INSTITUTE OF MATHEMATICS

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Ondřej Kurka

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ONDŘEJ KURKA

ABSTRACT. We consider the notion of Borel reducibility between pseudometrics on standard Borel spaces introduced and studied recently by Cúth, Doucha and Kurka, as well as the notion of an orbit pseudometric, a continuous version of the notion of an orbit equivalence relation. It is well known that the relation of isometry of Polish metric spaces is bireducible with a universal orbit equivalence relation. We prove a version of this result for pseudometrics, showing that the Gromov-Hausdorff distance of Polish metric spaces is bireducible with a universal element in a certain class of orbit pseudometrics.

1. INTRODUCTION AND MAIN RESULTS

Let E, F be equivalence relations on Polish spaces X, Y (see Section 2 for more details on the terminology). We say that E is *Borel reducible* to F if there exists a Borel mapping $f : X \to Y$ (so-called *reduction*) such that

$$f(x) \, F \, f(x') \quad \Leftrightarrow \quad x \, E \, x'$$

for all $x, x' \in X$. In some sense, this means that E is at most as complex as F, or that the problem of deciding whether two objects are E-equivalent can be reduced to the problem of deciding whether two objects are F-equivalent.

The study of reducibility between definable equivalence relations, often called invariant descriptive set theory, is an active area of current research, see e.g. [8]. Let us emphasize here the result of Clemens, Gao and Kechris [4, 9] which states that the relation of isometry of Polish metric spaces is bireducible with a universal orbit equivalence relation. Here, an *orbit equivalence relation* is the relation defined by $x E_G^X y \Leftrightarrow \exists g \in G : g \cdot x = y$ for some Borel action of a Polish group G on a Polish space X, and an orbit equivalence relation E_G^X is called *universal* if any other orbit equivalence relation E_H^Y is Borel reducible to E_G^X.

It is worth noting that the isometry of Polish metric spaces is Borel bireducible also with the linear isometry of separable Banach spaces [17], with the isomorphism of separable C*-algebras [18] and with the homeomorphism of compact metric spaces [19].

Recently, the notion of Borel reducibility has been investigated by Cúth, Doucha and Kurka in the setting of pseudometrics on Polish spaces, see [5, 6]. This is a more general setting, since an equivalence relation E can be viewed as the pseudometric ϱ_E defined by $\varrho_E(x, y) = 0$ if xEy and $\varrho_E(x, y) = 1$ otherwise. Moreover, in [5], it is shown that orbit equivalence relations have a natural generalization in the setting of pseudometrics, so-called orbit pseudometrics.

Let ϱ_X and ϱ_Y be pseudometrics on Polish spaces X and Y. We say that ϱ_X is *Borel-uniformly continuous (Borel-u.c.) reducible* to ϱ_Y if there exists a Borel mapping $f : X \to Y$ such that, for every $\varepsilon > 0$, there are $\delta_X > 0$ and $\delta_Y > 0$

satisfying

$$\varrho_X(x, x') < \delta_X \;\Rightarrow\; \varrho_Y(f(x), f(x')) < \varepsilon$$

and

$$\varrho_Y(f(x), f(x')) < \delta_Y \;\Rightarrow\; \varrho_X(x, x') < \varepsilon$$

for all $x, x' \in X$. This is a quantitative version of the Borel reducibility of the relation $\varrho_X(x, x') = 0$ to the relation $\varrho_Y(y, y') = 0$. Let us accentuate that pseudometrics in this work are allowed to attain ∞ as a value.

The central object studied in [5, 6] is the Gromov-Hausdorff distance of Polish metric spaces, which, in some sense, measures how far are two spaces from being isometric (see Section 2 for the precise definition, as well as for the definitions of the Urysohn space $\mathbb{U}$ and of the coding $F(\mathbb{U})$ of Polish metric spaces, enabling us to see the Gromov-Hausdorff distance as a pseudometric ϱ_{GH} on a Polish space). It is shown that the Gromov-Hausdorff distance is bireducible with several other distances, for instance the Lipschitz distance of Polish metric spaces or the Kadets and Banach-Mazur distances of separable Banach spaces. Also, it is bireducible with a simple looking orbit pseudometric, we recall this in Theorem 2.3.

We now formulate the main result of this work. This is a continuous version of the result of Clemens, Gao and Kechris, in which the Gromov-Hausdorff distance takes over the role of the isometry relation.

Theorem 1.1. *Let G be a Polish group acting continuously on a Polish space X. Let d be a lower semicontinuous pseudometric on X such that $d(x, y) = d(gx, gy)$ for any $x, y \in X$ and $g \in G$. Moreover, let there be a system Φ of continuous pseudometrics on X such that $d = \sup_{\varphi \in \Phi} \varphi$.*

Let $\varrho_{G,d}$ be the corresponding orbit pseudometric, i.e., the pseudometric defined by

$$\varrho_{G,d}(x, y) = \inf_{g \in G} d(gx, y).$$

Then $\varrho_{G,d}$ is Borel-u.c. reducible to the Gromov-Hausdorff distance ϱ_{GH}.

It is not clear if the theorem holds without the assumption of the existence of a system Φ of continuous pseudometrics (this is extensively discussed in Section 4). Actually, we prove a version of the theorem with a more general system Φ and a stronger conclusion, see Theorem 3.1. The constructed reduction $f : X \to F(\mathbb{U}) \setminus \{\emptyset\}$ satisfies

$$\varrho_{GH}(f(p), f(q)) \le \min\{\varrho_{G,d}(p, q), 1\} \le 2\varrho_{GH}(f(p), f(q)), \quad p, q \in X.$$

Thus, in the terminology of [5], the reduction is Borel-Lipschitz on small distances.

Let us recall a remarkable problem posed by Ben Yaacov, Doucha, Nies and Tsankov [3]. Let E_{GH} be the relation given by $ME_{GH}N \Leftrightarrow \varrho_{GH}(M, N) = 0$ for $M, N \in F(\mathbb{U}) \setminus \{\emptyset\}$. Then any orbit equivalence is Borel reducible to E_{GH}, but it is not known if E_{GH} is Borel reducible to an orbit equivalence, see [3, Question 8.6]. However, E_{GH} possesses some properties of orbit equivalences, as its equivalence classes are Borel, see [3, Corollary 8.3] (or [5, Corollary 52]), and the relation

$$xE_1y \;\Leftrightarrow\; \big(\exists n \,\forall m \ge n : x(m) = y(m)\big), \quad x, y \in (2^{\mathbb{N}})^{\mathbb{N}},$$

is not Borel reducible to it, see [5, Theorem 24]. Therefore, Theorem 1.1 has the following consequence.

Corollary 1.2. *Let $\varrho_{G,d}$ be as in Theorem 1.1, and let $E_{G,d}^X$ be the equivalence relation given by $xE_{G,d}^X y \Leftrightarrow \varrho_{G,d}(x, y) = 0$. Then $E_{G,d}^X$ is Borel reducible to E_{GH}, and thus*

- *the equivalence classes of $E_{G,d}^X$ are Borel,*
- *E_1 is not Borel reducible to $E_{G,d}^X$.*

It turns out that our methods moreover provide a positive answer to Question 41 from [5]. Let us recall that Gao and Kechris [9] proved that the relation induced by the canonical action of the isometry group $Iso(\mathbb{U})$ on $F(\mathbb{U})$ is a universal orbit equivalence relation (and so it is Borel bireducible with the isometry relation of Polish metric spaces). Considering the Hausdorff distance ϱ_H, we can obtain the following analogue for pseudometrics.

Theorem 1.3. *The Gromov-Hausdorff distance ϱ_{GH} is Borel-u.c. bireducible with the orbit pseudometric $\varrho_{Iso(\mathbb{U}),\varrho_H}$ on $F(\mathbb{U}) \setminus \{\emptyset\}$.*

This theorem is also a consequence of Theorem 3.1 which provides a reduction of an orbit pseudometric to both ϱ_{GH} and $\varrho_{Iso(\mathbb{U}),\varrho_H}$. Similarly as in Theorem 1.1, it is possible to construct both reductions in Theorem 1.3 to be Borel-Lipschitz on small distances.

Finally, we prove an analogue of Theorem 1.3 in the setting of Banach spaces, see Theorem A.2. The proof is more or less based on known methods, but it requires some additional work, and we postpone it to an appendix.

2. Preliminaries

Let $\varrho_H(A, B)$ denote the Hausdorff distance of two non-empty subsets A and B of a metric space. The *Gromov-Hausdorff distance* of non-empty metric spaces M and N is defined by

$$\varrho_{GH}(M, N) = \inf_{\substack{Z \text{ metric space} \\ i_M:M\hookrightarrow Z \\ i_N:N\hookrightarrow Z}} \varrho_H\big(i_M(M), i_N(N)\big),$$

where the symbol $\hookrightarrow$ denotes an isometric embedding. Actually, in this work, both the Hausdorff distance ϱ_H and the Gromov-Hausdorff distance ϱ_{GH} are considered mainly between elements of the space $F(\mathbb{U}) \setminus \{\emptyset\}$ defined below, and both distances are regarded as pseudometrics on this space.

