# Variational tools in analysis of multifunctions

Jiří V. Outrata

Institute of Information Theory and Automation,
Academy of Sciences of the Czech Republic

Variational analysis encompasses “a broad spectrum of mathematical theory that has grown in connection with the study of problems of optimization, equilibrium, control, and stability of linear and nonlinear systems”.

R.T. Rockafellar and R.J.-B. Wets

This lecture is focused on the tools which have been developed for local analysis of multifunctions (set-valued mappings) and on their usage in the investigation of Lipschitzian stability of “solution maps” to parameterized equilibria.

Motivations:

- Post-optimal analysis
- Optimal control/design
- Qualification conditions in generalized differential calculus

# Outline

(i) Basic notions from variational geometry and the theory of multifunctions
(ii) Generalized differential calculus
(iii) Lipschitzian single-valued localization
(iv) Aubin (Lipschitz-like) property
(v) Directional coderivatives and their applications
(vi) Conclusion

## Definition

Given a closed set $A \subset \mathbb{R}^n$ and $\bar{x} \in A$, we define

(i) the *tangent (Bouligand) cone* to $A$ at $\bar{x}$ by

$$T_A(\bar{x}) := \operatorname*{Lim\,sup}_{\vartheta \searrow 0} \frac{A - \bar{x}}{\vartheta} = \{h \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0 : \bar{x} + \vartheta_i h_i \in A \; \forall i\};$$

(ii) the *regular (Fréchet) normal cone* to $A$ at $\bar{x}$ by

$$\widehat{N}_A(\bar{x}) := (T_A(\bar{x}))^{\circ};$$

(iii) the *limiting (Mordukhovich) normal cone* to $A$ at $\bar{x}$ by

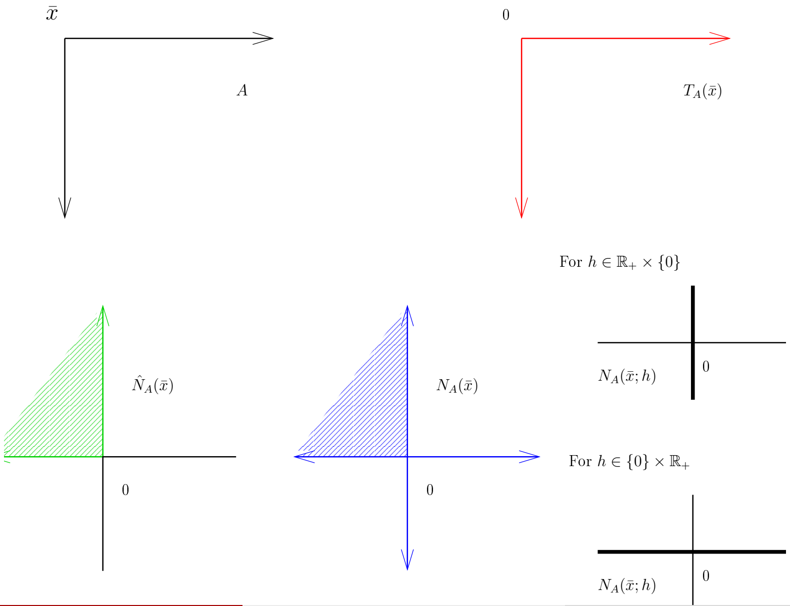
$$N_A(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{A} \bar{x}} \widehat{N}_A(x) = \{\xi \in \mathbb{R}^n | \exists x_i \xrightarrow{A} \bar{x}, \xi_i \to \xi : \xi_i \in \widehat{N}_A(x_i) \; \forall i\}.$$

(iv) Finally, given a direction $h \in \mathbb{R}^n$, the cone

$$N_A(\bar{x}; h) := \{\xi \in \mathbb{R}^n | \exists h_i \to h, \vartheta_i \searrow 0, \xi_i \to \xi : \xi_i \in \widehat{N}_A(\bar{x} + \vartheta_i h_i) \; \forall i\}$$

is called the *directional limiting normal cone* to $A$ at $\bar{x}$ in the direction $h$.

