INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

L^q-solution of the Robin problem for the Stokes system with Coriolis force

Dagmar Medková

Preprint No. 23-2018

PRAHA 2018

L^q-SOLUTION OF THE ROBIN PROBLEM FOR THE STOKES SYSTEM WITH CORIOLIS FORCE

DAGMAR MEDKOVÁ[†]

Abstract. We define single layer potential and double layer potential for the stationary Stokes system with Coriolis term and study properties of these potentials. Then using the integral equation method we study the Dirichlet problem, the Neumann problem and the Robin problem for the Stokes system with Coriolis term. We look for solutions of the problems such that the maximal functions of the velocity $\mathbf{u}$, of the pressure p and of $\nabla \mathbf{u}$ are q-integrable on the boundary, and the boundary conditions are fulfilled in the sense of a non-tangential limit. As a consequence we study solutions of the Dirichlet problem for an exterior domain in the homogeneous Sobolev spaces $D^{k,q}(\Omega,\mathbb{R}^3) \times D^{k-1,q}(\Omega)$ and in weighted Besov spaces.

1. Introduction

Problems with rotation in hydrodynamics lead to the Stokes system with Coriolis term $-\Delta\mathbf{u} - [\omega\times\mathbf{x}]\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p = \mathbf{f}$, $\nabla\cdot\mathbf{u} = 0$; to the Oseen system with Coriolis term $-\Delta\mathbf{u} + \psi\cdot\nabla\mathbf{u} - [\omega\times\mathbf{x}]\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p = \mathbf{f}$, $\nabla\cdot\mathbf{u} = 0$; and to the Navier-Stokes system with Coriolis term $-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \psi\cdot\nabla\mathbf{u} - [\omega\times\mathbf{x}]\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p = \mathbf{f}$, $\nabla\cdot\mathbf{u} = 0$. Solutions of these systems in the whole $\mathbb{R}^3$ are studied in [18], [20], [23], [24], [25], [26], [29], [30], [31], [32], [33], [64]. Solutions of the Dirichlet problem for these systems in exterior domains are studied in [19], [20], [23], [28], [29], [30], [33], [36], [62], [63], [100]. J. Bemelmans, G. P. Galdi and M. Kyed studied in [2] the mixed problem for the Navier-Stokes system with Coriolis term. They divided the boundary to two parts Γ_1 and Γ_2. The Dirichlet condition is given on Γ_1. The normal part of the Dirichlet condition and the tangential part of the Neumann condition are given on Γ_2.

One of the method used in hydrostatics is an integral equation method. This method enables to obtain another regularity results than the other methods. The boundary value problems for the stationary Stokes system are studied by this method in [3], [4], [5], [6], [7], [8], [9], [11], [15], [16], [40], [42], [43], [58], [61], [67], [66], [69], [70], [71], [72], [75], [76], [73], [87], [88], [89], [90], [92], [95], [98] [108], [110]. The boundary value problems for the stationary Brinkman system are studied by this method in [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], [56], [57], [83], [85], [99], [108], [109]. The boundary value problems for the stationary Oseen system are studied by this method in [59], [84], [96], [97]. It would

2000 *Mathematics Subject Classification.* 76U05;76D10;76D07;35Q35.

Key words and phrases. layer potential operators; Stokes system with Coriolis term; Dirichlet problem; Neumann problem; Robin problem.

[†]The work of Dagmar Medková was supported by RVO: 67985840 and grant GAČR no. 17-01747S.

be interesting to get similar results also for systems with Coriolis term. This paper represents the first step to this goal.

Let $\Omega \subset \mathbb{R}^3$ be a domain with compact Lipschitz boundary. Denote by $\mathbf{n}^\Omega(\mathbf{x})$ (or shortly $\mathbf{n}$) the outward unit normal of Ω at $\mathbf{x} \in \partial\Omega$. If $\mathbf{u} = (u_1, u_2, u_3)$ is a velocity, and p is a pressure, we define by

$$T(\mathbf{u}, p) = 2\hat{\nabla}\mathbf{u} - pI$$

the corresponding stress tensor, where I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the deformation tensor, with $(\nabla\mathbf{u})^T$ as the matrix transposed to $\nabla\mathbf{u}$. Let $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. We study the Stokes system with Coriolis term

$$-\Delta\mathbf{u} - [\omega \times \mathbf{x}] \cdot \nabla\mathbf{u} + \omega \times \mathbf{u} + \nabla p = 0, \qquad \nabla \cdot \mathbf{u} = 0 \tag{1.1}$$

with the Dirichlet condition

$$\mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega \tag{1.2}$$

and with the Robin condition

$$T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u} + h\mathbf{u} = \mathbf{f} \qquad \text{on } \partial\Omega. \tag{1.3}$$

(If $h \equiv 0$ we say about the Neumann problem.) We study such solutions of the problems that the non-tangential maximal functions of $\mathbf{u}$, $\nabla\mathbf{u}$ and p are in $L^q(\partial\Omega)$ and the boundary conditions are fulfilled in the sense of a non-tangential limit.

Let us gather what is known about the Dirichlet problem for the Stokes system with Coriolis term. It was shown the existence of a solution in $D^{2,q}(\Omega; \mathbb{R}^3) \times D^{1,q}(\Omega)$ for an exterior domain with boundary of class $\mathcal{C}^2$ and $1 < q < 3/2$ (see [29]). (Here $D^{k,q}(\Omega)$ denotes the homogeneous Sobolev space of order k.) G. P. Galdi and A. L. Silvestre proved in [33] this result for $q = 2$. G. P. Galdi studied in [28] the Dirichlet problem on a Lipschitz exterior domain in $D^{1,2}(\Omega, \mathbb{R}^3) \times L^2_{\mathrm{loc}}(\overline{\Omega})$. T. Hishida studied in [36] the Dirichlet problem in $D^{1,q}(\Omega, \mathbb{R}^3) \times L^q(\Omega)$, $3/2 < q < 3$, for an exterior domain with smooth boundary. R. Farwig and T. Hishida studied in [20] the Dirichlet problem for the Stokes system with Coriolis term in $\dot{W}^1_{q,r}(\Omega; \mathbb{R}^3) \times L^{q,r}(\Omega)$ in an exterior domain with smooth boundary. Here $L^{q,r}(\Omega)$ is a Lorentz space and $\dot{W}^{q,r}_1(\Omega)$ is defined by the real interpolation of homogeneous Sobolev spaces of the order 1.

To study boundary value problems by the integral equation method we need the existence of a fundamental solution of the system and its properties. The fundamental solution of the Stokes system with Coriolis term was given by E. A. Thomann and R. B. Guenther in [103]. Unfortunately, this fundamental solution has a very complicated formula, it is not of convolution type or symmetric. It is a bit problem because the simple formula of the fundamental solution for the Stokes system, symmetry and a convolution type of the fundamental solution play important role in the theory of the integral equtions for the Stokes system. Properties of the fundamental solution of the Stokes system with Coriolis term were studied by R. Farwig and T. Hishida in [21] and [22]. We need further properties, so we prove them. For this reason we first study regularity of solutions of the Dirichlet problem on bounded domains. These routine results enable us to compare the fundamental solution of the Stokes system with Coriolis term and the fundamental solution of the Stokes system in a neighbourhood of the diagonal. Then we define a single layer potential and a double layer potential, and prove some properties of these potentials similar to properties of the Stokes potentials. Especially useful is the representation of a solution of the homogeneous Stokes system with Coriolis term as the sum of the double layer potential corresponding to the trace of the velocity and the single layer potential corresponding to the Neumann condition. (This formula was proved both for a bounded and an unbounded domain.) Then we study so called regular L^q-solutions of the Dirichlet problem for a bounded and an exterior domain with Lipschitz boundary. The boundary condition is from $W^{1,q}(\partial\Omega,\mathbb{R}^3)$ with $1 < q \le 2$. We find necessary and sufficient condition for the solvability of the problem and characterize all solutions. If Ω is bounded then a solution is in the Besov space $B^{q,2}_{1+1/q}(\Omega,\mathbb{R}^3)\times B^{q,2}_{1/q}(\Omega)$. If Ω is unbounded and $\mathbf{u}(\mathbf{x})\to 0$, $p(\mathbf{x})\to 0$ at infinity, then the solution $(\mathbf{u},p)$ is in a similar Besov space but with a weight. As a consequence we prove the existence of a unique solution vanishing at infinity of the exterior Dirichlet problem (1.1), (1.2) in the homogeneous Sobolev space $D^{k+1,q}(\Omega,\mathbb{R}^3)\times D^{k,q}(\Omega)$ and also in weighted Besov spaces. Then we study the Neumann problem and the Robin problem on bounded and exterior domains with boundary conditions from $L^q(\partial\Omega,\mathbb{R}^3)$. Here $\partial\Omega$ is Lipschitz and $1 < q \le 2$, or $\partial\Omega$ is of class $\mathcal{C}^1$ and $1 < q < \infty$. We prove the solvability of the problem and characterize all solutions. If Ω is bounded then a solution is in the Besov space $B^{q,\max(q,2)}_{1+1/q}(\Omega,\mathbb{R}^3)\times B^{q,\max(q,2)}_{1/q}(\Omega)$. If Ω is unbounded and $\mathbf{u}(\mathbf{x})\to 0$, $p(\mathbf{x})\to 0$ at infinity, then the solution $(\mathbf{u},p)$ is in a similar Besov space but with a weight.

2. Formulation of problems

Let $\Omega\subset\mathbb{R}^3$ be an open set with compact Lipschitz boundary. If $\mathbf{x}\in\partial\Omega$, $\beta>0$ denote the nontangential approach region of opening β at the point $\mathbf{x}$ by

$$\Gamma_\beta(\mathbf{x})=\{\mathbf{y}\in\Omega; |\mathbf{x}-\mathbf{y}|<(1+\beta)\,\mathrm{dist}(\mathbf{y},\partial\Omega)\}.$$

If now $\mathbf{v}$ is a vector function defined in Ω, we denote the non-tangential maximal function of $\mathbf{v}$ on $\partial\Omega$ by

$$M_\beta(\mathbf{v})(\mathbf{x})=M^\Omega_\beta(\mathbf{v})(\mathbf{x}):=\sup\{|\mathbf{v}(\mathbf{y})|;\mathbf{y}\in\Gamma_\beta(\mathbf{x})\}.$$

It is well known that there exists $c>0$ such that for $\beta,b>c$ and $1\le q<\infty$ there exist $C_1,C_2>0$ such that

$$\|M_\beta v\|_{L^q(\partial\Omega)}\le C_1\|M_b v\|_{L^q(\partial\Omega)}\le C_2\|M_\beta v\|_{L^q(\partial\Omega)}$$

for any measurable function v in Ω. (See, e.g. [39] and [101, p. 62].) We suppose that $\beta>c$ and write $\Gamma(\mathbf{x})$ instead of $\Gamma_\beta(\mathbf{x})$. Next, define the non-tangential limit of $\mathbf{v}$ at $\mathbf{x}\in\partial\Omega$

$$\mathbf{v}(\mathbf{x})=\lim_{\Gamma(\mathbf{x})\ni\mathbf{y}\to\mathbf{x}}\mathbf{v}(\mathbf{y})$$

whenever the limit exists.

Let $a\in\mathbb{R}^1$, $\omega=(0,0,a)$, $h\in L^\infty(\partial\Omega)$, $1<q<\infty$, $\mathbf{f}\in L^q(\partial\Omega,\mathbb{R}^3)$. We say that $(\mathbf{u},p)$ *is an L^q-solution of the Robin problem for the Stokes system with Coriolis term* (1.1), (1.3) *if* $\mathbf{u}\in\mathcal{C}^2(\Omega,\mathbb{R}^3)$, $p\in\mathcal{C}^1(\Omega)$, $(\mathbf{u},p)$ *is a solution of* (1.1), $M_\beta(\mathbf{u})+M_\beta(\nabla\mathbf{u})+M_\beta(p)\in L^q(\partial\Omega)$, *there exist non-tangential limits of* $\mathbf{u}$, $\nabla\mathbf{u}$ *and* p *at almost all points of* $\partial\Omega$, *and the boundary condition* (1.3) *is fulfilled in the sense of a non-tangential limit a.e. on* $\partial\Omega$.

Let $\mathbf{g} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$. We say that $(\mathbf{u}, p)$ *is a regular L^q-solution of the Dirichlet problem for the Stokes system with Coriolis term* (1.1), (1.2) *if* $\mathbf{u} \in \mathcal{C}^2(\Omega, \mathbb{R}^3)$, $p \in \mathcal{C}^1(\Omega)$, $(\mathbf{u}, p)$ *is a solution of* (1.1), $M_\beta(\mathbf{u}) + M_\beta(\nabla\mathbf{u}) + M_\beta(p) \in L^q(\partial\Omega)$, *there exist nontangential limits of* $\mathbf{u}$, $\nabla\mathbf{u}$ *and* p *at almost all points of* $\partial\Omega$, *and* (1.2) *is fulfilled in the sense of a non-tangential limit a.e. on* $\partial\Omega$.

Remark that if $(\mathbf{u}, p)$ is a regular L^q-solution of the Dirichlet problem (1.1), (1.2) then $(\mathbf{u}, p) \in W^{1,q}_{\mathrm{loc}}(\overline{\Omega}, \mathbb{R}^3) \times L^q_{\mathrm{loc}}(\overline{\Omega})$ is a solution of this problem in the sense of traces. (See [79, Lemma 2].)

3. Spaces of functions

Let $\Omega \subset \mathbb{R}^3$ be an open set, $k \in \mathbb{N}_0$, $1 < p < \infty$. We denote the Sobolev space $W^{k,p}(\Omega) = \{u \in L^p(\Omega); \partial^\beta u \in L^p(\Omega) \text{ for } |\beta| \le k\}$. If $s = k + \lambda$ with $0 < \lambda < 1$ denote by $W^{s,p}(\Omega)$ the space of all $u \in W^{k,p}(\Omega)$ such that

$$\sum_{|\beta|=k} \int_\Omega \int_\Omega \frac{|\partial^\beta u(x) - \partial^\beta u(y)|^p}{|x-y|^{3+p\lambda}} \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

If $s \ge 0$ denote by $\mathring{W}^{s,p}(\Omega)$ the closure of $\mathcal{C}^\infty_c(\Omega)$ (the space of infinitely differentiable functions with compact support in Ω) in $W^{s,p}(\Omega)$ and by $W^{-s,p}(\Omega)$ the dual space of $\mathring{W}^{s,p/(p-1)}(\Omega)$. If $t > \tau$ then $W^{t,p}(\Omega) \subset W^{\tau,p}(\Omega)$.

If $s \in \mathbb{R}^1$ and $1 < p, q < \infty$, denote by $B^{p,q}_s(\mathbb{R}^3)$ the Besov space. (For the definition see for example [107].) If $k \in \mathbb{N}_0$, $s = k + \lambda$ with $0 < \lambda < 1$ then $u \in B^{p,q}_s(\mathbb{R}^3)$ if $u \in W^{k,p}(\mathbb{R}^3)$ and

$$\sum_{|\beta|=k} \int_0^\infty \left(\int_{\mathbb{R}^3} \int_{\{y \in \mathbb{R}^3; |x-y|<t\}} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|^p}{t^3} \, \mathrm{d}y \, \mathrm{d}x \right)^{q/p} \frac{\mathrm{d}t}{t^{\lambda q + 1}} < \infty.$$

By $B^{p,q}_s(\Omega)$ we denote the space of restrictions of functions from $B^{p,q}_s(\mathbb{R}^3)$ onto Ω. The norm on $B^{p,q}_s(\Omega)$ is defined by

$$\|u\|_{B^{p,q}_s(\Omega)} = \inf\{\|f\|_{B^{p,q}_s(\mathbb{R}^3)}; f = u \text{ on } \Omega\}.$$

If $s > t$ then $B^{p,q}_s(\Omega) \subset B^{p,q}_t(\Omega)$. If Ω has compact Lipschitz boundary and s is not integer then $B^{p,p}_s(\Omega) = W^{s,p}(\Omega)$.

For $\gamma \in \mathbb{R}^1$ define

$$\rho_\gamma(\mathbf{x}) = (1 + |\mathbf{x}|^2)^{\gamma/2}.$$

We define the weighted Besov space $B^{p,q}_s(\Omega; \rho_\gamma)$ as the set of tempered distributions w such that $\rho_\gamma w \in B^{p,q}_s(\Omega)$. The space $B^{p,q}_s(\Omega; \rho_\gamma)$ equipped with the norm

$$\|w\|_{B^{p,q}_s(\Omega;\rho_\gamma)} := \|\rho_\gamma w\|_{B^{p,q}_s(\Omega)}$$

is a Banach space (see [14, §4.2.2, pp. 156–160]). If Ω is bounded then $B^{p,q}_s(\Omega; \rho_\gamma) = B^{p,q}_s(\Omega)$ and both norms are equivalent.

