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Abstract

We study the energy conservation for the weak solutions to the compressible Primitive Equations (CPE) system with degenerate viscosity. We give sufficient conditions on the regularity of weak solutions for the energy equality to hold, even for solutions that may include vacuum. In this paper, we give two theorems, the first one gives regularity in the classical isotropic Sobolev and Besov spaces. The second one state result in the anisotropic spaces. We get new regularity results in the second theorem because of the special structure of CPE system, which are in contrast to compressible Navier-Stokes equations.

Key words: energy conservation, compressible, Primitive Equations, Onsager's conjecture

2010 Mathematics Subject Classifications: 35Q30, 35Q86.

1 Introduction

In this paper, we investigate the relationship between regularity and conservation of energy for the compressible Primitive Equations (CPE) system in the periodic domain $\mathbb{T}^3$. The system has the following form

$$\begin{cases} \partial_t\rho + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}) + \partial_z(\rho w) = 0, \\ \partial_t(\rho\mathbf{v}) + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}\otimes\mathbf{v}) + \partial_z(\rho\mathbf{v}w) + \nabla_{\mathbf{x}}p(\rho) = \mathrm{div}_{\mathbf{x}}(\rho\nabla_{\mathbf{x}}\mathbf{v}) + \partial_z(\rho\partial_z\mathbf{v}), \\ \partial_z p(\rho) = 0, \end{cases} \tag{1.1}$$

where $\rho, \mathbf{u}, p$ represent the density, velocity and pressure, respectively. The velocity can be defined as $\mathbf{u} = (\mathbf{v}, w)$, $\mathbf{v}(t,\mathbf{x},z) \in \mathbb{R}^2$ and $w(t,\mathbf{x},z) \in \mathbb{R}$ represent the horizonal velocity and vertical

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velocity respectively and where $\mathbf{x} \in \mathbb{R}^2$ denotes the horizontal direction and z denotes the vertical direction. From $(1.1)_3$, we can assume that the density is independent of z, which means $\rho = \rho(t, \mathbf{x})$. Such assumption can be seen also in [23, 31], where the authors use the change of variables to derive the density ρ is independent of z. Moreover we consider pressure as $p(\rho) = \rho^\gamma \ (\gamma > 1)$.

Primitive Equations (PE)[1] system is an important model which is widely used in the geophysical research to describe and analyze the phenomena of atmosphere and ocean. It is derived from the Navier-Stokes or Euler system by asymptotic analysis or numerical approximation. Let us briefly recall some important results for such a system.

During the last decades, there is a vast body of literature about rigourous mathematical justification of deriving PE model. More precisely, for incompressible (PE) system, Azérad and Guillén [2] proved that the incompressible Navier-Stokes equations converge to PE in the sense of weak solutions. Further, Li and Titi [40] proved convergence of the weak solutions of incompressible Navier-Stokes equations to the strong solutions of PE. Based on [2, 40], Donatelli and Juhasz [20] give a justification that PE model with the pollution effect is the hydrostatic limit of the Navier-Stokes equations with an advection-diffusion equation. Grenier [32] used the energy estimates and Brenier [7] used the relative entropy inequality to prove that the smooth solutions of incompressible Euler system converge to smooth solutions of inviscid PE. Precisely, Brenier [6] proved the existence of smooth solutions in two-dimensions under the convex horizontal velocity assumptions. Later, Masmoudi and Wong [47] extended Brenier's result, removing the convex horizontal velocity assumptions.

On the other hand, Ersoy et al. [23] used the asymptotic analysis to deduce the *compressible primitive system (CPE)* with degenerate viscosity coefficients. Gao, Nečasová and Tang [30] deduced the CPE from anisotropic Navier-Stokes equations with constant viscosity coefficient.

One of the typical features of the PE model is that there is no information for the vertical velocity in the momentum equation and the vertical velocity is determined by the horizontal velocity through the continuity equation that here reduces to incompressible constraint. Therefore, the mathematical and numerical study of the PE model was unsolved until 1990s when Lions, Teman and Wang [41, 42] were first to study the PE and received fundamental results in this field. Then Guillén-González, Masmoudi and Rodríguez-Bellido [33] proved the local existence of strong solutions and uniqueness with some interesting anisotropic estimates. The celebrated result was made by Cao and Titi [12], where they first proved the global well-posedness of PE in the three dimensional case. Then, by virtue of semigroup method, Hieber and Kashiwabara [35] extended this result relaxing the smoothness on the initial data.

Compressible primitive equations system (CPE) was studied recently. Gatapov and Kazhikhov [31] proved the global existence of weak solutions with the constant viscosity coefficients in 2D case. Liu and Titi [44, 46] proved the local existence of strong solutions in 3D case

[1] PE we mean incompressible primitive equations.

and considered the zero Mach number limit of CPE. Regarding with degenerate viscosity case, Ersoy et al. [23], and Tang and Gao [52] showed the stability of weak solutions. The stability means that a subsequence of weak solutions will converge to another weak solutions if it satisfies some uniform bounds. Liu and Titi [45] and Wang et al. [54], independently, used the B-D entropy to prove the global existence of weak solutions. Readers can refer to Bresch et al. [8], Cao et al. [13], Li and Titi [40], the book of Temam and Ziane [53] and references therein for more physical background and other interesting mathematical results.

The existence of weak solutions is a fundamental question in PDE, especially for fluids models. The case of the incompressible fluids was investigated already in thirties by famous works of Leray [38]. He proved the global existence of weak solutions to incompressible Navier-Stokes equations. On the other hand, the proof of existence for the compressible case is going back to nineties years by Lions [43] and Feireisl [25, 26, 27] which proved the existence of global weak solutions of compressible isentropic case and later to full system.

Generally, it is not clear known whether weak solutions satisfy the principle of conservation of energy for both incompressible and compressible fluids. It is a nature question how high regularity for weak solutions is needed to obtain the energy equality.

Onsager [49] gave a famous conjecture that the three dimensional *incompressible Euler equations* conserve energy if the velocity $\mathbf{u} \in L^3((0,T); C^{0,\alpha}(\mathbb{T}^3))$ with $\alpha > \frac{1}{3}$. The second part of conjecture said that there exist weak solutions of the Euler equation for $\alpha \leq \frac{1}{3}$ which do not conserve energy.

The second part of Onsager's conjecture has been underlined by the celebrated work of Scheffer [50] and Shnirelman [51]. And a series of breakthrough papers were done by De Lellis and Székelyhidi [17, 18, 19] by virtue of convex integration. Recently, this part is fully solved by Isett [36], Buckmaster et. al. [11].

The first part of Onsager's conjecture was expressed as energy conservation, which is a lively direction of research at the contemporary mathematical society. It was proved by Constantin, E, Titi et [15] (also by Eyink [24], and the work of Duchon and Robert [22]), stating that if $\mathbf{u}$ belongs to $L^3([0,T]; B_3^{\alpha,\infty}(\mathbb{T}^3)) \cap C([0,T]; L^2(\mathbb{T}^3))$ with $\alpha > \frac{1}{3}$, then the energy is conserved. Cheskidov et al. [14], and Fjordhlom and Wiedemann [29] made the sharpest result in optimal Besov spaces. Their main idea is using the suitable commutator estimates for incompressible Euler system. These results were extended to the bounded domain by Bardos, Titi and Wiedemann [4], Drivas and Nguyen [21]. Later this result was extended to incompressible *inhomogeneous* Euler equations by Feireisl et al. [28]. Precisely speaking, the authors in [28] mollified the weak solutions of the density ρ and velocity $\mathbf{u}$ and stated that if $\mathbf{u}$ belongs to $B_p^{\alpha,\infty}([0,T]; \times\mathbb{T}^3)$,$\varrho, \varrho\mathbf{u} \in B_q^{\beta,\infty}([0,T]; \times\mathbb{T}^3)$, $p \in L^{p^*}_{loc}((0,T) \times \mathbb{T}^3)$ for some $1 \leq p, q \leq \infty$, p^* is the conjugate of p, and $0 \leq \alpha, \beta \leq 1$, then the energy is locally conserved. On the other hand, Leslie and Shvydkoy [39] extended these results into the inhomogeneous Navier-Stokes case. Moreover, Gwiazda et al. [34]

proved the corresponding result to general system of first order conservation laws.

Concerning *compressible Euler system* such result goes to Feireisl et al [28]. Inspired by Constantin et al. [15] work, Feireisl et al. [28] proved the energy conservation for compressible Euler system with initial data containing vacuum. Yu [55] used the Lions's commutator estimates to show energy conservation for compressible Navier-Stokes equations with degenerate viscosity but without vacuum. Nguyen et al. [48] extended Yu's result with weaker regularity condition in bounded domain. Recently, Wiedmann et al. [1] proved the corresponding results for compressible Euler and Navier-Stokes system by removing the key assumptions that the pressure is a twice continuously differentiable function. One can refer to [5, 10, 16] for other related works.

Comparing with fruitful results for either incompressible Euler, Navier-Stokes system or compressible case, there is a few results about PE model due to its special structure. There are many differences at the mathematical structure between Navier-Stokes equations and PE model, due to the hydrostatic approximation. There is **no information for the vertical velocity in the momentum equation of PE model**, so it is very difficult to analyze the PE model. As far as we know, there is only one result due to Boutros, Markfelder and Titi [9], in which they showed the energy conservation for **the incompressible inviscid PE**. They gave three types of weak solutions, where their vertical velocity is determined by horizontal velocity by virtue of incompressible conditions. One of the sufficient condition for energy conservation is horizonal velocity $\mathbf{v} \in L^4((0,T); B^{\alpha}_{4,\infty}(\mathbb{T}^3))$ $(\alpha > \frac{1}{2})$ and vertical velocity $w \in L^2((0,T); L^2(\mathbb{T}^3))$. It is interesting to find that the index for velocity regularity is different from conventional Onsager's exponent, which coincides what stated in [10], "Several of these exponents were different from $\frac{1}{3}$, which is the Onsager exponent for the Euler equations. This means that the Onsager exponent is not universal and is plausibly determined by the regularity of the coefficient functions of the nonlinearity..." It also shows the PE has different mathematical structure and corresponding different results compared with Navier-Stokes equations.

