\|\phi\|\leqslant\varepsilon$ and $f\phi+g\phi^2=d$. Thus it suffices to set $z_1:=\beta_1\phi$ and $z_2:=\beta_2\phi$. □

Lemma 4.9. *Let $\varepsilon>0$ and $\psi\in C[a,b]$ with $\|\psi\|\leqslant\varepsilon^2$. Suppose there are $Z_a,W_a,\hat{Z}\in\mathbb{C}$ such that $Z_aW_a=\psi(a)$, $|Z_a|\,,|W_a|\leqslant\varepsilon$ and $\hat{Z}^2=\psi(b)$. Then may construct $Z_1,Z_2\in C[a,b]$ with the following properties:*

- *$Z_1(a)=Z_a, Z_2(a)=W_a$,*
- *$|Z_1(t)|\,,|Z_2(t)|\leqslant\varepsilon$ and $Z_1(t)Z_2(t)=\psi(t)$ for all t,*
- *$Z_1(b)=Z_2(b)=\hat{Z}$.*

A similar statement holds if Z_b,W_b are prescribed at b and $\hat{Z}$ at a.

Proof. Without loss of generality we may suppose that $|Z_a| \geqslant |W_a|$ so that $|Z_a| \geqslant \sqrt{|\psi(a)|}$. Choose $Z_1 \in [a,b]$ with $Z_1(a) = Z_a, Z_1(b) = \hat{Z}$ and $\varepsilon > |Z_1(t)| \geqslant \sqrt{|\psi(t)|}$ for all t; note that $|\hat{Z}| \geqslant \sqrt{|\psi(b)|}$[1].

We define

$$Z_2(t) := \begin{cases} 0 & \text{if } Z_1(t) = 0, \\ \psi(t)/Z_1(t) & \text{otherwise.} \end{cases}$$

Then Z_1 and Z_2 will have the claimed properties. Indeed, the continuity of Z_2 at points t_0 with $Z_1(t_0) = 0$ is proved as follows. If $Z_1(t_0) = 0$ then $\psi(t_0) = 0$. Thus, by continuity of ψ, if $t_n \to t_0$, then $\sqrt{|\psi(t_n)|} \to 0$. Hence $|Z_2(t_n)| = |\psi(t_n)/Z_1(t_n)| \leqslant \sqrt{|\psi(t_n)|}$ will tend to zero as well. □

Lemma 4.10. *Let $\varepsilon > 0$ and $\psi \in C[a,b]$. Suppose that $\inf_{t\in[a,b]} |\psi(t)| \leqslant \varepsilon^2$. If there are $Z_a, W_a, Z_b, W_b \in \mathbb{C}$ such that $Z_aW_a = \psi(a), Z_bW_b = \psi(b)$ and $|Z_a|, |W_a|, |Z_b|, |W_b| \leqslant \varepsilon$, then there are $Z_1, Z_2 \in C[a,b]$ with the following properties:*

- $Z_1(a) = Z_a, Z_2(a) = W_a, Z_1(b) = Z_b, Z_2(b) = W_b,$
- $|Z_1(t)|, |Z_2(t)| \leqslant \varepsilon$ *and* $Z_1(t)Z_2(t) = \psi(t)$ *for all* t.

Proof. Choose any b' between a and b and a $\hat{Z}$ with $\hat{Z}^2 = \psi(b')$. It remains to apply the preceding lemma to the intervals $[a,b']$ and $[b',b]$ (with $Z_1(b') = Z_2(b') = \hat{Z}$ in either case) and to glue the Z_1, Z_2 that are defined on these subintervals together. □

Proof of Theorem 4.1. Let $\varepsilon_0 > 0$. We have to find $\delta_0 > 0$ with the following property: whenever $d\colon [0,1] \to \mathbb{C}$ is a prescribed continuous function with $\|d\| \leqslant \delta_0$ it is possible to find $d_1, d_2 \in C[0,1]$ with $\|d_1\|, \|d_2\| \leqslant \varepsilon_0$ and $(f + d_1)(g + d_2) = fg + d$ (*i.e.*, $fd_2 + gd_1 + d_1d_2 = d$) for any $f, g \in C[0,1]$. Fix $f, g \in C[0,1]$.

The idea is to determine such d_1, d_2 by using Lemma 4.8 (Lemma 4.10, respectively) on the subintervals where the functions f and g are are jointly non-degenerate (respectively, jointly degenerate) and to glue the pieces together.