A binary relation $\mathcal{R} \subseteq X \times Y$ is called a *correspondence* between X and Y if $\forall x \in X \exists y \in Y : x\mathcal{R}y$ and $\forall y \in Y \exists x \in X : x\mathcal{R}y$. Let (M, δ_M) and (N, δ_N) be metric spaces and let $r > 0$. It is well known and easy to show that if $\varrho_{GH}(M, N) < r$, then there exists a correspondence $\mathcal{R}$ between M and N such that

$$m\mathcal{R}n \ \& \ m'\mathcal{R}n' \quad \Rightarrow \quad |\delta_M(m, m') - \delta_N(n, n')| < 2r.$$

A *Polish space (topology)* means a separable completely metrizable topological space (topology), and a *Polish metric space* means a separable complete metric space. A *Polish group* is a topological group whose topology is Polish. By an action of a group G on a set X we mean a mapping $(g, x) \in G \times X \mapsto g \cdot x \in X$ satisfying $1_G \cdot x = x$ and $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.

The *Urysohn space* is defined as the (up to isometry) only Polish metric space $\mathbb{U}$ with the property that for any finite metric space A and any isometric embedding $i : B \to \mathbb{U}$, where $B \subseteq A$, there exists an isometric embedding $\widetilde{i} : A \to \mathbb{U}$ extending i. It is well known that the Urysohn space contains an isometric copy of every Polish metric space, in fact, the following result holds.

Theorem 2.1 (Katětov [12])**.** *Let X be a Polish metric space. Then there exists an isometric embedding $i : X \to \mathbb{U}$ such that any surjective isometry on $i(X)$ can be extended to a surjective isometry on $\mathbb{U}$.*

For a Polish metric space (X, δ_X), we define

$$F(X) = \big\{F \subseteq X : F \text{ is closed}\big\},$$

and we equip $F(X)\setminus\{\emptyset\}$ with the *Wijsman topology*, defined as the coarsest topology for which the function

$$F \mapsto \delta_X(x, F),$$

is continuous for each $x \in X$. If we moreover add $\emptyset$ as an isolated point, then $F(X)$ is a Polish space whose Borel σ-algebra is the *Effros Borel structure* of X, defined as the σ-algebra generated by the sets $\{F \in F(X) : F \cap U \neq \emptyset\}$, $U \subseteq X$ open, see e.g. [1]. Let us note that if X is universal (in the sense that it contains an isometric copy of every Polish metric space), then $F(X)$ is a coding of all Polish metric spaces up to isometry. For this purpose, we will employ the Urysohn space and the coding $F(\mathbb{U})$. Although this is not the same coding as the coding considered in the preceding works [5] and [6], it makes no important difference.

For a Polish metric space X, we denote by $Iso(X)$ the group of all surjective isometries on X with the topology of pointwise convergence. This is a Polish group, and by its canonical action on $F(X)$ we mean the action $I \cdot F = I(F)$. Let us prove a simple lemma.

Lemma 2.2. *Let $F(X) \setminus \{\emptyset\}$ be equipped with the Wijsman topology. Then the canonical action of $Iso(X)$ on $F(X) \setminus \{\emptyset\}$ is continuous. Moreover, the Hausdorff distance ϱ_H is lower semicontinuous on $F(X) \setminus \{\emptyset\}$ and there is a system Φ of continuous pseudometrics such that $\varrho_H = \sup_{\varphi \in \Phi} \varphi$.*

Proof. Assuming $I_n \to I$ and $F_n \to F$, we show that $I_n(F_n) \to I(F)$. Given $x \in X$, we need to check that $\delta_X(x, I_n(F_n)) \to \delta_X(x, I(F))$. We have

$$|\delta_X(x, I_n(F_n)) - \delta_X(x, I(F))| = |\delta_X(I_n^{-1}(x), F_n) - \delta_X(I^{-1}(x), F)|$$
$$\leq |\delta_X(I_n^{-1}(x), F_n) - \delta_X(I^{-1}(x), F_n)| + |\delta_X(I^{-1}(x), F_n) - \delta_X(I^{-1}(x), F)|$$
$$\leq \delta_X(I_n^{-1}(x), I^{-1}(x)) + |\delta_X(I^{-1}(x), F_n) - \delta_X(I^{-1}(x), F)|,$$

and it remains to note that $\delta_X(I_n^{-1}(x), I^{-1}(x)) \to 0$ since $I_n \to I$ and $\delta_X(I^{-1}(x), F_n) \to \delta_X(I^{-1}(x), F)$ since $F_n \to F$.

Concerning the moreover part, it is sufficient to notice that

$$\varrho_H(A, B) = \sup_{x \in X} |\delta_X(x, A) - \delta_X(x, B)|$$

for all $A, B \in F(X) \setminus \{\emptyset\}$. $\square$

Let us consider the action of the group S_∞ of all permutations of $\mathbb{N}$ on the space $[1/2, 1]^{[\mathbb{N}]^2}$ given by

$$(\pi \cdot x)(m, n) = x(\pi^{-1}(m), \pi^{-1}(n)), \quad \{m, n\} \in [\mathbb{N}]^2,$$

for $\pi \in S_\infty, x \in [1/2, 1]^{[\mathbb{N}]^2}$, and let a pseudometric σ on $[1/2, 1]^{[\mathbb{N}]^2}$ be defined by

$$\sigma(x, y) = \sup_{\{m, n\} \in [\mathbb{N}]^2} |x(m, n) - y(m, n)|$$

for $x, y \in [1/2, 1]^{[\mathbb{N}]^2}$. Let us remark that the points of $[1/2, 1]^{[\mathbb{N}]^2}$ represent the metrics on $\mathbb{N}$ with values in $\{0\} \cup [1/2, 1]$. We will apply the following result from the preceding work [5], for more details, see [5, Theorem 11] and the proof of [5, Theorem 26].

Theorem 2.3 ([5])**.** *The Gromov-Hausdorff distance ϱ_{GH} is Borel-u.c. bireducible with the orbit pseudometric $\varrho_{S_\infty, \sigma}$ on $[1/2, 1]^{[\mathbb{N}]^2}$.*

Let us recall one more result which will play an important role in our construction.

Theorem 2.4 (Melleray [16])**.** *Let X be a Polish metric space of diameter at most 1 and let G be a closed subgroup of $Iso(X)$. Then there exists an extension Y of X such that*

- *Y is a Polish metric space,*
- *any member of G can be extended in a unique way to a surjective isometry on Y,*
- *any surjective isometry on Y is an extension of a member of G.*

Since this theorem is not stated explicitly in [16], it is necessary to give an explanation. Theorem 2.4 can be proved in the same way as [16, Theorem 1.1] with the following two differences:

- The part before *Claim* is to be ignored.
- We want to preserve the distance, so d should not be replaced by $\frac{d}{1+d}$ after the proof of [16, Lemma 3.2]. The purpose of this replacement is to fulfill $\mathrm{diam}(Z) \leq 1$. However, it holds for the original distance that $\mathrm{diam}(Z) \leq 3$, which is sufficient for the final step of the construction. Therefore, d can be preserved.

3. The Construction

The aim of this section is to prove the following theorem from which the results introduced above follow.

Theorem 3.1. *Let G be a Polish group acting continuously on a Polish space X. Let d be a lower semicontinuous pseudometric on X with $d \leq 1$ such that $d(x,y) = d(gx, gy)$ for any $x, y \in X$ and $g \in G$. Moreover, let there be a system Φ of lower semicontinuous pseudometrics on X such that $d = \sup_{\varphi \in \Phi} \varphi$ and every $\varphi \in \Phi$ has the property that the mapping*

$$p \mapsto H_\varphi(p) := \big\{ (x, u) : x \in X, \varphi(p, x) \leq u \leq 1 \big\}$$

from X to $F(X \times [0,1])$ is Borel.

Then there exists a Borel mapping $p \mapsto Y_p$ from X to $F(\mathbb{U}) \setminus \{\emptyset\}$ such that

$$\varrho_{GH}(Y_p, Y_q) \leq \varrho_{Iso(\mathbb{U}), \varrho_H}(Y_p, Y_q) \leq \varrho_{G,d}(p, q) \leq 2\varrho_{GH}(Y_p, Y_q)$$

for all $p, q \in X$.

Before proving this theorem, let us show first that Theorems 1.1 and 1.3 are its consequences. It is sufficient to prove Theorem 1.1 in the case $d \leq 1$, as one can consider $\min\{d, 1\}$ instead of d. So, we just need to show that any continuous pseudometric φ on X satisfies the requirement from Theorem 3.1. For an open $U \subseteq X \times [0,1]$, we show that $\{p \in X : H_\varphi(p) \cap U \neq \emptyset\}$ is Borel (actually open). Let us assume that $p \in X$ is such that $H_\varphi(p)$ intersects U, and let us pick $(x, u) \in H_\varphi(p) \cap U$. If $u < 1$, consider $v > u$ such that $(x, v) \in U$, and if $u = 1$, consider $v = 1$. As φ is continuous, for every q from a neighborhood of p, we have $\varphi(q, x) \leq v$, and so $(x, v) \in H_\varphi(q) \cap U$.