There is no result concerning the energy equality for CPE system. The goal of this work is to investigate the energy equality for CPE system under some regularity conditions.

The paper is organized as follows: In Section 2, we introduce the definition of weak solutions, give some useful lemmas and state the main theorems. Section 3 is devoted to the proof of the energy conservation.

2 Definition and Preliminaries

2.1 Definition of weak solution

Definition 2.1. *The $(\rho, \mathbf{u})$ is called a weak solution to CPE system (1.1), if $(\rho, \mathbf{u})$ satisfies the following assumptions for any time $t \in [0,T]$*

- *ρ and $\mathbf{v}$ have the following regularity:*

$$\rho \geq 0,\ \rho \in L^\infty(0,T;L^1(\mathbb{T}^3)\cap L^\gamma(\mathbb{T}^3)),\ \nabla_{\mathbf{x}}\sqrt{\rho}\in L^\infty(0,T;L^2(\mathbb{T}^3)),\ \nabla_{\mathbf{x}}\rho^{\frac{\gamma}{2}}\in L^2((0,T)\times\mathbb{T}^3),$$
$$\sqrt{\rho}\nabla_{\mathbf{x}}\mathbf{v}\in L^2(0,T;L^2(\mathbb{T}^3)),\ \sqrt{\rho}\partial_z\mathbf{v}\in L^2(0,T;L^2(\mathbb{T}^3)),\ \sqrt{\rho}\partial_z w\in L^2(0,T;L^2(\mathbb{T}^3));$$

- *the continuity equation is satisfied in the sense of distributions as follows*

$$\left[\int_{\mathbb{T}^3}\rho\varphi\, d\mathbf{x}dz\right]\Big|_{t=0}^{t=\tau} = \int_0^\tau\int_{\mathbb{T}^3}(\rho\partial_t\varphi+\rho\mathbf{v}\nabla_{\mathbf{x}}\varphi+\rho w\partial_z\varphi)\, d\mathbf{x}dzdt,$$

for all $\varphi\in C_c^\infty((0,T)\times\mathbb{T}^3)$;

- *the momentum equation is satisfied as*

$$\begin{aligned}\left[\int_{\mathbb{T}^3}\rho\mathbf{v}\varphi\, d\mathbf{x}dz\right]\Big|_{t=0}^{t=\tau} = &\int_0^\tau\int_{\mathbb{T}^3}(\rho\mathbf{v}\partial_t\varphi+\rho\mathbf{v}\otimes\mathbf{v}:\nabla\varphi+\rho\mathbf{v}w\partial_z\varphi+p(\rho)\mathrm{div}_{\mathbf{x}}\varphi)\, d\mathbf{x}dzdt\\ &-\int_0^\tau\int_{\mathbb{T}^3}(\rho\nabla_{\mathbf{x}}\mathbf{v}:\nabla_{\mathbf{x}}\varphi+\rho\partial_z\mathbf{v}\partial_z\varphi)\, d\mathbf{x}dzdt=0,\end{aligned}$$

for all $\varphi\in C_c^\infty([0,T)\times\Omega)$;

- *the energy inequality*

$$\frac{d}{dt}\int_{\mathbb{T}^3}\left(\frac{1}{2}\rho|\mathbf{v}|^2+P(\rho)\right)\, d\mathbf{x}dz\Big|_{t=0}^{t=\tau}+\int_0^\tau\int_{\mathbb{T}^3}\left(\rho|\nabla_{\mathbf{x}}\mathbf{v}|^2+\rho|\partial_z\mathbf{v}|^2\right)\, d\mathbf{x}dz\leq 0$$

for a.a $\tau\in(0,T)$, where $P(\rho)=\rho\int_1^\rho\frac{p(s)}{s^2}ds$.

Remark 2.1. *By virtue of the B-D entropy, Liu and Titi [45] obtained the extra estimates and the existence of global weak solutions to CPE with degenerate viscosity. However, it is still an open problem the existence of weak solutions to CPE with constant viscosity.*

2.2 Function spaces

Let $\Omega := (0,T)\times\mathbb{T}^3$, and we define the Besov space: $B_p^{\alpha,\infty}(\Omega)$ with $1\leq p<\infty$, $0<\alpha<1$, normed by

$$\|w\|_{B_p^{\alpha,\infty}(\Omega)} = \|w\|_{L^p(\Omega)}+\sup_{\boldsymbol{\xi}\in\Omega}\{|\boldsymbol{\xi}|^{-\alpha}\|w(\cdot+\boldsymbol{\xi})-w\|_{L^p(\Omega\cap(\Omega-\boldsymbol{\xi}))}\}$$

is finite. In order to understand the anisotropic structure in CPE, inspired by [9, 33], we introduce the following anisotropic spaces:

$$\|w\|_{H^1_{\mathbf{x}}(\mathbb{T}^2)L^2_z(\mathbb{T}^1)} = \|\|w(x,\cdot)\|_{L^2(\mathbb{T}^1)}\|_{H^1_{\mathbf{x}}(\mathbb{T}^2)}$$

and considering $\boldsymbol{\xi}=(\tau,\boldsymbol{\xi}_h,\xi_z)$,

$$\|w\|_{B_p^{\alpha,\infty}((0,T);B_p^{\alpha,\infty}(\mathbb{T};B_p^{\beta,\infty}(\mathbb{T}^2))} = \|w\|_{L^p(\Omega)}+\sup_{\boldsymbol{\xi}\in\Omega}\{|\boldsymbol{\xi}|^{-\alpha}\|w(\cdot+\boldsymbol{\xi})-w\|_{L^p(\Omega)\cap(\Omega-\boldsymbol{\xi}))}\}$$

$$+\sup_{\boldsymbol{\xi}_h\in\mathbb{T}^2}\{|\boldsymbol{\xi}_h|^{-\beta}\|w(\cdot+(0,\boldsymbol{\xi}_h,0))-w\|_{L^p(\Omega)\cap(\Omega-(0,\boldsymbol{\xi}_h,0))}\},$$

where $0<\alpha<\beta<1$. The space $H^1_{\mathbf{x}}(\mathbb{T}^2)L^2_z(\mathbb{T}^1)$ means its horizontal regularity is H^1 and vertical regularity is L^2. On the other hand, the space $B^{\alpha,\infty}_p(\mathbb{T};B^{\beta,\infty}_p(\mathbb{T}^2))$ means that it is Besov regular in the $z-$ direction with exponent α while Besov regular in the horizontal direction with exponent β.

Let $\eta\in C^\infty_c(\mathbb{R}^{d+1})$ (d is the number of the space dimension) be a standard mollification kernel and set

$$\eta^\varepsilon(x)=\frac{1}{\varepsilon^{d+1}}\eta\left(\frac{x}{\varepsilon}\right),\ w^\epsilon=\eta^\varepsilon*w,\ f^\varepsilon(w)=f(w)*\eta^\varepsilon.$$

We should notice that w^ε is well-defined on $\Omega^\varepsilon=\{x\in\Omega:d(x,\partial\Omega)>\varepsilon\}$. We first recall some useful lemmas which will be frequently used throughout the paper.

Lemma 2.1. *([28, 16]) For any function $u\in B^{\alpha,\infty}_p(\Omega)$, we have*

$$\|u(\cdot+\boldsymbol{\xi})-u\|_{L^p(\Omega\cap(\Omega-\boldsymbol{\xi}))}\le|\boldsymbol{\xi}|^\alpha\|u\|_{B^{\alpha,\infty}_p(\Omega)},$$
$$\|u^\varepsilon-u\|_{L^p(\Omega)}\le\varepsilon^\alpha\|u\|_{B^{\alpha,\infty}_p(\Omega)},$$
$$\|\nabla u^\varepsilon\|_{L^p(\Omega)}\le\varepsilon^{\alpha-1}\|u\|_{B^{\alpha,\infty}_p(\Omega)},$$

where ∇ stands for space-time gradient.

Remark 2.2. *It is easy to prove the following equality [28]*

$$f^\varepsilon g^\varepsilon-(fg)^\varepsilon=(f^\varepsilon-f)(g^\varepsilon-g)$$
$$-\int_{-\varepsilon}^{\varepsilon}\int_{\mathbb{T}^3}\eta^\varepsilon(\tau,\boldsymbol{\xi}_h,\xi_z)\left(f(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z)-f(t,x.,z)\right)\left(g(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z)-g(t,\mathbf{x},\mathbf{z})\right)d\boldsymbol{\xi}_h d\xi_z d\tau.$$

Lemma 2.2. *([3] Besov space embedding) Let $s\in\mathbb{R}$, $1\le p,r\le\infty$, then*

$$B^{s,r}_p\hookrightarrow B^{s-d(\frac{1}{p}-\frac{1}{p_1}),r}_{p_1},\ p_1\ge p.$$

It is easy to obtain the following remark

Remark 2.3. *$H^1(\mathbb{T}^3)\hookrightarrow B^{1,\infty}_2(\mathbb{T}^3)\hookrightarrow B^{\alpha,\infty}_3(\mathbb{T}^3)$ ($\alpha>\frac{1}{2}$).*

The following two lemmas can be proved as [16]

Lemma 2.3. *Let $u\in B^{\alpha,\infty}_p(\Omega,\mathbb{R}^m)$. Suppose $f:\mathbb{R}^m\to\mathbb{R}$ is a C^1 function with $\frac{\partial f}{\partial u_i}\in L^\infty$ $i=1...m$. Then*

$$\|\nabla f(u^\varepsilon)\|_{L^p(\Omega)}\le C\varepsilon^{\alpha-1}\|u\|_{B^{\alpha,\infty}_p(\Omega)},$$

where ∇ stands for space-time gradient

Lemma 2.4. *Let $1 \leq q < \infty$ and suppose $v \in L^{2q}((0,T)\times\mathbb{T}^d;\mathbb{R}^k)$ and $f \in C^2(\mathbb{R}^k;\mathbb{R}^N)$. If*

$$\sup_{i,j} \| \frac{\partial^2 f}{\partial v_i \partial v_j} \|_{L^\infty} < \infty,$$

then there exists a constant $C > 0$ such that

$$\|f(v^\varepsilon) - f^\varepsilon(v)\|_{L^q} < C\big(\|v^\varepsilon - v\|_{L^{2q}} + \sup_{(s,\mathbf{y})\in \mathrm{supp}\eta_\varepsilon} \|v(\cdot,\cdot) - v(\cdot - s, \cdot - \mathbf{y})\|^2_{L^{2q}}\big).$$

The following lemma, known as Lions's commutator lemma ([55] Lemma 2.3 and [37] Lemma 3.2) will be very useful for the estimates below

Lemma 2.5. *Let ∂ be a partial derivative in space or time. Let f, $\partial f \in L^p(\mathbb{R}^+ \times \Omega)$, $g \in L^q(\mathbb{R}^+ \times \Omega)$ with $1 \leq p, q \leq \infty$, and $\frac{1}{p} + \frac{1}{q} \leq 1$. Then, we have*

$$\|\partial(fg)^\varepsilon - \partial(fg^\varepsilon)\|_{L^r} \leq C\|\partial f\|_{L^p}\|g\|_{L^q}$$

for some constant $C > 0$ independent of ε, f and g, with $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. In addition,

$$\partial(fg)^\varepsilon - \partial(fg^\varepsilon) \to 0, \text{ in } L^r,$$

as $\varepsilon \to 0$, if $r < \infty$.