# Ad (i) - Example: $A$ is the graph of $N_{\mathbb{R}_+} \subset \mathbb{R}^2$



# Ad (i) - Basic notions

Consider now a multifunction $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$. The *domain* and the *range* of $F$ are defined by

$$\text{dom}\, F := \{u \in \mathbb{R}^k | F(u) \neq \emptyset\}, \ \text{rge}\, F := \{v \in \mathbb{R}^s | \exists\, u \text{ with } v \in F(u)\},$$

respectively, and the *graph* of $F$ is the set

$$\text{gph}\, F := \{(u, v) \in \mathbb{R}^k \times \mathbb{R}^s | v \in F(u)\}.$$

### Definition

Consider a point $(\bar{u}, \bar{v}) \in \operatorname{gph} F$. Then

(i) the multifunction $DF(\bar{u}, \bar{v}) : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$, defined by

$$DF(\bar{u}, \bar{v})(h) := \{k \in \mathbb{R}^s | (h, k) \in T_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, h \in \mathbb{R}^k,$$

is called the *graphical derivative* of $F$ at $(\bar{u}, \bar{v})$;

(ii) the multifunction $\hat{D}^* F(\bar{u}, \bar{v}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^k$, defined by

$$\hat{D}^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^k | (u^*, -v^*) \in \hat{N}_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^s,$$

is called the *regular (Fréchet) coderivative* of $F$ at $(\bar{u}, \bar{v})$.

(iii) the multifunction $D^* F(\bar{u}, \bar{v}) : \mathbb{R}^s \rightrightarrows \mathbb{R}^k$, defined by

$$D^* F(\bar{u}, \bar{v})(v^*) := \{u^* \in \mathbb{R}^k | (u^*, -v^*) \in N_{\operatorname{gph} F}(\bar{u}, \bar{v})\}, v^* \in \mathbb{R}^s,$$

is called the *limiting (Mordukhovich) coderivative* of $F$ at $(\bar{u}, \bar{v})$.

If $F$ is a continuously differentiable single-valued mapping, then
$DF(\bar{u}, \bar{v})(h) = \nabla F(\bar{u})h$ and $\hat{D}^* F(\bar{u}, \bar{v})(v^*) = D^* F(\bar{u}, \bar{v})(v^*) = \nabla F(\bar{u})^T v^*$.

# Ad (ii) - Generalized differential calculus

R1) Let $f : \mathbb{R}^s \to \mathbb{R}$ be a continuously differentiable function, $F : \mathbb{R}^k \to \mathbb{R}^s$ be a locally Lipschitz mapping, $A \subset \mathbb{R}^k$ be closed, and consider the minimization problem

$$\begin{array}{ll} \text{minimize} & (f \circ F)(x) \\ \text{subject to} & \\ & x \in A. \end{array} \tag{1}$$

Assume that $\bar{x}$ is a (local) solution of (1). Then one has

$$0 \in D^* F(\bar{x})(\nabla f(F(\bar{x})) + N_A(\bar{x}). \tag{2}$$

R2) Let $A = A_1 \cap A_2$ and $\bar{x} \in A$. Then

$$\widehat{N}_A(\bar{x}) \supset \widehat{N}_{A_1}(\bar{x}) + \widehat{N}_{A_2}(\bar{x}).$$

However, for the inclusion

$$N_A(\bar{x}) \subset N_{A_1}(\bar{x}) + N_{A_2}(\bar{x}) \tag{3}$$

to be valid we need a *qualification condition*.

# Ad (ii) - Generalized differential calculus

**Definition**

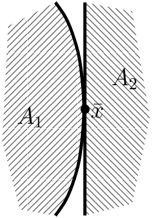
We say that $C : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$ is *calm* at $(\bar{u}, \bar{v}) \in \operatorname{gph} C$ provided there exist a neighborhood $V$ of $\bar{v}$ and a modulus $\kappa > 0$ such that

$$C(u) \cap V \subset C(\bar{u}) + \kappa \|u - \bar{u}\| \mathbb{B} \; \forall u.$$

Inclusion (3) in R2 holds provided the *perturbation mapping*

$$\Psi(p) := \{x \in A_1 | x - p \in A_2\}$$

is calm at $(0, \bar{x})$.




$$N_A(\bar{x}) = \mathbb{R}^2$$
$$N_{A_1}(\bar{x}) + N_{A_2}(\bar{x}) = \mathbb{R} \times \{0\}$$

Figure: $\Psi$ is not calm at $(0, \bar{x})$.

In what follows we will consider a closed-graph multifunction $M : \mathbb{R}^l \times \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and examine the two mentioned stability properties of the *solution map* $S : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ defined implicitly via the relationship

$$S(p) := \{x \in \mathbb{R}^n | 0 \in M(p, x)\}. \tag{4}$$

around a given *reference pair* $(\bar{p}, \bar{x}) \in \operatorname{gph} S$. In (4), $p$ plays the role of a *parameter/control* and $x$ is the *decision/state* variable.

This model has a distinguished special case called *parameterized variational system*:

$$M(p, x) = H(p, x) + Q(x), \tag{5}$$

where $H : \mathbb{R}^l \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $Q : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ has a closed graph. The relation $0 \in H(p, x) + Q(x)$ is called *generalized equation* (GE). $Q$ amounts typically to $\hat{N}_\Gamma(x)$ with some closed set $\Gamma \subset \mathbb{R}^m$. If $\Gamma$ is convex, then $S$ is the solution map to a *parameterized variational inequality* (VI) of the first kind.