We denote the homogeneous Sobolev space $D^{k,p}(\Omega) = \{u \in L^1_{loc}(\Omega); \partial^\beta u \in L^p(\Omega) \text{ for } |\beta| = k\}$. Then $D^{k,p}(\Omega) \subset W^{k,p}_{loc}(\Omega)$. Let Ω be a domain. Fix a bounded open set G such that $\overline{G} \subset \Omega$. Then $D^{k,p}(\Omega)$ is a Banach space with the norm

$$\|u\|_{D^{k,p}(\Omega)} = \|u\|_{L^p(G)} + \| \, |\nabla^k u| \, \|_{L^p(\Omega)}.$$

Moreover, different choices of G give equivalent norms. (See [74], §1.5.3, Corollary 2.) If Ω is a bounded domain with Lipschitz boundary then $D^{k,p}(\Omega) = W^{k,p}(\Omega)$ and the corresponding norms are equivalent. (See [74, §1.5.2–§1.5.4].)

4. Green's formula

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $\mathbf{\Phi}, \mathbf{u} \in W^{2,2}(\Omega, \mathbb{R}^3)$, $p \in W^{1,2}(\Omega)$, $h \in L^\infty(\partial\Omega)$, $\mathbf{f} \in L^2(\partial\Omega, \mathbb{R}^3)$. If $(\mathbf{u}, p)$ is a solution of the Robin problem (1.1), (1.3) (in the sense of traces) then*

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma - \int_{\partial\Omega} h\mathbf{u} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma$$

$$= \int_\Omega \{\mathbf{\Phi} \cdot [\omega \times \mathbf{u} - \frac{1}{2}(\omega \times \mathbf{x}) \cdot \nabla \mathbf{u}] + 2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p\nabla \cdot \mathbf{\Phi} + \mathbf{u} \cdot \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \nabla \mathbf{\Phi}]\} \, \mathrm{d}\mathbf{x},$$

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{u} \, \mathrm{d}\sigma = 2\int_\Omega |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} h|\mathbf{u}|^2 \, \mathrm{d}\sigma. \tag{4.1}$$

Proof. If $h \equiv 0$ then the Green formula and (1.1) give

$$\int_{\partial\Omega} \mathbf{f} \cdot \mathbf{\Phi} \, \mathrm{d}\sigma = \int_\Omega \{\mathbf{\Phi} \cdot [\Delta \mathbf{u} - \nabla p + \frac{1}{2}(\omega \times \mathbf{x}) \cdot \nabla \mathbf{u}] + 2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p\nabla \cdot \mathbf{\Phi} + \mathbf{u} \cdot \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \nabla \mathbf{\Phi}]\} \, \mathrm{d}\mathbf{x}$$

$$= \int_\Omega \{\mathbf{\Phi} \cdot [\omega \times \mathbf{u} - \frac{1}{2}(\omega \times \mathbf{x}) \cdot \nabla \mathbf{u}] + 2\hat{\nabla}\mathbf{u} \cdot \hat{\nabla}\mathbf{\Phi} - p\nabla \cdot \mathbf{\Phi} + \mathbf{u} \cdot \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \nabla \mathbf{\Phi}]\} \, \mathrm{d}\mathbf{x}.$$

Since $\mathbf{u} \cdot (\omega \times \mathbf{u}) = 0$, $\nabla \cdot \mathbf{u} = 0$, we obtain (4.1). □

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $1 < q < \infty$, $q' = q/(q-1)$. Let $h \in L^\infty(\partial\Omega)$, $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^3)$, $(\mathbf{u}, p)$ be an L^q-solution of the Robin problem (1.1), (1.3). If $M_\beta(\mathbf{u}) \in L^{q'}(\partial\Omega)$ then the equality (4.1) holds.*

Proof. Let Ω_j be the sequence of sets from Lemma 14.1. According to Lemma 4.1

$$\int_{\partial\Omega_j} \mathbf{u} \cdot \left\{T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u}\right\} \, \mathrm{d}\sigma = 2\int_{\Omega_j} |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{x}.$$

Letting $j \to \infty$ we obtain (4.1) by the Lebesgue lemma. □

5. Liouville's theorem

Lemma 5.1. *Let $\omega \in \mathbb{R}^3$ and $\mathbf{u} = (u_1, u_2, u_3)$, p be tempered distributions in $\mathbb{R}^3$ satisfying the equations (1.1) in $\mathbb{R}^3$. Then u_j, p are polynomials.*

(See [1, p. 614].)

6. Solutions in the whole space

Lemma 6.1. *Let $3/2 < q < \infty$, $\omega = (0, 0, a) \in \mathbb{R}^3 \setminus \{0\}$ and let $\mathbf{f} \in L^q(\mathbb{R}^3, \mathbb{R}^3)$. If $g \in \mathcal{C}^\infty(\mathbb{R}^3)$ has compact support then there exists a solution $(\mathbf{u}, p) \in D^{2,q}(\mathbb{R}^3, \mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)$ of*

$$\Delta \mathbf{u} - (\omega \times x) \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} + \nabla p = \mathbf{f}, \ \nabla \cdot \mathbf{u} = g \quad \text{in } \mathbb{R}^3. \tag{6.1}$$

Proof. Denote $h_\Delta(\mathbf{x}) = [4\pi|\mathbf{x}|]^{-1}$ the fundamental solution of the Laplace equation, i.e. $-\Delta h_\Delta = \delta_0$ in the sense of distributions. Put $\mathbf{v} = \nabla(h_\Delta * g)$, where $*$ denotes the convolution. Then $\mathbf{v} \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R}^3)$ and $\nabla \cdot \mathbf{v} = \Delta(h_\Delta * g) = -g$. If β is a multiindex then $\partial^\beta \mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{-2-|\beta|})$ as $|\mathbf{x}| \to \infty$. So, $\partial^\beta \mathbf{v} \in L^q(\mathbb{R}^3, \mathbb{R}^3)$. Put $\tilde{\mathbf{f}} = \mathbf{f} + \Delta \mathbf{v} - (\omega \times x) \cdot \nabla \mathbf{v} + \omega \times \mathbf{v}$. According to [17, Theorem 4] there exists a solution $(\mathbf{w}, p) \in D^{2,q}(\mathbb{R}^3, \mathbb{R}^3) \times D^{1,q}(\mathbb{R}^3)$ of

$$\Delta \mathbf{w} - (\omega \times x) \cdot \nabla \mathbf{w} + \omega \times \mathbf{w} + \nabla p = \tilde{\mathbf{f}}, \ \nabla \cdot \mathbf{w} = 0 \quad \text{in } \mathbb{R}^3.$$

Put $\mathbf{u} = \mathbf{w} - \mathbf{v}$. Then $\mathbf{u}$, p satisfy the proposition of the Lemma. □

7. Solution of the Dirichlet problem in Sobolev spaces

Lemma 7.1. *Let $\Omega \subset \mathbb{R}^3$ be open, $\omega \in \mathbb{R}^3$ and $\mathbf{u} = (u_1, u_2, u_3)$, p solve (1.1) in Ω in the sense of distributions. Then $u_j, p \in \mathcal{C}^\infty(\Omega)$ and $\Delta p = 0$ in Ω.*

.

Proof. Since $\nabla \cdot \mathbf{u} = 0$ we obtain

$$\Delta p = \nabla \cdot \nabla p = \nabla \cdot \{\Delta \mathbf{u} + [\omega \times \mathbf{x}] \cdot \nabla \mathbf{u} - \omega \times \mathbf{u}\} = 0$$

in the sense of distributions. So, $p \in \mathcal{C}^\infty(\Omega)$ by [27, Appendix B,Theorem B.6]. Fix bounded domains Ω_1, Ω_2 such that $\overline{\Omega}_1 \subset \Omega_2$, $\overline{\Omega}_2 \subset \Omega$. According to [12, Satz 9.11] and [13, Chapter VI, §6, Theorem 2] there exist a nonnegative integer k and finite real measures $\mu_{j\beta}$ with compact support in Ω such that

$$u_j = \sum_{|\beta| \leq k} \partial^\beta \mu_{j\beta} \quad \text{in } \Omega_2.$$

Since $W^{k+2,2}(\Omega_2) \hookrightarrow \mathcal{C}^k(\overline{\Omega}_2)$ by the Sobolev imbedding theorem, we see that $u \in W^{-k-2,2}(\Omega_2, \mathbb{R}^3)$. Since $\Delta \mathbf{u} - (\omega \times x) \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} \in W^{t,2}(\Omega_2, \mathbb{R}^3)$ for all $t \in \mathbb{R}^1$, [77, Theorem 6.4] gives that $\mathbf{u} \in W^{t+2,2}(\Omega_1, \mathbb{R}^3)$ for all $t \in \mathbb{R}^1$. So, $\mathbf{u} \in \mathcal{C}^\infty(\Omega_1, \mathbb{R}^3)$ by the Sobolev imbedding theorem. □

Proposition 7.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$.*

- *If $\mathbf{f} \in W^{-1,2}(\Omega, \mathbb{R}^3)$, $\mathbf{g} \in W^{1/2,2}(\partial\Omega, \mathbb{R}^3)$, $h \in L^2(\Omega)$, $\mathbf{g}$ and h satisfy the compatibility condition*

$$\int_\Omega h \, \mathrm{d}\mathbf{x} + \int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma = 0, \tag{7.1}$$

  then there exists $(\mathbf{u}, p) \in W^{1,2}(\Omega, \mathbb{R}^3) \times L^2(\Omega)$ such that

$$-\Delta \mathbf{u} - [\omega \times \mathbf{x}] \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = h \quad \text{in } \Omega, \quad \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega. \tag{7.2}$$

  p is unique up to an additive constant, $\mathbf{u}$ is unique. Moreover,

$$\|\mathbf{u}\|_{W^{1,2}(\Omega)} \leq C \left(\|\mathbf{f}\|_{W^{-1,2}(\Omega)} + \|h\|_{L^2(\Omega)} + \|\mathbf{g}\|_{W^{1/2,2}(\partial\Omega)} \right)$$

  where C depends only on Ω.
- *If $k \in \mathbb{N}_0$, $\partial\Omega$ is of class $\mathcal{C}^{k+1}$, $1 < q < \infty$, $\mathbf{f} \in W^{k-1,q}(\Omega, \mathbb{R}^3)$, $\mathbf{g} \in W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3)$, $h \in W^{k,q}(\Omega)$ satisfy the compatibility condition(7.1), then there exists a solution $(\mathbf{u}, p) \in W^{k+1,q}(\Omega, \mathbb{R}^3) \times W^{k,q}(\Omega)$ of (7.2). Here p is unique up to an additive constant, $\mathbf{u}$ is unique and*

$$\|\mathbf{u}\|_{W^{k+1,q}(\Omega)} \leq C \left(\|\mathbf{f}\|_{W^{k-1,q}(\Omega)} + \|h\|_{W^{k,q}(\Omega)} + \|\mathbf{g}\|_{W^{k+1-1/q,q}(\partial\Omega)} \right)$$

  where C depends only on Ω, k and q.

Proof. The Divergence theorem gives that (7.1) is a necessary condition for the solvability of the problem (7.2).

Let now $[\mathbf{u}, p] \in W^{1,2}(\Omega, \mathbb{R}^3) \times L^2(\Omega)$ be a solution of the problem (7.2) with $\mathbf{f} \equiv 0$, $h \equiv 0$, $\mathbf{g} \equiv 0$. Since $\mathbf{u} \in \mathring{W}^{1,2}(\Omega, \mathbb{R}^3)$, there exists a sequence of functions $\mathbf{\Phi}_j \in \mathcal{C}^\infty(\Omega, \mathbb{R}^3)$ with compact support in Ω such that $\mathbf{\Phi}_j \to \mathbf{u}$ in $W^{1,2}(\Omega, \mathbb{R}^3)$.

Choose open sets $\Omega(j)$ with smooth boundary such that $\mathbf{\Phi}_j$ is supported in $\Omega(j)$ and $\overline{\Omega(j)} \subset \Omega$. According to Lemma 7.1 and Lemma 4.1

$$0 = \lim_{j\to\infty} \int_{\partial\Omega(j)} \mathbf{\Phi}_j \{T(\mathbf{u},p)\mathbf{n} + \frac{1}{2}[(\omega\times\mathbf{x})\cdot\mathbf{n}]\mathbf{u}\}\ \mathrm{d}\sigma = \lim_{j\to\infty}\int_{\Omega(j)} \{\mathbf{\Phi}_j\cdot[\omega\times\mathbf{u}$$

$$-\frac{1}{2}(\omega\times\mathbf{x})\cdot\nabla\mathbf{u}] + 2\hat\nabla\mathbf{u}\cdot\hat\nabla\mathbf{\Phi}_j - p\nabla\cdot\mathbf{\Phi}_j + \mathbf{u}\cdot\frac{1}{2}[(\omega\times\mathbf{x})\cdot\nabla\mathbf{\Phi}_j]\}\ \mathrm{d}\mathbf{x} = \int_\Omega 2|\hat\nabla\mathbf{u}|^2\ \mathrm{d}\mathbf{x}.$$

Since $\hat\nabla\mathbf{u}\equiv 0$, the function $\mathbf{u}$ is linear by [80, Lemma 3.1]. Since $u = 0$ on $\partial\Omega$, the maximum principle for the Laplace equation gives that $\mathbf{u}\equiv 0$. Therefore $\nabla p\equiv 0$ and p is constant.

Denote the space of $[\mathbf{f},h,\mathbf{g}] \in W^{l-1,q}(\Omega,\mathbb{R}^3)\times W^{l,q}(\Omega)\times W^{l+1-1/q,q}(\partial\Omega,\mathbb{R}^3)$ satisfying (7.1) by $Y_{l,q}$,

$$X_{l,q} = \{[\mathbf{u},p]\in W^{l+1,q}(\Omega,\mathbb{R}^3)\times W^{l,q}(\Omega); \int_\Omega p\ \mathrm{d}\mathbf{x} = 0\},$$

$$U_a(\mathbf{u},p) = [-\Delta\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p, \nabla\cdot\mathbf{u}, \mathbf{u}|_{\partial\Omega}].$$

Then $U_0 : X_{0,2}\to Y_{0,2}$ is an isomorphism by [35, Theorem 2.1]. Since $U_a - U_0 : X_{l,q}\to Y_{l,q}$ is a compact operator by [102, Lemma 18.4], the operator $U_a : X_{0,2}\to Y_{0,2}$ is Fredholm with index 0. If $[\mathbf{u},p]\in X_{0,2}$, $U_a[\mathbf{u},p] = 0$ then $\mathbf{u}\equiv 0$, p is constant. $\int_\Omega p\ \mathrm{d}\mathbf{x} = 0$ gives that $p\equiv 0$. So, $U_a : X_{0,2}\to Y_{0,2}$ is an isomorphism.

Suppose that $\partial\Omega$ is of class $\mathcal{C}^{k+1}$. Then $U_0 : X_{k,q}\to Y_{k,q}$ is an isomorphism by [34, Theorem 2.1] and [35, Theorem 2.1]. Since $U_a - U_0$ is compact, $U_a : X_{k,q}\to Y_{k,q}$ is a Fredholm operator with index 0. Since $U_a : X_{0,2}\to Y_{0,2}$ is an isomorphism, [92, Lemma 11.9.21] gives that $U_a : X_{k,q}\to Y_{k,q}$ is an isomorphism.