Remark 2.4. *The original version in [55] is $\|(\partial(fg))^\varepsilon - \partial(fg^\varepsilon)\|_{L^r} \leq C\|\partial f\|_{L^p}\|g\|_{L^q}$. Using Lemma 3.1 in [37], $\partial(fg)^\varepsilon = (\partial(fg))^\varepsilon$, we get the statement of Lemma 2.5 as*

$$\|\partial(fg)^\varepsilon - \partial(fg^\varepsilon)\|_{L^r} = \|(\partial(fg))^\varepsilon - \partial(fg^\varepsilon)\|_{L^r}.$$

2.3 Main result

Our main results can be stated as follows:

Theorem 2.1. *Let $(\rho, \mathbf{u})$ be a weak solution of CPE (1.1) with $\rho > 0$. Let us assume $\mathbf{v} \in B_3^{\alpha,\infty}((0,T); B_3^{\alpha,\infty}(\mathbb{T}^3))$, $\rho, \rho\mathbf{v} \in B_3^{\beta,\infty}((0,T); B_3^{\beta,\infty}(\mathbb{T}^3))$, and*

$$0 \leq \underline{\rho} \leq \rho \leq \overline{\rho}, \text{ a.e.in } (0,T)\times\Omega$$

for some constants $\underline{\rho}, \overline{\rho}$ and $0 \leq \alpha, \beta \leq 1$, $\alpha > \frac{3}{4}$ and $\alpha + \beta > \frac{3}{2}$. Moreover, we assume the additional regularity $w \in L^2(0,T; L^2(\mathbb{T}^3))$, $\mathbf{v} \in L^2((0,T); H^1(\mathbb{T}^3))$, $\rho\mathbf{v} \in L^3((0,T); H^1(\mathbb{T}^3))$ and $p \in C[\underline{\rho}, \overline{\rho}]$. Then the energy is locally conserved i.e.

$$\begin{aligned}\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2 + P(\rho)\right) + \rho|\nabla_{\mathbf{x}}\mathbf{v}|^2 + \rho|\partial_z\mathbf{v}|^2 + \mathrm{div}_{\mathbf{x}}\left[(\frac{1}{2}\rho|\mathbf{v}|^2 + p(\rho) + P(\rho) - \rho\nabla_{\mathbf{x}}\mathbf{v})\mathbf{v}\right] \\ +\partial_z\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2 + p(\rho) + P(\rho)\right)w - \rho\partial_z\mathbf{v}\cdot\mathbf{v}\right] = 0.\end{aligned} \tag{2.1}$$

Remark 2.5. *Our result is still valid for the constant viscosity coefficient whose corresponding energy is*

$$\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2+P(\rho)\right)+|\nabla_{\mathbf{x}}\mathbf{v}|^2+|\partial_z\mathbf{v}|^2+\operatorname{div}_{\mathbf{x}}\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2+p(\rho)+P(\rho)-\nabla_{\mathbf{x}}\mathbf{v}\right)\mathbf{v}\right]$$
$$+\partial_z\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2+p(\rho)+P(\rho)\right)w-\partial_z\mathbf{v}\cdot\mathbf{v}\right]=0.$$

Remark 2.6. There are some major differences between our work and Feireisl et al. [28], Wiedemann et al. [1], Titi et al.[9]. *Theorem 2.1 seems similar and parallel to the results in [28, 1]. However, in contrast with compressible Euler and Navier-Stokes system, we have only information on the horizontal velocity in CPE. It means it is a hurdle to control the estimate of vertical velocity in the isotropic space. In order to overcome this difficulty, we need the additional assumption $\rho\mathbf{v}\in L^3((0,T);H^1)$ in Theorem 2.1. Fortunately, by deep investigation of the structure of CPE, we find that we can work with the anisotropic space to solve the corresponding problems. Therefore, we are able to remove this restriction by virtue of the anisotropic regularity in Theorem 2.2.*

Compared with [9], we deal with the compressible situation which completely changes the behaviour of fluid: Firstly, the only way to obtain the information of vertical velocity is through continuity equation instead of incompressible constraint. Moreover, the pressure depends on density and needs to be controlled. Last but not least, we have dissipative term and the viscosity depends on density, which brings the new commutator estimates. Therefore, the conventional method used in incompressible PE [9] can not be applied straightforwardly to the CPE model. Here, we find a new observation and use the Lions's commutator lemma (Lemma 2.5) to control the convergence of vertical viscosity commutator which is different to [1].

Inspired by [9, 33], we introduce the horizontal and vertical regularity and obtain the following theorem:

Theorem 2.2. *Let $(\rho,\mathbf{u})$ be a weak solution of CPE (1.1) with $\rho>0$. Let us assume $\mathbf{v}\in B_3^{\alpha,\infty}((0,T);B_3^{\alpha,\infty}(\mathbb{T};B_3^{\beta,\infty}(\mathbb{T}^2)))$, $\rho,\rho\mathbf{v}\in B_3^{\alpha,\infty}((0,T);B_3^{\alpha,\infty}(\mathbb{T};B_3^{\beta,\infty}(\mathbb{T}^2)))$, where $0<\alpha<\beta\leq 1$, $\alpha>\frac{1}{3}$ and $2\alpha+\beta>2$. Moreover, we assume future*

$$0\leq\underline{\rho}\leq\rho\leq\overline{\rho},\ a.e.in\,(0,T)\times\Omega$$

for some constants $\underline{\rho},\overline{\rho}$. Moreover, we assume $w\in L^2(0,T;L^2(\mathbb{T}^3))$ $\mathbf{v}\in L^2((0,T);H^1_{\mathbf{x}}(\mathbb{T}^2)L^2_z(\mathbb{T}^1))$ and $p\in C[\underline{\rho},\overline{\rho}]$. Then the energy is locally conserved i.e.

$$\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2+P(\rho)\right)+\rho|\nabla_{\mathbf{x}}\mathbf{v}|^2+\rho|\partial_z\mathbf{v}|^2+\operatorname{div}_{\mathbf{x}}\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2+p(\rho)+P(\rho)-\rho\nabla_{\mathbf{x}}\mathbf{v}\right)\mathbf{v}\right]$$
$$+\partial_z\left[\left(\frac{1}{2}\rho|\mathbf{v}|^2+p(\rho)+P(\rho)\right)w-\rho\partial_z\mathbf{v}\cdot\mathbf{v}\right]=0.$$

Remark 2.7. *The corresponding constant viscosity coefficient result is also valid as in Remark 2.5.*

Remark 2.8. *Notice that our assumption $\beta > \alpha$, in the definition of the anisotropic Besov space and Theorem 2.2, satisfies the special structure of CPE, which means that the horizontal regularity is higher than vertical one. Moreover, from the assumptions $\beta > \alpha$ and $2\alpha + \beta > 2$, we can deduce $\beta > \frac{2}{3}$. It is important to point out that our result could be extended to different regularity with $\mathbf{v}$ and ρ, such as $\mathbf{v} \in B_3^{\alpha,\infty}((0,T); B_3^{\alpha_1,\infty}(\mathbb{T}; B_3^{\beta_1,\infty}(\mathbb{T}^2)))$ and $\rho \in B_3^{\beta,\infty}((0,T); B_3^{\alpha_2,\infty}(\mathbb{T}; B_3^{\beta_2,\infty}(\mathbb{T}^2)))$. But it will produce many complex index relationships. In order not to bring confuse to readers, we just uniform the regularity and assume they are in the same regularity space.*

Remark 2.9. *It should be noticed that vertical regularity of $\mathbf{v}$ in the Theorem 2.2 is lower than in [1], in which they assume $\mathbf{v} \in L^2((0,T); H^1)$, by virtue of the anisotropic regularity and the special structure of CPE.*

3 Proof of Theorem 2.1 and Theorem 2.2

3.1 Energy equality

We follow the strategy from [1, 28]. Precisely speaking, we mollify the CPE in both space and time, then derive the local energy equality for regularized quantities, estimate commutator errors generated by nonlinear terms and, passing $(\varepsilon, \delta) \to 0$ (see below), commutators tend to zero and the result is obtained in the original quantities.