With an $\varepsilon_1 > 0$ that will be fixed later we apply Lemma 4.6 with $h := |f|^2 + |g|^2$ and $\eta_1 := \varepsilon_1^2, \eta_2 := 4\varepsilon_1^2$. Write the intervals J_j $(j = 1, \dots, k)$ as $J_j = [a_j, b_j]$, where, without loss of generality, $0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_k < b_k$. Note that $h(t) \leqslant 4\varepsilon_1^2$ on each $[a_j, b_j]$ and $h(t) > \varepsilon_1^2$ on the intervals $[b_j, a_{j+1}]$.

Let us consider the intervals $[b_j, a_{j+1}]$ and apply Lemma 4.8 with $[a_0, b_0] := [b_j, a_{j+1}]$, $\eta := \varepsilon_1$, and $\varepsilon := \varepsilon_1$. Choose δ as in the lemma; without loss of generality we may assume that $\delta \leqslant \varepsilon_1^2$. We consider any $d \in C[0,1]$ with $\|d\| \leqslant \delta$. Lemma 4.8 provides continuous $z_1, z_2\colon [b_j, a_{j+1}] \to \mathbb{C}$ with $f(t)z_1(t) + g(t)z_2(t) + z_1z_2 = d(t)$ and $|z_1(t)|, |z_2(t)| \leqslant \varepsilon_1$ for $t \in [b_j, a_{j+1}]$. We define d_1 (d_2, respectively) on $[b_j, a_{j+1}]$ by z_2 (z_1, respectively). Then $(f + d_1)(g + d_2) = fg + d$ on these subintervals. (It should be noted here that the δ in Lemma 4.8 does only depend on η and ε but not on a_0, b_0.)

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[1]Here again it is important that we work in $\mathbb{C}$ and not in $\mathbb{R}$.

Now d_1, d_2 are suitably defined on the union of the $[b_j, a_{j+1}]$. The gaps will be filled with the help of Lemma 4.10. Consider any $[a_j, b_j]$. For a t in such an interval we know that $|f(t)|, |g(t)| \leqslant 2\varepsilon_1$ so that $|f(t)g(t)| \leqslant 4\varepsilon_1^2$.

It follows that $\psi\colon [a_j, b_j] \to \mathbb{C}, t \mapsto f(t)g(t) + d(t)$ satisfies $|\psi(t)| \leqslant 5\varepsilon_1^2 \leqslant (5\varepsilon_1)^2$. We apply Lemma 4.10 with this function ψ and

$$Z_a := (f + d_1)(a_j), W_a := (g + d_2)(a_j), Z_b := (f + d_1)(b_j), W_b := (g + d_2)(b_j)$$

and $\varepsilon := 5\varepsilon_1$. It remains to use the functions Z_1, Z_2 found by the lemma to define d_1, d_2 on $[a_j, b_j]$. Here Z_1 (respectively Z_2) plays the rôle of $f + d_1$ $(g + d_2)$ so that we may set $d_1(t) := Z_1(t) - f(t)$ and $d_2(t) := Z_2(t) - g(t)$ for $t \in [a_j, b_j]$. At the endpoints this assignment is compatible with the previous one: at a_j, e.g., d_1 was already defined, but as a consequence of $Z_1(a) = Z_a = f(a_j) + d(a_j)$ the new definition of $d_1(a_j)$ as $(Z_1 - f)(a_j)$ leads to the same value.

We observe that $|d_j(t)| \leqslant (2 + 5)\varepsilon_1 = 7\varepsilon_1$ for $j = 1, 2$ so that we may summarise the above calculations as follows: if one starts with $\varepsilon_1 := \varepsilon_0/7$, then $\delta_0 := \delta$ with the δ that we have just found has the desired properties.

It should be noted that our proof is not yet complete since, when considering the $[a_j, b_j]$, our argument used the fact that the functions d_1, d_2 were already defined at a_j and b_j, so we are to consider the cases $a_1 = 0$ or $b_k = 1$. If, *e.g.*, $a_1 = 0$ we choose any Z_a, W_a with $|Z_a|, |W_a| \leqslant \varepsilon$ and $Z_a W_a = \psi(a)$; we proceed similarly for $b_k = 1$. □

4.1. **Uniform openness of multiplication in** $C(X)$**.** The next result is crucial for establishing the only non-trivial implication in Theorem C.

Theorem 4.11. *Let X be a compact space of covering dimension at most 1. Then multiplication in $C(X)$ is uniformly open.*

Proof. Case 1: X is a topological realisation of a graph in the complex plane.

We *claim* that $C(X)$ has uniformly open multiplication and $\delta(\varepsilon)$ does not depend on X in the class of such graphs, that is, multiplications in $C(X)$ are equi-uniformly open for all graphs X.