Since $U_a : X_{k,q}\to Y_{k,q}$ is an isomorphism, $U_a : W^{k+1,q}(\Omega,\mathbb{R}^3)\times W^{k,q}(\Omega)\to W^{k-1,q}(\Omega,\mathbb{R}^3)\times W^{k,q}(\Omega)\times W^{k+1-1/q,q}(\partial\Omega,\mathbb{R}^3)$ is a Fredholm operator with index 0. Since $U_a : W^{1,2}(\Omega,\mathbb{R}^3)\times L^2(\Omega)\to W^{-1,2}(\Omega,\mathbb{R}^3)\times L^2(\Omega)\times W^{1/2,2}(\partial\Omega,\mathbb{R}^3)$ is also a Fredholm operator with index 0, [92, Lemma 11.9.21] gives that the kernel of U_a on $W^{k+1,q}(\Omega,\mathbb{R}^3)\times W^{k,q}(\Omega)$ is the same like the kernel of U_a on $W^{1,2}(\Omega,\mathbb{R}^3)\times L^2(\Omega)$. So, if $(\mathbf{u},p)$ and $(\mathbf{w},\pi)$ are two solutions of (7.2) in $W^{k+1,q}(\Omega,\mathbb{R}^3)\times W^{k,q}(\Omega)$ then $\mathbf{u} = \mathbf{w}$ and $p-\pi$ is constant. □

Corollary 7.3. *Let $\Omega\subset\mathbb{R}^3$ be a bounded domain with boundary of class $\mathcal{C}^{k+1}$, $k\in\mathbb{N}$. Let $a\in\mathbb{R}^1$, $\omega\in(0,0,a)$, $1<q,r<\infty$.*

- *If $1\le s\le k+1$ then there exists a solution $(\mathbf{u},p)\in W^{s,q}(\Omega,\mathbb{R}^3)\times W^{s-1,q}(\Omega)$ of the Dirichlet problem (7.2) if and only if $\mathbf{f}\in W^{s-2,q}(\Omega,\mathbb{R}^3)$, $h\in W^{s-1,q}(\Omega)$, $\mathbf{g}\in W^{s-1/q,q}(\partial\Omega,\mathbb{R}^3)$ and (7.1) holds.*
- *If $1<s<k+1$ then there exists a solution $(\mathbf{u},p)\in B^{q,r}_s(\Omega,\mathbb{R}^3)\times B^{q,r}_{s-1}(\Omega)$ of the Dirichlet problem (7.2) if and only if $\mathbf{f}\in B^{q,r}_{s-2}(\Omega,\mathbb{R}^3)$, $h\in B^{q,r}_{s-1}(\Omega)$, $\mathbf{g}\in B^{q,r}_{s-1/q}(\partial\Omega,\mathbb{R}^3)$ and (7.1) holds.*

Proof. Define

$$U(\mathbf{u},p) = [-\Delta\mathbf{u} - (\omega\times\mathbf{x})\cdot\nabla\mathbf{u} + \omega\times\mathbf{u} + \nabla p, \nabla\cdot\mathbf{u}, \mathbf{u}|_{\partial\Omega}].$$

If $s>1$, $(\mathbf{u},p)\in B^{q,r}_s(\Omega,\mathbb{R}^3)\times B^{q,r}_{s-1}(\Omega)$ and $U(\mathbf{u},p) = 0$, then $\mathbf{u}\equiv 0$ and p is constant by Proposition 7.2, because $B^{q,r}_s(\Omega,\mathbb{R}^3)\times B^{q,r}_{s-1}(\Omega)\subset W^{1,q}(\Omega,\mathbb{R}^3)\times L^q(\Omega)$.

Denote the space of $[\mathbf{f}, h, \mathbf{g}] \in W^{t-2,q}(\Omega, \mathbb{R}^3) \times W^{t-1,q}(\Omega) \times W^{t-1/q,q}(\partial\Omega, \mathbb{R}^3)$ satisfying (7.1) by Y_t^q, the space of $[\mathbf{f}, h, \mathbf{g}] \in B_{t-2}^{q,r}(\Omega, \mathbb{R}^3) \times B_{t-1}^{q,r}(\Omega) \times B_{t-1/q}^{q,r}(\partial\Omega, \mathbb{R}^3)$ satisfying (7.1) by $Y_t^{q,r}$,

$$X_t^q = \{[\mathbf{u}, p] \in W^{t,q}(\Omega, \mathbb{R}^3) \times W^{t-1,q}(\Omega); \int_\Omega p \ \mathrm{d}\mathbf{x} = 0\},$$

$$X_t^{q,r} = \{[\mathbf{u}, p] \in B_t^{q,r}(\Omega, \mathbb{R}^3) \times B_{t-1}^{q,r}(\Omega); \int_\Omega p \ \mathrm{d}\mathbf{x} = 0\}.$$

If $t > 1$ then $U(B_t^{q,r}(\Omega, \mathbb{R}^3) \times B_{t-1}^{q,r}(\Omega)) \subset Y_t^{q,r}$ by the Divergence theorem. If we denote by $(\ , \ )_{\theta,r}$ the real interpolation, then

$$(X_1^q, X_{k+1}^q)_{\theta,r} = X_{t+1}^{q,r}, \quad (Y_1^q, Y_{k+1}^q)_{\theta,r} = Y_{t+1}^{q,r}$$

for $0 < \theta < 1$, $t = \theta k$ by [106, Theorem 2.13], [105, §3.3.6, Proposition] and [104, §1.17.1, Theorem 1]. Proposition 7.2 gives that $U : X_1^q \to Y_1^q$ and $U : X_{k+1}^q \to Y_{k+1}^q$ are isomorphisms. So, $U : X_s^{q,r} \to Y_s^{q,r}$ is an isomorphism for $1 < s < k+1$. (See [102, Lemma 22.3].) The rest is a consequence of the fact that $W^{t,q}(\Omega) = B_t^{p,p}(\Omega)$, $W^{t,q}(\partial\Omega) = B_t^{p,p}(\partial\Omega)$ for non-integer t. (See [10, Theorem 6.7].) □

8. Fundamental solution

Let $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$. Let $Z^a(\mathbf{x}, \mathbf{y}) = \{Z_{ij}^a(\mathbf{x}, \mathbf{y})\}_{i \le 3, j \le 4}$ and $Q^a(\mathbf{x}, \mathbf{y}) = (Q_1^a(\mathbf{x}, \mathbf{y}), \dots, Q_4^a(\mathbf{x}, \mathbf{y}))$ be matrix functions. Denote $Z_j^a = (Z_{1j}^a, Z_{2j}^a, Z_{3j}^a)$. We say that (Z^a, Q^a) is a fundamental solution of the Stokes system with Coriolis term (1.1) in $\mathbb{R}^3$ if for each $\mathbf{y} \in R^3$

$$-\Delta Z_j^a(\mathbf{x}, \mathbf{y}) - [\omega \times \mathbf{x}] \cdot \nabla Z_j^a(\mathbf{x}, \mathbf{y}) + \omega \times Z_j^a(\mathbf{x}, \mathbf{y}) + \nabla Q_j^a(\mathbf{x}, \mathbf{y}) = \mathbf{e}_j \delta_{\mathbf{y}}(\mathbf{x}), \tag{8.1a}$$

$$\sum_{k=1}^3 \nabla \cdot Z_j^a(\mathbf{x}, \mathbf{y}) = \delta_{j4} \delta_{\mathbf{y}}(\mathbf{x}). \tag{8.1b}$$

Here $\delta_{\mathbf{y}}$ is the unit mass concentrated at the point $\mathbf{y}$, $\mathbf{e}_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$, and $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ otherwise. It is known that there exists a fundamental solution of (1.1) such that for each $\mathbf{y} \in \mathbb{R}^3$ one has

$$Z_{ij}^a(\mathbf{x}, \mathbf{y}) \to 0, \quad Q_j^a(\mathbf{x}, \mathbf{y}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty.$$

(See [22].) This fundamental solution is unique by Lemma 5.1. For $a = 0$, i.e. for the Stokes system

$$Z_{ij}^0(\mathbf{x}, \mathbf{y}) = \delta_{ij} \frac{1}{8\pi|\mathbf{x} - \mathbf{y}|} + \frac{(x_i - y_i)(x_j - y_j)}{8\pi|\mathbf{x} - \mathbf{y}|^3}, \quad i, j \le 3,$$

$$Z_{i4}^0(\mathbf{x}, \mathbf{y}) = Q_i^0(\mathbf{x}, \mathbf{y}) = \frac{(x_i - y_i)}{4\pi|\mathbf{x} - \mathbf{y}|^3}, \quad i \le 3,$$

$$Q_4^0(\mathbf{x}, \mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}).$$

(See for example [108].) For $i \le 3$ clearly $Q_i^0(\mathbf{x}, \mathbf{y}) = -\partial_i h_\Delta(\mathbf{x} - \mathbf{y})$, where $h_\Delta(\mathbf{x}) = [4\pi|\mathbf{x}|]^{-1}$ is the fundamental solution of the Laplace equation, i.e. $-\Delta h_\Delta = \delta_0$ in the sense of distributions.

If $a \neq 0$ then $Q_i^a(\mathbf{x}, \mathbf{y}) = Q_i^0(\mathbf{x}, \mathbf{y})$ for $i \le 3$ by [22, Proposition 2.1]. Denote $\tilde{Z}^a(\mathbf{x}, \mathbf{y}) = \{Z_{ij}^a(\mathbf{x}, \mathbf{y})\}_{i,j \le 3}$. According to [22, Proposition 2.1]

$$\tilde{Z}^a(\mathbf{x}, \mathbf{y}) = \Gamma^0(\mathbf{x}, \mathbf{y}) + \Gamma^1(\mathbf{x}, \mathbf{y}),$$

where

$$\Gamma^0(\mathbf{x},\mathbf{y}) = \int_0^\infty O(at)^T(4\pi t)^{-3/2}\exp(-|O(at)x-y|^2/(4t))\ \mathrm{d}t,$$

$$\Gamma^1(\mathbf{x},\mathbf{y}) = \int_0^\infty (4\pi s)^{-3/2}\int_0^s \{\exp[-|O(at)\mathbf{x}-\mathbf{y}|^2/(4s)]\}$$

$$\cdot\left\{\frac{[\mathbf{x}-O(at)^T\mathbf{y}]\times[O(at)\mathbf{x}-\mathbf{y}]}{4s^2} - \frac{1}{2s}O(at)^T\right\}\ \mathrm{d}t\ \mathrm{d}s,$$

$$O(t) = \begin{pmatrix} \cos t, & -\sin t, & 0\\ \sin t, & \cos t, & 0\\ 0, & 0, & 1\end{pmatrix}.$$

Easy calculation yields

$$Z^a_{i4}(\mathbf{x},\mathbf{y}) = Q^0_i(\mathbf{x},\mathbf{y}) = \frac{(x_i-y_i)}{4\pi|\mathbf{x}-\mathbf{y}|^3},\quad i\le 3,$$

$$Q^a_4(\mathbf{x},\mathbf{y}) = \delta_{\mathbf{y}}(\mathbf{x}) - [\omega\times\mathbf{x}]\cdot\nabla h_\Delta(\mathbf{x}-\mathbf{y}).$$

It was shown in [22] that there exists a positive constant C such that

$$|\tilde Z^a(\mathbf{x},\mathbf{y})| \le \frac{C}{|\mathbf{x}|},\quad |\nabla_{\mathbf{x}}\tilde Z^a(\mathbf{x},\mathbf{y})| \le \frac{C}{|\mathbf{x}|^2}\quad \text{for } |\mathbf{x}|\ge 2|\mathbf{y}|. \tag{8.2}$$

Moreover,

$$[\tilde Z^a(\mathbf{x},\mathbf{y})]^T = \tilde Z^{-a}(\mathbf{y},\mathbf{x}). \tag{8.3}$$

Lemma 8.1. *Let $a\in\mathbb{R}^1$, and α, β be multiindices. Then there exists a constant C such that*

$$|\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\tilde Z^a(\mathbf{x},\mathbf{y})| \le C|\mathbf{x}|^{-1-|\alpha|-|\beta|}\quad \textit{for } |\mathbf{x}|\ge 2|\mathbf{y}|.$$

Proof. The proof is the same like the proof of (8.2) in [22]. If $|\mathbf{x}|\ge 2|\mathbf{y}|$ then $\exp[-|O(at)\mathbf{x}-\mathbf{y}|^2/(4s)] \le \exp[-|\mathbf{x}|^2/(16s)]$. From this together with the equality

$$\int_0^\infty s^{-m/2}\exp[-t|\mathbf{x}|^2/s]\ \mathrm{d}s = \frac{c^{-m/2+1}\gamma(m/2-1)}{|\mathbf{x}|^{m-2}},\quad m>2, c>0$$

(where $\gamma(\cdot)$ denotes the gamma function), it follows

$$|\partial^\alpha_{\mathbf{x}}\partial^\beta_{\mathbf{y}}\tilde\Gamma^1(\mathbf{x},\mathbf{y})| \le C|\mathbf{x}|^{-1-|\alpha|-|\beta|}.$$

Similarly for Γ^0. $\square$

Proposition 8.2. *Let $a\in\mathbb{R}^1$, $R\in(0,\infty)$. Then $Z^a_{ij}(\cdot,\mathbf{y})\in\mathcal{C}^\infty(\mathbb{R}^3\setminus\{\mathbf{y}\})$ and there exists a positive constant C such that*

$$|\tilde Z^a(\mathbf{x},\mathbf{y}) - \tilde Z^0(\mathbf{x},\mathbf{y})| \le C,\quad |\nabla_{\mathbf{x}}\tilde Z^a(\mathbf{x},\mathbf{y}) - \nabla_{\mathbf{x}}\tilde Z^0(\mathbf{x},\mathbf{y})| \le C|\mathbf{x}-\mathbf{y}|^{-1},$$

for $\mathbf{x},\mathbf{y}\in B(0;R)$ where $B(\mathbf{z};r) = \{\mathbf{x}\in\mathbb{R}^3; |\mathbf{x}-\mathbf{z}|<r\}$.

Proof. Lemma 7.1 gives that $Z^a_{ij}(\cdot,\mathbf{y})\in\mathcal{C}^\infty(\mathbb{R}^3\setminus\{\mathbf{y}\})$.

If $\mathbf{z}\in\mathbb{R}^3$ then there exists a matrix function $E^{\mathbf{z}}(\mathbf{x})$ of the type 3×3 such that $E^{\mathbf{z}}_{ij}\in\mathcal{C}^\infty(\mathbb{R}^3\setminus\{0\})$ and

$$-\Delta E^{\mathbf{z}}_{ij}(\mathbf{x}) - \mathbf{z}\cdot\nabla E^{\mathbf{z}}_{ij}(\mathbf{x}) + \partial_i Q^0_j(\mathbf{x},0) = \delta_{ij}\delta_0(\mathbf{x}),$$

$$\sum_{k=1}^{3} \partial_k E^{\mathbf{z}}_{kj}(\mathbf{x}) = 0,$$
$$E^{\mathbf{z}}_{kj}(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty.$$

(For the explicit prescription of $E^{\mathbf{z}}$ see [29] or [94].) According to [29] there exists a constant C_1 such that

$$|E^{\mathbf{z}}(\mathbf{x}) - \tilde{Z}^0(\mathbf{x},0)| \le C_1,\ |\nabla E^{\mathbf{z}}(\mathbf{x}) - \nabla \tilde{Z}^0(\mathbf{x},0)| \le C_1 |\mathbf{x}|^{-1} \tag{8.4}$$

for all $\mathbf{x}, \mathbf{z} \in B(0;10R)$.

Fix $\mathbf{y} \in B(0;R)$. Denote $E^{\mathbf{z}}_j = (E^{\mathbf{z}}_{1j}, E^{\mathbf{z}}_{2j}, E^{\mathbf{z}}_{3j})$. Clearly

$$\nabla_{\mathbf{x}} \cdot [Z^a_j(\mathbf{x},\mathbf{y}) - E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y})] = 0. \tag{8.5}$$

Since

$$\begin{aligned} 0 = &-\Delta_{\mathbf{x}} Z^a_j(\mathbf{x},\mathbf{y}) - [\omega\times\mathbf{x}]\cdot\nabla_{\mathbf{x}} Z^a_j(\mathbf{x},\mathbf{y}) + \omega\times Z^a_j(\mathbf{x},\mathbf{y}) + \nabla_{\mathbf{x}} Q^0_j(\mathbf{x},\mathbf{y}) \\ &-[-\Delta_{\mathbf{x}} E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y}) - [\omega\times\mathbf{y}]\cdot\nabla_{\mathbf{x}} E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y}) + \nabla_{\mathbf{x}} Q^0_j(\mathbf{x},\mathbf{y})] \end{aligned}$$

we have

$$\begin{aligned} &-\Delta_{\mathbf{x}}[Z^a_j(\mathbf{x},\mathbf{y}) - E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y})] - [\omega\times\mathbf{x}]\cdot\nabla_{\mathbf{x}}[Z^a_j(\mathbf{x},\mathbf{y}) \\ &\quad - E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y})] + \omega\times[Z^a_j(\mathbf{x},\mathbf{y}) - E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y})] \\ &= [\omega\times(\mathbf{y}-\mathbf{x})]\cdot\nabla_{\mathbf{x}} E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y}) - \omega\times E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{8.6}$$

Choose $\psi \in \mathcal{C}^\infty(\mathbb{R}^3)$ supported in $B(0;4R)$ such that $\psi = 1$ in $B(0;3R)$. Fix $j \in \{1,2,3\}$ and define

$$\mathbf{v}(\mathbf{x}) = \psi(\mathbf{x})[\mathcal{Z}^a_j(\mathbf{x},\mathbf{y}) - E^{\omega\times\mathbf{y}}_j(\mathbf{x}-\mathbf{y})],$$
$$\mathbf{f} = -\Delta\mathbf{v} - [\omega\times\mathbf{x}]\cdot\nabla\mathbf{v} + \omega\times\mathbf{v}, \quad g = \nabla\cdot\mathbf{v}.$$

Then $\mathbf{v}$, $\mathbf{f}$ and g are supported in $B(0;4R)$ and $g \in \mathcal{C}^\infty(\mathbb{R}^3)$. According to (8.5), (8.6), (8.2) and (8.4)

$$|g| + |\nabla g| \le C_2, \quad |\mathbf{f}(\mathbf{x})| + |\nabla^2 g(\mathbf{x})| \le C_2|\mathbf{x}-\mathbf{y}|^{-1}, \quad |\nabla\mathbf{f}(\mathbf{x})| \le C_2|\mathbf{x}-\mathbf{y}|^{-2} \tag{8.7}$$

where C_2 does not depend on $\mathbf{y}$. According to Lemma 6.1 there exists a solution $(\mathbf{u},p) \in W^{2,2}_{loc}(\mathbb{R}^3,\mathbb{R}^3)\times W^{1,2}_{loc}(\mathbb{R}^3)$ of (6.1) such that $|\nabla p| \in L^2(\mathbb{R}^3)$, $|\nabla^2\mathbf{u}| \in L^2(\mathbb{R}^3)$. Since p, v_j and u_j are tempered distributions and $(\mathbf{u}-\mathbf{v},p)$ is a solution of the homogeneous Stokes system with Coriolis term, Lemma 5.1 yields that $v_j - u_j$ are polynomials. Therefore $\mathbf{v} \in W^{2,2}(\mathbb{R}^3,\mathbb{R}^3)$.