Concerning Theorem 1.3, the reducibility of ϱ_{GH} to $\varrho_{Iso(\mathbb{U}), \varrho_H}$ follows from Theorem 3.1 and Theorem 2.3, and the reducibility of $\varrho_{Iso(\mathbb{U}), \varrho_H}$ to ϱ_{GH} follows from Theorem 1.1 and Lemma 2.2.

Let us now turn to the proof of Theorem 3.1. We notice first that there is a countable $\Phi' \subseteq \Phi$ such that still $d = \sup_{\varphi \in \Phi'} \varphi$. Indeed, the open sets $\{(x, y, u) \in X \times X \times [0,1] : u < \varphi(x,y)\}, \varphi \in \Phi$, form a covering of $\{(x, y, u) \in X \times X \times [0,1] : u < d(x,y)\}$, and due to the Lindelöf property, countably many of them form a covering as well. We choose a sequence $\varphi_1, \varphi_2, \ldots$ in Φ' such that every element of Φ' appears infinitely many times.

Let γ be a compatible right-invariant metric on G with $\gamma \leq 1$ (let us note that the space (G, γ) needs not to be complete). Let δ_X be a compatible complete metric on X with $\delta_X \leq 1$. Let us consider the maximum metric on $G \times X$, i.e., the metric $\delta_{G \times X}((g, x), (h, y)) = \max\{\gamma(g, h), \delta_X(x, y)\}$. It is easy to check that the mapping $I_h : (g, x) \mapsto (gh, x)$ is an isometry on $G \times X$ for every $h \in G$. We obtain the following claim based on Melleray's result.

Claim 3.2. *There is an extension Z of $G \times X$ such that*
- *Z is a Polish metric space of diameter at most 2,*
- *for any $h \in G$, the isometry $I_h : (g, x) \mapsto (gh, x)$ can be extended in a unique way from $G \times X$ to a surjective isometry on Z,*
- *any surjective isometry on Z is an extension of I_h for some $h \in G$.*

Proof. Let $\overline{G}$ denote the completion of (G, γ). For every $h \in G$, let I_h^* be the unique surjective isometry on $\overline{G} \times X$ extending I_h. We show that the set $\{I_h^* : h \in G\}$ forms a closed subgroup of $Iso(\overline{G} \times X)$. This is clearly a subgroup, as $I_{h'}^* \circ I_h^* = I_{hh'}^*$, and by the fact that any Polish subgroup of a Polish group is closed (see e.g. [13, Exercise 9.6]), it is sufficient to show that $h \mapsto I_h^*$ is a homeomorphism from G into $Iso(\overline{G} \times X)$.

First, let $h_n \to h$. Then $I_{h_n}^*(g, x) = (gh_n, x) \to (gh, x) = I_h^*(g, x)$ for every $(g, x) \in G \times X$. So, $I_{h_n}^*$ converges pointwise to I_h^* on a dense subset of $\overline{G} \times X$. As the mappings $I_{h_n}^*$ and I_h^* are isometries, they are equicontinuous, and the pointwise convergence on the whole $\overline{G} \times X$ follows. Conversely, let $I_{h_n}^* \to I_h^*$. If we pick $x \in X$ arbitrarily, then $(h_n, x) = I_{h_n}^*(1_G, x) \to I_h^*(1_G, x) = (h, x)$, which means that $h_n \to h$.

So, $\{I_h^* : h \in G\}$ is a closed subgroup of $Iso(\overline{G} \times X)$ indeed, and we obtain from Theorem 2.4 an extension Z of $\overline{G} \times X$ satisfying all desired properties with the only possible exception of the diameter requirement. However, such property can be easily arranged by changing the metric appropriately (we can take an increasing concave function $\xi : [0, \infty) \to [0, 2)$ with $\xi(t) = t$ for $t \in [0, 1]$ and consider $\xi \circ \delta_Z$ instead of δ_Z). $\square$

Now, let us consider

$$Y = (Z \times \mathbb{N}) \cup (Z \times [0, 1] \times \mathbb{N}).$$

We define a compatible complete metric m on Y in three steps. Concerning the verification of the triangle inequality, some details will be left to the reader. The first step, perhaps the least obvious one, uses a similar idea as the proofs of [6, Theorem 39] and [6, Theorem 41]. The choice of the distances between points in $Z \times \mathbb{N}$ will play a crucial role in the proof of Claim 3.7.

(i) On the subset $Z \times \mathbb{N}$ of Y, we put

$$m((z_1, k_1), (z_2, k_2)) = 100 \cdot |2^{k_1} - 2^{k_2}| + 2^{\min\{k_1, k_2\}} \delta_Z(z_1, z_2).$$

The triangle inequality

$$m((z_1, k_1), (z_3, k_3)) \leq m((z_1, k_1), (z_2, k_2)) + m((z_2, k_2), (z_3, k_3))$$

can be easily checked when $k_2 \geq \min\{k_1, k_3\}$ (it is possible to deal separately with the terms $100 \cdot |2^{k_1} - 2^{k_2}|$ and $2^{\min\{k_1, k_2\}} \delta_Z(z_1, z_2)$). We can suppose that $k_1 \leq k_3$, so we deal now with the case $k_2 < k_1 \leq k_3$, in which the inequality can be simplified to

$$200 \cdot 2^{k_2} + 2^{k_1} \delta_Z(z_1, z_3) \leq 200 \cdot 2^{k_1} + 2^{k_2} \delta_Z(z_1, z_2) + 2^{k_2} \delta_Z(z_2, z_3).$$

It is sufficient to use $200 \cdot 2^{k_2} \leq 100 \cdot 2^{k_1}$ and $2^{k_1} \delta_Z(z_1, z_3) \leq 2^{k_1} \cdot 2$.

(ii) Let $k \in \mathbb{N}$ be fixed. To attach $Z \times [0, 1] \times \{k\}$ to $Z \times \mathbb{N}$, we put

$$m((z_1, u_1, k), (z_2, u_2, k)) = |u_1 - u_2| + 2^k \delta_Z(z_1, z_2)$$

and
$$m((z_1, l), (z_2, u, k)) = u + 10 \cdot 2^k + m((z_1, l), (z_2, k)).$$
The triangle inequality can be verified easily, since $(Z \times \mathbb{N}) \cup (Z \times [0, 1] \times \{k\})$ with m can be isometrically embedded into $Z \times \mathbb{N} \times (\{-10 \cdot 2^k\} \cup [0, 1])$ with the metric
$$((z_1, k_1, u_1), (z_2, k_2, u_2)) \mapsto |u_1 - u_2| + m((z_1, k_1), (z_2, k_2)).$$
(iii) It remains to define distances between points in $Z \times [0, 1] \times \{k_1\}$ and points in $Z \times [0, 1] \times \{k_2\}$ for distinct k_1 and k_2. Assuming $k_1 \neq k_2$, we put

$$m((z_1, u_1, k_1), (z_2, u_2, k_2)) = u_1 + 10 \cdot 2^{k_1} + u_2 + 10 \cdot 2^{k_2} + m((z_1, k_1), (z_2, k_2)).$$

To check the triangle inequality, one can for instance proceed similarly as in the previous step, considering a suitable subset of $(Z \times \mathbb{N}) \times \prod_{k \in \mathbb{N}}(\{-10 \cdot 2^k\} \cup [0, 1])$ with the sum metric.

Once we have defined the space Y and its metric m, we can consider for every $p \in X$ the subspace

$$W_p = (Z \times \mathbb{N}) \cup \big\{(g, x, u, k) : g \in G, x \in X, k \in \mathbb{N}, \varphi_k(gp, x) \leq u \leq 1\big\}.$$

For every $p \in X$, let Y_p be the closure of W_p in Y. We now provide a series of claims concerning the spaces Y_p.

Claim 3.3. *The mapping $p \mapsto Y_p$ from X to $F(Y)$ is Borel.*

Proof. Given an open $U \subseteq Y$, we need to check that the set $\{p \in X : Y_p \cap U \neq \emptyset\}$ is Borel. Let us show first that for each $k \in \mathbb{N}$, the set

$$V_k = \big\{(g^{-1}x, u) : g \in G, x \in X, u \in [0, 1], (g, x, u, k) \in U\big\}$$

is open. Let us pick $(g^{-1}x, u) \in V_k$ (where $(g, x, u, k) \in U$). Let $O \subseteq X$ be an open neighborhood of x and $J \subseteq [0, 1]$ be a (relatively) open neighborhood of u such that $\{g\} \times O \times J \times \{k\} \subseteq U$. Then $(g^{-1} \cdot O) \times J$ is an open neighborhood of $(g^{-1}x, u)$ contained in V_k.