For this, let us consider a partition of X into finitely many intervals, $\bigcup_{j=1}^{k} [a_j, b_j]$. We define a finer partition of this graph into intervals as follows. If the intervals $[a_j, b_j]$ and $[a_i, b_i]$ intersect at c for some $j, i \in \{1, \dots k\}$, then c must be the endpoint of the intervals, *i.e.*, we replace the interval $[a_j, b_j]$ by sub-intervals $[a_j, c]$ and $[c, b_j]$ whenever $c \in (a_j, b_j)$ (analogously for the interval $[a_i, b_i]$). For each interval in the new partition $P = \bigcup_{j=1}^{K} [a_j, b_j]$ we apply analogous procedure as in the proof of Theorem 4.1.

More precisely, denote for any function $F\colon P \to \mathbb{C}$ its restriction to the interval $[a_j, b_j]$ by F^j. Then for $\varepsilon_0 > 0$ find a positive δ_0 with the following property: whenever $d \in C(P)$, $\|d\| \leqslant \delta_0$ for every restriction $d^j \in C[a_j, b_j]$ $(j \in \{1, \dots, K\})$ we may $d_1^j, d_2^j \in C[a_j, b_j]$ with $\|d_1^j\|, \|d_2^j\| \leqslant \varepsilon_0$ and $\left(f^j + d_1^j\right)\left(g^j + d_2^j\right) = f^j g^j + d^j$ for any $f, g\colon P \to \mathbb{C}$. We glue the functions d_1^j for all $j \in \{1, \dots, K\}$ to obtain function $d_1\colon P \to \mathbb{C}$ (analogously for function d_2). Note that due to the choice of partition P, these functions are uniquely defined at the endpoints of the intervals, because at the intersection points of the intervals we always take the same value of the function. It should be noted also (again) that the δ in Lemma 4.8 does only depend on η and ε and not on a_0, b_0.

Case 2: X is a compact metric space of covering dimension at most 1.

It is known that for a zero-dimensional (not necessarily metrisable) compact space X, $C(X)$ has uniformly open multiplication with $\delta(\varepsilon) = \epsilon^2/4$ ([16, Proposition 4.6]). In the light of Case 1, by taking minimum if necessary, we may suppose that $\delta(\varepsilon)$ is the same for all zero-dimensional spaces as well as all graphs in the plane. However, every one-dimensional compact metric space X is the projective limit of an inverse sequence (K_i, π_i^j) of at most one-dimensional 'polyhedra' (this is a theorem of Freudenthal [19]; see [18, Theorem 1.13.2] for modern exposition), *i.e.*, finite sets and graphs in the plane. Such an inverse sequence gives rise to a direct system $(C(K_i), h_{\pi_i^j})$, where $h_{\pi_i^j}$ is a *-homomorphic embedding of $C(K_i)$ into $C(K_j)$ $(i \leqslant j)$ given by

$$h_{\pi_i^j} f = f \circ \pi_i^j \quad (f \in C(K_i)).$$

As $C(X)$ is naturally *-isomorphic to the completion of the chain $(C(K_i), h_{\pi_i^j})$ (*i.e.*, the C*-direct limit; see [32, Section 1] for more details) in which multiplications are equi-uniformly open, by [16, Corollary 3.6], $C(X)$ has uniformly open multiplication and $\delta(\varepsilon)$ depends only on ε but not the compact metric space X considered.

Case 3: X is an arbitrary compact space of covering dimension at most 1.

By [28, Theorem 1] every compact space X is an inverse limit of a well-ordered system of metrisable compacta X_α with $\dim X_\alpha \leqslant \dim X$. As proved in Claim 2, $C(X_\alpha)$ have equi-uniformly open multiplications, meaning that $\delta(\varepsilon)$ is the same for all items of the inverse system considered, so multiplication in $C(X)$ is uniformly open ([16, Corollary 3.6]).

□

5. Open problems

In the light of Theorem A let us pose the following question.

Question 1. What are further examples of (dual) Banach algebras that are approximable by jointly non-degenerate elements? What about algebras of Lipschitz functions on zero-dimensional compact spaces?

As the case of convolution algebras on discrete groups having at most one-dimensional dual groups, we ask the following question.

Question 2. Can the group algebra of a group with bounded exponent have (uniformly) open convolution?

More generally:

Question 3. Is there an infinite group G for which $\ell_1(G)$ has open convolution?

References

[1] R. Arens. The adjoint of a bilinear operation. *Proc. Amer. Math. Soc.*, 2(6):839–848, 1951.

[2] M. Balcerzak, E. Behrends, and F. Strobin. On certain uniformly open multilinear mappings. *Banach J. Math. Anal.*, 10(3):482–494, 2016.

[3] M. Balcerzak and A. Maliszewski. On multiplication in spaces of continuous functions. *Colloq. Math.*, 122:247–253, 2011.