Define $p \equiv 0$. Then $(\mathbf{v},p) \in W^{2,2}(B(0;4R);\mathbb{R}^3)\times W^{1,2}(B(0;4R))$ is a solution of

$$-\Delta\mathbf{v} - [\omega\times\mathbf{x}]\cdot\nabla\mathbf{v} + \omega\times\mathbf{v} + \nabla p = \mathbf{f},\ \nabla\cdot\mathbf{v} = g \text{ in } B(0;4R),$$
$$\mathbf{v} = 0 \text{ on } \partial B(0;4R).$$

Proposition 7.2 and (8.7) give

$$\|\mathbf{v}\|_{W^{2,2}(B(0;4R))} \le C_3, \tag{8.8}$$

where C_3 depends only on R. Sobolev's imbedding theorem forces $|\mathbf{v}| \le C_4$ on $B(0;4R)$ where C_4 depends only on R. This and (8.4) gives $|\tilde{Z}^a(\mathbf{x},\mathbf{y}) - \tilde{Z}^0(\mathbf{x},\mathbf{y})| \le C_5$ for $\mathbf{x} \in B(0;R)$, where C_5 does not depend on $\mathbf{y}$.

Fix $i,k \in \{1,2,3\}$ and define $\tilde{\mathbf{v}}(\mathbf{x}) = (x_k - y_k)\partial_i\mathbf{v}(\mathbf{x})$. Then $(\tilde{\mathbf{v}},p)$ is a solution of

$$-\Delta\tilde{\mathbf{v}} - [\omega\times\mathbf{x}]\cdot\nabla\tilde{\mathbf{v}} + \omega\times\tilde{\mathbf{v}} + \nabla p = \tilde{\mathbf{f}},\ \nabla\cdot\tilde{\mathbf{v}} = \tilde{g} \text{ in } B(0;4R)$$

$$\tilde{\mathbf{v}} = 0 \text{ on } \partial B(0; 4R)$$

in $W^{1,2}(B(0;4R);\mathbb{R}^3) \times L^2(B(0;4R))$. We now estimate $\tilde{g}$ and $\tilde{\mathbf{f}}$. Since

$$\tilde{g} = (x_k - y_k)\partial_i g + \partial_i v_k \in W^{1,2}(B(0;4R)),$$

(8.8) and (8.7) give

$$\|\tilde{g}\|_{W^{1,2}(B(0;4R))} \leq C_6, \tag{8.9}$$

where C_6 does not depend on $\mathbf{y}$. Since

$$\tilde{\mathbf{f}} = (x_k - y_k)\{\partial_i \mathbf{f} + [\partial_i(\omega \times \mathbf{x})] \cdot \nabla \mathbf{v}\} - (\omega \times \mathbf{x}) \cdot (\delta_{1k}, \delta_{2k}, \delta_{3k})\partial_i \mathbf{v} - 2\partial_k \partial_i \mathbf{v},$$

(8.8) and (8.7) give

$$\|\tilde{\mathbf{f}}\|_{L^2(B(0;4R);\mathbb{R}^3)} \leq C_7, \tag{8.10}$$

where C_7 does not depend on $\mathbf{y}$. Proposition 7.2, (8.10) and (8.9) give that $\tilde{\mathbf{v}} \in W^{2,2}(B(0;4R),\mathbb{R}^3)$ and

$$\|\tilde{\mathbf{v}}\|_{W^{2,2}(B(0;4R))} \leq C_8,$$

where C_8 depends only on R. Sobolev's imbedding theorem forces $|\tilde{\mathbf{v}}| \leq C_9$ on $B(0;4R)$ where C_9 depends only on R. This and (8.4) give that $|\mathbf{x}-\mathbf{y}||\nabla_{\mathbf{x}}(\tilde{Z}^a(\mathbf{x},\mathbf{y}) - \tilde{Z}^0(\mathbf{x},\mathbf{y}))| \leq C_{10}$, where C_{10} depends only on R. □

Corollary 8.3. *Let $a \in \mathbb{R}^1$. Denote $G = \{[\mathbf{x},\mathbf{y}] \in \mathbb{R}^3 \times \mathbb{R}^3; \mathbf{x} \neq \mathbf{y}\}$. Then $Z^a_{ij} \in \mathcal{C}^\infty(G)$.*

Proof. We can suppose that $i, j \leq 3$. We prove by the induction that $Z^a_{ij} \in W^{k,4/3}_{loc}(G)$ for each $k \in \mathbb{N}$. For $k = 1$ this is a consequence of Proposition 8.2 and the explicit formula for Z^0.

Let now $k \in \mathbb{N}$ and $Z^a_{rs} \in W^{k,4/3}_{loc}(G)$ for all r, s. According to (8.1a)

$$\Delta_{\mathbf{x}} \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) + \mathbf{b}(\mathbf{x}) \cdot \nabla_{\mathbf{x}} \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) + A\tilde{Z}^a(\mathbf{x},\mathbf{y}) = \phi(\mathbf{x},\mathbf{y}), \tag{8.11}$$

where A is a matrix of the type 3×3, $\mathbf{b} \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and $\phi \in \mathcal{C}^\infty(G)$. Since $\tilde{Z}^a(\mathbf{x},\mathbf{y}) = \tilde{Z}^{-a}(\mathbf{y},\mathbf{x})^T$ by (8.3), we have

$$\Delta_{\mathbf{y}} \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) + \tilde{\mathbf{b}}(\mathbf{y}) \cdot \nabla_{\mathbf{y}} \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) + \tilde{A}\tilde{Z}^a(\mathbf{x},\mathbf{y}) = \tilde{\phi}(\mathbf{x},\mathbf{y}), \tag{8.12}$$

where $\tilde{A}$ is a matrix of the type 3×3, $\tilde{\mathbf{b}} \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^3)$ and $\tilde{\phi} \in \mathcal{C}^\infty(G)$. Adding (8.11) and (8.12) we obtain

$$\Delta \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) = \mathbf{B}(\mathbf{x},\mathbf{y}) \cdot \nabla \tilde{Z}^a_{ij}(\mathbf{x},\mathbf{y}) - (A + \tilde{A})\tilde{Z}^a(\mathbf{x},\mathbf{y}) + \Phi(\mathbf{x},\mathbf{y}), \tag{8.13}$$

where $\mathbf{B} \in \mathcal{C}^\infty(\mathbb{R}^3,\mathbb{R}^6)$ and $\Phi \in \mathcal{C}^\infty(G)$. Since the right side of (8.13) is in $W^{k-1,4/3}_{loc}(G)$, [68, Chapter 2, Théorème 3.2] gives that $Z^a_{ij} = \tilde{Z}^a_{ij} \in W^{k+1,4/3}_{loc}(G)$.

Since $Z^a_{ij} \in W^{k,4/3}_{loc}(G)$ for each $k \in \mathbb{N}$, the Sobolev imbedding theorem gives that $Z^a_{ij} \in \mathcal{C}^\infty(G)$. □

9. Single layer potential

Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary. Let $1 < q < \infty$, $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$. We define the velocity part of the single layer potential with density $\mathbf{\Phi}$ by

$$Z^a_\Omega \mathbf{\Phi}(\mathbf{x}) = \int_{\partial\Omega} \tilde{Z}^a(\mathbf{x}, \mathbf{y})\mathbf{\Phi}(\mathbf{y}) \; \mathrm{d}\sigma(\mathbf{y}),$$

the pressure part of the single layer potential with density $\mathbf{\Phi}$ by

$$Q_\Omega \mathbf{\Phi}(\mathbf{x}) = \int_{\partial\Omega} \tilde{Q}(\mathbf{x}, \mathbf{y})\mathbf{\Phi}(\mathbf{y}) \; \mathrm{d}\sigma(\mathbf{y}),$$

where $\tilde{Q} = (Q_1, Q_2, Q_3)$.

If $\mathbf{x} \in \partial\Omega$ and there exists the outward unit normal $\mathbf{n}^\Omega(\mathbf{x})$ of Ω at $\mathbf{x}$ then we define

$$\tilde{K}^a_\Omega(\mathbf{x}, \mathbf{y}) = T_{\mathbf{x}}(\tilde{Z}^a(\mathbf{x}, \mathbf{y}), \tilde{Q}(\mathbf{x}, \mathbf{y}))\mathbf{n}^\Omega(\mathbf{x}) + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{x})]\tilde{Z}^a(\mathbf{x}, \mathbf{y})$$

for $\mathbf{y} \in \mathbb{R}^3 \setminus \{\mathbf{x}\}$. Define the following integral whenever it makes sense

$$\tilde{K}^a_\Omega \mathbf{\Phi}(\mathbf{x}) = \lim_{\epsilon \downarrow 0} \int_{\partial\Omega \setminus B(\mathbf{x};r)} \tilde{K}^a_\Omega(\mathbf{x}, \mathbf{y})\mathbf{\Phi}(\mathbf{y}) \; \mathrm{d}\sigma(\mathbf{y}).$$

Theorem 9.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$. Denote by $[\mathbf{u}(\mathbf{x})]_+$ the nontangential limit of $\mathbf{u}$ with respect to Ω at $\mathbf{x}$, and by $[\mathbf{u}(\mathbf{x})]_-$ the nontangential limit of $\mathbf{u}$ with respect to $\mathbb{R}^3 \setminus \Omega$ at $\mathbf{x}$.*

i) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then $(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi}) \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \partial\Omega, \mathbb{R}^4)$ is a solution of (1.1) in $\mathbb{R}^3 \setminus \partial\Omega$.

ii) There exists a constant C such that

$$M_\beta(Z^a_\Omega \mathbf{\Phi}) + M_\beta(\nabla Z^a_\Omega \mathbf{\Phi}) + M_\beta(Q_\Omega \mathbf{\Phi}) \le C\|\mathbf{\Phi}\|_{L^q(\partial\Omega)}$$

for all $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$.

iii) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then $Z^a_\Omega \mathbf{\Phi}(\mathbf{x})$ is finite for almost all $\mathbf{x} \in \partial\Omega$. Denote by $\mathcal{Z}^a_\Omega \mathbf{\Phi}$ the restriction of $Z^a_\Omega \mathbf{\Phi}$ onto $\partial\Omega$. Then $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a bounded linear operator and $\mathcal{Z}^a_\Omega - \mathcal{Z}^0_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a compact operator.

iv) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then $\mathcal{Z}^a_\Omega \mathbf{\Phi}(\mathbf{x}) = [Z^a_\Omega \mathbf{\Phi}(\mathbf{x})]_+ = [Z^a_\Omega \mathbf{\Phi}(\mathbf{x})]_-$ for almost all $\mathbf{x} \in \partial\Omega$.

v) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then there exists a finite non-tangential limit of $\nabla \mathcal{Z}^a_\Omega \mathbf{\Phi}$ at almost all $\mathbf{x} \in \partial\Omega$.

vi) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then there exists a finite non-tangential limit of $Q_\Omega \mathbf{\Phi}$ at almost all $\mathbf{x} \in \partial\Omega$.

vii) $\tilde{K}^a_\Omega$ is a bounded linear operator on $L^q(\partial\Omega; \mathbb{R}^3)$; $\tilde{K}^a_\Omega - \tilde{K}^0_\Omega$ is a compact operator on $L^q(\partial\Omega; \mathbb{R}^3)$. If $\partial\Omega$ is of class $\mathcal{C}^1$ then $\tilde{K}^a_\Omega$ is a compact operator on $L^q(\partial\Omega; \mathbb{R}^3)$.

viii) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ then

$$[T_{\mathbf{x}}(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})\mathbf{n}^\Omega(\mathbf{x}) + \frac{1}{2}(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega(\mathbf{x}) Z^a_\Omega \mathbf{\Phi}]_\pm = \pm\frac{1}{2}\mathbf{\Phi}(\mathbf{x}) + \tilde{K}^a_{\mathbf{\Omega}} \mathbf{\Phi}(\mathbf{x})$$

for almost all $\mathbf{x} \in \partial\Omega$.

ix) If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ *and* α *is a multiindex then*

$$|\partial^\alpha Z^a_\Omega \mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-1-|\alpha|}), \quad |\partial^\alpha Q^a_\Omega \mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-2-|\alpha|}) \quad \text{as } |\mathbf{x}| \to \infty.$$

Proof. $\tilde{Z}^a \in \mathcal{C}^\infty(\mathbb{R}^3 \times \mathbb{R}^3 \setminus \{[\mathbf{x},,\mathbf{x}]; \mathbf{x} \in \mathbb{R}^3\})$ by Corollary 8.3. So, $(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi}) \in \mathcal{C}^\infty(\mathbb{R}^3 \setminus \partial\Omega, \mathbb{R}^4)$. The definition of a fundamental solution gives that $(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})$ is a solution of (1.1) in $\mathbb{R}^3 \setminus \partial\Omega$.

ix) is a consequence of Lemma 8.1 and the explicit formula for Q^a.

If $a = 0$ then ii) holds true by [92, Proposition 4.2.3]; iii) is [92, Corollary 4.2.4]; iv) is in [92, Corollary 4.3.2]; v) follows from [92, Proposition 4.2.1]; viii) is in [92, Corollary 4.3.2]. The operator $\tilde{K}^0_\Omega$ is bounded on $L^q(\partial\Omega; \mathbb{R}^3)$ by [92, Corollary 4.2.4]. If $\partial\Omega$ is of class $\mathcal{C}^1$ then the adjoint operator $[\tilde{K}^0_\Omega]'$ is compact on $L^{q/(q-1)}(\partial\Omega; \mathbb{R}^3)$ by [73, p. 232], and thus $\tilde{K}^0_\Omega$ is compact on $L^q(\partial\Omega; \mathbb{R}^3)$.

vi) is a consequence of [81, Lemma 3.2], [37] and [38, Theorem 1].

Denote

$$R^a_0(\mathbf{x},\mathbf{y}) = Z^a(\mathbf{x},\mathbf{y}) - Z^0(\mathbf{x},\mathbf{y}), \quad R^a_j(\mathbf{x},\mathbf{y}) = \frac{\partial [Z^a(\mathbf{x},\mathbf{y}) - Z^0(\mathbf{x},\mathbf{y})]}{\partial x_j}, \ j = 1, \ldots, m,$$

$$R^a_k \mathbf{\Phi}(\mathbf{x}) = \int_{\partial\Omega} R^a_k(\mathbf{x},\mathbf{y}) \mathbf{\Phi}(\mathbf{y}) \, \mathrm{d}\sigma(\mathbf{y}) \qquad k = 0, \ldots, m.$$

The estimates of R^a_k in (8.3) and in Proposition 8.2, and [84, Proposition 1] give that there exists a constant C_1 such that for all $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ and $k = 0, \ldots, m$

$$\|M_\beta(R^a_k \mathbf{\Phi})\|_{L^q(\partial\Omega)} \le C_1 \|\mathbf{\Phi}\|_{L^q(\partial\Omega)},$$

$R^a_k \mathbf{\Phi}$ is finite almost everywhere on $\partial\Omega$, $R^a_k \mathbf{\Phi}(\mathbf{x})$ is the non-tangential limit of $R^a_k \mathbf{\Phi}$ at $\mathbf{x}$ for almost all $\mathbf{x} \in \partial\Omega$ and $\|R^a_k \mathbf{\Phi}\|_{L^q(\partial\Omega)} \le C_1 \|\mathbf{\Phi}\|_{L^q(\partial\Omega)}$. So, ii), iv), v), viii) hold. These reasoning and estimates of R^a_k in Proposition 8.2 give iii). (See [60, Chapter II, Theorem 8.1 and Theorem 8.6].)