Now, for $p \in X$, we have $Y_p \cap U \neq \emptyset$ if and only if $W_p \cap U \neq \emptyset$, and this is equivalent to

$$(Z \times \mathbb{N}) \cap U \neq \emptyset \quad \text{or} \quad \exists k : H_{\varphi_k}(p) \cap V_k \neq \emptyset.$$

By the assumption on the pseudometrics in Φ, the set of all $p \in X$ with this property is Borel. $\square$

Claim 3.4. *For all $p \in X$ and $h \in G$, there is $I \in Iso(Y)$ that maps Y_p onto Y_{hp}.*

Proof. If we denote by I_h^{**} the unique isometry on Z extending I_h, then the mapping

$$(z, k) \mapsto (I_h^{**}(z), k), \quad (z, u, k) \mapsto (I_h^{**}(z), u, k), \quad (\text{in part. } (g, x, u, k) \mapsto (gh, x, u, k)),$$

is an isometry on Y which maps W_{hp} onto W_p. It follows that it maps Y_{hp} onto Y_p as well. $\square$

Claim 3.5. *For all $p, q \in X$,*

$$\varrho_{Iso(Y), \varrho_H}(Y_p, Y_q) \leq \varrho_{G, d}(p, q).$$

Proof. Let us note that the Hausdorff distance between Y_p and Y_q is at most $d(p, q)$, which follows from the fact that $|\varphi_k(gp, x) - \varphi_k(gq, x)| \leq \varphi_k(gp, gq) \leq d(gp, gq) = d(p, q)$ for every $g \in G, x \in X$ and $k \in \mathbb{N}$. So, using Claim 3.4, we obtain for each $h \in G$ that $\varrho_{Iso(Y), \varrho_H}(Y_p, Y_q) \leq \varrho_H(Y_{hp}, Y_q) \leq d(hp, q)$, and it is sufficient to consider the infimum over $h \in G$. $\square$

Claim 3.6. *Let $p \in X, w \in W_p$, and let $A(w, p) = \{m(w', w) : w' \in W_p\}$. Let further $B \subseteq [0, \infty)$ be such that $\varrho_H(B, A(w, p)) \leq 1$ and β_l be defined by $\beta_l = \inf(B \cap (80 \cdot 2^l, \infty))$ for $l \in \mathbb{N}$.*

If $w \in Z \times \{k\}$, then $\liminf_{l \to \infty}(100 \cdot 2^l - \beta_l) \in [100 \cdot 2^k - 1, 100 \cdot 2^k + 1]$.

If $w \in Z \times [0, 1] \times \{k\}$, then $\liminf_{l \to \infty} (100 \cdot 2^l - \beta_l) \in [90 \cdot 2^k - 2, 90 \cdot 2^k + 1]$.

Consequently, if $\varrho_H(A(w, p), A(w', q)) \leq 1$ for some $p, q \in X, w \in W_p, w' \in W_q$, then $w \in Z \times \{k\}$ is equivalent to $w' \in Z \times \{k\}$ and $w \in Z \times [0, 1] \times \{k\}$ is equivalent to $w' \in Z \times [0, 1] \times \{k\}$.

Proof. Let us first consider $w = (z, k)$. For every $l \geq k + 3$,

- if $w' \in (Z \times \{1, \dots, l - 1\}) \cup (Z \times [0, 1] \times \{1, \dots, l - 1\})$, then $m(w', w) \leq 1 + 110 \cdot 2^{l-1} + 2^k \cdot 2 \leq 80 \cdot 2^l - 2$,
- if $w' \in (Z \times \{l, l + 1, \dots\}) \cup (Z \times [0, 1] \times \{l, l + 1, \dots\})$, then $m(w', w) \geq 100 \cdot (2^l - 2^k)$,
- $w' = (z, l)$ belongs to W_p and $m(w', w) = 100 \cdot (2^l - 2^k)$.

So, the set $A(w, p)$ does not intersect the interval $(80 \cdot 2^l - 2, 100 \cdot (2^l - 2^k))$ and contains $100 \cdot (2^l - 2^k)$. It follows that $\beta_l \in [100 \cdot (2^l - 2^k) - 1, 100 \cdot (2^l - 2^k) + 1]$, and so $100 \cdot 2^l - \beta_l \in [100 \cdot 2^k - 1, 100 \cdot 2^k + 1]$.

Now, let us consider $w = (z, u, k)$. For every $l \geq k + 3$,

- if $w' \in (Z \times \{1, \dots, l - 1\}) \cup (Z \times [0, 1] \times \{1, \dots, l - 1\})$, then $m(w', w) \leq 1 + 110 \cdot 2^{l-1} + 2^k \cdot 2 + 1 + 10 \cdot 2^k \leq 80 \cdot 2^l - 2$,
- if $w' \in (Z \times \{l, l + 1, \dots\}) \cup (Z \times [0, 1] \times \{l, l + 1, \dots\})$, then $m(w', w) \geq 100 \cdot (2^l - 2^k) + u + 10 \cdot 2^k$,
- $w' = (z, l)$ belongs to W_p and $m(w', w) = 100 \cdot (2^l - 2^k) + u + 10 \cdot 2^k$.

So, the set $A(w, p)$ does not intersect the interval $(80 \cdot 2^l - 2, 100 \cdot (2^l - 2^k) + 10 \cdot 2^k)$ and intersects the interval $[100 \cdot (2^l - 2^k) + 10 \cdot 2^k, 100 \cdot (2^l - 2^k) + 1 + 10 \cdot 2^k]$. It follows that $\beta_l \in [100 \cdot (2^l - 2^k) + 10 \cdot 2^k - 1, 100 \cdot (2^l - 2^k) + 1 + 10 \cdot 2^k + 1]$, and so $100 \cdot 2^l - \beta_l \in [90 \cdot 2^k - 2, 90 \cdot 2^k + 1]$.

Concerning the consequence part, we can choose the same set B for both points w, w' (the choice $B = A(w, p)$ works, as well as the choice $B = A(w', q)$). It is then sufficient to note that B determines to which one of the sets $Z \times \{k\}$ or $Z \times [0, 1] \times \{k\}$ the points w, w' belong, as the intervals $[100 \cdot 2^k - 1, 100 \cdot 2^k + 1]$ together with the intervals $[90 \cdot 2^k - 2, 90 \cdot 2^k + 1]$ are pairwise disjoint.   □

Claim 3.7. *For all $p, q \in X$,*

$$\varrho_{G,d}(p, q) \leq 2\varrho_{GH}(Y_p, Y_q).$$

Proof. Given $r > \varrho_{GH}(Y_p, Y_q)$, we need to show that $2r \geq \varrho_{G,d}(p, q)$. Since $r > \varrho_{GH}(Y_p, Y_q) = \varrho_{GH}(W_p, W_q)$, there is a correspondence $\mathcal{R}$ between W_p and W_q such that

$$a\mathcal{R}a' \ \& \ b\mathcal{R}b' \quad \Rightarrow \quad |m(a, b) - m(a', b')| \leq 2r.$$

As $\varrho_{G,d}(p, q) \leq 1$, we can assume that $2r < 1$. In such a case, due to Claim 3.6, elements of $Z \times \{k\}$ may correspond only to elements of $Z \times \{k\}$ itself, and the same holds for $Z \times [0, 1] \times \{k\}$. Consequently, the relations

$$\mathcal{R}_k = \big\{ (z, z') \in Z \times Z : (z, k)\mathcal{R}(z', k) \big\}, \quad k \in \mathbb{N},$$

are correspondences on Z.

It is easy to check that

$$z\mathcal{R}_{k_1}z_1 \ \& \ z\mathcal{R}_{k_2}z_2 \quad \Rightarrow \quad \delta_Z(z_1, z_2) \leq 2^{-\min\{k_1, k_2\}} \cdot 2r,$$

as $(z, k_1)\mathcal{R}(z_1, k_1), (z, k_2)\mathcal{R}(z_2, k_2)$ implies $2r \geq |m((z, k_1), (z, k_2)) - m((z_1, k_1), (z_2, k_2))| = |100 \cdot |2^{k_1} - 2^{k_2}| + 2^{\min\{k_1, k_2\}}\delta_Z(z, z) - 100 \cdot |2^{k_1} - 2^{k_2}| - 2^{\min\{k_1, k_2\}}\delta_Z(z_1, z_2)| = 2^{\min\{k_1, k_2\}}\delta_Z(z_1, z_2)$.

It follows that for every $z \in Z$, there is $I(z) \in Z$ such that

$$z\mathcal{R}_k z' \quad \Rightarrow \quad \delta_Z(I(z), z') \leq 2^{-k} \cdot 2r$$

(if we consider $S_j = \bigcup_{k=j}^{\infty} z\mathcal{R}_k$, then the diameter of S_j is at most $2^{-j} \cdot 2r$, and we can take $\{I(z)\} = \bigcap_{j=1}^{\infty} \overline{S_j}$). Similarly, for every $z \in Z$, there is $J(z) \in Z$ such that

$$z'\mathcal{R}_k z \quad \Rightarrow \quad \delta_Z(z', J(z)) \leq 2^{-k} \cdot 2r.$$

We claim that I and J are isometries. It is sufficient to consider I only. Let $z_1, z_2 \in Z$, and let $k \in \mathbb{N}$ be arbitrary. If we choose z_1', z_2' such that $z_1 \mathcal{R}_k z_1', z_2 \mathcal{R}_k z_2'$, then $\delta_Z(I(z_1), z_1') \leq 2^{-k} \cdot 2r, \delta_Z(I(z_2), z_2') \leq 2^{-k} \cdot 2r$. At the same time, from $(z_1, k)\mathcal{R}(z_1', k), (z_2, k)\mathcal{R}(z_2', k)$ we get $|m((z_1, k), (z_2, k)) - m((z_1', k), (z_2', k))| \leq 2r$, i.e., $2^k \cdot |\delta_Z(z_1, z_2) - \delta_Z(z_1', z_2')| \leq 2r$. By the triangle inequality, $|\delta_Z(z_1, z_2) - \delta_Z(I(z_1), I(z_2))| \leq 3 \cdot 2^{-k} \cdot 2r$. Since $k \in \mathbb{N}$ was arbitrary, we finally see that $\delta_Z(I(z_1), I(z_2)) = \delta_Z(z_1, z_2)$.