# Ad(iii) - Lipschitzian single-valued localization

## Definition.

We say that $C : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$ has a *Lipschitzian single-valued localization* around $(\bar{u}, \bar{v}) \in \operatorname{gph} C$ provided there exist neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$, and a Lipschitzian function $\sigma : \mathcal{U} \to \mathbb{R}^s$ such that

$$\sigma(\bar{u}) = \bar{v} \text{ and } C(u) \cap \mathcal{V} = \{\sigma(u)\}\ \forall\, u \in \mathcal{U}.$$

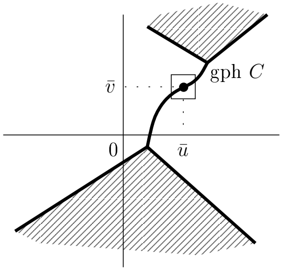



Figure: $C$ has a Lipschitzian single-valued localization around $(\bar{u}, \bar{v})$.

# Ad(iii) - Lipschitzian single-valued localization

This type of Lipschitzian continuity plays an important rule above all in

- control of equilibria governed by variational systems or optimization problems via control/design variables (optimum design problems in mechanics, optimal production strategies of firms acting in an oligopolistic market, optimal design of transportations networks etc.);
- post-optimal analysis, i.e., analysis of the influence of "small" changes of some problem data on equilibria computed with nominal values of these data. Can a "small" change of problem data lead to an unproportional change of the respective equilibrium?

# Ad (iii) - Lipschitzian single-valued localization

Consider now the case of $M$ given by (5).

## Theorem 1.

(Robinson 1980, Dontchev, Hager 1994). Let the mapping $\Sigma : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\Sigma(b) := \{x | b \in H(\bar{p}, \bar{x}) + \nabla_x H(\bar{p}, \bar{x})(x - \bar{x}) + Q(x)\} \tag{6}$$

have a Lipschitzian single-valued localization around $(0, \bar{x})$. Then $S$ has a Lipschitzian single-valued localization around $(\bar{p}, \bar{x})$. Moreover, one has for all $w \in \mathbb{R}^n$ the inclusion

$$D^* S(\bar{p}, \bar{x})(w) \subset \{(\nabla_p H(\bar{p}, \bar{x}))^T b | 0 \in w + (\nabla_x H(\bar{p}, \bar{x}))^T b + D^* Q(\bar{p}, -H(\bar{p}, \bar{x}))(b)\}. \tag{7}$$

## Remark 1.

If $H$ is not continuously differentiable, then the term $H(\bar{p}, \bar{x}) + \nabla_x H(\bar{p}, \bar{x})(x - \bar{x})$ can be replaced by another so-called strict estimator of $H$ at $(\bar{p}, \bar{x})$ satisfying certain conditions.

## Remark 2.

This is an example of the so-called implicit function paradigm saying that if $\Sigma$ possesses a certain property around $(0, \bar{x})$, then this property is inherited by $S$ around $(\bar{p}, \bar{x})$.

# Ad (iii) - Application of Theorem 1

**Theorem 2.**

(Robinson 1980). Assume that $Q(\cdot) = N_\Gamma(\cdot)$, where $\Gamma$ is a convex polyhedron. Further assume that the mapping $\Xi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by

$$\Xi(b) := \{x | b \in \nabla_x H(\bar{p}, \bar{x})y + N_K(y)\} \text{ with } K = T_\Gamma(\bar{x}) \cap [H(\bar{p}, \bar{x})]^\perp \tag{8}$$

is single-valued. Then $S$ has a Lipschitzian single-valued localization around $(\bar{p}, \bar{x})$.

# Ad (iii) - Implicit programming (ImP) approach

Consider the *mathematical program with equilibrium constraints* (MPEC) given by

$$\begin{array}{ll} \text{minimize} & f(p,x) \\ \text{subject to} & \\ & 0 \in H(p,x) + Q(x) \\ & p \in \omega, \end{array} \tag{9}$$

where $p$ is the control variable, $x$ is *decision* variable, $f$ is a continuously differentiable objective, $\omega$ is a closed set of *admissible controls* and mappings $H, Q$ fulfill the conditions from slide 11.

Let $S : p \mapsto x$ be the solution map defined by the GE in (9) and $(\bar{p}, \bar{x})$ be a local solution of (9). Suppose that the GE in (9) satisfies the assumptions of Theorem 1. Then (9) amounts locally (close to $(\bar{p}, \bar{x})$) to the optimization problem

$$\begin{array}{ll} \text{minimize} & J(p) \\ \text{subject to} & \\ & p \in \omega, \end{array} \tag{10}$$

where $J(p) := f(p, \sigma(p))$ with $\sigma$ being the Lipschitzian single-valued localization of $S$ around $(\bar{p}, \bar{x})$.