Firstly, we mollify the momentum equation $(1.1)_2$ and obtain

$$\partial_t(\rho\mathbf{v})^\varepsilon + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}\otimes\mathbf{v})^\varepsilon + \partial_z(\rho\mathbf{v}w)^\varepsilon + \nabla_{\mathbf{x}}p^\varepsilon(\rho) = \mathrm{div}_{\mathbf{x}}(\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon + \partial_z(\rho\partial_z\mathbf{v})^\varepsilon. \tag{3.1}$$

Notice that the pressure term in (3.1) does not belong to C^2, thus it does not satisfy the conditions in [28]. Therefore, we need mollify the pressure as the following (based on [1]): Take a sequence $p^\delta \in C^2[\underline{\rho}, \overline{\rho}]$ that converges uniformly to $p \in C[\underline{\rho}, \overline{\rho}]$, that is for each $\delta > 0$

$$\|p^\delta - p\|_{L^\infty} \le \delta.$$

Replacing p^δ in (3.1), we obtain

$$\begin{aligned}\partial_t(\rho\mathbf{v})^\varepsilon + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v}\otimes\mathbf{v})^\varepsilon + \partial_z(\rho\mathbf{v}w)^\varepsilon + \nabla_{\mathbf{x}}\left(p^\delta(\rho)\right)^\varepsilon \\ = \mathrm{div}_{\mathbf{x}}(\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon + \partial_z(\rho\partial_z\mathbf{v})^\varepsilon + \nabla_{\mathbf{x}}\Big((p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)\Big).\end{aligned}$$

It is easy to obtain

$$\partial_t(\rho^\varepsilon\mathbf{v}^\varepsilon) + \mathrm{div}_{\mathbf{x}}\big((\rho\mathbf{v})^\varepsilon\otimes\mathbf{v}^\varepsilon\big) + \partial_z\big((\rho w)^\varepsilon\mathbf{v}^\varepsilon\big) + \nabla_{\mathbf{x}}p^\delta(\rho^\varepsilon) - \mathrm{div}_{\mathbf{x}}\left(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon\right) - \partial_z\left(\rho^\varepsilon\partial_z\mathbf{v}^\varepsilon\right)$$

$$
\begin{aligned}
&= \partial_t\Big(\rho^\varepsilon \mathbf{v}^\varepsilon - (\rho\mathbf{v})^\varepsilon\Big) + \mathrm{div}_{\mathbf{x}}\big((\rho\mathbf{v})^\varepsilon \otimes \mathbf{v}^\varepsilon - (\rho\mathbf{v}\otimes\mathbf{v})^\varepsilon\big)\\
&\quad + \partial_z\Big((\rho w)^\varepsilon \mathbf{v}^\varepsilon - (\rho\mathbf{v}w)^\varepsilon\Big)\\
&\quad - \mathrm{div}_{\mathbf{x}}\Big(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon - (\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon\Big) - \partial_z\big(\rho^\varepsilon\partial_z\mathbf{v}^\varepsilon - (\rho\partial_z\mathbf{v})^\varepsilon\big)\\
&\quad + \nabla_{\mathbf{x}}\Big(p^\delta(\rho^\varepsilon) - (p^\delta(\rho))^\varepsilon\Big) + \nabla_{\mathbf{x}}\big((p^\delta(\rho))^\varepsilon - p^\varepsilon(\rho)\big)\\
&= R^\varepsilon = \sum_{i=1}^{7} R_i^\varepsilon. \qquad (3.2)
\end{aligned}
$$

Multiplying with $\mathbf{v}^\varepsilon$, we get

$$
\begin{aligned}
&\partial_t\rho^\epsilon|\mathbf{v}^\epsilon|^2 + \rho^\epsilon\partial_t\frac{|\mathbf{v}^\epsilon|^2}{2} + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v})^\varepsilon|\mathbf{v}^\epsilon|^2 + (\rho\mathbf{v})^\varepsilon\nabla_{\mathbf{x}}\frac{|\mathbf{v}^\epsilon|^2}{2} + \partial_z(\rho w)^\epsilon|\mathbf{v}^\epsilon|^2 + (\rho w)^\varepsilon\partial_z\frac{|\mathbf{v}^\epsilon|^2}{2}\\
&\qquad + \rho^\epsilon\mathbf{v}^\epsilon\cdot\nabla_{\mathbf{x}}(P^\delta(\rho^\varepsilon))' + \rho^\varepsilon|\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon|^2 + \rho^\varepsilon|\partial_z\mathbf{v}^\varepsilon|^2 - \mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\epsilon\cdot\mathbf{v}^\epsilon) - \partial_z(\rho^\epsilon\partial_z\mathbf{v}^\epsilon\cdot\mathbf{v}^\varepsilon)\\
&\qquad = R^\varepsilon\mathbf{v}^\varepsilon \qquad (3.3)
\end{aligned}
$$

where

$$
\nabla_{\mathbf{x}}p^\delta(\rho^\varepsilon) = (p^\delta(\rho^\varepsilon))'\nabla_{\mathbf{x}}\rho^\epsilon = \rho^\varepsilon(P^\delta(\rho^\varepsilon))''\nabla_{\mathbf{x}}\rho^\epsilon, = \rho^\varepsilon\nabla_{\mathbf{x}}(P^\delta(\rho^\varepsilon))'.
$$

Mollifying the continuity equation $(1.1)_1$ with space and time, we have

$$
\partial_t\rho^\varepsilon + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v})^\varepsilon + \partial_z(\rho w)^\varepsilon = 0. \qquad (3.4)
$$

It is easy to obtain the following equality

$$
(\partial_t\rho^\varepsilon + \mathrm{div}_{\mathbf{x}}(\rho\mathbf{v})^\varepsilon + \partial_z(\rho w)^\varepsilon)\frac{|\mathbf{v}^\varepsilon|^2}{2} = 0. \qquad (3.5)
$$

We rewrite (3.5) and get

$$
\partial_t\rho^\varepsilon + \mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\mathbf{v}^\varepsilon) + \partial_z(\rho w)^\varepsilon = \mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\mathbf{v}^\varepsilon - (\rho\mathbf{v})^\varepsilon),
$$

then

$$
\partial_t P^\delta(\rho^\varepsilon) + \mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\mathbf{v}^\varepsilon)(P^\delta(\rho^\varepsilon))' + \partial_z(\rho w)^\varepsilon(P^\delta(\rho^\varepsilon))' = \mathrm{div}_{\mathbf{x}}\Big(\rho^\varepsilon\mathbf{v}^\varepsilon - (\rho\mathbf{v})^\varepsilon\Big)(P^\delta(\rho^\varepsilon))'. \qquad (3.6)
$$

We define $s^\varepsilon = \mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\mathbf{v}^\varepsilon - (\rho\mathbf{v})^\varepsilon)(P^\delta(\rho^\varepsilon))'$, and put (3.3)-(3.6) together to obtain

$$
\begin{aligned}
&\partial_t\left(\frac{1}{2}\rho^\varepsilon|\mathbf{v}^\varepsilon|^2 + P^\delta(\rho^\varepsilon)\right) + \rho^\varepsilon|\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon|^2 + \rho^\varepsilon|\partial_z\mathbf{v}^\varepsilon|^2\\
&\qquad + \mathrm{div}_{\mathbf{x}}\left((\rho\mathbf{v})^\varepsilon\frac{1}{2}|\mathbf{v}^\varepsilon|^2 + (\rho^\varepsilon\mathbf{v}^\varepsilon)(P^\delta(\rho^\varepsilon))' - \rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon\mathbf{v}^\varepsilon\right)\\
&\qquad + \partial_z\left((\rho w)^\varepsilon\frac{1}{2}|\mathbf{v}^\varepsilon|^2 + (\rho w)^\varepsilon(P^\delta(\rho^\varepsilon))' - \rho^\epsilon\partial_z\mathbf{v}^\epsilon\cdot\mathbf{v}^\varepsilon\right) = R^\varepsilon\mathbf{v}^\varepsilon + s^\varepsilon. \qquad (3.7)
\end{aligned}
$$

Therefore, we just need to prove the limit $\varepsilon \to 0$ for each fixed $\delta > 0$, then consider $\delta \to 0$ in the following section.

3.2 Commutator estimates

Section 3.1 can be seen as the prior estimates of energy equality. We will deduce the local energy equality for the regularized quantities as follows:

Let $\varphi \in C_c^\infty(\Omega)$ be a test function and take $\varepsilon > 0$ small enough so that $supp(\varphi) \subset (\varepsilon, T-\varepsilon)\times\mathbb{T}^3$. Multiplying (3.2) with $\varphi\mathbf{v}^\varepsilon$ and integrating in time and space, we have

$$\begin{aligned}\int_0^\tau\int_{\mathbb{T}^3} \partial_t(\rho^\varepsilon\mathbf{v}^\varepsilon)\cdot\varphi\mathbf{v}^\varepsilon\, d\mathbf{x}dzdt + \int_0^\tau\int_{\mathbb{T}^3} \operatorname{div}_{\mathbf{x}}\big((\rho\mathbf{v})^\varepsilon\otimes\mathbf{v}^\varepsilon\big)\cdot\varphi\mathbf{v}^\varepsilon + \partial_z\big((\rho w)^\varepsilon\mathbf{v}^\varepsilon\big)\cdot\varphi\mathbf{v}^\epsilon\, d\mathbf{x}dzdt \\ + \int_0^\tau\int_{\mathbb{T}^3} \nabla_{\mathbf{x}}p^\delta(\rho^\varepsilon)\cdot\varphi\mathbf{v}^\varepsilon\, d\mathbf{x}dzdt \\ = \int_0^\tau\int_{\mathbb{T}^3} \operatorname{div}_{\mathbf{x}}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon)\cdot\varphi\mathbf{v}^\varepsilon + \partial_z(\rho^\varepsilon\partial_z\mathbf{v}^\varepsilon)\varphi\mathbf{v}^\varepsilon\, d\mathbf{x}dzdt \\ + \sum_{i=1}^{7}\int_0^\tau\int_{\mathbb{T}^3} R_i^\varepsilon\cdot\varphi\mathbf{v}^\varepsilon\, d\mathbf{x}dzdt.\end{aligned}$$

Then, multiplying (3.4) by $\dfrac{1}{2}\varphi|\mathbf{v}^\varepsilon|^{\mathbf{2}}$, multiplying (3.6) by φ and combining together, so that we obtain the form of (3.7) in the distribution sense. By the same taken as [1, 28], we just need to show that each commutator error term converges to zero as

$$\int_0^\tau\int_{\mathbb{T}^3} R_i^\varepsilon\varphi\, d\mathbf{x}dzdt \to 0,\quad \int_0^\tau\int_{\mathbb{T}^3} s^\varepsilon\varphi\, d\mathbf{x}dzdt \to 0.$$

In the following sections, we will divide into two parts to prove the above convergence. Therefore, we complete the proof of both theorems.