In order to show that I is surjective, we further claim that J is inverse to I. Let $z \in Z$, and let $k \in \mathbb{N}$ be arbitrary. If we choose z' such that $z\mathcal{R}_k z'$, then $\delta_Z(I(z), z') \leq 2^{-k} \cdot 2r$ and $\delta_Z(z, J(z')) \leq 2^{-k} \cdot 2r$. As $\delta_Z(J(I(z)), J(z')) = \delta_Z(I(z), z')$, the triangle inequality provides $\delta_Z(J(I(z)), z) \leq 2 \cdot 2^{-k} \cdot 2r$. Since $k \in \mathbb{N}$ was arbitrary, we see that $\delta_Z(J(I(z)), z) = 0$.

So, we have shown that I is a surjective isometry on Z, and it follows that I is the extension of I_h for some $h \in G$.

Now, let $k \in \mathbb{N}$ be arbitrary. If we choose z such that $(1_G, p)\mathcal{R}_k z$ and g, x, u such that $(1_G, p, 0, k)\mathcal{R}(g, x, u, k)$, then

$$\left| m((1_G, p, k), (1_G, p, 0, k)) - m((z, k), (g, x, u, k)) \right| \leq 2r.$$

The involved distances are $10 \cdot 2^k$ and $u + 10 \cdot 2^k + 2^k \delta_Z(z, (g, x))$, hence we get

$$u + 2^k \delta_Z(z, (g, x)) \leq 2r.$$

As $(g, x, u, k) \in W_q$, we have $\varphi_k(gq, x) \leq u \leq 1$, and thus

$$\varphi_k(gq, x) \leq 2r.$$

Since $\delta_Z((h, p), z) = \delta_Z(I(1_G, p), z) \leq 2^{-k} \cdot 2r$, the triangle inequality provides $\delta_Z((h, p), (g, x)) \leq 2^{-k} \cdot 2r + 2^{-k} \cdot 2r$, i.e.,

$$\gamma(h, g) \leq 2^{-k} \cdot 4r, \quad \delta_X(p, x) \leq 2^{-k} \cdot 4r.$$

So, we have seen that for every $k \in \mathbb{N}$, there are $g_k \in G$ and $x_k \in X$ such that

$$\varphi_k(g_k q, x_k) \leq 2r, \quad \gamma(h, g_k) \leq 2^{-k} \cdot 4r, \quad \delta_X(p, x_k) \leq 2^{-k} \cdot 4r.$$

We have $g_k \to h$, $x_k \to p$, and for every $\varphi \in \Phi'$, we get $\varphi(hq, p) \leq 2r$, as it is lower semicontinuous and $\varphi = \varphi_k$ infinitely many times. It follows that $\varrho_{G,d}(q, p) \leq d(hq, p) \leq 2r$. $\square$

To finish the proof of Theorem 3.1, we apply Theorem 2.1. We may suppose that Y is actually a subspace of $\mathbb{U}$ such that any surjective isometry on Y can be extended to a surjective isometry on $\mathbb{U}$. It is easy to check that

$$\varrho_{GH}(Y_p, Y_q) \leq \varrho_{Iso(\mathbb{U}), \varrho_H}(Y_p, Y_q) \leq \varrho_{Iso(Y), \varrho_H}(Y_p, Y_q)$$

for all $p, q \in X$. Together with Claims 3.5, 3.7 and 3.3, this shows that the mapping $p \mapsto Y_p$ works, which completes the proof of Theorem 3.1.

Remark 3.8. Let us note here that for all $p \in X$ and $h \in G$, there is $I \in Iso(\mathbb{U})$ that maps Y_p onto Y_{hp} (this follows from Claim 3.4). Hence, our reduction has the property that

$$pE_G^X q \quad \Rightarrow \quad Y_p E_{Iso(\mathbb{U})}^{F(\mathbb{U})} Y_q \quad \Rightarrow \quad Y_p \text{ and } Y_q \text{ are isometric} \quad \Rightarrow \quad \varrho_{GH}(Y_p, Y_q) = 0.$$

Moreover, if d is the discrete metric, then this holds with equivalences. In this way, we obtain the result of Clemens, Gao and Kechris that E_G^X is Borel reducible to the relation of isometry on $F(\mathbb{U})$, as well as to the relation $E_{Iso(\mathbb{U})}^{F(\mathbb{U})}$, see [4, 9].

4. Comments and questions

An immediate question concerning Theorem 1.1 remains open.

Question 4.1. *Does Theorem 1.1 hold without the requirement of the existence of the system Φ of continuous pseudometrics?*

We will show in the next remark that the existence of such a system is not guaranteed for a lower semicontinuous pseudometric d. However, we do not know if the more general system from Theorem 3.1 needs to exist. We just know that the mapping $p \mapsto H_\varphi(p)$ needs not to be Borel for a lower semicontinuous pseudometric φ, see Remark 4.3. Later, we will show that Question 4.1 has a positive answer in the case of a countable group action, see Proposition 4.5.

Let us recall that an equivalence relation E on a Polish space X is called *smooth* if there exists a Borel reduction of E into the relation of equality on a Polish space Y. It is known that every closed (actually every G_δ) equivalence relation is smooth, see [10, Corollary 1.2]. It turns out that the question if a system Φ from Theorem 1.1 exists for any lower semicontinuous pseudometric is connected with the question if the reduction witnessing smoothness of a general closed equivalence relation can be found not only Borel, but even continuous. We show in the next remark that the question has a negative answer. Although this is not a surprising finding, we were not able to find a counterexample in the literature.

Remark 4.2. There is a closed equivalence relation E on a Polish space X and two points $a, b \in X$ which are not E-equivalent but have the property that any continuous mapping $f : X \to Y$ into a Polish space Y with $xEy \Rightarrow f(x) = f(y)$ satisfies $f(a) = f(b)$. It follows that

- there is no continuous reduction to the relation of equality on a Polish space Y (although a Borel reduction necessarily exists),
- the pseudometric ϱ_E given by $\varrho_E(x, y) = 0$ if xEy and $\varrho_E(x, y) = 1$ otherwise does not satisfy the assumption of Theorem 1.1, as any continuous pseudometric φ with $\varphi \leq \varrho_E$ satisfies $\varphi(a, b) = 0 < 1 = \varrho_E(a, b)$.

Let us provide such an example. Let A be the subset of the Banach space ℓ_1 consisting of the point $x = 0$ and the points $x_{n,m} = \frac{1}{m}e_n, n, m \in \mathbb{N}$. Let B be the set of real numbers consisting of the point $p = 0$ and the points $p_k = \frac{1}{k}, k \in \mathbb{N}$. Further, let X be the space consisting of two copies of A and infinitely many copies of B. More precisely,

$$X = \left(\{x^{(0)}\} \cup \{x_{n,m}^{(0)} : n, m \in \mathbb{N}\}\right) \cup \left(\{x^{(1)}\} \cup \{x_{n,m}^{(1)} : n, m \in \mathbb{N}\}\right) \cup \bigcup_{m=1}^{\infty} \left(\{p^{(m)}\} \cup \{p_k^{(m)} : k \in \mathbb{N}\}\right).$$

The distance of two points from different copies of A or B can be defined as 10, say. Finally, we consider the equivalence relation E with the equivalence classes

$$\{x^{(0)}\}, \quad \{x^{(1)}\}, \quad \{p^{(m)}\}, m \in \mathbb{N}, \quad \{x_{n,m}^{(0)}, p_{2n-1}^{(m)}\}, n, m \in \mathbb{N}, \quad \{x_{n,m}^{(1)}, p_{2n}^{(m)}\}, n, m \in \mathbb{N}.$$

Let us check that E is closed. Suppose that $a_i \to a$ and $b_i \to b$ in X and that $a_i E b_i$ for every $i \in \mathbb{N}$. We need to check that aEb. We may suppose that each a_i (each b_i) belongs to the same copy of A or B as a (as b). In the case that a and b belong to the same copy of A or B, then $a_i = b_i$ for $i \in \mathbb{N}$, and so $a = b$. In the opposite case, there are four possibilities, but the argument is the same in each of them. If, say, there is m such for every i, we have $a_i = x_{n(i),m}^{(0)}$ and $b_i = p_{2n(i)-1}^{(m)}$ for some $n(i)$, then there is n such that $n(i) = n$ for all but finitely many i's (as the distance of $x_{n_1,m}^{(0)}$ and $x_{n_2,m}^{(0)}$ is always $2/m$ for $n_1 \neq n_2$). Hence, $a = x_{n,m}^{(0)}$, $b = p_{2n-1}^{(m)}$, and aEb.