# Ad (iii) - Optimality conditions

**Theorem 3.**

Under the posed assumptions there exists an MPEC multiplier $\bar{b}$ such that

$$0 \in \nabla_p f(\bar{p}, \bar{x}) + (\nabla_p H(\bar{p}, \bar{x}))^T \bar{b} + N_\omega(\bar{x}) \quad (11)$$

$$0 \in \nabla_x f(\bar{p}, \bar{x}) + (\nabla_x H(\bar{p}, \bar{x}))^T \bar{b} + D^* Q(\bar{p}, -H(\bar{p}, \bar{x}))(b). \quad (12)$$

Sketch of the proof.
From the second statement of Thm.1 we know that

$$D^* S(\bar{p}, \bar{x})(w) \subset \{(\nabla_p H(\bar{p}, \bar{x}))^T b | 0 \in w + (\nabla_x H(\bar{p}, \bar{x}))^T b + D^* Q(\bar{p}, H(\bar{p}, \bar{x}))(b)\}.$$

Then it suffices to apply R1. □

If $\nabla_p H(\bar{p}, \bar{x})$ is surjective, then we say that the GE in (9) is *amply parameterized*. In this case,

$$D^* S(\bar{p}, \bar{x})(w) = \{(\nabla_p H(\bar{p}, \bar{x}))^T b | 0 \in w + (\nabla_x H(\bar{p}, \bar{x}))^T b + D^* Q(\bar{p}, -H(\bar{p}, \bar{x}))(b)\}. \quad (13)$$

It follows that in case of ample parameterization the optimality conditions of Theorem 4 are generally sharper (more selective).

# Ad (iii) - Computation of $D^* N_\Gamma$

Given a convex polyhedral cone $\mathcal{K} \subset \mathbb{R}^n$, a set $\mathcal{F} \subset \mathcal{K}$ is called a *face* of Γ if $\mathcal{F} = \mathcal{K} \cap [v^*]^\perp$ for some $v^* \in \mathcal{K}^0$.

Let $K$ be the critical cone to Γ at $\bar{x}$ with respect to $H(\bar{p}, \bar{x})$, i.e., $K = T_\Gamma(\bar{x}) \cap [H(\bar{p}, \bar{x})]^\perp$.

**Theorem 4.**

(Dontchev, Rockafellar 1996). Let $(z, z^*) \in \operatorname{gph} N_\Gamma$. Then $N_{\operatorname{gph} N_\Gamma}(z, z^*)$ is the union of all product sets $V^0 \times V$ associated with cones $V$ of the form $F_1 - F_2$, where $F_1, F_2$ are closed faces of the critical cone $K$ satisfying

$$F_2 \subset F_1.$$

Example

$\Gamma = \mathbb{R}_+, (z, z^*) = (0, 0)$,

$K = T_{\mathbb{R}_+}(z) \cap [z^*]^\perp = \mathbb{R}_+,\ F_1 = \mathbb{R}_+, F_2 = \{0\}$. By Thm. 4 it follows that

$$\begin{aligned} & N_{\operatorname{gph} N_\Gamma}(z, z^*) \\ & = (F_1 - F_1)^\circ \times (F_1 - F_1) \cup (F_1 - F_2)^\circ \times (F_1 - F_2) \cup (F_2 - F_2)^\circ \times (F_2 - F_2) \\ & = (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R} \times \{0\}). \end{aligned}$$

△

# Ad (iv) - Aubin property

## Definition

(Aubin 1984). We say that $C : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$ has the *Aubin property* around $(\bar{u}, \bar{v}) \in \operatorname{gph} C$ provided there exist neighborhoods $\mathcal{U}$ of $\bar{u}$, $\mathcal{V}$ of $\bar{v}$ and a modulus $\kappa > 0$ such that

$$C(u_1) \cap \mathcal{V} \subset C(u_2) + \kappa \|u_1 - u_2\| \mathbb{B} \quad \forall u_1, u_2 \in \mathcal{U}.$$

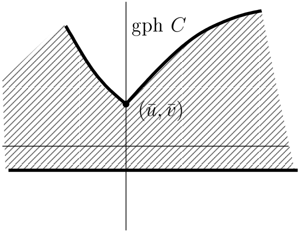



Figure: $C$ has the Aubin property around $(\bar{u}, \bar{v})$.