The estimates of R^a_k in (8.3) and in Proposition 8.2 give that there exists a constant C_2 such that

$$|\tilde{K}^a_\Omega(\mathbf{x},\mathbf{y}) - \tilde{K}^0_\Omega(\mathbf{x},\mathbf{y})| \le C_2 |\mathbf{x} - \mathbf{y}|^{-1} \qquad \forall \mathbf{x}, \mathbf{y} \in \partial\Omega.$$

[91, Lemma 3.4] gives that $\tilde{K}^a_\Omega - \tilde{K}^0_\Omega$ is a compact operator on $L^q(\partial\Omega; \mathbb{R}^3)$. $\square$

Proposition 9.2. *Let* $\Omega \subset \mathbb{R}^3$ *be an open set with compact Lipschitz boundary,* $a \in \mathbb{R}^1$*,* $1 < q < \infty$*,* $0 < r < \infty$*. Denote* $G := \Omega \cap B(0; r)$*. Then*

$$Z^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to B^{q,\max(q,2)}_{1+1/q}(G; \mathbb{R}^3), \qquad Q_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to B^{q,\max(q,2)}_{1/q}(G)$$

are bounded linear operators.

Proof. We can suppose that Ω is bounded and $G = \Omega$.

For $a = 0$ see [90, Proposition 3.3] or [90, Theorem 3.1].

Let $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$. Since $M_\beta(Z^a_\Omega \mathbf{\Phi}) + M_\beta(\nabla Z^a_\Omega \mathbf{\Phi}) \in L^q(\partial\Omega)$ by Theorem 9.1, [112, Lemma 3.3] gives that $Z^a_\Omega \mathbf{\Phi} \in W^{1,q}(\Omega; \mathbb{R}^3)$. Define $\mathbf{f} = [\omega \times \mathbf{x}] \cdot \nabla Z^a_\Omega \mathbf{\Phi} - \omega \times Z^a_\Omega \mathbf{\Phi}$ in Ω, $\mathbf{f} = 0$ elsewhere, where $\omega = (0, 0, a)$. Then $\mathbf{f} \in L^q(\mathbb{R}^3, \mathbb{R}^3)$. Let $\Omega(k)$ be a sequence of sets from Lemma 14.1. Since $(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi}) \in \mathcal{C}^\infty(\Omega, \mathbb{R}^4)$ by Theorem 9.1 and

$$-\Delta Z^a_\Omega \mathbf{\Phi} + \nabla Q_\Omega \mathbf{\Phi} = \mathbf{f}, \quad \nabla \cdot Z^a_\Omega \mathbf{\Phi} = 0 \qquad \text{in } \Omega,$$

[65, Chapter 3, §2] gives

$$Z^a_\Omega \mathbf{\Phi}(\mathbf{x}) = \int_{\Omega(k)} Z^0(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\ \mathrm{d}\mathbf{y} + Z^0_{\Omega(k)} T(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})n + \mathcal{D}^0_{\Omega(k)} Z^a_\Omega \mathbf{\Phi},$$

where $\mathcal{D}^0_V \mathbf{g}$ is the velocity part of the Stokes double layer potential corresponding to a domain V and a velocity $\mathbf{g}$. Letting $k \to \infty$ we get by Theorem 9.1 and Lebesgue's lemma

$$Z^a_\Omega \mathbf{\Phi}(\mathbf{x}) = \int_{\mathbb{R}^3} Z^0(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\ \mathrm{d}\mathbf{y} + Z^0_\Omega [T(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})n] + \mathcal{D}^0_\Omega (\mathcal{Z}^a_\Omega \mathbf{\Phi}).$$

Since $T(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})n \in L^q(\partial\Omega, \mathbb{R}^3)$ by Theorem 9.1, one has $Z^0_\Omega[T(Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi})n] \in B^{q,\max(q,2)}_{1+1/q}(\Omega;\mathbb{R}^3)$.

[92, pp. 61–62] express $\partial_j[\mathcal{D}^0_\Omega \mathbf{g}]_i$ for $\mathbf{g} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$ as a linear combination of $\partial_l \int_{\partial\Omega} \tilde Z^0_{st}(\mathbf{x},\mathbf{y})\partial_{\tau_{bd}} g_k(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y})$ and $\partial_l \int_{\partial\Omega} h_\Delta(\mathbf{x}-\mathbf{y})\partial_{\tau_{bd}} g_k(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y})$, where h_Δ is a fundamental solution of the Laplace equation and the non-tangential derivative $\partial_{\tau_{bd}} g_k = n_b\partial_d g_k - n_d \partial_b g_k$. Since $\mathcal{Z}^a_\Omega \mathbf{\Phi} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$ by Theorem 9.1, [41, Theorem 2.2.13] and [86, Corollary 4.4] give $\mathcal{D}^0_\Omega(\mathcal{Z}^a_\Omega \mathbf{\Phi}) \in B^{q,\max(q,2)}_{1+1/q}(\Omega;\mathbb{R}^3)$.

$\int_{\mathbb{R}^3} Z^0(\mathbf{x},\mathbf{y})\mathbf{f}(\mathbf{y})\ \mathrm{d}\mathbf{y} \in W^{2,q}(\Omega, \mathbb{R}^3)$ by [29, §IV.2]. Since $W^{2,q}(\Omega) \hookrightarrow B^{q,\max(q,2)}_{1+1/q}(\Omega)$ by [104, §4.6.1, Theorem], we infer that $Z^a_\Omega \mathbf{\Phi} \in B^{q,\max(q,2)}_{1+1/q}(\Omega;\mathbb{R}^3)$. Since $\tilde Z^a \in \mathcal{C}^\infty(\{[\mathbf{x},\mathbf{y}]; \mathbf{x} \neq \mathbf{y}\})$, we infer that $Z^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to B^{q,\max(q,2)}_{1+1/q}(G;\mathbb{R}^3)$ is a closed operator. So, the Closed graph theorem gives that $Z^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to B^{q,\max(q,2)}_{1+1/q}(\Omega;\mathbb{R}^3)$ is bounded. □

10. Double layer potential

The goal of this section is to define a double layer potential for the Stokes system with Coriolis term such that the classical integral representation formula holds. Then we study properties of a double layer potential.

If $\mathbf{y} \in \partial\Omega$ and there exists the outward unit normal $\mathbf{n}^\Omega(\mathbf{y})$ of Ω at $\mathbf{y}$ then we define

$$\begin{aligned} K^{\Omega,a}_{i,j}(\mathbf{x},\mathbf{y}) &= \frac{1}{2}(\omega \times \mathbf{y})\cdot \mathbf{n}^\Omega(\mathbf{y}) Z^a_{i,j}(\mathbf{x},\mathbf{y}) - \frac{1}{2}\sum_{k=1}^3 n^\Omega_k(\mathbf{y})\Big(\frac{\partial Z^a_{i,j}(\mathbf{x},\mathbf{y})}{\partial y_k} \\ &\quad + \frac{\partial Z^a_{i,k}(\mathbf{x},\mathbf{y})}{\partial y_j}\Big) - Z^a_{i,4}(\mathbf{x},\mathbf{y}) n^\Omega_j(\mathbf{y}), \end{aligned}$$

$$\begin{aligned} \Pi^{\Omega,a}_{j}(\mathbf{x},\mathbf{y}) &= \frac{1}{2}(\omega \times \mathbf{y})\cdot \mathbf{n}^\Omega(\mathbf{y}) Q^a_{j}(\mathbf{x},\mathbf{y}) - \frac{1}{2}\sum_{k=1}^3 n^\Omega_k(\mathbf{y})\Big(\frac{\partial Q^a_{j}(\mathbf{x},\mathbf{y})}{\partial y_k} \\ &\quad + \frac{\partial Q^a_{k}(\mathbf{x},\mathbf{y})}{\partial y_j}\Big) - Q^a_{4}(\mathbf{x},\mathbf{y}) n^\Omega_j(\mathbf{y}), \end{aligned}$$

for $\omega = (0,0,a)$, $\mathbf{x} \in \mathbb{R}^3 \setminus \{\mathbf{y}\}$ and $1 \le i,j \le 3$. Define the velocity part of the double layer potential with density $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ by

$$\mathcal{D}^a_\Omega \mathbf{\Phi}(\mathbf{x}) = \int_{\partial\Omega} K^{\Omega,a}(\mathbf{x},\mathbf{y})\mathbf{\Phi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega$$

and the pressure part of the double layer potential with density $\mathbf{\Phi}$ by

$$\Pi_\Omega^a \mathbf{\Phi}(\mathbf{x}) = \int_{\partial\Omega} \Pi^{\Omega,a}(\mathbf{x},\mathbf{y})\mathbf{\Phi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \partial\Omega.$$

(Remark that for $a = 0$ this definition agrees with the usual definition of a double layer potential for the Stokes system - see [52], [92], [108].)

Define the following integral on $\partial\Omega$ whenever it makes sense

$$K_\Omega^a \mathbf{\Phi}(\mathbf{x}) = \lim_{\epsilon\downarrow 0} \int_{\partial\Omega\setminus B(\mathbf{x};\epsilon)} K^{\Omega,a}(\mathbf{x},\mathbf{y})\mathbf{\Phi}(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}).$$

Theorem 10.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$, $1 < q < \infty$.*

i) If $\mathbf{\Phi} \in L^q(\partial\Omega,\mathbb{R}^3)$ then $(\mathcal{D}_\Omega^a\mathbf{\Phi}, \Pi_\Omega^a\mathbf{\Phi}) \in \mathcal{C}^\infty(\mathbb{R}^3\setminus\partial\Omega,\mathbb{R}^4)$ is a solution of (1.1) in $\mathbb{R}^3\setminus\partial\Omega$.

ii) There exists a constant C such that

$$\|M_\beta(\mathcal{D}_\Omega^a)\mathbf{\Phi})\|_{L^q(\partial\Omega)} \le C_2\|\mathbf{\Phi}\|_{L^q(\partial\Omega)} \qquad \forall \mathbf{\Phi} \in L^q(\partial\Omega,\mathbb{R}^3).$$

iii) If $\mathbf{\Phi} \in L^q(\partial\Omega,\mathbb{R}^3)$ then

$$[\mathcal{D}_\Omega^a\mathbf{\Phi}(\mathbf{x})]_\pm = \pm\frac{1}{2}\mathbf{\Phi}(\mathbf{x}) + K_\Omega^a\mathbf{\Phi}(\mathbf{x}) \quad \textit{for a.a. } \mathbf{x} \in \partial\Omega.$$

iv) K_Ω^a is a bounded linear operator on $L^q(\partial\Omega;\mathbb{R}^3)$; $K_\Omega^a - K_\Omega^0$ is a compact operator on $L^q(\partial\Omega;\mathbb{R}^3)$.

v) If $\mathbf{\Phi} \in L^q(\partial\Omega,\mathbb{R}^3)$ and α is a multiindex then

$$|\partial^\alpha \mathcal{D}_\Omega^a\mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-1-|\alpha|}), |\partial^\alpha \Pi_\Omega^a\mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-2-|\alpha|}) \quad \textit{as } |\mathbf{x}| \to \infty.$$

Proof. $Z^a \in \mathcal{C}^\infty(\mathbb{R}^3\times\mathbb{R}^3\setminus\{[\mathbf{x},,\mathbf{x}];\mathbf{x}\in\mathbb{R}^3\})$ by Corollary 8.3. So, $(\mathcal{D}_\Omega^a\mathbf{\Phi}, \Pi_\Omega^a\mathbf{\Phi}) \in \mathcal{C}^\infty(\mathbb{R}^3\setminus\partial\Omega,\mathbb{R}^4)$. The definition of a fundamental solution gives that $(\mathcal{D}_\Omega^a\mathbf{\Phi}, \Pi_\Omega^a\mathbf{\Phi})$ is a solution of (1.1) in $\mathbb{R}^3\setminus\partial\Omega$.

K_Ω^0 is a bounded linear operator on $L^q(\partial\Omega;\mathbb{R}^3)$ by [73, Corollary 3.3]. According to (8.2) and Proposition 8.2 there exists a constant C_1 such that

$$|K^{a,\Omega}(\mathbf{x},\mathbf{y}) - K^{0,\Omega}(\mathbf{x},\mathbf{y})| \le C_1|\mathbf{x}-\mathbf{y}|^{-1} \quad \forall \mathbf{y}\in\partial\Omega, \forall \mathbf{x}\in\mathbb{R}^3. \tag{10.1}$$

So, $K_\Omega^a - K_\Omega^0$ is a compact operator on $L^q(\partial\Omega;\mathbb{R}^3)$ by [91, Lemma 3.4].

If $a = 0$ then iii) follows from [73, Proposition 3.2], and ii) is in [92, Proposition 4.2.3].

(10.1) and [84, Proposition 1] give that there exists a constant C_2 such that for all $\mathbf{\Phi} \in L^q(\partial\Omega,\mathbb{R}^3)$

$$\|M_\beta((\mathcal{D}_\Omega^a - \mathcal{D}_\Omega^0)\mathbf{\Phi})\|_{L^q(\partial\Omega)} \le C_2\|\mathbf{\Phi}\|_{L^q(\partial\Omega)},$$

and $(K_\Omega^a - K_\Omega^0)\mathbf{\Phi}(\mathbf{x})$ is the non-tangential limit of $(\mathcal{D}_\Omega^a - \mathcal{D}_\Omega^0)\mathbf{\Phi}$ at $\mathbf{x}$ for almost all $\mathbf{x}\in\partial\Omega$.

Lemma 8.1 gives $|\partial^\alpha\mathcal{D}_\Omega^a\mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-1-|\alpha|})$ as $|\mathbf{x}| \to \infty$.

Fix $\rho > 0$ such that $\partial\Omega \subset B(0;\rho)$. Easy calculation yields that there is a constant C_1 such that

$$|\partial^\alpha\{\omega\times\mathbf{x}\cdot[\nabla h_\Delta(\mathbf{x}-\mathbf{y}) - \nabla h_\Delta(\mathbf{x})]\}| \le C_1|\mathbf{x}|^{-2-|\alpha|}, \quad |\mathbf{x}| \ge \rho, \mathbf{y}\in\partial\Omega.$$

(Remember that $h_\Delta(\mathbf{x}-\mathbf{y}) = c|\mathbf{x}-\mathbf{y}|^{-1}$.) But $\omega\times\mathbf{x}\cdot\nabla h_\Delta(\mathbf{x}) = 0$. This and the explicit formula for Q^a gives that there exists a constant C_2 such that

$$|\partial_{\mathbf{x}}^\alpha\Pi^{\Omega,a}(\mathbf{x},\mathbf{y})| \le C_2|\mathbf{x}|^{-2-|\alpha|}, \quad |\mathbf{x}| \ge \rho, \mathbf{y}\in\partial\Omega.$$

So, $|\partial^\alpha \Pi^a_\Omega \mathbf{\Phi}(\mathbf{x})| = O(|\mathbf{x}|^{-2-|\alpha|})$ as $|\mathbf{x}| \to \infty$. □

11. Integral representation

Theorem 11.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$. Let $\mathbf{u} \in \mathcal{C}^\infty(\overline{\Omega}, \mathbb{R}^3)$, $p \in \mathcal{C}^\infty(\overline{\Omega})$,*

$$-\Delta \mathbf{u} - [\omega \times \mathbf{x}] \cdot \nabla \mathbf{u} + \omega \times \mathbf{u} + \nabla p = \mathbf{F}, \qquad \nabla \cdot \mathbf{u} = g \quad \text{in } \Omega,$$

$$T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u} = \mathbf{f} \qquad \text{on } \partial\Omega.$$

Then

$$\mathbf{u}(\mathbf{x}) = \mathcal{D}^a_\Omega \mathbf{u}(\mathbf{x}) + Z^a_\Omega \mathbf{f}(\mathbf{x}) + \int_\Omega Z^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y}, \tag{11.1}$$

$$p(\mathbf{x}) = \Pi^a_\Omega \mathbf{u}(\mathbf{x}) + Q_\Omega \mathbf{f}(\mathbf{x}) + \int_\Omega Q^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y}. \tag{11.2}$$

Proof. Suppose first that $\mathbf{u}$ and p have compact supports in Ω. Since Z^a, Q^a is a fundamental solution of (1.1) in $\mathbb{R}^3$, we obtain

$$\int_\Omega Z^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y} = \mathbf{u}(\mathbf{x}), \quad \int_\Omega Q^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y} = p(\mathbf{x}).$$

Let now $\mathbf{u} = 0$, $p = 0$ on a neighborhood of $\mathbf{x}$. Denote $\check{Z}^a_j = (Z^a_{j1}, Z^a_{j2}, Z^a_{j3})$. Since $Z^a_{i4} = Q^a_i$ for $i \le 3$ and Z^a, Q^a are of class $\mathcal{C}^\infty$ outside the diagonal by Corollary 8.3, Green's formula gives

$$\left[Z^a_\Omega \mathbf{f}(\mathbf{x}) + \int_\Omega Z^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y} \right]_i = \int_\Omega \{ -\mathbf{u}(\mathbf{y}) \cdot [\Delta_{\mathbf{y}} \check{Z}_i(\mathbf{x}, \mathbf{y})$$
$$+ (\omega \times \mathbf{y}) \cdot \nabla_{\mathbf{y}} \check{Z}^a_i(\mathbf{x}, \mathbf{y}) - \omega \times \check{Z}^a_i(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{y}} Q^a_i(\mathbf{x}, \mathbf{y})] - p(\mathbf{y}) \nabla_{\mathbf{y}} \cdot \check{Z}^a_i(\mathbf{x}, \mathbf{y}) \} \, \mathrm{d}\mathbf{y}$$
$$+ \int_{\partial\Omega} \{ -\frac{1}{2}[(\omega \times \mathbf{y}) \cdot \mathbf{n}^\Omega(\mathbf{y})][\check{Z}^a_i(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}(\mathbf{y})] + \mathbf{u}(\mathbf{y}) \cdot \hat{\nabla}_{\mathbf{y}} \check{Z}^a_i(\mathbf{x}, \mathbf{y}) \mathbf{n}^\Omega(\mathbf{y})$$
$$+ Q^a_i(\mathbf{x}, \mathbf{y}) \mathbf{n}^\Omega(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \} \, \mathrm{d}\sigma(\mathbf{y}).$$