[4] M. Balcerzak, A. Wachowicz, and W. Wilczyński. Multiplying balls in the space of continuous functions on [0, 1]. *Studia Math.*, 2(170):203–209, 2005.

[5] E. Behrends. Products of n open subsets in the space of continuous functions on $[0, 1]$. *Stud. Math.*, 204(1):73–95, 2011.

[6] E. Behrends. Walk the dog, or: products of open balls in the space of continuous functions. *Funct. et Approx. Comment. Math.*, 44(1):153–164, 2011.

[7] E. Behrends. Where is matrix multiplication locally open? *Linear Alg. Appl.*, 517:167–176, 2017.

[8] E. Behrends. Where is pointwise multiplication in real CK-spaces locally open? *Fund. Math.*, 236:51–69, 2017.

[9] B. Blackadar and J. Cuntz. Differential Banach algebra norms and smooth subalgebras of C*-algebras. *J. Oper. Theory*, 26:255–282, 1991.

[10] F. Botelho and H. Renaud. Topological properties of operations on spaces of differentiable functions. *Adv. Oper. Theory*, 4(1):305–320, 2019.

[11] T. Canarias, A. Karlovich, and E. Shargorodsky. Multiplication is an open bilinear mapping in the Banach algebra of functions of bounded Wiener p-variation. *Real Anal. Exch.*, 46(1):121–148, 2021.

[12] H.G. Dales, A.T-M. Lau, and D. Strauss. Second duals of measure algebras. *Dissertationes Math. (Rozprawy Matematyczne)*, 481:1–121, 2012.

[13] M. Daws. Ultrapowers of Banach algebras and modules. *Glasgow Math. J.*, 50(3):539–555, 2008.

[14] M. Daws. Amenability of ultrapowers of Banach algebras. *Proc. Edinb. Math. Soc.*, 52(2):307–338, 2009.

[15] R. Doran. *Characterizations of C*-Algebras: the Gelfand Naimark Theorems*, volume 101. CRC Press, 1986.

[16] S. Draga and T. Kania. When is multiplication in a Banach algebra open? *Linear Alg. Appl.*, 538:149–165, 2018.

[17] O. El-Fallah, N.K. Nikol'skii, and M. Zarrabi. Estimates for resolvents in Beurling–Sobolev algebras. *Algebra i Analiz*, 10(6):1–92, 1998.

[18] R. Engelking. *Dimension theory*, volume 19. North-Holland Publishing Company Amsterdam, 1978.

[19] H. Freudenthal. Entwicklungen von Räumen und ihren Gruppen. *Compositio Math.*, 4:145–234, 1937.

[20] I. Gelfand. Normierte ringe. *Mat. Sbornik*, 9(1):3–24, 1941.

[21] K. Gröchenig and A. Klotz. Norm-controlled inversion in smooth Banach algebras, I. *J. London Math. Soc.*, 88(1):49–64, 2013.

[22] K. Gröchenig and A. Klotz. Norm-controlled inversion in smooth Banach algebras, II. *Math. Nachr.*, 287(8-9):917–937, 2014.

[23] E. Hewitt and K.A. Ross. *Abstract Harmonic Analysis: Volume I Structure of Topological Groups Integration Theory Group Representations*, volume 115. Springer Science & Business Media, 2012.

[24] E. Kissin and V.S. Shulman. Differential properties of some dense subalgebras of C*-algebras. *Proc. Edinb. Math. Soc.*, 37(3):399–422, 1994.

[25] A. Komisarski. A connection between multiplication in $C(X)$ and the dimension of X. *Fund. Math*, 189(2):149–154, 2006.

[26] S. Kowalczyk and M. Turowska. Multiplication in the space of functions of bounded variation. *J. Math. Anal. Appl.*, 472(1):696–704, 2019.

[27] A. G. Kurosh. *The theory of groups*, volume 1. Chelsea Publishing Company, 1955.

[28] S. Mardešić. On covering dimension and inverse limits of compact spaces. *Illinois J. Math.*, 4(1):278–291, 1960.

[29] N. Nikolski. In search of the invisible spectrum. In *Annales de l'Institut Fourier*, volume 49, pages 1925–1998, 1999.

[30] H. Renaud. *Topological properties of the standard operations on spaces of continuous functions and integrable functions*. PhD thesis, The University of Memphis, 2019.

[31] E. Samei and V. Shepelska. Norm-controlled inversion in weighted convolution algebras. *J. Fourier Anal. Appl.*, 25(6):3018–3044, 2019.

[32] Z. Takeda. Inductive limit and infinite direct product of operator algebras. *Tohoku Math. J., Second Series*, 7(1-2):67–86, 1955.

[33] N.J. Young. The irregularity of multiplication in group algebras. *Quarterly J. Math.*, 24(1):59–62, 1973.

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