# Ad (iv) - Aubin property

Applications

- Post-optimal analysis: In contrast to the existence of Lipschitzian single-valued localization, the Aubin property only ensures that for slightly perturbed problem data there is an equilibrium whose distance from the original one is bounded by a multiple of the norm of the difference between the data.
- Construction of easily verifiable CQs in the generalized differential calculus. For instance, in R3, inclusion (3) holds true provided the *perturbation mapping*

  $$\Psi(p) := \{x \in A_1 | x - p \in A_2\}$$

  has the Aubin property around $(0, \bar{x})$.
- Close relationship with some nonlinear and set-valued generalizations of the Banach open mapping theorem (e.g. Lyusternik 1934, Graves 1950). This relationship facilitates significantly the respective proofs.
- Close relationship with convergence theory for some numerical methods for the solution of GEs and optimization problems (Newton method, SQP etc.).

# Ad (iv) - Aubin property

$C$ has a Lipschitzian single valued localization around $(\bar{u}, \bar{v})$.

$\Downarrow$

$C$ has the Aubin property around $(\bar{u}, \bar{v})$.

$\Downarrow$

$C$ is calm at $(\bar{u}, \bar{v})$.

# Ad (iv) - Mordukhovich criterion

## Theorem 5.

(Mordukhovich 1992). Let $C : \mathbb{R}^l \rightrightarrows \mathbb{R}^n$ be a closed graph multifunction and $(\bar{u}, \bar{v}) \in \operatorname{gph} C$. Then $C$ has the Aubin property around $(\bar{u}, \bar{v})$ iff $D^*C(\bar{u}, \bar{v})(0) = \{0\}$.

To apply this criterion to $S$ given by (4) we associate with $M$ the multifunction $\Phi : \mathbb{R}^m \rightrightarrows \mathbb{R}^l \times \mathbb{R}^n$ defined by

$$\Phi(v) := \{(p, x) | v \in M(p, x)\}.$$

## Theorem 6.

Let $(\bar{p}, \bar{x}) \in \operatorname{gph} S$ and assume that

1) $\Phi$ is calm at $(0, \bar{p}, \bar{x})$;
2) The implication
$$(q^*, 0) \in D^*M(\bar{p}, \bar{x}, 0)(b^*) \Rightarrow q^* = 0$$
holds true.

Then $S$ has the Aubin property around $(\bar{p}, \bar{x})$.
Assumptions 1), 2) can be replaced by the implication

$$(q^*, 0) \in D^*M(\bar{p}, \bar{x}, 0)(b^*) \Rightarrow q^* = 0, b^* = 0. \tag{14}$$

# Ad (iv) - Mordukhovich criterion

**Corollary.**

Let $M(p,x) = p - F(x)$, where $F$ is continuously differentiable. Assume that $\nabla F(\bar{x})$ is surjective. Then $S$ has the Aubin property around $(\bar{p}, \bar{x})$.

Indeed, (14) attains the form

$$0 = (\nabla F(\bar{x}))^T b^* \Rightarrow b^* = 0 \text{ which amounts to } 0 \in (\mathcal{R}(\nabla F(\bar{x})))^{\perp}.$$

**Remark.**

It follows that $\exists$ neighborhoods $\mathcal{U}$ of $\bar{p}$ and $\mathcal{V}$ of $\bar{x}$ such that for $p \in \mathcal{U}$ the equation $p = F(x)$ has a solution $x \in \mathcal{V}$. This is a simplified variant of the Graves theorem.

# Ad (iv) - Mordukhovich criterion

Next we apply Theorem 6 to the parameterized variational system with $M$ given by (5).

## Theorem 7.

Let $(\bar{p}, \bar{x}) \in \operatorname{gph} S$ and assume that

1) The respective $\Phi$ is calm at $(0, \bar{p}, \bar{x})$;
2) The implication

$$0 \in (\nabla_x H(\bar{p}, \bar{x}))^T b^* + D^* Q(\bar{x}, -H(\bar{p}, \bar{x}))(b^*) \Rightarrow b^* \in \ker(\nabla_p H(\bar{p}, \bar{x}))^T \tag{15}$$

holds true.

Then $S$ has the Aubin property around $(\bar{p}, \bar{x})$.
Assumptions 1), 2) can be replaced by the stronger requirement that the GE

$$0 \in (\nabla_x H(\bar{p}, \bar{x}))^T b^* + D^* Q(\bar{x}, -H(\bar{p}, \bar{x}))(b^*) \tag{16}$$

has only the trivial solution $b^* = 0$. This condition is also necessary for $S$ to have the Aubin property provided $\nabla_p H(\bar{p}, \bar{x})$ is surjective.

The GE on the left-hand side of (15) is called *adjoint GE*.