Now, we show that the choice $a = x^{(0)}, b = x^{(1)}$ works. Let $f : X \to Y$ be a continuous mapping into a Polish space Y such that $xEy \Rightarrow f(x) = f(y)$. We show first that $f(p^{(m)}) \to f(x^{(0)})$. Let U be a neighborhood of $f(x^{(0)})$. There is m_0 such that $f(x) \in U$ whenever the distance of x to $x^{(0)}$ is less than $1/m_0$. For all $m > m_0$ and $n \in \mathbb{N}$, we have $f(p_{2n-1}^{(m)}) = f(x_{n,m}^{(0)}) \in U$. It follows that $f(p^{(m)}) \in \overline{U}$ for all $m > m_0$. As U was an arbitrary neighborhood of $f(x^{(0)})$, we finally arrive at $f(p^{(m)}) \to f(x^{(0)})$. By the same argument, we can obtain $f(p^{(m)}) \to f(x^{(1)})$. Therefore, $f(x^{(0)}) = f(x^{(1)})$.

Remark 4.3. Concerning the assumptions of Theorem 3.1, we can ask if the mapping $p \mapsto H_\varphi(p)$ is Borel for a general lower semicontinuous pseudometric φ. If we restrict our attention on pseudometrics attaining only values 0 and 1, we actually ask if the mapping $x \mapsto [x]_E$ from X to $F(X)$ is Borel for a general closed equivalence relation E on a Polish space X. This question has a negative answer. We can consider any closed equivalence without a Borel selector, or we can consider the following example due to Noé de Rancourt.

Let A be an analytic non-Borel set in a Polish space Y. Then A is the projection of a closed subset B of $Y \times \mathbb{N}^\mathbb{N}$. Let us add an isolated point p to $\mathbb{N}^\mathbb{N}$ and consider the space $X = B \cup (Y \times \{p\})$ and the equivalence $(y_1, \nu_1)E(y_2, \nu_2) \Leftrightarrow y_1 = y_2$. Then B is an open subset of X but the set $\{y \in Y : [(y, p)]_E \cap B \neq \emptyset\} = A$ is not Borel.

It is known and easy to show that an equivalence relation E on a Polish space X is smooth if and only if there exist Borel subsets A_n of X such that xEy iff $\{n : x \in A_n\} = \{n : y \in A_n\}$. For this reason, the following lemma can be seen as a continuous version of the above mentioned result from [10] that every closed equivalence is smooth. Its proof is inspired by the proof of [10, Lemma 5.1] and a proof of Urysohn's lemma.

Lemma 4.4. *Let d be a lower semicontinuous pseudometric on a Polish space X. Then there are Borel functions $s_n : X \to \mathbb{R}, n \in \mathbb{N}$, such that*
$$d(p, q) = \sup_{n \in \mathbb{N}} |s_n(p) - s_n(q)|, \quad p, q \in X.$$

Proof. For $A \subseteq X$ and $r > 0$, let us denote
$$(A)_r = \{x \in X : d(x, A) < r\},$$
where $d(x, A)$ means the distance of x to A in the pseudometric space (X, d). It holds that
$$A \text{ is analytic} \quad \Rightarrow \quad (A)_r \text{ is analytic},$$
since $(A)_r$ is a projection of the intersection of $X \times A$ with the F_σ set $\{(x, y) \in X \times X : d(x, y) < r\}$.

Let us pick $\eta > 0$ and $p, q \in X$ with $d(p, q) > \eta$. We can choose open neighborhoods $U \ni p$ and $V \ni q$ such that $d(x, y) > \eta$ for $x \in U$ and $y \in V$. By a recursive procedure, we will find for every rational number $u \in [0, 1]$ a Borel set B_u in the way that

- $B_0 = U, B_1 = X \setminus V$,
- $B_u \subseteq B_v$ for $u < v$,
- $d(B_u, X \setminus B_v) \geq \eta \cdot (v - u)$ for $u < v$,

where $d(B_u, X \setminus B_v)$ means the distance of these two sets in the pseudometric space (X, d). Note first that $d(B_0, X \setminus B_1) = d(U, V) \geq \eta = \eta \cdot (1 - 0)$. Let $u_1 = 0, u_2 = 1$, and let the rational numbers in $(0, 1)$ be enumerated by $u_3, u_4, \ldots$. Suppose that $n \geq 2$ and that $B_{u_1}, \ldots, B_{u_n}$ are already chosen. Let us consider
$$C = \bigcup_{\substack{1 \leq k \leq n \\ u_k < u_{n+1}}} (B_{u_k})_{\eta \cdot (u_{n+1} - u_k)}, \quad D = \bigcup_{\substack{1 \leq k \leq n \\ u_k > u_{n+1}}} (X \setminus B_{u_k})_{\eta \cdot (u_k - u_{n+1})}.$$

As C and D are disjoint analytic sets, using the Lusin separation principle (see e.g. [13, Theorem 14.7]), we can find a Borel set $B_{u_{n+1}}$ which separates C from D, that is, $C \subseteq B_{u_{n+1}} \subseteq X \setminus D$. It is easy to check that the procedure works.

Once the sets B_u are found, we define

$$s(x) = \eta \cdot \inf \left(\{1\} \cup \{u \in [0,1] \cap \mathbb{Q} : x \in B_u\} \right), \quad x \in X.$$

It is easy to show that s is a Borel function and that $s(y) - s(x) = \eta - 0 = \eta$ for $x \in U$ and $y \in V$. Let us check that $|s(x) - s(y)| \leq d(x,y)$ for all $x, y \in X$. We can suppose that $s(x) > s(y)$. Let u, v be rational with $s(y)/\eta < u < v < s(x)/\eta$. Then $y \in B_u$ and $x \notin B_v$, and so $d(x,y) \geq d(B_u, X \setminus B_v) \geq \eta \cdot (v - u)$. Since u can be arbitrarily close to $s(y)/\eta$ and v can be arbitrarily close to $s(x)/\eta$, we obtain $d(x,y) \geq s(x) - s(y)$.

Now, let $\eta > 0$ be fixed. The construction above provides a covering of the set $\{(p,q) \in X \times X : d(p,q) > \eta\}$ by open sets of the form $U \times V$, for each of which there is a Borel function $s : X \to \mathbb{R}$ such that $|s(x) - s(y)| \leq d(x,y)$ for all $x, y \in X$ and $s(y) - s(x) = \eta$ for $x \in U$ and $y \in V$. Due to the Lindelöf property, there is a countable system S_η of such functions satisfying $d(p,q) > \eta \Rightarrow \sup_{s \in S_\eta} |s(p) - s(q)| \geq \eta$.

Finally, the system $S = \bigcup \{S_\eta : \eta > 0, \eta \in \mathbb{Q}\}$ has the desired property $d(p,q) = \sup_{s \in S} |s(p) - s(q)|$ for all $p, q \in X$. $\qquad\square$

In the next proposition, we give a positive answer to Question 4.1 in the case that the group G is countable.

Proposition 4.5. *Let G be a countable group acting on a Polish space X, and let the action be Borel. Let d be a lower semicontinuous pseudometric on X such that $d(x,y) = d(gx, gy)$ for any $x, y \in X$ and $g \in G$. Then $\varrho_{G,d}$ is Borel-u.c. reducible to the Gromov-Hausdorff distance ϱ_{GH}.*

Proof. Let $\mathbb{R}^{\mathbb{N} \times G}$ be equipped with the distance $\sigma(a, b) = \sup_{(n,g) \in \mathbb{N} \times G} |a(n,g) - b(n,g)|$ and with the action $(h \cdot a)(n,g) = a(n, gh)$. By Theorem 1.1, the pseudometric $\varrho_{G,\sigma}$ on $\mathbb{R}^{\mathbb{N} \times G}$ is Borel-u.c. reducible to ϱ_{GH}, so it is sufficient to show that $\varrho_{G,d}$ is Borel-u.c. reducible to $\varrho_{G,\sigma}$. In fact, there is a Borel-isometric reduction.

Indeed, if $s_n, n \in \mathbb{N}$, are as in Lemma 4.4 and $f : X \to \mathbb{R}^{\mathbb{N} \times G}$ is defined by $f(x)(n,g) = s_n(gx)$, then we have

$$\varrho_{G,\sigma}(f(x), f(y)) = \inf_h \sigma(h \cdot f(x), f(y)) = \inf_h \sup_{n,g} \left| (h \cdot f(x))(n,g) - f(y)(n,g) \right|$$

$$= \inf_h \sup_{n,g} \left| f(x)(n, gh) - f(y)(n,g) \right| = \inf_h \sup_{n,g} \left| s_n(ghx) - s_n(gy) \right|$$

$$= \inf_h \sup_g d(ghx, gy) = \inf_h \sup_g d(hx, y) = \inf_h d(hx, y) = \varrho_{G,d}(x,y)$$

for all $x, y \in X$. $\qquad\square$

Analogously as in Corollary 1.2, we obtain in the setting of Proposition 4.5 that the equivalence relation given by $x E_{G,d}^X y \Leftrightarrow \varrho_{G,d}(x,y) = 0$ is reducible to E_{GH}, and so E_1 is not reducible to it. Let us note that $E_{G,d}^X$ is automatically Borel in such setting, so it possess much lower complexity than E_{GH}. Despite of that, we still do not know if $E_{G,d}^X$ is reducible to an orbit equivalence relation, even for the following simple example.