3.2.1 Proof of Theorem 2.1

The terms R_i^ε $(i = 1, 2)$ and s^ε is dealt by the same way as [28]. For readers' convenience, we recall the main steps and take R_2^ε as an example to show the proof. Using Remark 2.2, we have

$$\begin{aligned}(\rho\mathbf{v})^\varepsilon\otimes\mathbf{v}^\varepsilon - (\rho\mathbf{v}\otimes\mathbf{v})^\varepsilon = \big((\rho\mathbf{v})^\varepsilon - \rho\mathbf{v}\big)\otimes(\mathbf{v}^\varepsilon - \mathbf{v}) \\ - \int_{-\varepsilon}^{\varepsilon}\int_{\mathbb{T}^3} \eta^\varepsilon(\tau,\boldsymbol{\xi}_h,\xi_z)\left(\rho\mathbf{v}(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z) - \rho\mathbf{v}(t,\mathbf{x},z)\right) \\ \otimes\left(\mathbf{v}(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z) - \mathbf{v}(t,\mathbf{x},z)\right)\, d\boldsymbol{\xi}_h d\xi_z d\tau.\end{aligned}\tag{3.8}$$

Therefore, for the first term we split it into two parts according to (3.8),

$$\begin{aligned}&\int_0^\tau\int_{\mathbb{T}^3} \operatorname{div}_{\mathbf{x}}\big(((\rho\mathbf{v})^\varepsilon - \rho\mathbf{v})\otimes(\mathbf{v}^\varepsilon - \mathbf{v})\big)\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt \\ &\quad\le \|\varphi\|_{C^1}\|(\rho\mathbf{v})^\varepsilon - \rho\mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon\|_{L^3} + \|\varphi\|_{C^0}\|(\rho\mathbf{v})^\varepsilon - \rho\mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L^3}\|\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon\|_{L^3} \\ &\quad\le C\left(\varepsilon^{\alpha+\beta} + \varepsilon^{2\alpha+\beta-1}\right)\|\rho\mathbf{v}\|_{B_3^{\beta,\infty}}\|\mathbf{v}\|^2_{B_3^{\alpha,\infty}}.\end{aligned}$$

From the assumption in Theorem 2.1 $\alpha + \beta > \frac{3}{2}$, it is easy to deduce $2\alpha + \beta > 1$, so that we can obtain the term vanishes when $\varepsilon \to 0$. For another part of R_2^ε, we use the Fubini's theorem and

Holder's inequality, and get

$$\int_0^\tau\int_{\mathbb{T}^3}\mathrm{div}\left(\int_\varepsilon^\varepsilon\int_{\mathbb{T}^3}\eta^\varepsilon(\tau,\xi)\left(\rho\mathbf{v}(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z)-\rho\mathbf{v}(t,\mathbf{x},z)\right)\right.$$
$$\left.\cdot\left(\mathbf{v}(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi)-\mathbf{v}(t,\mathbf{x},z)\right)d\boldsymbol{\xi}_h\,d\xi_z\,d\tau\right)\cdot\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt$$
$$\leq C\left(\varepsilon^{\alpha+\beta}+\varepsilon^{2\alpha+\beta-1}\right)\|\rho\mathbf{v}\|_{B_3^{\beta,\infty}}\|\mathbf{v}\|^2_{B_3^{\alpha,\infty}}. \tag{3.9}$$

Next, we turn to estimate the commutators R_6^ε and R_7^ε, the proof is inspired by [1] and we include the proof here for completeness

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}\nabla_{\mathbf{x}}\big(p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\big)\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad=-\int_0^\tau\int_{\mathbb{T}^3}\big(p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\big)(\varphi\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon+\nabla_{\mathbf{x}}\varphi\mathbf{v}^\varepsilon)d\mathbf{x}dzdt\\
&\quad\leq\|\varphi\|_{C^0}\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\|\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon\|_{L^3}+C\|\varphi\|_{C^1}\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\|\mathbf{v}^\varepsilon\|_{L^3}\\
&\quad\leq C\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\varepsilon^{\alpha-1}\|\mathbf{v}\|_{B_3^{\alpha,\infty}}+C\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\|\mathbf{v}^\varepsilon\|_{L^3}\\
&\quad\leq C\big(\|\rho^\varepsilon-\rho\|^2_{L^3}+\sup_{\mathbf{y}\in\mathrm{supp}\eta^\varepsilon}\|\rho(\cdot)-\rho(\cdot-\mathbf{y})\|^2_{L^3}\big)(\varepsilon^{\alpha-1}+1)\|\mathbf{v}\|_{B_3^{\alpha,\infty}}\\
&\quad\leq C\,\varepsilon^{2\beta}(\varepsilon^{\alpha-1}+1)\|\rho\|^2_{B_3^{\beta,\infty}}\|\mathbf{v}\|_{B_3^{\alpha,\infty}},
\end{aligned}$$

where we have used Lemma 2.4 in the second inequality and

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}\nabla_{\mathbf{x}}\big((p^\delta(\rho))^\varepsilon-p^\varepsilon(\rho)\big)\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad=-\int_0^\tau\int_{\mathbb{T}^3}\big(p^\delta(\rho)-p(\rho)\big)^\varepsilon\varphi\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon d\mathbf{x}dzdt-\int_0^\tau\int_{\mathbb{T}^3}\big(p^\delta(\rho)-p(\rho)\big)^\varepsilon\nabla_{\mathbf{x}}\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad\leq C\,\|\varphi\|_{C^0}\|\big(p^\delta(\rho)-p(\rho)\big)^\varepsilon\|_{L^\infty}\|\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon\|_{L^2}+C\|\varphi\|_{C^1}\|\big(p^\delta(\rho)-p(\rho)\big)^\varepsilon\|_{L^\infty}\|\mathbf{v}^\varepsilon\|_{L^3}\\
&\quad\leq C\delta.
\end{aligned}$$

The remaining is to estimate R_4^ε and R_5^ε. We give a different proof compared with [1] as the following:

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}\mathrm{div}_{\mathbf{x}}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon-(\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon)\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad=-\int_0^\tau\int_{\mathbb{T}^3}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon-(\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon)\nabla_{\mathbf{x}}\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt-\int_0^\tau\int_{\mathbb{T}^3}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon-(\rho\nabla_{\mathbf{x}}\mathbf{v})^\varepsilon)\varphi\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon d\mathbf{x}dzdt.
\end{aligned}\tag{3.10}$$

Using Remark 2.2, we split the two terms on the right hand side of (3.10) into four terms:

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}(\rho^\varepsilon\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon-(\rho\nabla\mathbf{v})^\varepsilon)\nabla_{\mathbf{x}}\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&=\int_0^\tau\int_{\mathbb{T}^3}(\rho^\varepsilon-\rho)(\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon-\nabla_{\mathbf{x}}\mathbf{v})\mathbf{v}^\epsilon\cdot\nabla_{\mathbf{x}}\varphi\,d\mathbf{x}dzdt\\
&+\int_{-\varepsilon}^\varepsilon\int_{\mathbb{T}^3}\eta^\varepsilon(\tau,\xi)\Big(\rho(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h)-\rho(t,\mathbf{x})\Big)\Big(\nabla_{\mathbf{x}}\mathbf{v}(t-\tau,\mathbf{x}-\boldsymbol{\xi}_h,z-\xi_z)-\nabla_{\mathbf{x}}\mathbf{v}(t,x,z)\Big)\mathbf{v}^\epsilon\cdot\nabla_{\mathbf{x}}\varphi\,d\boldsymbol{\xi}_hd\xi_zd\tau
\end{aligned}$$

$$= J_1 + J_2,$$

and

$$\int_0^\tau \int_{\mathbb{T}^3} (\rho^\varepsilon \nabla_{\mathbf{x}} \mathbf{v}^\varepsilon - (\rho \nabla_{\mathbf{x}} \mathbf{v})^\varepsilon) \varphi \nabla_{\mathbf{x}} \mathbf{v}^\varepsilon d\mathbf{x} dz dt$$
$$= \int_0^\tau \int_{\mathbb{T}^3} (\rho^\varepsilon - \rho)(\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon - \nabla_{\mathbf{x}} \mathbf{v}) : \nabla_{\mathbf{x}} \mathbf{v}^\epsilon \varphi \, d\mathbf{x} dz dt$$
$$+ \int_{-\varepsilon}^{\varepsilon} \int_{\mathbb{T}^3} \eta^\varepsilon(\tau, \xi) \Big(\rho(t - \tau, \mathbf{x} - \boldsymbol{\xi}_h) - \rho(t, \mathbf{x}) \Big) \Big(\nabla_{\mathbf{x}} \mathbf{v}(t - \tau, \mathbf{x} - \boldsymbol{\xi}_h, z - \xi_z) - \nabla_{\mathbf{x}} \mathbf{v}(t, \mathbf{x}, z) \Big) : \nabla_{\mathbf{x}} \mathbf{v}^\epsilon \varphi \, d\boldsymbol{\xi}_h d\xi_z d\tau$$
$$= J_3 + J_4.$$

Observe that J_2 and J_4 has been written taking into account that ρ is independent of z.