Since $[\tilde{Z}^a(\mathbf{x}, \mathbf{y})]^T = \tilde{Z}^{-a}(\mathbf{y}, \mathbf{x})$ we have $\nabla_{\mathbf{y}} \cdot \check{Z}^a_i(\mathbf{x}, \mathbf{y}) = 0$, $-\Delta_{\mathbf{y}} \check{Z}^a_i(\mathbf{x}, \mathbf{y}) + [\omega \times \mathbf{y}] \cdot \nabla_{\mathbf{y}} \check{Z}^a_i(\mathbf{x}, \mathbf{y}) - \omega \times \check{Z}^a_i(\mathbf{x}, \mathbf{y}) - \nabla_{\mathbf{y}} Q^a_i(\mathbf{x}, \mathbf{y}) = 0$ for $\mathbf{y} \neq \mathbf{x}$. Thus

$$\left[Z^a_\Omega \mathbf{f}(\mathbf{x}) + \int_\Omega Z^a(\mathbf{x}, \mathbf{y}) \begin{pmatrix} \mathbf{F}(\mathbf{y}) \\ g(\mathbf{y}) \end{pmatrix} \, \mathrm{d}\mathbf{y} \right]_i = \int_{\partial\Omega} \{ -\frac{1}{2}[(\omega \times \mathbf{y}) \cdot \mathbf{n}^\Omega(\mathbf{y})][\check{Z}^a_i(\mathbf{x}, \mathbf{y}) \cdot \mathbf{u}(\mathbf{y})]$$
$$+ \mathbf{u}(\mathbf{y}) \cdot \hat{\nabla}_{\mathbf{y}} \check{Z}^a_i(\mathbf{x}, \mathbf{y}) \mathbf{n}^\Omega(\mathbf{y}) + Q^a_i(\mathbf{x}, \mathbf{y}) \mathbf{n}^\Omega(\mathbf{y}) \cdot \mathbf{u}(\mathbf{y}) \} \, \mathrm{d}\sigma(\mathbf{y}) = -\left[\mathcal{D}^a_\Omega \mathbf{u}(\mathbf{x}) \right]_i.$$

The relation (11.2) can be proved by the same way.

Let now $\mathbf{u}$, p be general. For $\mathbf{x} \in \Omega$ choose $\varphi \in \mathcal{C}^\infty(\Omega)$ with compact support in Ω such that $\varphi = 1$ on a neighborhood of $\mathbf{x}$. We have proved the relations (11.1), (11.2) for $\varphi \mathbf{u}$, φp and also for $(1-\varphi)\mathbf{u}$, $(1-\varphi)p$. Adding we get (11.1), (11.2) for $\mathbf{u}$, p. □

Corollary 11.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$, $1 < q < \infty$, $h \equiv 0$. If $(\mathbf{u}, p)$ is an L^q-solution of the Neumann problem (1.1), (1.3) then*

$$\mathbf{u}(\mathbf{x}) = \mathcal{D}^a_\Omega \mathbf{u}(\mathbf{x}) + Z^a_\Omega \mathbf{f}(\mathbf{x}), \qquad p(\mathbf{x}) = \Pi^a_\Omega \mathbf{u}(\mathbf{x}) + Q_\Omega \mathbf{f}(\mathbf{x}). \tag{11.3}$$

Proof. Let $\Omega(j)$ be sets from Lemma 14.1. Then Lemma 7.1 and Theorem 11.1 give

$$\mathbf{u}(\mathbf{x}) = \mathcal{D}^a_{\Omega(j)}\mathbf{u}(\mathbf{x}) + Z^a_{\Omega(j)}\mathbf{f}_j(\mathbf{x}), \qquad p(\mathbf{x}) = \Pi^a_{\Omega(j)}\mathbf{u}(\mathbf{x}) + Q_{\Omega(j)}\mathbf{f}_j(\mathbf{x})$$

with $\mathbf{f}_j := T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u}$ on $\partial\Omega(j)$. If $j \to \infty$ then Lebesque's lemma gives (11.3). $\square$

Proposition 11.3. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$, $h \equiv 0$. If $(\mathbf{u}, p)$ is an L^q-solution of the Neumann problem (1.1), (1.3) then there exist constants b, c such that*

$$\mathbf{u}(\mathbf{x}) = \mathcal{D}^a_\Omega\mathbf{u}(\mathbf{x}) + Z^a_\Omega\mathbf{f}(\mathbf{x}) + (0, 0, b), \qquad p(\mathbf{x}) = \Pi^a_\Omega\mathbf{u}(\mathbf{x}) + Q_\Omega\mathbf{f}(\mathbf{x}) + c. \tag{11.4}$$

If α is a multiindex then

$$|\partial^\alpha[\mathbf{u}(\mathbf{x}) - (0, 0, b)]| = O(|\mathbf{x}|^{-1-|\alpha|}), \quad |\partial^\alpha[p(\mathbf{x}) - c]| = O(|\mathbf{x}|^{-2-|\alpha|}), \quad |\mathbf{x}| \to \infty.$$

Proof. Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$ and put $G = \Omega \cap B(0; r)$. Define $\mathbf{f}(\mathbf{x}) := T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u}(\mathbf{x})$ on $\partial B(0; r)$. Then

$$\mathbf{u}(\mathbf{x}) = \mathcal{D}^a_G\mathbf{u}(\mathbf{x}) + Z^a_G\mathbf{f}(\mathbf{x}), \qquad p(\mathbf{x}) = \Pi^a_G\mathbf{u}(\mathbf{x}) + Q_G\mathbf{f}(\mathbf{x}) \quad \text{in } G \tag{11.5}$$

by Corollary 11.2. Define

$$\mathbf{v}(\mathbf{x}) = \left\{ \begin{array}{ll} \mathcal{D}^a_\Omega\mathbf{u}(\mathbf{x}) + Z^a_\Omega\mathbf{f}(\mathbf{x}) - \mathbf{u}(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \mathcal{D}^a_{B(0;r)}\mathbf{u}(\mathbf{x}) + Z^a_{B(0;r)}\mathbf{f}(\mathbf{x}), & \mathbf{x} \in B(0; r), \end{array} \right.$$

$$\rho(\mathbf{x}) = \left\{ \begin{array}{ll} \Pi^a_\Omega\mathbf{u}(\mathbf{x}) + Q^a_\Omega\mathbf{f}(\mathbf{x}) - p(\mathbf{x}), & \mathbf{x} \in \Omega, \\ \Pi^a_{B(0;r)}\mathbf{u}(\mathbf{x}) + Q^a_{B(0;r)}\mathbf{f}(\mathbf{x}), & \mathbf{x} \in B(0; r). \end{array} \right.$$

$\mathbf{v}$ and ρ are well defined by (11.5). Theorem 9.1 and Theorem 10.1 give that $(\mathbf{v}, \rho)$ is a bounded solution of (1.1) in $\mathbb{R}^3$. So, $(\mathbf{v}, \rho)$ is constant by Lemma 5.1, i.e. $\rho = -c$, $\mathbf{v} = (b_1, b_2, -b)$. Since $(\mathbf{v}, \rho)$ is a solution of (1.1), we obtain $\omega \times (b_1, b_2, -b) = 0$. This gives $b_2 = 0 = b_1$.

If α is a multiindex then $|\partial^\alpha[\mathbf{u}(\mathbf{x}) - (0, 0, b)]| = O(|\mathbf{x}|^{-1-|\alpha|})$, $|\partial^\alpha[p(\mathbf{x}) - c]| = O(|\mathbf{x}|^{-2-|\alpha|})$ as $|\mathbf{x}| \to \infty$ by Theorem 9.1 and Theorem 10.1. $\square$

Corollary 11.4. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $1 < q < \infty$. If $\mathbf{\Phi} \in L^q(\partial\Omega)$ then $K^a_\Omega\mathcal{Z}^a_\Omega\mathbf{\Phi} = -\mathcal{Z}^a_\Omega\tilde{K}^a_\Omega\mathbf{\Phi}$.*

Proof. Put $\mathbf{u} = Z^a_\Omega\mathbf{\Phi}$, $p = Q^a_\Omega\mathbf{\Phi}$ in Ω. According to Theorem 9.1, Corollary 11.2 and Proposition 11.3

$$Z^a_\Omega\mathbf{\Phi} = \mathbf{u} = \mathcal{D}^a_\Omega\mathcal{Z}^a_\Omega\mathbf{\Phi} + Z^a_\Omega\left(\frac{1}{2}I + \tilde{K}^a_\Omega\right)\mathbf{\Phi}.$$

Coming to $\partial\Omega$ we obtain by Theorem 9.1 and Theorem 10.1

$$\mathcal{Z}^a_\Omega\mathbf{\Phi} = \left(\frac{1}{2}I + K^a_\Omega\right)\mathcal{Z}^a_\Omega\mathbf{\Phi} + \mathcal{Z}^a_\Omega\left(\frac{1}{2}I + \tilde{K}^a_\Omega\right)\mathbf{\Phi}.$$

This gives $K^a_\Omega\mathcal{Z}^a_\Omega\mathbf{\Phi} = -\mathcal{Z}^a_\Omega\tilde{K}^a_\Omega\mathbf{\Phi}$. $\square$

Corollary 11.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary, $a \in \mathbb{R}^1$. Then $Z^a_\Omega\mathbf{n}^\Omega \equiv 0$, and $Q_\Omega\mathbf{n}^\Omega = -1$ in Ω, $Q_\Omega\mathbf{n}^\Omega = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$.*

Proof. $Q_\Omega \mathbf{n}^\Omega = -1$ in Ω, $Q_\Omega \mathbf{n}^\Omega = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$ by [80] and [82, Corollary 6.2].

Define $\mathbf{u} := 0$, $p = 1$. Then $T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u}] = -\mathbf{n}$. Corollary 11.3 gives that $Z^a_\Omega \mathbf{n} = 0$ in Ω. Proposition 11.3 used for $\mathbb{R}^3 \setminus \overline{\Omega}$ and the fact that $Z^a_\Omega \mathbf{n}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ (Theorem 9.1) give that $Z^a_\Omega \mathbf{n} = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega}$. According to Theorem 9.1 we obtain $Z^a_\Omega \mathbf{n} = 0$ on $\partial\Omega$. □

12. L^q-regular Dirichlet problem

To study the uniqueness of a solution of the problem we need the following two propositions:

Proposition 12.1. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$, $q' = q/(q-1)$. Let $h \in L^\infty(\partial\Omega)$, $\mathbf{f} \in L^p(\partial\Omega, \mathbb{R}^3)$, $(\mathbf{u}, p)$ be an L^q-solution of the Robin problem (1.1), (1.3) such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If $M_\beta(\mathbf{u}) \in L^{q'}(\partial\Omega)$ then the equality (4.1) holds.*

Proof. If $\partial\Omega \subset B(0; r)$ denote $\Omega(r) = \Omega \cap B(0; r)$. Corollary 4.2 gives

$$\int_{\partial\Omega(r)} \mathbf{u} \cdot \left\{ T(\mathbf{u}, p)\mathbf{n} + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}]\mathbf{u} \right\} \, \mathrm{d}\sigma = 2 \int_{\Omega(r)} |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{x}.$$

$|\mathbf{u}(\mathbf{x}) \cdot T(\mathbf{u}, p)\mathbf{n}(\mathbf{x})| = O(|\mathbf{x}|^{-3})$ as $|\mathbf{x}| \to \infty$ by Proposition 11.3. Since $(\omega \times \mathbf{x}) \cdot \mathbf{n} = 0$ on $\partial B(0; r)$ we obtain (4.1) letting $r \to \infty$. □

Proposition 12.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q \le 2$. Then $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0.*

Proof. For $a = 0$ see [92, Theorem 9.1.4]. Since $\mathcal{Z}^a_\Omega - \mathcal{Z}^0_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a compact operator by Theorem 9.1, we obtain the proposition. □

Proposition 12.3. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$. Let $(\mathbf{u}, p)$ be a regular L^q-solution of the Dirichlet problem (1.1), (1.2) with $\mathbf{g} \equiv 0$. If Ω is unbounded suppose moreover that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then $\mathbf{u} \equiv 0$ and p is constant. If Ω is unbounded then $p \equiv 0$.*

Proof. If Ω is unbounded then $\nabla \mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ by Proposition 11.3. Thus, we can suppose that $q \le 2$. Define $\mathbf{f} := T(\mathbf{u}, p)\mathbf{n}^\Omega + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega]\mathbf{u}$ on $\partial\Omega$. Then $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^3)$. Since $\mathbf{u} = 0$ on $\partial\Omega$, Corollary 11.2 and Proposition 11.3 give $\mathbf{u} = Z^a_\Omega \mathbf{f}$, $p = Q_\Omega \mathbf{f}$. So, $\mathcal{Z}^a_\Omega \mathbf{f} \equiv 0$ by Theorem 9.1. Proposition 12.2 gives that $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ and $\mathcal{Z}^a_\Omega : L^2(\partial\Omega, \mathbb{R}^3) \to W^{1,2}(\partial\Omega, \mathbb{R}^3)$ are Fredholm operators with index 0. Thus $\mathbf{f} \in L^2(\partial\Omega, \mathbb{R}^3)$ by [92, Lemma 11.9.21]. Theorem 9.1 gives that $(\mathbf{u}, p)$ is a regular L^2-solution of the Dirichlet problem (1.1), (1.2). According to Corollary 4.2 and Proposition 12.1

$$0 = \int_{\partial\Omega} \mathbf{f}\mathbf{u} \, \mathrm{d}\sigma = 2 \int_\Omega |\hat{\nabla}\mathbf{u}| \, \mathrm{d}\mathbf{x}.$$

Since $\hat{\nabla}\mathbf{u} \equiv 0$, $\mathbf{u}$ is linear by [80, Lemma 3.1]. Since $\mathbf{u} = 0$ on $\partial\Omega$, $\mathbf{u} \equiv 0$ by the maximum principle for the Laplace equation. Thus $\nabla p = \Delta \mathbf{u} + [\omega \times \mathbf{x}] \cdot \nabla \mathbf{u} - \omega \times \mathbf{u} \equiv 0$. Since Ω is connected, p is constant. If Ω is unbounded then $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ forces $p \equiv 0$. □

The following lemma states a necessary condition for the sovability of the Dirichlet problem on bounded domains.