Question 4.6. *Is the equivalence relation*

$$E_{\mathbb{Z},\sigma}^{[0,1]^{\mathbb{Z}}} = \left\{ (x,y) \in [0,1]^{\mathbb{Z}} : \left(\forall \varepsilon > 0 \, \exists k \in \mathbb{Z} \, \forall l \in \mathbb{Z} : |x(l+k) - y(l)| < \varepsilon \right) \right\}$$

Borel reducible to an orbit equivalence relation?

It is conjectured that for any Borel equivalence relation E, either E is reducible to an orbit equivalence relation, or E_1 is reducible to E, see [11, Conjecture 1]. Thus, a negative answer to Question 4.6 would be an unexpected and remarkable result.

Appendix A. Banach spaces setting

The purpose of this appendix is to prove a version of Theorem 1.3 in the framework of Banach spaces. We need to recall some definitions first, starting with an analogue of the Urysohn space.

A separable Banach space $\mathbb{G}$ is called a *Gurariy space* if, for every $\varepsilon > 0$, every finite-dimensional Banach spaces X and Y with $X \subseteq Y$ and every linear isometry $f : X \to \mathbb{G}$, there exists an extension $g : Y \to \mathbb{G}$ of f such that $(1 + \varepsilon)^{-1}\|y\| \leq \|g(y)\| \leq (1+\varepsilon)\|y\|$ for every $y \in Y$. It is known that there exists only one Gurariy space up to isometry ([15], see also [14]). We will need the following analogue of Theorem 2.1 which follows from [2, Lemma 3.8].

Theorem A.1 (Ben Yaacov [2]). *Let X be a separable Banach space. Then there exists a linear isometric embedding $i : X \to \mathbb{G}$ such that any surjective linear isometry on $i(X)$ can be extended to a surjective linear isometry on $\mathbb{G}$.*

We define
$$\mathcal{SE}(\mathbb{G}) = \{X \in F(\mathbb{G}) : X \text{ is linear}\}.$$

This is a closed subset of $F(\mathbb{G})$ (with the Wijsman topology), since it consists of elements X of $F(\mathbb{G})$ such that $\text{dist}(x + y, X) \leq \text{dist}(x, X) + \text{dist}(y, X)$ and $\text{dist}(\lambda x, X) = |\lambda|\text{dist}(x, X)$ for $x, y \in \mathbb{G}$ and $\lambda \in \mathbb{R}$. Further, by $Iso_L(\mathbb{G})$ we denote the subgroup of $Iso(\mathbb{G})$ of all surjective linear isometries on $\mathbb{G}$.

For two linear subspaces E, F of a normed linear space Z, we define their modified Hausdorff distance by
$$\tilde{\varrho}_H(E, F) = \varrho_H(B_E, B_F),$$

where B_E and B_F denote the closed unit balls of E and F. The *Kadets distance* of normed linear spaces X and Y is defined by
$$\varrho_K(X, Y) = \inf_{\substack{Z \text{ normed linear space} \\ i_X:X\hookrightarrow Z \\ i_Y:Y\hookrightarrow Z}} \tilde{\varrho}_H\big(i_X(X), i_Y(Y)\big),$$

where the symbol $\hookrightarrow$ denotes a linear isometric embedding.

Now, we can formulate the main result of this section. Of course, the distances $\tilde{\varrho}_H$ and ϱ_K are regarded as pseudometrics on $\mathcal{SE}(\mathbb{G})$ here.

Theorem A.2. *The Kadets distance ϱ_K is Borel-u.c. bireducible with the orbit pseudometric $\varrho_{Iso_L(\mathbb{G}),\tilde{\varrho}_H}$ on $\mathcal{SE}(\mathbb{G})$.*

Let us notice first that the pseudometric $\varrho_{Iso_L(\mathbb{G}),\tilde{\varrho}_H}$ is reducible to ϱ_{GH}. The action of $Iso_L(\mathbb{G})$ on $\mathcal{SE}(\mathbb{G})$ is continuous by Lemma 2.2, and the pseudometric
$$\tilde{\tilde{\varrho}}_H(E, F) = \sup_{x\in B_\mathbb{G}} \big|\text{dist}(x, E) - \text{dist}(x, F)\big|, \quad E, F \in \mathcal{SE}(\mathbb{G}),$$

satisfies the requirement from Theorem 1.1, hence the orbit pseudometric $\varrho_{Iso_L(\mathbb{G}),\tilde{\tilde{\varrho}}_H}$ is reducible to ϱ_{GH}. It remains to note that $\tilde{\tilde{\varrho}}_H \leq 2\tilde{\varrho}_H$ and $\tilde{\varrho}_H \leq 2\tilde{\tilde{\varrho}}_H$.

Further, we know from Theorem 2.3 that the pseudometrics ϱ_{GH} and $\varrho_{S_\infty,\sigma}$ on $[1/2, 1]^{[\mathbb{N}]^2}$ are bireducible. It is also known that both of them are bireducible with the Kadets distance ϱ_K (see [6, Theorems 46 and 48]).

So, concerning the proof of Theorem A.2, it is sufficient to find a reduction of $\varrho_{S_\infty,\sigma}$ to $\varrho_{Iso_L(\mathbb{G}),\tilde{\varrho}_H}$. To find such a reduction, we use a construction from [6, Section 4]. Let us fix numbers α and δ such that

$$1 < \alpha < \alpha + \delta \leq \frac{200}{199}.$$

For every $f : [\mathbb{N}]^2 \to [1/2, 1]$, we define an equivalent norm $\|\cdot\|_f$ on ℓ_2 by

$$\|x\|_f = \sup\left(\{\|x\|_{\ell_2}\} \cup \left\{\frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(m,n)) \cdot |x_m + x_n| : \{m,n\} \in [\mathbb{N}]^2\right\}\right)$$

for $x = \{x_k\}_{k=1}^\infty \in \ell_2$. It is easy to show that this is an equivalent norm indeed, as $\|x\|_{\ell_2} \leq \|x\|_f \leq \frac{200}{199}\|x\|_{\ell_2}$ (see [6]).

The following result follows from [6, Lemma 51] (for an explanation, see the proof of [6, Theorem 48], step (5)).

Lemma A.3 ([6]). *There is a constant $C > 0$ such that*

$$\varrho_{S_\infty,\sigma}(f,g) \leq C\varrho_K\big((\ell_2, \|\cdot\|_f), (\ell_2, \|\cdot\|_g)\big), \quad f, g \in [1/2, 1]^{[\mathbb{N}]^2}.$$

The next step uses a similar idea as the proof of $(2) \Rightarrow (1)$ in [7, Proposition 3.15]. We define a subset of $c_{00}([1/2,1]^{[\mathbb{N}]^2} \times \mathbb{N})$ by

$$\Omega = \mathrm{co}\bigg(\bigcup_f \Big\{\sum_k x_k e_{f,k} : x \in c_{00}(\mathbb{N}), \|x\|_f \leq 1\Big\}$$

$$\cup \bigcup_{f,g}\Big\{\sum_k x_k e_{f,k} - \sum_k x_k e_{g,k} : x \in c_{00}(\mathbb{N}), \delta \cdot \sigma(f,g) \cdot \|x\|_{\ell_2} \leq 1\Big\}$$

$$\cup \bigcup_{f,g,\{m,n\}}\Big\{c \cdot (e_{f,m} + e_{f,n} - e_{g,m} - e_{g,n}) : |c| \cdot \delta \cdot |f(m,n) - g(m,n)| \cdot \sqrt{2} \leq 1\Big\}\bigg),$$

where $\{e_{f,k}\}_{f,k}$ stands for the canonical basis of $c_{00}([1/2,1]^{[\mathbb{N}]^2} \times \mathbb{N})$. Let us denote the corresponding Minkowski functional by γ. Before we prove several properties of Ω and γ, let us note that it follows immediately from the definition that

- $\gamma(\sum_k x_k e_{f,k}) \leq \|x\|_f$,
- $\gamma(\sum_k x_k e_{f,k} - \sum_k x_k e_{g,k}) \leq \delta \cdot \sigma(f,g) \cdot \|x\|_{\ell_2}$,
- $\gamma(e_{f,m} + e_{f,n} - e_{g,m} - e_{g,n}) \leq \delta \cdot |f(m,n) - g(m,n)| \cdot \sqrt{2}$.

Claim A.4. *We have $\gamma(\sum_k x_k e_{f,k}) = \|x\|_f$ for $f \in [1/2,1]^{[\mathbb{N}]^2}$ and $x = \{x_k\}_k \in c_{00}(\mathbb{N})$.*

Proof. For notational purposes, we change the notation f and x in the claim to h and y. We already know that $\gamma(\sum_k y_k e_{h,k}) \leq \|y\|_h$. The verification of the opposite inequality consists of two tasks. Let us note first that any linear functional u^* on $c_{00}([1/2,1]^{[\mathbb{N}]^2} \times \mathbb{N})$ for which $u^* \leq 1$ on Ω satisfies $u^* \leq \gamma$.