We just concern on the terms J_1 and J_3, the remaining ones can be estimated similarly by applying the Fubini's theorem (see in [1, 28]).

$$|J_1| + |J_3|$$
$$= \left| \int_0^\tau \int_{\mathbb{T}^3} (\rho^\varepsilon - \rho)(\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon - \nabla_{\mathbf{x}} \mathbf{v}) \mathbf{v}^\epsilon \cdot \nabla_{\mathbf{x}} \varphi \, d\mathbf{x} dz dt \right| + \left| \int_0^\tau \int_{\mathbb{T}^3} (\rho^\varepsilon - \rho)(\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon - \nabla_{\mathbf{x}} \mathbf{v}) : \nabla_{\mathbf{x}} \mathbf{v}^\epsilon \varphi \, d\mathbf{x} dz dt \right|$$
$$\leq \|\varphi\|_{C^1} \|\rho^\varepsilon - \rho\|_{L^3} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\mathbf{v}^\varepsilon\|_{L^6} + \|\varphi\|_{C^0} \|\rho^\varepsilon - \rho\|_{L^3} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon\|_{L^6}$$
$$\leq \|\varphi\|_{C^1} \varepsilon^\beta \|\rho\|_{B_3^{\beta,\infty}} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\mathbf{v}\|_{H^1} + \|\varphi\|_{C^0} \varepsilon^{\beta + \alpha - 3/2} \|\rho\|_{B_3^{\beta,\infty}} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\mathbf{v}\|_{B_6^{\alpha - 1/2,\infty}}$$
$$\leq \|\varphi\|_{C^1} \varepsilon^\beta \|\rho\|_{B_3^{\beta,\infty}} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\mathbf{v}\|_{H^1} + \|\varphi\|_{C^0} \varepsilon^{\beta + \alpha - 3/2} \|\rho\|_{B_3^{\beta,\infty}} \|\nabla_{\mathbf{x}}(\mathbf{v}^\varepsilon - \mathbf{v})\|_{L^2} \|\mathbf{v}\|_{B_3^{\alpha,\infty}}.$$

where we have used Lemma 2.1 and Lemma 2.2 to get

$$\|\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon\|_{L^6} \leq \varepsilon^{\alpha - \frac{3}{2}} \|\mathbf{v}\|_{B_6^{\alpha - \frac{1}{2},\infty}} \leq \varepsilon^{\alpha - \frac{3}{2}} \|\mathbf{v}\|_{B_3^{\alpha,\infty}}. \tag{3.11}$$

Because we deduce $\mathbf{v} \in B_6^{\alpha - \frac{1}{2},\infty}$ by virtue of the assumption $\mathbf{v} \in B_3^{\alpha,\infty}$ and the embedding $B_3^{\alpha,\infty} \hookrightarrow B_6^{\alpha - \frac{1}{2},\infty}$ from Lemma 2.2.

Remark 3.1. *In (3.11) we applied Lemma $(2.1)_3$ with ω*

$$\|\nabla_{\mathbf{x}} \mathbf{v}^\varepsilon\|_{L^6} \leq \varepsilon^{\omega - 1} \|\mathbf{v}\|_{B_6^{\omega,\infty}}.$$

Setting $\omega = \alpha - 1/2$ we get (3.11). Since $\alpha > \frac{3}{4}$, it converges to zero.

Recalling $\mathbf{v} \in L^2((0,T); H^1)$, we have $R_4^\varepsilon \to 0$ as $\varepsilon \to 0$. For the term R_5^ε, it is similar as $R_4^\varepsilon \to 0$, hence we omit the detail.

Finally, we turn to analysis the difficult commutator error R_3^ε. As usual, the decomposition to R_3^ε is similar to R_2^ε, we divide it into two parts as the following:

$$(\rho w)^\varepsilon \mathbf{v}^\varepsilon - (\rho w \mathbf{v})^\varepsilon = ((\rho w)^\varepsilon - \rho w)(\mathbf{v}^\varepsilon - \mathbf{v})$$
$$- \int_\varepsilon^\varepsilon \int_{\mathbb{T}^3} \eta^\varepsilon(\tau, \xi) \big(\rho w(t - \tau, \mathbf{x} - \boldsymbol{\xi}_h, z - \xi_z) - \rho w(t, \mathbf{x}, z) \big) \big(\mathbf{v}(t - \tau, \mathbf{x} - \boldsymbol{\xi}_h, z - \xi_z) - \mathbf{v}(t, \mathbf{x}, z) \big) \, d\boldsymbol{\xi}_h d\xi_z d\tau.$$

We only focus on the first part only, as the second part produce the same estimates after applying Fubin's theorem. It is difficult to estimate this term directly, because we do not have enough regularity for the regularity of vertical velocity. This is the essential difficulty and distinction between Navier-Stokes system and CPE system.

Remark 3.2. *The following equality plays a key role in the existence of CPE [45], which is also helpful and important in our analysis,*

$$
\begin{aligned}
(\rho w) &= -\int_0^z \mathrm{div}_{\mathbf{x}}[\rho(v(x,z',t) - \int_0^1 v(x,s,t)ds)]dz'. \\
&= -\int_0^z \mathrm{div}_{\mathbf{x}}(\rho\widetilde{\mathbf{v}})dz', \qquad (3.12)
\end{aligned}
$$

where $\widetilde{\mathbf{v}} = \mathbf{v} - \overline{\mathbf{v}} = v - \int_0^1 \mathbf{v}(x,s,t)ds$. *Observe that here it is crucial the fact that* ρ *is independent of the* z*-variable (see [45]).*

Since $\rho\mathbf{v} \in L^3((0,T);H^1(\mathbb{T}^3))$, which means $\rho w \in L^3((0,T);L^2(\mathbb{T}^3))$ from (3.12), thus we get

$$
\begin{aligned}
\int_0^\tau \int_{\mathbb{T}^3} &((\rho w)^\varepsilon - \rho w)(\mathbf{v}^\varepsilon - \mathbf{v})(\partial_z\varphi\mathbf{v}^\varepsilon + \varphi\partial_z\mathbf{v}^\varepsilon)\, d\mathbf{x}dzdt \\
&\le \|(\rho w)^\varepsilon - \rho w\|_{L^2}\|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L^3}\left(\|\varphi\|_{C^1}\|\mathbf{v}^\varepsilon\|_{L^6} + \|\varphi\|_{C^0}\|\partial_z\mathbf{v}^\varepsilon\|_{L^6}\right) \\
&\le C\|\rho w\|_{L^2}\varepsilon^\alpha\|\mathbf{v}\|_{B_3^{\alpha,\infty}}\left(\|\mathbf{v}\|_{B_6^{\alpha-\frac{1}{2},\infty}} + \varepsilon^{\alpha-\frac{3}{2}}\|\mathbf{v}\|_{B_6^{\alpha-\frac{1}{2},\infty}}\right) \\
&\le C\left(\varepsilon^\alpha + \varepsilon^{2\alpha-\frac{3}{2}}\right)\|\mathbf{v}\|^2_{B_3^{\alpha,\infty}},
\end{aligned}
$$

where we use similar method from (3.11) to get

$$
\|\partial_z\mathbf{v}^\varepsilon\|_{L^6} \le \varepsilon^{\alpha-\frac{3}{2}}\|\mathbf{v}\|_{B_6^{\alpha-\frac{1}{2},\infty}} \le \varepsilon^{\alpha-\frac{3}{2}}\|\mathbf{v}\|_{B_3^{\alpha,\infty}}. \qquad (3.13)
$$

Recalling the assumption $\alpha > \frac{3}{4}$, we could deduce this term converges to zero.

Remark 3.3. *We used similar argument as in* (3.11)*. From Lemma 2.1, it is easy to obtain that* $\|\partial_z\mathbf{v}^\varepsilon\|_{L^6} \le \varepsilon^{\omega-1}\|\mathbf{v}\|_{B_6^{\omega,\infty}}$*. Then, setting again* $\omega = \alpha - 1/2$ *(since* $\alpha > 3/4$*), we can deduce* (3.13).

In order to complete the proof of Theorem 2.1, we need to show the final level of convergence for δ. For each fixed $\delta > 0$, we have the in the limit (3.7) as $\varepsilon \to 0$

$$
\begin{aligned}
\partial_t\left(\frac{1}{2}\rho|\mathbf{v}|^2 + P^\delta(\rho)\right) + \rho|\nabla_{\mathbf{x}}\mathbf{v}|^2 + \rho|\partial_z\mathbf{v}|^2 + \mathrm{div}_{\mathbf{x}}\left((\rho\mathbf{v})\frac{1}{2}|\mathbf{v}|^2 + (p^\delta(\rho) + P^\delta(\rho))\mathbf{v} - \rho\nabla_{\mathbf{x}}\mathbf{v}\mathbf{v}\right) \\
+\partial_z\left((\rho w)\frac{1}{2}|\mathbf{v}|^2 + (p^\delta(\rho) + P^\delta(\rho))w - \rho\partial_z\mathbf{v}\cdot\mathbf{v}\right),
\end{aligned}
\qquad (3.14)
$$

where $P^\delta(\rho)=\rho^\delta\displaystyle\int_1^\rho \frac{p^\delta(r)}{r^2}dr$.

We will prove (3.14) converges in the sense of distributions on Ω as $\delta\to 0$ to (2.1). This limit process of δ is similar to Wiedemann et al. [1]. We just show the slight difference in contrast to [1], since our system has different behavior in vertical direction, that is $w\in L^2(0,T;L^2(\mathbb{T}^3))$.

Taking $\varphi\in C_c^\infty(\Omega)$, we have

$$\Big|\int_0^\tau\int_{\mathbb{T}^3}\partial_z\varphi(p^\delta(\rho)-p(\rho))w\,d\mathbf{x}dzdt\Big|\le C\,\|\varphi\|_{C^1}\|p^\delta(\rho)-p(\rho)\|_{L^\infty}\|w\|_{L^2}\le C\delta.$$

Recalling the estimates from [1]

$$|P^\delta(\rho)-P(\rho)|\le\rho\int_1^\rho\frac{|p^\delta(r)-p(r)|}{r^2}dr\le\|p^\delta-p\|_{L^\infty}\rho\left|\int_1^\rho\frac{1}{r^2}dr\right|\le(1+\rho)\|p^\delta-p\|_{L^\infty},$$

we can obtain that

$$\Big|\int_0^\tau\int_{\mathbb{T}^3}\partial_z\varphi(P^\delta(\rho)-P(\rho))w\,d\mathbf{x}dzdt\Big|\le\|\varphi\|_{C^1}\Big(1+\|\rho\|_{L^2}\Big)\|p^\delta-p\|_{L^\infty}\|w\|_{L^2}\le C\delta.$$

3.2.2 Proof for Theorem 2.2

Comparing with the Theorem 2.1, we delicately combine the anisotropic regularity with the special structure of CPE system, and observe some new results. The process of proving Theorem 2.2 is similar to Theorem 2.1. We just focus on some different estimates for commutators R_2^ε and R_6^ε. First we observe for R_2^ε