Lemma 12.4. *Let $\Omega \subset \mathbb{R}^m$ be a bounded open set with Lipschitz boundary and $1 \le q \le \infty$. If $\mathbf{u} \in \mathcal{C}^1(\Omega, \mathbb{R}^m)$, $M_\beta(\mathbf{u}) \in L^q(\partial\Omega$ and there exists a non-tangential limit of $\mathbf{u}$ at almost all points of $\partial\Omega$ then*

$$\int_{\partial\Omega} \mathbf{n}^\Omega \cdot \mathbf{u} \, \mathrm{d}\sigma = 0. \tag{12.1}$$

Proof. Let $\Omega(k)$ be a sequence of sets from Lemma 14.1. The Divergence theorem gives

$$\int_{\partial\Omega(k)} \mathbf{n} \cdot \mathbf{u} \, \mathrm{d}\sigma = 0.$$

Letting $k \to \infty$ we obtain (12.1) by Lebesgue's lemma. □

Corollary 12.5. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$. If $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ and S is a component of $\partial\Omega$ then*

$$\int_S \mathbf{n}^\Omega \cdot \mathcal{Z}^a_\Omega \mathbf{\Phi} \, \mathrm{d}\sigma = 0. \tag{12.2}$$

Proof. Let G be a bounded open set such that $\partial G = S$. The corollary is a consequence of Theorem 9.1 and Lemma12.4 used for G. □

We shall look for a particular solution of the Dirichlet problem (1.1), (1.2) in a special form. Denote by $G(1), \dots, G(k)$ bounded components of $\mathbb{R}^3 \setminus \overline{\Omega}$. If Ω is bounded denote by $G(0)$ the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}$. If Ω is bounded and $\mathbb{R}^3 \setminus \overline{\Omega} = G(0)$ we look for a solution in the form of a single layer potential

$$(\mathbf{u}, p) = (Z^a_\Omega \mathbf{\Phi}, Q_\Omega \mathbf{\Phi}) \tag{12.3}$$

with $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$. If Ω has holes then a solution of the Dirichlet problem cannot be expressed in this form. (Compare Corollary 12.5.) So, we must modify this form. Choose $\mathbf{x}^j \in G(j)$, $j = 1, \dots, k$, and $r \in (0, \infty)$ such that $\overline{B(\mathbf{x}^j, r)} \subset G_j$. Denote $B(j) = B(\mathbf{x}^j; r)$. We shall look for a solution of the Dirichlet problem (1.1), (1.2) in the form

$$\mathbf{u} = Z^a_\Omega \mathbf{\Phi} + \sum_{j=1}^k \left[\int_{\partial G(j)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, \mathrm{d}\sigma \right] \mathcal{D}^a_{B(j)} \mathbf{n}^{B(j)}, \tag{12.4}$$

$$p = Q_\Omega \mathbf{\Phi} + \sum_{j=1}^k \left[\int_{\partial G(j)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, \mathrm{d}\sigma \right] \Pi^a_{B(j)} \mathbf{n}^{B(j)} \tag{12.5}$$

with $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$. Then $(\mathbf{u}, p)$ given by (12.4), (12.5) is a regular L^q-solution of the Dirichlet problem (1.1), (1.2) if and only if $\mathcal{Z}_{\mathrm{mod}} \mathbf{\Phi} = \mathbf{g}$, where

$$\mathcal{Z}_{\mathrm{mod}} \mathbf{\Phi} := \mathcal{Z}^a_\Omega \mathbf{\Phi} + \sum_{j=1}^k \left[\int_{\partial G(j)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, \mathrm{d}\sigma \right] \mathcal{D}^a_{B(j)} \mathbf{n}^{B(j)}. \tag{12.6}$$

(If Ω is a bounded domain with connected boundary then $\mathcal{Z}_{\mathrm{mod}} = \mathcal{Z}^a_\Omega$.)

Lemma 12.6. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$, $1 < q < \infty$. Let $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$, $\mathcal{Z}_{\rm mod}\mathbf{\Phi} = 0$ on $\partial\Omega$. If Ω is unbounded then $\mathbf{\Phi} \equiv 0$. If Ω is bounded then there exists a constant c such that $\mathbf{\Phi} = c\mathbf{n}^\Omega \chi_{\partial G(0)}$.*

Proof. First we show that $\mathcal{Z}_{\rm mod}\mathbf{\Phi} = \mathcal{Z}^a_\Omega\mathbf{\Phi}$. Fix $i \in \{1,\dots,k\}$. If $j \in \{1,\dots,k\}$, $j \neq i$ then $\mathcal{D}^a_{B(j)}\mathbf{n}^{B(j)} \in \mathcal{C}^\infty(\overline{G(i)}, \mathbb{R}^3)$ is a solution of (1.1) in $G(i)$ by Theorem10.1. Lemma 12.4 gives

$$\int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathcal{D}^a_{B(j)}\mathbf{n}^{B(j)} \, {\rm d}\sigma = 0.$$

According to Corollary 12.5

$$0 = \int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathcal{Z}_{\rm mod}\mathbf{\Phi} \, {\rm d}\sigma = \left[\int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, {\rm d}\sigma\right] \int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathcal{D}^a_{B(i)}\mathbf{n}^{B(i)} \, {\rm d}\sigma. \tag{12.7}$$

Using Lemma 12.4 on $G(i)\setminus\overline{B(i)}$ and $\mathcal{D}^a_{B(i)}\mathbf{n}^{B(i)}$ we obtain by virtue of Theorem10.1

$$0 = \int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathcal{D}^a_{B(i)}\mathbf{n}^{B(i)} \, {\rm d}\sigma + \int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[-\frac{1}{2}\mathbf{n}^{B(i)} + K^a_\Omega \mathbf{n}^{B(i)}\right] \, {\rm d}\sigma.$$

This and (12.7) gives

$$\left[\int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, {\rm d}\sigma\right] \int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[-\frac{1}{2}\mathbf{n}^{B(i)} + K^a_\Omega \mathbf{n}^{B(i)}\right] \, {\rm d}\sigma = 0. \tag{12.8}$$

Using Lemma 12.4 on $B(i)$ and $\mathcal{D}^a_{B(i)}\mathbf{n}^{B(i)}$ we obtain by virtue of Theorem10.1

$$0 = \int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[\frac{1}{2}\mathbf{n}^{B(i)} + K^a_\Omega \mathbf{n}^{B(i)}\right] \, {\rm d}\sigma =$$

$$\int_{\partial B(i)} \mathbf{n}^{B(i)} \cdot \left[-\frac{1}{2}\mathbf{n}^{B(i)} + K^a_\Omega \mathbf{n}^{B(i)}\right] \, {\rm d}\sigma + \int_{\partial B(i)} 1 \, {\rm d}\sigma.$$

Now we get from (12.8) that

$$\int_{\partial G(i)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, {\rm d}\sigma = 0. \tag{12.9}$$

Thus $\mathcal{Z}_{\rm mod}\mathbf{\Phi} = \mathcal{Z}^a_\Omega\mathbf{\Phi}$.

$(Z^a_\Omega\mathbf{\Phi}, Q_\Omega\mathbf{\Phi})$ is an L^q-regular solution of the Dirichlet problem (1.1), (1.2) with $\mathbf{g} \equiv 0$ in Ω and in $\mathbb{R}^3 \setminus \overline{\Omega}$. Moreover, $|Z^a_\Omega\mathbf{\Phi}(\mathbf{x})| + |Q_\Omega\mathbf{\Phi}(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$. (See Theorem 9.1.) Proposition 12.3 gives that $Z^a_\Omega\mathbf{\Phi} \equiv 0$ and there exist constants d, $d(j)$ such that $Q_\Omega\mathbf{\Phi} = d$ in Ω, $Q_\Omega\mathbf{\Phi} = d(j)$ in $G(j)$. On $\partial G(j)$

$$\mathbf{\Phi} = \left[\frac{1}{2}\mathbf{\Phi} + \tilde{K}^a_\Omega\mathbf{\Phi}\right] - \left[-\frac{1}{2}\mathbf{\Phi} + \tilde{K}^a_\Omega\mathbf{\Phi}\right] = [T(Z^a_\Omega\mathbf{\Phi}, Q_\Omega\mathbf{\Phi})\mathbf{n}^\Omega + \frac{1}{2}(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega Z^a_\Omega\mathbf{\Phi}]_+$$

$$-[T(Z^a_\Omega\mathbf{\Phi}, Q_\Omega\mathbf{\Phi})\mathbf{n}^\Omega + \frac{1}{2}(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega Z^a_\Omega\mathbf{\Phi}]_- = [d(j) - d]\mathbf{n}^\Omega.$$

by Theorem 9.1. (12.9) gives that $\mathbf{\Phi} = 0$ on $\partial G(j)$ for $j = 1,\dots,k$. So, if Ω is unbounded then $\mathbf{\Phi} \equiv 0$. If Ω is bounded then $\mathbf{\Phi} = [d(0) - d]\mathbf{n}^\Omega \chi_{\partial G(0)}$. $\square$

Theorem 12.7. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$, $1 < q \le 2$, $\mathbf{g} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$. Then there exists an L^q-regular solution $(\mathbf{u}, p)$ of the Dirichlet problem (1.1), (1.2) if and only if*

$$\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{n}^\Omega \, \mathrm{d}\sigma = 0. \tag{12.10}$$

A velocity $\mathbf{u}$ is unique, a pressure p is unique up to an additive constant. A fixed solution $(\mathbf{u}, p)$ is given by the modified single layer potential corresponding to some $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$ (the formula (12.3) for $\partial\Omega$ connected, and the formulas (12.4), (12.5) for a domain with holes.) Moreover, $\mathbf{u} \in B^{q,2}_{1+1/q}(\Omega, \mathbb{R}^3)$, $p \in B^{q,2}_{1/q}(\Omega)$ and

$$M_\alpha(\mathbf{u}) + M_\alpha(\nabla \mathbf{u}) + \|\mathbf{u}\|_{B^{q,2}_{1+1/q}(\Omega,\mathbb{R}^3)} \le C \|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}, \tag{12.11}$$

$$\|M_\alpha(p)\|_{L^q(\partial\Omega)} + \|p\|_{B^{q,2}_{1/q}(\Omega)} \le C \left(\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)} + \left| \int_\Omega p \, \mathrm{d}\mathbf{x} \right| \right). \tag{12.12}$$

Proof. (12.10) is a necessary condition for the existence of an L^q-regular solution of the Dirichlet problem (1.1), (1.2) by Lemma 12.4.

$\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0 by Proposition 12.2. Since $\mathcal{Z}^a_\Omega - \mathcal{Z}_{\mathrm{mod}} : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is compact, $\mathcal{Z}_{\mathrm{mod}} : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0. The kernel of $\mathcal{Z}_{\mathrm{mod}}$ is equal to $X := \{c\chi_{\partial G(0)}\mathbf{n}^\Omega; c \in \mathbb{R}^1\}$ by Lemma 12.6 and Corollary 11.5. Denote by Y the space of $\mathbf{g} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$ satisfying (12.10). If $\mathbf{\Phi} \in L^q(\Omega, \mathbb{R}^3)$ then $(\mathbf{u}, p)$ given by the formula (12.3) for $\partial\Omega$ connected and by the formulas (12.4), (12.5) for a domain with holes is a regular L^q-solution of the Dirichlet problem (1.1), (1.2) with $\mathbf{g} = \mathcal{Z}_{\mathrm{mod}}\mathbf{\Phi}$. Thus the range of $\mathcal{Z}_{\mathrm{mod}}$ is a subset of Y. Since $\mathcal{Z}_{\mathrm{mod}}$ is a Fredholm operator with index 0 its range has co-dimension 1. Hence $\mathcal{Z}_{\mathrm{mod}}(L^q(\partial\Omega, \mathbb{R}^3)) = Y$.

Let now $\mathbf{g} \in Y$. We have proved that there exists an L^q-solution $(\mathbf{u}, p)$ of the Dirichlet problem (1.1), (1.2) in the form of a modified single layer potential with a density $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$. If c is a constant then $(\mathbf{u}, p + c)$ is an L^q-solution of the problem, too. Let now $(\mathbf{v}, \pi)$ is another L^q-solution of the problem. According to Proposition12.3 there exists a constant c such that $\mathbf{v} \equiv \mathbf{u}$ and $\pi = p - c$. Corollary 11.5 gives that $(\mathbf{v}, \pi)$ is a modified single layer potential with a density $\mathbf{\Phi} + c\mathbf{n}^\Omega \in L^q(\partial\Omega, \mathbb{R}^3)$. Proposition 9.2 and Theorem 10.1 give $\mathbf{v} \in B^{q,2}_{1+1/q}(\Omega, \mathbb{R}^3)$, $\pi \in B^{q,2}_{1/q}(\Omega)$.

Define

$$W\mathbf{\Phi} = \left[\mathcal{Z}_{\mathrm{mod}}\mathbf{\Phi}, \int_\Omega \left(Q_\Omega \mathbf{\Phi} + \sum_{j=1}^k \Pi^a_{B(j)} \mathbf{n}^{B(j)} \int_{\partial G(j)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, \mathrm{d}\sigma \right) \mathrm{d}\mathbf{x} \right].$$

Then $W : L^q(\partial\Omega, \mathbb{R}^3) \to Y \times \mathbb{R}^1$ is an isomorphism. So, the inequalities (12.11), (12.12) are consequence of Proposition 9.2 and Corollary 8.3. $\square$

Theorem 12.8. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0,0,a)$, $1 < q \le 2$, $\mathbf{g} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$.*

- *$\mathcal{Z}_{\mathrm{mod}} : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is an isomorphism.*
- *Put $\mathbf{\Phi} = \mathcal{Z}^{-1}_{\mathrm{mod}}\mathbf{g}$. Let $\mathbf{u}$, p be given by the formulas (12.4), (12.5). Then $(\mathbf{u}, p)$ is a unique regular L^q-solution of the Dirichlet problem (1.1), (1.2)*

such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ *as* $|\mathbf{x}| \to \infty$. *Moreover,* $u_j \in B^{q,2}_{1+1/q}(\Omega, \rho_{-3})$, $p \in B^{q,2}_{1/q}(\Omega, \rho_{-3})$ *and*

$$M_\alpha(\mathbf{u}) + M_\alpha(\nabla \mathbf{u}) + \sum_{j=1}^{3} \|u_j\|_{B^{q,2}_{1+1/q}(\Omega,\rho_{-3})} \le C\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}, \tag{12.13}$$

$$\|M_\alpha(p)\|_{L^q(\partial\Omega)} + \|p\|_{B^{q,2}_{1/q}(\Omega,\rho_{-3})} \le C\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}. \tag{12.14}$$

- *If* $(\mathbf{u}, p)$ *is a regular* L^q*-solution of the Dirichlet problem* (1.1)*,* (1.2) *then there exist constants* b *and* c *such that* $\mathbf{u}(\mathbf{x}) \to (0, 0, b)$*,* $p(\mathbf{x}) \to c$ *as* $|\mathbf{x}| \to \infty$*.*
- *Let constants* b *and* c *be given. Then there exists a unique regular* L^q*-solution* $(\mathbf{u}, p)$ *of the Dirichlet problem* (1.1)*,* (1.2) *such that* $\mathbf{u}(\mathbf{x}) \to (0, 0, b)$*,* $p(\mathbf{x}) \to c$ *as* $|\mathbf{x}| \to \infty$*. Moreover,*

$$M_\alpha(\mathbf{u}) + M_\alpha(\nabla \mathbf{u}) \le C\left(\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)} + |b|\right),$$

$$\|M_\alpha(p)\|_{L^q(\partial\Omega)} \le C\left(\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)} + |b| + |c|\right).$$

Proof. $\mathcal{Z}_{\rm mod} : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is an isomorphism by Proposition 12.2 and Lemma 12.6.

Put $\mathbf{\Phi} = \mathcal{Z}^{-1}_{\rm mod}\mathbf{g}$. Let $\mathbf{u}$, p be given by the formulas (12.4), (12.5). Then $(\mathbf{u}, p)$ is a regular L^q-solution of the Dirichlet problem (1.1), (1.2) by Theorem 9.1 and Theorem 10.1. The uniqueness follows from Proposition 12.3. According to Theorem 9.1, Theorem 10.1 and Corollary 8.3 there exists a constant C_1 such that

$$M_\alpha(\mathbf{u}) + M_\alpha(\nabla \mathbf{u}) + \|M_\alpha(p)\|_{L^q(\partial\Omega)} \le C_1\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}.$$

Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Proposition 9.2 and Theorem 10.1 give $u_j \in B^{q,2}_{1+1/q}(B(0; 2r) \cap \Omega, \rho_{-3})$, $p \in B^{q,2}_{1/q}(\Omega \cap B(0; 2r); \rho_{-3})$ and

$$\sum_{j=1}^{3} \|u_j\|_{B^{q,2}_{1+1/q}(\Omega\cap B(0;2r),\rho_{-3})} + \|p\|_{B^{q,2}_{1/q}(\Omega\cap B(0;2r),\rho_{-3})} \le C_2\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}.$$

Theorem 9.1, Theorem 10.1 and Lemma 8.1 give $\rho_{-3}u_j, \rho_{-3}p \in W^{2,q}(\mathbb{R}^3 \setminus B(0; r))$ and

$$\|\mathbf{u}\|_{W^{2,q}(\mathbb{R}^3\setminus B(0;r))} + \|p\|_{W^{2,q}(\mathbb{R}^3\setminus B(0;r))} \le C_3\|\mathbf{g}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}.$$

Since $W^{2,q}(\mathbb{R}^3 \setminus B(0; r)) \hookrightarrow B^{q,2}_{1+1/q}(\mathbb{R}^3 \setminus B(0; r))$, $W^{2,q}(\mathbb{R}^3 \setminus B(0; r)) \hookrightarrow B^{q,2}_{1/q}(\mathbb{R}^3 \setminus B(0; r))$ by [104, §2.3.3, Remark 4], we obtain (12.13), (12.14).

Let $(\mathbf{u}, p)$ be a regular L^q-solution of the Dirichlet problem (1.1), (1.2). Then $(\mathbf{u}, p)$ is an L^q-solution of the Neumann problem (1.1), (1.3) with $h \equiv 0$ and some $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^3)$. According to Proposition 11.3 there exist constants b and c such that $\mathbf{u}(\mathbf{x}) \to (0, 0, b)$, $p(\mathbf{x}) \to c$ as $|\mathbf{x}| \to \infty$.