**Example**

Consider the GE given by GE (5) with $\Gamma = \mathbb{R}^2_+$ and

$$H(p,x) = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ p+2 \end{bmatrix},$$

which amounts to the *linear complementarity problem* (LCP)

$$0 \leq \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ p+2 \end{bmatrix} \perp \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq 0.$$

Put $(\bar{p}, \bar{x}) = (0,(1,0))$. The corresponding multifunction $M$ is metrically subregular at $(\bar{p}, \bar{x})$ and so we can make use of implication (14) which attains the form

$$0 \in \begin{bmatrix} 0 - 2b_2^* \\ b_1^* + 3b_2^* \end{bmatrix} + D^* N_{\mathbb{R}^2_+}((1,0),-(0,0))(b^*) \Rightarrow b^* \in \ker([0,1]).$$

Since all solutions of the adjoint GE belong to $\mathbb{R}_- \times \{0\}$, this implication holds true and the respective $S$ has the Aubin property around $(\bar{p}, \bar{x})$. This cannot be concluded on the basis of the second statement in Theorem 7. $\triangle$

If we examine the Aubin property of variational systems given by (5) and $\nabla_p H(\bar{p}, \bar{x})$ is not surjective, then the conditions of Theorem 5 may be far from necessity. It turns out that on the basis of the directional limiting normal cone one can derive a weaker yet sufficient criterion.
Consider a multifunction $F : \mathbb{R}^k \rightrightarrows \mathbb{R}^s$ and a point $(\bar{u}, \bar{v}) \in \mathrm{gph}\, F$.

**Definition.**

(Gfrerer 2011). Given a pair of directions $(h, k) \in \mathbb{R}^k \times \mathbb{R}^s$, the multifunction $D^* F((\bar{u}, \bar{v}); (h, k)) : \mathbb{R}^s \rightrightarrows \mathbb{R}^k$, defined by

$$D^* F((\bar{u}, \bar{v}); (h, k))(v^*) := \{u^* \in \mathbb{R}^k | (u^*, -v^*) \in N_{\mathrm{gph}\, F}((\bar{u}, \bar{v}); (h, k))\}, v^* \in \mathbb{R}^s,$$

is called the *directional limiting coderivative* of $F$ at $(\bar{u}, \bar{v})$ in direction $(h, k)$.

It holds that $D^* F((\bar{u}, \bar{v}); (0, 0)) = D^* F(\bar{u}, \bar{v})$ and for any $v^* \in \mathbb{R}^s$

$$D^* F((\bar{u}, \bar{v}); (h, k))(v^*) \neq \emptyset,$$

whenever $(h, k) \notin T_{\mathrm{gph}\, F}(\bar{u}, \bar{v})$.

# Ad (v) - Application to the Aubin property

This new approach relies on the possibility to express the Mordukhovich criterion in terms of the directional limiting coderivatives. The next result concerns $S$ given by (4).

**Theorem 8.**

Assume that

- $\Phi$ is calm at $(0, \bar{p}, \bar{x})$;
- $$\{u | 0 \in DM(\bar{p}, \bar{x}, 0)(v, u)\} \neq \emptyset \text{ for all } v \in \mathbb{R}^l. \tag{17}$$
- For every nonzero $(v, u) \in \mathbb{R}^l \times \mathbb{R}^n$ such that $0 \in DM(\bar{p}, \bar{x}, 0)(v, u)$ the implication
$$(q^*, 0) \in D^*M((\bar{p}, \bar{x}, 0); (v, u, 0))(b^*) \Rightarrow q^* = 0. \tag{18}$$
holds true.

Then $S$ has the Aubin property around $(\bar{p}, \bar{x})$ and $DS(\bar{x}, \bar{y})(\cdot)$ admits the representation

$$DS(\bar{p}, \bar{x})(v) = \{u | 0 \in DM(\bar{p}, \bar{x}, 0)(v, u)\}, \ v \in \mathbb{R}^l. \tag{19}$$

# Ad (v) - Application to the Aubin property

## Remark

Equality (19) means that the graphical derivative of $S$ at $(\bar{p}, \bar{x})$ is implicitly given by the graphical derivative of $M$ at $(\bar{p}, \bar{x}, 0)$. This directly generalizes the classical formula for the derivative of the implicit functions (U. Dini, 1877).

## Remark

Since condition (17) is necessary for $S$ to have the Aubin property and the directional limiting coderivatives are typically much smaller than the standard ones, the conditions of Theorem 8 are typically less restrictive than the conditions of Theorem 6.

## Theorem 9.

Let us omit the first assumption of Theorem 8 and strengthen the implication (18) to

$$(q^*, 0) \in D^* M((\bar{p}, \bar{x}, 0); (v, u, 0))(b^*) \Rightarrow q^* = 0, b^* = 0. \tag{20}$$

Then the assertions of Theorem 8 remain valid.