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}\mathrm{div}_{\mathbf{x}}\big((\rho\mathbf{v})^\varepsilon-\rho\mathbf{v})\otimes(\mathbf{v}^\varepsilon-\mathbf{v})\big)\varphi\mathbf{v}^\varepsilon\,d\mathbf{x}dzdt\\
&\quad\le\|\varphi\|_{C^1}\|(\rho\mathbf{v})^\varepsilon-\rho\mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon-\mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon\|_{L^3}+\|\varphi\|_{C^0}\|(\rho\mathbf{v})^\varepsilon-\rho\mathbf{v}\|_{L^3}\|\mathbf{v}^\varepsilon-\mathbf{v}\|_{L^3}\|\nabla_{\mathbf{x}}\mathbf{v}^\varepsilon\|_{L^3}\\
&\quad\le C\varepsilon^{2\alpha}\|\rho\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&\qquad+C\varepsilon^{2\alpha}\|\rho\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|_{B^\alpha_{3,t},B^\alpha_{3,z}(B^\beta_{3,h})}\varepsilon^{\alpha-1}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&\quad\le C\left(\varepsilon^{2\alpha}+\varepsilon^{3\alpha-1}\right)\|\rho\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}.
\end{aligned}$$

We can use the same the idea on the commutator of horizontal dissipative term R_4^ε. For the vertical dissipative commutator term R_5^ε, we utilize the anisotropic regularity to give an different and interesting proof. Recalling the density is independent of z, we integrate R_5^ε by parts and rewrite it as

$$\begin{aligned}
&\int_0^\tau\int_{\mathbb{T}^3}\partial_z\Big(\rho^\varepsilon\partial_z\mathbf{v}^\varepsilon-(\rho\partial_z\mathbf{v})^\varepsilon\Big)\cdot\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad=\int_0^\tau\int_{\mathbb{T}^3}\partial_z\Big(\partial_z(\rho^\varepsilon\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\Big)\cdot\varphi\mathbf{v}^\varepsilon d\mathbf{x}dzdt\\
&\quad=-\int_0^\tau\int_{\mathbb{T}^3}\Big(\partial_z(\rho^\varepsilon\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt\\
&\quad=-\int_0^\tau\int_{\mathbb{T}^3}\Big(\partial_z(\rho\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt
\end{aligned}$$

$$-\int_0^\tau\int_{\mathbb{T}^3}\Big(\partial_z(\rho^\varepsilon-\rho)\mathbf{v}^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt.$$

For the first term, we use Holder's inequality to obtain

$$\begin{aligned}
&\Big|\int_0^\tau\int_{\mathbb{T}^3}\Big(\partial_z(\rho\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt\Big|\\
&\quad\le\|\partial_z(\rho\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\|_{L^3}\Big(\|\mathbf{v}^\epsilon\|_{L^3}\|\partial_z\varphi^\epsilon\|_{L^3}+\|\partial_z\mathbf{v}^\epsilon\|_{L^3}\|\varphi^\epsilon\|_{L^3}\Big).
\end{aligned}$$

Recalling Lemma 2.5, we have $\|\partial_z(\rho\mathbf{v}^\varepsilon)-\partial_z(\rho\mathbf{v})^\varepsilon\|_{L^3}\to 0$ as $\varepsilon\to 0$, where we take $\rho=f$ and $\mathbf{v}=\mathbf{g}$.

Due to the independence of ρ with respect to z, Besov estimates for ρ only depends on variations in the β-index, which is the index related to the horizontal components. Based on this fact, we observe the second term as the following

$$\begin{aligned}
&\Big|\int_0^\tau\int_{\mathbb{T}^3}\Big(\partial_z(\rho^\varepsilon-\rho)\mathbf{v}^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt\Big|\\
&=\Big|\int_0^\tau\int_{\mathbb{T}^3}\Big(\rho^\varepsilon-\rho)(\partial_z\mathbf{v}^\varepsilon\Big)\cdot\Big(\mathbf{v}^\epsilon\partial_z\varphi+\partial_z\mathbf{v}^\epsilon\varphi\Big)\,d\mathbf{x}dzdt\Big|\\
&\le C\|\rho^\epsilon-\rho\|_{L^3}\|\partial_z\mathbf{v}^\epsilon\|_{L^3}\Big(\|\mathbf{v}^\epsilon\|_{L^3}+\|\partial_z\mathbf{v}^\epsilon\|_{L^3}\Big)\\
&\le C\varepsilon^\beta\|\rho\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\varepsilon^{\alpha-1}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\Big(\|\mathbf{v}\|_{L^3}+\varepsilon^{\alpha-1}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\Big)\\
&\le C\Big(\varepsilon^{\alpha+\beta-1}+\varepsilon^{2\alpha+\beta-2}\Big)\|\rho\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}.
\end{aligned}$$

Then for R_6^ε, it is similar to obtain

$$\begin{aligned}
&\Big|\int_0^\tau\int_{\mathbb{T}^3}\nabla_{\mathbf{x}}\big(p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\big)\varphi\mathbf{v}^\varepsilon\,d\mathbf{x}dzdt\Big|\\
&=\Big|-\int_0^\tau\int_{\mathbb{T}^3}\big(p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\big)(\varphi\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon+\nabla_{\mathbf{x}}\varphi\mathbf{v}^\varepsilon)\,d\mathbf{x}dzdt\Big|\\
&\le\|\varphi\|_{C^0}\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\|\mathrm{div}_{\mathbf{x}}\mathbf{v}^\varepsilon\|_{L^3}+C\|\varphi\|_{C^1}\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{\frac32}}\|\mathbf{v}^\varepsilon\|_{L^3}\\
&\le C\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{3/2}}\varepsilon^{\alpha-1}\|\mathbf{v}^\varepsilon\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}+C\|p^\delta(\rho^\varepsilon)-(p^\delta(\rho))^\varepsilon\|_{L^{3/2}}\|\mathbf{v}\|_{L^3}\\
&\le C\big(\|\rho^\varepsilon-\rho\|^2_{L^3}+\sup_{y\in\mathrm{supp}\eta^\varepsilon}\|\rho(\cdot)-\rho(\cdot-y)\|^2_{L^3}\big)\Big(\varepsilon^{\alpha-1}+1\Big)\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&\le C\varepsilon^{2\beta}\Big(\varepsilon^{\alpha-1}+1\Big)\|\rho\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&\le C\Big(\varepsilon^{2\beta+\alpha-1}+\varepsilon^{2\beta}\Big)\|\rho\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))},
\end{aligned}$$

where we use the assumption $2\alpha+\beta-2>0$ and $\beta>\alpha$ to deduce $\alpha+2\beta-1>0$.

Let us stress that for the term R_3^ε, we need to apply the important property of (CPE), see Remark 3.2 and the equality (3.12). Due to the assumption $\rho\mathbf{v}\in B_3^{\alpha,\infty}((0,T);B_3^{\alpha,\infty}(\mathbb{T};B_3^{\beta,\infty}(\mathbb{T}^2)))$, then we can deduce $\rho w\in B_3^{\alpha,\infty}((0,T);B_3^{\alpha+1,\infty}(\mathbb{T};B_3^{\beta-1,\infty}(\mathbb{T}^2))$. It means ρw has $B_3^{\alpha+1,\infty}$ regularity in the $z-$ direction, $B_3^{\beta-1,\infty}$ regularity in the horizontal directions, and $B_3^{\alpha,\infty}$ regularity in time. Therefore, we can estimate R_3^ε as follows

$$\int_0^\tau\int_{\mathbb{T}^3}((\rho w)^\varepsilon-\rho w)(\mathbf{v}^\varepsilon-\mathbf{v})(\partial_z\varphi\mathbf{v}^\varepsilon+\varphi\partial_z\mathbf{v}^\varepsilon)\,d\mathbf{x}dzdt$$

$$
\begin{aligned}
&\le \left(\int_0^t \|(\rho w)^\varepsilon - \rho w\|^3_{L^3(\mathbb{T}^3)} dt\right)^{\frac{1}{3}} \cdot \|\mathbf{v}^\varepsilon - \mathbf{v}\|_{L^3} \Big(\|\varphi\|_{C^1}\|\|\mathbf{v}^\varepsilon\|_{L^3} + \|\varphi\|_{C^0}\|\|\partial_z \mathbf{v}^\varepsilon\|_{L^3}\Big)\\
&\le C\varepsilon^{\beta-1}\left(\int_0^t \|\rho w\|^3_{B^{\alpha+1}_{3,z}(B^{\beta-1}_{3,h})} dt\right)^{\frac{1}{3}} \varepsilon^\alpha \|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&\quad + C\varepsilon^{\beta-1}\left(\int_0^t \|\rho w\|^3_{B^{\alpha+1}_{3,z}(B^{\beta-1}_{3,h})} dt\right)^{\frac{1}{3}} \varepsilon^\alpha \|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\varepsilon^{\alpha-1}\|\mathbf{v}\|_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&= C\left(\varepsilon^{\alpha+\beta-1} + \varepsilon^{2\alpha+\beta-2}\right)\Big(\int_0^t \|\rho w\|^3_{B^{\alpha+1}_{3,z}(B^{\beta-1}_{3,h})} dt\Big)^{\frac{1}{3}} \|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}\\
&= \Big(\varepsilon^{\alpha+\beta-1} + \varepsilon^{2\alpha+\beta-2}\Big)\|\rho w\|_{B^\alpha_{3,t}(B^{\alpha+1}_{3,z}(B^{\beta-1}_{3,h}))}\|\mathbf{v}\|^2_{B^\alpha_{3,t}(B^\alpha_{3,z}(B^\beta_{3,h}))}
\end{aligned}
$$

4 Conclusion

Theorem 2.1 can be seen as a kind of generalization of Feireisl et al. [28] and Wiedeman et al. [1], by different proof of some part of estimates (like R_4^ε) together by observation of the special structure of CPE. But the conventional isotropic spaces in Theorem 2.1 do not reflect the different regularity in the horizontal and the vertical directions of the velocity field. Since the regularity of the w follows from the continuity equation under assumptions that the density is independent on the vertical direction, we need additional regularity condition about $\rho\mathbf{v}$ to control the convergence of R_3^ε. In Theorem 2.2, we show that we can solve the corresponding problem and remove this additional regularity condition by virtue of anisotropic space. Theorem 2.2 not only considers the continuity of the pressure but also reflects the anisotropy of the velocity field in the vertical and horizontal directions of the velocity field, the density and also the momentum. Moreover, our CPE model also contains dissipation term, vacuum and it is compressible which means different from the structure of the problem investigated by Titi et al. [9].