(i) We check that $\gamma(\sum_k y_k e_{h,k}) \geq \|y\|_{\ell_2}$. We can suppose that $\|y\|_{\ell_2} = 1$. Let us consider $u^* \in c_{00}([1/2,1]^{[\mathbb{N}]^2} \times \mathbb{N})^\#$ given by $u^*(e_{f,k}) = y_k$. Then $u^* \leq 1$ on Ω, since

- $u^*(\sum_k x_k e_{f,k}) = \sum_k x_k u^*(e_{f,k}) = \sum_k x_k y_k \leq \|x\|_{\ell_2}\|y\|_{\ell_2} = \|x\|_{\ell_2} \leq \|x\|_f \leq 1$ whenever $\|x\|_f \leq 1$,
- $u^*(\sum_k x_k e_{f,k} - \sum_k x_k e_{g,k}) = \sum_k x_k y_k - \sum_k x_k y_k = 0 \leq 1$,
- $u^*(c \cdot (e_{f,m} + e_{f,n} - e_{g,m} - e_{g,n})) = c \cdot (y_m + y_n - y_m - y_n) = 0 \leq 1$.

It follows that $\gamma(\sum_k y_k e_{h,k}) \geq u^*(\sum_k y_k e_{h,k}) = \sum_k y_k u^*(e_{h,k}) = \sum_k y_k^2 = \|y\|_{\ell_2}^2 = \|y\|_{\ell_2}$.

(ii) We check that $\gamma(\sum_k y_k e_{h,k}) \geq \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot h(m,n)) \cdot |y_m + y_n|$ for any $\{m,n\} \in [\mathbb{N}]^2$. We can suppose that $y_m + y_n \geq 0$. Let us consider $u^* \in c_{00}([1/2,1]^{[\mathbb{N}]^2} \times \mathbb{N})^\#$

given by $u^*(e_{f,m}) = u^*(e_{f,n}) = \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(m,n))$ and $u^*(e_{f,k}) = 0$ for $k \notin \{m,n\}$. Then $u^* \leq 1$ on Ω, since the points in the formula for Ω satisfy

- $u^*(\sum_k x_k e_{f,k}) = \sum_k x_k u^*(e_{f,k}) = \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(m,n))(x_m + x_n) \leq \|x\|_f \leq 1$,
- $u^*(\sum_k x_k e_{f,k} - \sum_k x_k e_{g,k}) = \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot f(m,n))(x_m + x_n) - \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot g(m,n))(x_m + x_n) = \frac{1}{\sqrt{2}} \cdot \delta \cdot (f(m,n) - g(m,n))(x_m + x_n) \leq \delta \cdot \sigma(f,g) \cdot \|x\|_{\ell_2} \leq 1$,
- $u^*(c \cdot (e_{f,m'} + e_{f,n'} - e_{g,m'} - e_{g,n'})) = c \cdot \frac{1}{\sqrt{2}} \cdot \delta \cdot (f(m,n) - g(m,n))|\{m',n'\} \cap \{m,n\}| \leq |c| \cdot \delta \cdot |f(m,n) - g(m,n)| \cdot \sqrt{2} \leq 1$.

It follows that $\gamma(\sum_k y_k e_{h,k}) \geq u^*(\sum_k y_k e_{h,k}) = \sum_k y_k u^*(e_{h,k}) = \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot h(m,n)) \cdot (y_m + y_n) = \frac{1}{\sqrt{2}} \cdot (\alpha + \delta \cdot h(m,n)) \cdot |y_m + y_n|$. $\qquad \square$

Claim A.5. *For $\pi \in S_\infty$, the linear operator T_π given by $e_{f,k} \mapsto e_{\pi \cdot f, \pi(k)}$ satisfies $T_\pi(\Omega) = \Omega$, and so it is an isometry on $(c_{00}([1/2,1]^{\mathbb{N}^2} \times \mathbb{N}), \gamma)$.*

Proof. It is straightforward to check that T_π maps the set of the points $\sum_k x_k e_{f,k}$, $f \in [1/2,1]^{\mathbb{N}^2}$, $x \in c_{00}(\mathbb{N})$, $\|x\|_f \leq 1$, onto itself. The same holds for the other two collections of points in the definition of Ω. $\qquad \square$

Claim A.6. *Let $f, g \in [1/2,1]^{\mathbb{N}^2}$ and let $u \in \mathrm{span}\{e_{f,k} : k \in \mathbb{N}\}$ satisfy $\gamma(u) \leq 1$. Then there is $v \in \mathrm{span}\{e_{g,k} : k \in \mathbb{N}\}$ satisfying $\gamma(v) \leq 1$ such that $\gamma(v - u) \leq 2\delta \cdot \sigma(f,g)$.*

Proof. Let u be expressed by $u = \sum_k x_k e_{f,k}$, where $x = \{x_k\}_k \in c_{00}(\mathbb{N})$, and let $u' = \sum_k x_k e_{g,k}$. We know that $\gamma(u - u') = \gamma(\sum_k x_k e_{f,k} - \sum_k x_k e_{g,k}) \leq \delta \cdot \sigma(f,g) \cdot \|x\|_{\ell_2}$. By Claim A.4, we have $\|x\|_{\ell_2} \leq \|x\|_f = \gamma(u) \leq 1$, and so

$$\gamma(u - u') \leq \delta \cdot \sigma(f,g),$$

in particular $\gamma(u') \leq 1 + \delta \cdot \sigma(f,g)$. Choosing $v = \frac{1}{1+\delta \cdot \sigma(f,g)} u'$, we arrive at $\gamma(v) \leq 1$ and $\gamma(v - u) \leq \gamma(u' - v) + \gamma(u - u') = \gamma(\delta \cdot \sigma(f,g) \cdot v) + \gamma(u - u') \leq 2\delta \cdot \sigma(f,g)$. $\quad \square$

Claim A.7. *The mapping $f \mapsto e_{f,k}$ is continuous for every k (in the sense that $\gamma(e_{g,k} - e_{f,k}) \to 0$ as $g \to f$ in $[1/2,1]^{\mathbb{N}^2}$).*

Proof. For all $\{m,n\} \in [\mathbb{N}]^2$, the mapping $f \mapsto e_{f,m} + e_{f,n}$ is continuous, as $\gamma(e_{f,m} + e_{f,n} - e_{g,m} - e_{g,n}) \leq \delta \cdot |f(m,n) - g(m,n)| \cdot \sqrt{2}$ for each f and g. It remains to note that $e_{f,k} = \frac{1}{2}[(e_{f,k} + e_{f,k+1}) + (e_{f,k} + e_{f,k+2}) - (e_{f,k+1} + e_{f,k+2})]$ for each f. $\quad \square$

Now, let W be the completion of the quotient $c_{00}([1/2,1]^{\mathbb{N}^2} \times \mathbb{N})/M$, where $M = \{u \in c_{00}([1/2,1]^{\mathbb{N}^2} \times \mathbb{N}) : \gamma(u) = 0\}$. Let $Q : c_{00}([1/2,1]^{\mathbb{N}^2} \times \mathbb{N}) \to W$ denote the quotient mapping. Define

$$W_f = \overline{\mathrm{span}}\{Q e_{f,k} : k \in \mathbb{N}\}, \quad f \in [1/2,1]^{\mathbb{N}^2}.$$

We see that

- W_f is isometric to $(\ell_2, \|\cdot\|_f)$ (Claim A.4),
- for $f \in [1/2,1]^{\mathbb{N}^2}$ and $\pi \in S_\infty$, there is a surjective linear isometry on W which maps W_f onto $W_{\pi \cdot f}$ (Claim A.5),
- $\tilde{\varrho}_H(W_f, W_g) \leq 2\delta \cdot \sigma(f,g)$ (Claim A.6),
- the mapping $f \mapsto Q e_{f,k}$ is continuous for every $k \in \mathbb{N}$ (Claim A.7).

From the last property, we obtain that W is separable. By Theorem A.1, we can suppose that W is a subspace of $\mathbb{G}$ and that any surjective linear isometry on W can be extended to a surjective linear isometry on $\mathbb{G}$. So, for $f \in [1/2,1]^{\mathbb{N}^2}$ and

$\pi \in S_\infty$, there is a surjective linear isometry on $\mathbb{G}$ which maps W_f onto $W_{\pi \cdot f}$. It follows that for $f, g \in [1/2, 1]^{[\mathbb{N}]^2}$, we have

$$\varrho_{Iso_L(\mathbb{G}), \tilde{\varrho}_H}(W_f, W_g) \leq 2\delta \cdot \varrho_{S_\infty, \sigma}(f, g).$$

By Lemma A.3,

$$\varrho_{S_\infty, \sigma}(f, g) \leq C \varrho_K(W_f, W_g),$$

and it is clear that

$$\varrho_K(W_f, W_g) \leq \varrho_{Iso_L(\mathbb{G}), \tilde{\varrho}_H}(W_f, W_g).$$

To show that $f \mapsto W_f$ is the desired reduction, it remains to note that it is a Borel mapping, which is a consequence of the continuity of the mappings $f \mapsto Qe_{f,k}$.

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Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

E-mail address: kurka.ondrej@seznam.cz