Let Γ be convex and closed and

$$M(p,x) = H(p,x) + N_\Gamma(x). \tag{21}$$

### Theorem 10.

Assume that $(\bar{p},\bar{x}) \in \operatorname{gph} S$ and

- The respective Φ is calm at $(0,\bar{p},\bar{x})$;
- $\{u | 0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u)\} \neq \emptyset$ for all $v \in \mathbb{R}^l$;
- For every nonzero $(v,u) \in \mathbb{R}^l \times \mathbb{R}^n$ such that

$$0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u) \tag{22}$$

the implication

$$\left.\begin{array}{l} 0 \in (\nabla_x H(\bar{p},\bar{x}))^T b^* + \\ D^* N_\Gamma((\bar{x}, -H(\bar{p},\bar{x})); (u, -\nabla_p H(\bar{p},\bar{x}))v - \nabla_x H(\bar{p},\bar{x})u))(b^*) \end{array}\right\} \Rightarrow \\ \Rightarrow b^* \in \ker(\nabla_p H(\bar{p},\bar{x}))^T \tag{23}$$

holds true. Then $S$ has the Aubin property around $(\bar{p},\bar{x})$ and

$$DS(\bar{p},\bar{x})(v) = \{u | 0 \in \nabla_p H(\bar{p},\bar{x})v + \nabla_x H(\bar{p},\bar{x})u + DN_\Gamma(\bar{x}, -H(\bar{p},\bar{x}))(u)\}.$$

# Ad (v) - Variational systems

The relation on the left-hand side of (23) is called *directional adjoint GE*.

## Remark

If $\Gamma$ is polyhedral, then

$$DN_\Gamma(\bar{x}, -H(\bar{p}, \bar{x}))(u) = N_K(u),$$

where $K := \mathcal{K}_\Gamma(\bar{x}, H(\bar{p}, \bar{x})) = T_\Gamma(\bar{x}) \cap [H(\bar{p}, \bar{x})]^\perp$ (critical cone to $\Gamma$ at $\bar{x}$ with respect to $H(\bar{p}, \bar{x})$).

## Theorem 11.

Let $(z, z^*) \in \mathrm{gph}\, N_\Gamma$ and $(v, u) \in T_{\mathrm{gph}\, N_\Gamma}(z, z^*)$ be given. Then $N_{\mathrm{gph}\, N_\Gamma}((z, z^*); (v, u))$ is the union of all product sets $V^0 \times V$ associated with cones $V$ of the form $F_1 - F_2$, where $F_1, F_2$ are closed faces of the critical cone $\mathcal{K}_\Gamma(z, z^*)$ satisfying

$$v \in F_2 \subset F_1 \subset [u]^\perp. \tag{24}$$

Clearly for $(v, u) = (0, 0)$, Theorem 11 reduces to Theorem 4.

**Example**

$\Gamma = \mathbb{R}_+, (z, z^*) = (0, 0) \in \mathrm{gph}\, N_\Gamma$
$\mathcal{K}_\Gamma(z, z^*) = T_\Gamma(z) \cap [z^*]^\perp = \mathbb{R}_+,\ F_1 = \mathbb{R}_+, F_2 = \{0\}$

By virtue of Theorem 4,

$$\begin{aligned}&N_{\mathrm{gph}\, N_\Gamma}(z, z^*)\\&= (F_1 - F_1)^\circ \times (F_1 - F_1) \cup (F_1 - F_2)^\circ \times (F_1 - F_2) \cup (F_2 - F_2)^\circ \times (F_2 - F_2)\\&= (\{0\} \times \mathbb{R}) \cup (\mathbb{R}_- \times \mathbb{R}_+) \cup (\mathbb{R} \times \{0\}).\end{aligned}$$

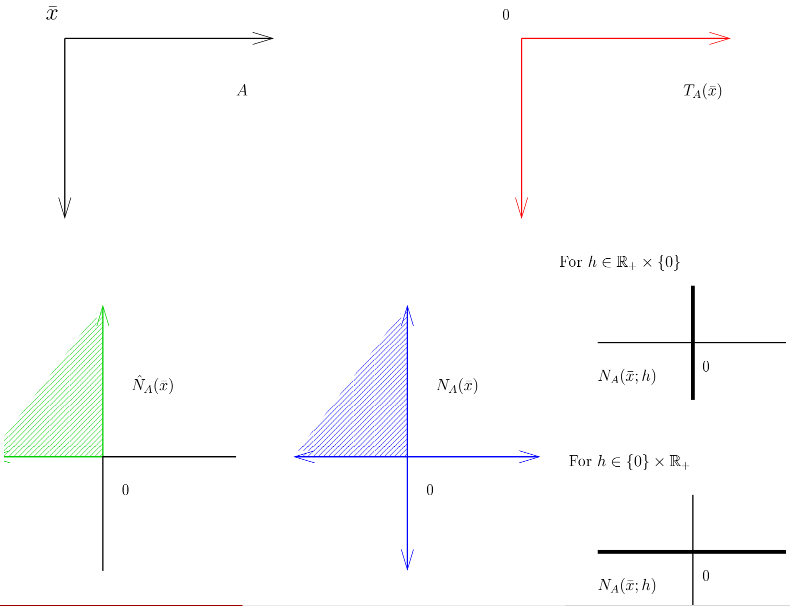
For $(v, u) = (1, 0)$, by Theorem 11, one obtains