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References

[1] I. Akramov, T. Debiec, J. Skipper and E. Wiedemann, Energy conservation for the compressible Euler and Navier-Stokes equations with vacuum, *Anal. PDE*, 13 (2020), 789-811.

[2] P. Azérad and F. Guillén, Mathematical justification of the hydrostatic approximation in the primitive equations of geophysical fluid dynamics, *SIAM J. Math. Anal.*, 33 (2001), 847-859.

[3] H. Bahouri, J.Y. Chemin, and R. Danchin, Fourier analysis and nonlinear partial differential equations, Grundlehren der Math. Wissenschaften 343, Springer, 2011.

[4] C. Bardos and E. S. Titi, Onsager's conjecture for the incompressible Euler equations in bounded domains,*Arch. Ration. Mech. Anal.*, 228 (2018), 197-207.

[5] C. Bardos, E. S. Titi and E. Wiedemann, Onsager's conjecture with physical boundaries and an application to the vanishing viscosity limit, *Comm. Math. Phys.*, 370 (2019), 291-310.

[6] Y. Brenier, Homogeneous hydrostatic flows with convex velocity profiles, *Nonlinearity*, 12 (1999), 495-512.

[7] Y. Brenier, Remarks on the derivation of the hydrostatic Euler equations, *Bull. Sci. Math.*, 127 (2003), 585-595.

[8] D. Bresch, F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido, On the uniqueness of weak solutions of the two-dimensional primitive equations, *Differential Integral Equations*, 16 (2003), 77-94.

[9] D. Boutros, S. Markfelder and E. S. Titi, On Energy Conservation for the Hydrostatic Euler Equations: An Onsager Conjecture, *arXiv:2208.08334*.

[10] D. Boutros and E. S. Titi, Onsager's conjecture for subgrid scale α-models of turbulence, *Phys. D*, 443 (2023), Paper No. 133553.

[11] T. Buckmaster, C. De Lellis, L. Székelyhidi and V. Vicol, Onsager's conjecture for admissible weak solutions,*Comm. Pure Appl. Math.*, 72 (2019), 229-274.

[12] C. S. Cao and E. S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, *Ann. of Math.*, 166 (2007), 245-267.

[13] C. S. Cao, J. K. Li and E. S. Titi, Global well-posedness of the three-dimensional primitive equations with only horizontal viscosity and diffusion, *Comm. Pure Appl. Math.*, 69 (2016), 1492-1531.

[14] A. Cheskidov, P. Constantin, S. Friedlander, and R. Shvydkoy, Energy conservation and Onsager's conjecture for the Euler equations,*Nonlinearity*, 21 (2008), 1233-1252.

[15] P. Constantin, W. E, and E. S. Titi, Onsager's conjecture on the energy conservation for solutions of Euler's equation, *Comm. Math. Phys.*, 165 (1994), 207-209.

[16] T. Debiec, P. Gwiazda, A. Swierczewska-Gwiazda, and A. Tzavaras, Conservation of energy for the Euler-Korteweg equations, *Calc. Var. Partial Differential Equations*, 57 (2018), art. id. 160.

[17] C. De Lellis and L. Székelyhidi, On Admissibility criteria for weak solutions of the euler equations, *Arch. Ration. Mech. Anal.*, 195 (2010), 225-260.

[18] C. De Lellis and L. Székelyhidi, Dissipative continuous Euler flows, *Invent. Math.*, 193 (2013), 377–407.

[19] C. De Lellis and L. Székelyhidi, Dissipative euler flows and Onsager's conjecture, *J. Eur. Math. Soc.*, 16 (2014), 1467-1505.

[20] D. Donatelli and N. Juhasz, The primitive equations of the polluted atmosphere as a weak and strong limit of the 3D Navier-Stokes equations in downwind-matching coordinates, *Discrete Contin. Dyn. Syst.*, 42 (2022), 2859-2892.

[21] T. D. Drivas and H. Q. Nguyen, Onsager's conjecture and anomalous dissipation on domains with boundary, *SIAM J. Math. Anal.*, 50 (2018), 4785-4811.

[22] J. Duchon and R. Robert, Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations,*Nonlinearity*, 13 (2000), 249-255.

[23] M. Ersoy, T. Ngom and M. Sy, Compressible primitive equations: formal derivation and stability of weak solutions, *Nonlinearity*, 24 (2011), 79-96.

[24] G. L. Eyink, Energy dissipation without viscosity in ideal hydrodynamics, I: Fourier analysis and local energy transfer, *Phys. D*, 78 (1994), 222-240.

[25] E. Feireisl, A. Novotný, and H. Petzeltova, On the existence of globally defined weak solutions to the Navier-Stokes equations of compressible isentropic fluids, J. Math. Fluid Mech., 3 (2001), 358-392.

[26] E. Feireisl, Dynamics of Viscous Compressible Fluids, Oxford University Press, Oxford, 2004

[27] E. Feireisl and A. Novotný, Singular limits in thermodynamics of viscous fluids, Advances in Mathematical Fluid Mechanics, Birkhäuser, Basel, 2009.

[28] E. Feireisl, P. Gwiazda, A. Swierczewska-Gwiazda, and E. Wiedemann, Regularity and energy conservation for the compressible Euler equations,*Arch. Ration. Mech. Anal.*, 223 (2017), 1375-1395.

[29] U. S. Fjordholm and E. Wiedemann, Statistical solutions and Onsager's conjecture, *Phys. D*, 376/377 (2018), 259-265.

[30] H. J. Gao, Š. Nečasová and T. Tang, On the hydrostatic approximation of compressible anisotropic Navier-Stokes equations Justification, J. Math. Fluid Mech., 24 (2022), Paper No. 86.

[31] B. V. Gatapov and A. V. Kazhikhov, Existence of a global solution of a model problem of atmospheric dynamics, *Siberian Math. J.*, 46 (2005), 805-812.

[32] E. Grenier, On the derivation of homogeneous hydrostatic equations, *M2AN Math. Model. Numer. Anal.*, 33 (1999), 965-970.

[33] F. Guillén-González, N. Masmoudi and M. A. Rodríguez-Bellido, Anisotropic estimates and strong solutions of the primitive equations, *Differential Integral Equations*, 14 (2001), 1381-1408.

[34] P. Gwiazda, M. Michalek, and A. Swierczewska-Gwiazda, A note on weak solutions of conservation laws and energy/entropy conservation, *Arch. Ration. Mech. Anal.*, 229 (2018), 1223-1238.

[35] M. Hieber and T. Kashiwabara, Global strong well-posedness of the three dimensional primitive equations in L^p-spaces, *Arch. Ration. Mech. Anal.*, 221 (2016), 1077-1115.

[36] P. Isett, A proof of Onsager's conjecture, *Ann. of Math.*, 188 (2018), 871-963.

[37] I. Lacroix-Violet and A. Vasseur, Global weak solutions to the compressible quantum Navier-Stokes equation and its semi-classical limit, *J. Math. Pures Appl.*, 114 (2018), 191-210.

[38] J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace. (French) *Acta Math.* 63 (1934), no. 1, 193–248.

[39] T. M. Leslie and R. Shvydkoy, The energy balance relation for weak solutions of the density-dependent Navier-Stokes equations, *J. Differential Equations*, 261 (2016), 3719-3733.

[40] J. K. Li, and E. S. Titi, The primitive equations as the small aspect ratio limit of the Navier-Stokes equations: rigorous justification of the hydrostatic approximation, *J. Math. Pures Appl.*, 124 (2019), 30-58.

[41] J. L. Lions, R. Temam and S. H. Wang, On the equations of the large-scale ocean, *Nonlinearity*, 5 (1992), 1007-1053.

[42] J. L. Lions, R. Temam and S. H. Wang, New formulations of the primitive equations of atmosphere and applications, *Nonlinearity*, 5 (1992), 237-288.

[43] P.-L. Lions. Mathematical topics in fluid dynamics, Compressible models, Oxford Science Publication, Oxford, 1998

[44] X. Liu and E. S. Titi, Local well-posedness of strong solutions to the three-dimensional compressible Primitive Equations, *Arch. Ration. Mech. Anal.*, 241 (2021), 729-764.

[45] X. Liu and E. S. Titi, Global existence of weak solutions to the compressible Primitive Equations of atmosphereic dynamics with degenerate viscositites, *SIAM J. Math. Anal.*, 51 (2019), 1913-1964.

[46] X. Liu and E. S. Titi, Zero Mach number limit of the compressible Primitive Equations Part I: well-prepared initial data, *Arch. Ration. Mech. Anal.*, 238 (2020), 705-747.

[47] N. Masmoudi and T. K. Wong, On the H^s theory of hydrostatic Euler equations, *Arch. Ration. Mech. Anal.*, 204 (2012), 231-271.

[48] Q. Nguyen, P. Nguyen and B. Tang, Energy equalities for compressible Navier-Stokes equations, *Nonlinearity*, 32 (2019), 4206-4231.

[49] L. Onsager, Statistical hydrodynamics, *Nuovo Cimento*, 6 (1949), 279-287.

[50] V. Scheffer, An inviscid flow with compact support in space-time, *J. Geom. Anal.*, 3 (1993), 343–401.

[51] A. Shnirelman, Weak solutions with decreasing energy of incompressible Euler equations, *Commun. Math. Phys.*, 210(2000), 541-603.

[52] T. Tang and H. J. Gao, On the stability of weak solution for compressible primitive equations, *Acta Appl. Math.*, 140 (2015), 133-145.

[53] R. Temam and M. Ziane, *Some mathematical problems in geophysical fluid dynamics*, Handbook of Mathematical Fluid Dynamics, 2004.

[54] F. C. Wang, C. S. Dou and Q. S. Jiu, Global weak solutions to 3D compressible primitive equations with density-dependent viscosity, *J. Math. Phys.*, 61 (2020), 021507, 33 pp.

[55] C. Yu, Energy conservation for the weak solutions of the compressible Navier-Stokes equations, *Arch. Ration. Mech. Anal.*, 225 (2017), 1073-1087.