$$N_{\mathrm{gph}\, N_\Gamma}((z, z^*); (v, u)) = (F_1 - F_1)^\circ \times (F_1 - F_1) = \{0\} \times \mathbb{R},$$

because $F_2$ does not contain $v$.
Likewise for $(v, u) = (0, 1)$ one has

$$N_{\mathrm{gph}\, N_\Gamma}((z, z^*); (v, u)) = (F_2 - F_2)^\circ \times (F_2 - F_2) = \mathbb{R} \times \{0\},$$

because $F_1$ is not contained in $\{u\}^\perp$. $\triangle$


$\bar{x}$
$A$
0
$T_A(\bar{x})$
$\hat{N}_A(\bar{x})$
0
$N_A(\bar{x})$
0
For $h \in \mathbb{R}_+ \times \{0\}$
$N_A(\bar{x}; h)$
0
For $h \in \{0\} \times \mathbb{R}_+$
$N_A(\bar{x}; h)$
0

# Ad (v) - Variational systems

Consider the case (21), where $N_\Gamma$ is replaced by $\hat{N}_\Gamma$, $\Gamma = g^{-1}(D)$ with $g : \mathbb{R}^n \to \mathbb{R}^s$ twice continuously differentiable and $D \subset \mathbb{R}^s$ closed. We assume that

(A1): There exists a closed set $\Theta \subset \mathbb{R}^d$ along with a twice continuously differentiable mapping $h : \mathbb{R}^s \to \mathbb{R}^d$ and a neighborhood $\mathcal{V}$ of $g(\bar{x})$ such that $\nabla h(g(\bar{x}))$ is surjective and

$$D \cap \mathcal{V} = \{z \in \mathcal{V} | h(z) \in \Theta\};$$

(A2):

$$\mathrm{rge} \nabla g(\bar{x}) + \ker \nabla h(g(\bar{x})) = \mathbb{R}^l. \tag{25}$$

Note that conditions (A1), (A2) amount to the reducibility of $D$ to $\Theta$ at $g(\bar{x})$ and the nondegeneracy of $\bar{x}$ with respect to $\Gamma$ and the mapping $h$ in the sense of Bonnans, Shapiro (2001), provided the sets $D, \Theta$ are convex.

## Theorem 12.

Let assumptions (A1), (A2) be fulfilled, $\bar{x}^* \in \hat{N}_\Gamma(\bar{x})$ and $\bar{\lambda}$ be the (unique) multiplier satisfying

$$\bar{\lambda} \in \hat{N}_D(g(\bar{x})), \ \nabla g(\bar{x})^T \bar{\lambda} = \bar{x}^*. \tag{26}$$

Then

$$T_{\mathrm{gph}\, \hat{N}_\Gamma}(\bar{x}, \bar{x}^*) = \{(u, u^*) | \exists \xi : (\nabla g(\bar{x}) u, \xi) \in T_{\mathrm{gph}\, \hat{N}_D}(g(\bar{x}), \bar{\lambda}), u^* = \nabla g(\bar{x})^T \xi + \nabla^2 \langle \bar{\lambda}, g \rangle (\bar{x}) u\}. \tag{27}$$

# Ad (vi) - Conclusion

We have introduced the graphical derivative, the limiting coderivative, and the directional limiting coderivative and shown how these notions can be used in analysis of Lipschitzian behavior of implicitly defined multifunctions. In addition to the Lipschitzian single-valued localization and to the Aubin property these notions can be employed also in analysis of other Lipschitzian properties like calmness and tilt/full stability.

Among further research goals in this area one could list

1) a deeper study of calmness which seems to be the most important qualification condition in generalized differential calculus;
2) an investigation of more complicated parameterized equilibria like problems with conic constraints (SDPs), QVIs, semi-infinite programming, evolutionary equilibria etc;
3) application to various concrete models of practical relevance (Cournot-Nash-Walras equilibria modeling, e.g., the market with emission permits).

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*THANK YOU*