Let b, c be given. We have proved the existence of a regular L^q-solution $(\mathbf{v}, \pi)$ of the Dirichlet problem (1.1), $\mathbf{v} = \mathbf{g} - (0, 0, b)$ on $\partial\Omega$ such that $\mathbf{v}(\mathbf{x}) \to 0$, $\pi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Put $\mathbf{u} = \mathbf{v} + (0, 0, b)$, $p = \pi + c$. Then $(\mathbf{u}, p)$ is a unique regular L^q-solution of the Dirichlet problem (1.1), (1.2) such that $\mathbf{u}(\mathbf{x}) \to (0, 0, b)$, $p(\mathbf{x}) \to c$ as $|\mathbf{x}| \to \infty$. □

Corollary 12.9. *Let* $\Omega \subset \mathbb{R}^3$ *be an unbounded domain with compact boundary of class* $\mathcal{C}^{k+1}$*,* $k \in \mathbb{N}$*. Let* $a \in \mathbb{R}^1$*,* $\omega = (0, 0, a)$*,* $1 < q, r < \infty$*.*

- *If* $\mathbf{g} \in W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3)$ *then there exists a unique solution* $(\mathbf{u}, p) \in D^{k+1,q}(\Omega, \mathbb{R}^3) \times D^{k,q}(\Omega)$ *of the Dirichlet problem* (1.1), (1.2) *such that* $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ *as* $|\mathbf{x}| \to \infty$. *Moreover,*
$$\|\mathbf{u}\|_{D^{k+1,q}(\Omega)} + \|p\|_{D^{k,q}(\Omega)} \leq C\|\mathbf{g}\|_{W^{k+1-1/q,q}(\partial\Omega)}$$
where C *does not depend on* $\mathbf{g}$. *If* $q > 3/2$ *then* $p \in W^{k,q}(\Omega)$ *and*
$$\|p\|_{W^{k,q}(\Omega)} \leq C\|\mathbf{g}\|_{W^{k+1-1/q,q}(\partial\Omega)}.$$
If $q > 3$ *then* $\mathbf{u} \in W^{k+1,q}(\Omega; \mathbb{R}^3)$ *and*
$$\|\mathbf{u}\|_{W^{k+1,q}(\Omega)} \leq C\|\mathbf{g}\|_{W^{k+1-1/q,q}(\partial\Omega)}.$$
- *If* $1 + 1/q < s < k + 1$ *and* $\mathbf{g} \in B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3)$, *then there exists a unique solution* $(\mathbf{u}, p)$ *of the Dirichlet problem* (1.1), (1.2) *such that* $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ *as* $|\mathbf{x}| \to \infty$ *and* $u_j \in B^{q,r}_s(\Omega, \rho_{-3})$, $p \in B^{q,r}_{s-1}(\Omega, \rho_{-3})$. *Moreover,*
$$\|\mathbf{u}\|_{B^{q,r}_s(\Omega,\rho_{-3})} + \|p\|_{B^{q,r}_{s-1}(\Omega,\rho_{-3})} \leq C\|\mathbf{g}\|_{B^{q,r}_{s-1/q}(\partial\Omega)}$$
where C *does not depend on* $\mathbf{g}$. *If* $q > 3/2$ *then* $p \in B^{q,r}_{s-1}(\Omega)$ *and*
$$\|p\|_{B^{q,r}_{s-1}(\Omega)} \leq C\|\mathbf{g}\|_{B^{q,r}_{s-1/q}(\partial\Omega)}.$$
If $q > 3$ *then* $\mathbf{u} \in B^{q,r}_s(\Omega; \mathbb{R}^3)$ *and*
$$\|\mathbf{u}\|_{B^{q,r}_s(\Omega)} \leq C\|\mathbf{g}\|_{B^{q,r}_{s-1/q}(\partial\Omega)}.$$

Proof. Put $\tau = \min(q, 2)$. According to Theorem 12.8 there exists a unique regular L^τ-solution $(\mathbf{u}, p)$ of the Dirichlet problem (1.1), (1.2) such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. (Remember that $u_j \in B^{\tau,2}_{1+1/\tau}(\Omega, \rho_{-3})$, $p \in B^{\tau,2}_{1/\tau}(\Omega, \rho_{-3})$.) Fix $R > 0$ such that $\partial\Omega \subset B(0; R)$. Then $\mathbf{u} \in D^{k+1,q}(\Omega \setminus \overline{B(0; R)}, \mathbb{R}^3)$, $u_j \in B^{q,r}_t(\Omega \setminus \overline{B(0; R)}, \rho_{-3})$, $p \in D^{k,q}(\Omega \setminus \overline{B(0; R)}) \cap B^{q,r}_{t-1}(\Omega \setminus \overline{B(0; R)}, \rho_{-3})$ for all $t \in (1+1/q, k+1)$ by Lemma 7.1 and Proposition 11.3. According to [104, §4.6.1, Theorem] one has $(\mathbf{u}, p) \in W^{1,\tau}(\Omega \cap B(0; 2r), \mathbb{R}^3) \times L^\tau(\Omega \cap B(0; 2R))$. Put $\mathbf{g} := \mathbf{u}$ on $\partial B(0; 2R)$. Then $(\mathbf{u}, p)$ is a solution of the Dirichlet problem (1.1), (1.2) in $\Omega \cap B(0; 2R)$. If $\mathbf{g} \in W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3)$ then $(\mathbf{u}, p) \in W^{k+1,q}(\Omega \cap B(0; 2R), \mathbb{R}^3) \times W^{k,q}(\Omega \cap B(0; 2R))$ by Corollary 7.3, and thus $(\mathbf{u}, p) \in D^{k+1,q}(\Omega, \mathbb{R}^3) \times D^{k,q}(\Omega)$. If $\mathbf{g} \in B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3)$ then $(\mathbf{u}, p) \in B^{q,r}_s(\Omega \cap B(0; 2R), \mathbb{R}^3) \times B^{q,r}_{s-1}(\Omega \cap B(0; 2R))$ by Corollary 7.3, and therefore $u_j \in B^{q,r}_s(\Omega, \rho_{-3})$, $p \in B^{q,r}_{s-1}(\Omega, \rho_{-3})$. If α is a multiindex that $|\partial^\alpha \mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{-1-|\alpha|})$, $|\partial^\alpha p(\mathbf{x})| = O(|\mathbf{x}|^{-2-|\alpha|})$ as $|\mathbf{x}| \to \infty$ by Proposition 11.3. If $q > 3/2$ then $p \in W^{k,q}(\Omega)$ for $\mathbf{g} \in W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3)$, and $p \in B^{q,r}_{s-1}(\Omega)$ for $\mathbf{g} \in B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3)$. If $q > 3$ then $\mathbf{u} \in W^{k+1,q}(\Omega; \mathbb{R}^3)$ for $\mathbf{g} \in W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3)$, and $\mathbf{u} \in B^{q,r}_s(\Omega; \mathbb{R}^3)$ for $\mathbf{g} \in B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3)$.

We now show the uniqueness. Let $(\mathbf{u}, p)$ be a solution of the Dirichlet problem (1.1), (1.2) for $\mathbf{g} \equiv 0$ such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Proposition 11.3 gives that $\nabla\mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Fix $R > 0$ such that $\partial\Omega \subset B(0; R)$. Then $\mathbf{u} \in W^{1,q}(\Omega \cap B(0; R), \mathbb{R}^3)$, $p \in L^q(\Omega \cap B(0; R))$ by [104, §4.6.1, Theorem]. Put $\mathbf{g} := \mathbf{u}$ on $\partial B(0; R)$. Then $\mathbf{g} \in \mathcal{C}^\infty(\partial[\Omega \cap B(0; R)]; \mathbb{R}^3)$. Proposition 7.2 gives (12.10). According to Theorem 12.7 there exists a regular L^2-solution $(\mathbf{w}, \pi)$ of the Dirichlet problem (1.1), (1.2) for $\Omega \cap B(0; R)$. Since $(\mathbf{w}, \pi) \in W^{1,2}(\Omega \cap B(0; R); \mathbb{R}^3) \times L^2(\Omega \cap B(0; R))$, Proposition 7.2 gives $\mathbf{u} = \mathbf{w}$ and $p - \pi$ is constant in $\Omega \cap B(0; R)$. Thus $(\mathbf{u}, p)$ is an L^2-regular solution of the Dirichlet problem (1.1), (1.2). Theorem 12.8 gives that $\mathbf{u} \equiv 0$, $p \equiv 0$.

Denote by $\mathbf{u_g}$, $p_{\mathbf{g}}$ the solution of the problem (1.1), (1.2) such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. For the operator $F : \mathbf{g} \mapsto [\mathbf{u_g}, p_{\mathbf{g}}]$ one has

$$F : W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3) \to D^{k+1,q}(\Omega, \mathbb{R}^3) \times D^{k,q}(\Omega),$$

$$F : B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3) \to B^{q,r}_s(\Omega, \rho_{-3})^3 \times B^{q,r}_{s-1}(\Omega, \rho_{-3}).$$

If $q > 3/2$ then

$$F : W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3) \to D^{k+1,q}(\Omega, \mathbb{R}^3) \times W^{k,q}(\Omega),$$

$$F : B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3) \to B^{q,r}_s(\Omega, \rho_{-3})^3 \times B^{q,r}_{s-1}(\Omega).$$

If $q > 3$ then

$$F : W^{k+1-1/q,q}(\partial\Omega, \mathbb{R}^3) \to W^{k+1,q}(\Omega, \mathbb{R}^3) \times W^{k,q}(\Omega),$$

$$F : B^{q,r}_{s-1/q}(\partial\Omega, \mathbb{R}^3) \to B^{q,r}_s(\Omega, \mathbb{R}^3) \times B^{q,r}_{s-1}(\Omega).$$

These operators are closed by Theorem 12.8. The Closed graph theorem gives that they are continuous. □

13. Robin problem

In this section we study the Robin problem (1.1), (1.3). To study uniqueness of the problem we need the following auxiliary results:

Lemma 13.1. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $1 < q \le 2$. If $\mathbf{\Phi} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$ then there exists a finite non-tangential limits of $\nabla \mathcal{D}^a_\Omega \mathbf{\Phi}$ and $\Pi^a_\Omega \mathbf{\Phi}$ at almost all points of $\partial\Omega$ and*

$$\|M_\beta(\nabla \mathcal{D}^a_\Omega \mathbf{\Phi})\|_{L^q(\partial\Omega)} + \|M_\beta(\Pi^a_\Omega \mathbf{\Phi})\|_{L^q(\partial\Omega)} \le C \|\mathbf{\Phi}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}$$

where C does not depend on $\mathbf{\Phi}$.

Proof. According to Corollary 8.3 we can suppose that Ω is connected. Fix $\mathbf{z} \in \Omega$ and $r > 0$ such that $B(\mathbf{z}; 2r) \subset \Omega$ and put $G = \Omega \setminus \overline{B(\mathbf{z}; r)}$. Define $\mathbf{g} = \mathbf{\Phi}$ on $\partial\Omega$, $\mathbf{g} = c\mathbf{n}^G$ on $\partial B(\mathbf{z}; r)$ where c is a constant chosen in a such way that $\int_{\partial G} \mathbf{g} \cdot \mathbf{n}^G \, \mathrm{d}\sigma = 0$. Clearly,

$$\|\mathbf{g}\|_{W^{1,q}(\partial G,\mathbb{R}^3)} \le C_1 \|\mathbf{\Phi}\|_{W^{1,q}(\partial\Omega,\mathbb{R}^3)}. \tag{13.1}$$

Put $\omega = (0, 0, a)$. According to Theorem 12.7 and Theorem 12.8 there exists a regular L^q-solution $(\mathbf{u}, p)$ of the Dirichlet problem (1.1), (1.2) for G such that

$$\|M_\beta(\mathbf{u})\|_{L^q(\partial G)} + \|M_\beta(\nabla \mathbf{u})\|_{L^q(\partial G)} + \|M_\beta(p)\|_{L^q(\partial G)} \le C_2 \|\mathbf{g}\|_{W^{1,q}(\partial G,\mathbb{R}^3)}. \tag{13.2}$$

If Ω is unbounded we can suppose that $|\mathbf{u}(\mathbf{x})| \to 0$, $|p(\mathbf{x})| \to 0$ as $|\mathbf{x}| \to \infty$. Define $\mathbf{f}(\mathbf{x}) := T(\mathbf{u}, p)\mathbf{n}^G + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}^G]\mathbf{u}$. Then

$$\|\mathbf{f}\|_{L^q(\partial G)} \le C_3 \left(\|M_\beta(\mathbf{u})\|_{L^q(\partial G)} + \|M_\beta(\nabla \mathbf{u})\|_{L^q(\partial G)} + \|M_\beta(p)\|_{L^q(\partial G)} \right). \tag{13.3}$$

Corollary 11.2 and Proposition 11.3 give $\mathcal{D}^a_G \mathbf{g} = \mathbf{u} - Z^a_G \mathbf{f}$, $\Pi^a_\Omega \mathbf{g} = p - Q_\Omega \mathbf{f}$. Theorem 9.1, (13.3) and (13.2) give that there exist non-tangential limits of $\nabla \mathcal{D}^a_G \mathbf{g}$ and $\Pi^a_\Omega \mathbf{g}$ at almost all points of ∂G and

$$\|M_\beta(\nabla \mathcal{D}^a_G \mathbf{g})\|_{L^q(\partial G)} + \|M_\beta(\Pi^a_G \mathbf{g})\|_{L^q(\partial G)} \le C_4 \|\mathbf{g}\|_{W^{1,q}(\partial G,\mathbb{R}^3)}.$$

The proposition of the Lemma is a consequence of the fact that $\mathcal{D}^a_\Omega \mathbf{\Phi} = \mathcal{D}^a_G \mathbf{g} + c\mathcal{D}^a_{B(\mathbf{z};r)} \mathbf{n}^{B(\mathbf{z};r)}$ and $\Pi^a_\Omega \mathbf{\Phi} = \Pi^a_G \mathbf{g} + c\Pi^a_{B(\mathbf{z};r)} \mathbf{n}^{B(\mathbf{z};r)}$ in G, (13.1), Theorem 10.1 and Corollary 8.3. □

Lemma 13.2. *Let $\Omega \subset \mathbb{R}^3$ be an open set with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $1 < q \leq 2$. Then $\frac{1}{2}I \pm \tilde{K}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to L^q(\partial\Omega, \mathbb{R}^3)$, $\frac{1}{2}I \pm K^a_\Omega : W^{1,q}(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ are bounded Fredholm operators with index 0.*

Proof. $\frac{1}{2}I \pm \tilde{K}^0_\Omega$ is a Fredholm operator on $L^q(\partial\Omega, \mathbb{R}^3)$ with index 0 by [92, Theorem 9.1.11]. Since $\tilde{K}^a_\Omega - \tilde{K}^0_\Omega$ is a compact operator on $L^q(\partial\Omega, \mathbb{R}^3)$ by Theorem 9.1, the operator $\frac{1}{2}I \pm \tilde{K}^a_\Omega$ is Fredholm on $L^q(\partial\Omega, \mathbb{R}^3)$ with index 0.

If $\mathbf{\Phi} \in W^{1,q}(\partial\Omega, \mathbb{R}^3)$ then $(\frac{1}{2}I + K^a_\Omega)\mathbf{\Phi}$ is the non-tangential limit of $\mathcal{D}^a_\Omega \mathbf{\Phi}$ by Theorem 10.1. So, $\frac{1}{2}I + K^a_\Omega$ is a bounded operator on $W^{1,q}(\partial\Omega, \mathbb{R}^3)$ by Lemma 13.1. Corollary 11.4 gives $(\frac{1}{2}I + K^a_\Omega)\mathcal{Z}^a_\Omega = \mathcal{Z}^a_\Omega(\frac{1}{2}I - \tilde{K}^a_\Omega)$. The operator $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0 by Proposition 12.2. Thus $(\frac{1}{2}I + K^a_\Omega)\mathcal{Z}^a_\Omega = \mathcal{Z}^a_\Omega(\frac{1}{2}I - \tilde{K}^a_\Omega) : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0. Since $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0 and $(\frac{1}{2}I + K^a_\Omega)$ is a bounded linear operator on $W^{1,q}(\partial\Omega, \mathbb{R}^3)$, the operator $(\frac{1}{2}I + K^a_\Omega)$ is Fredholm with index 0 on $W^{1,q}(\partial\Omega, \mathbb{R}^3)$ by [93, § 16].

Put $G = \mathbb{R}^3 \setminus \overline{\Omega}$. Then $(1/2)I + K^a_G = (1/2)I - K^a_\Omega$ is a Fredholm operator with index 0 on $W^{1,q}(\partial\Omega, \mathbb{R}^3)$. □

Proposition 13.3. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$, $h \in L^\infty(\partial\Omega)$, $h \geq 0$. Let $(\mathbf{u}, p)$ be an L^q-solution of the Robin problem (1.1), (1.3) with $\mathbf{f} \equiv 0$. If Ω is bounded suppose that $\int_{\partial\Omega} h \, \mathrm{d}\sigma > 0$. If Ω is unbounded suppose that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then $\mathbf{u} \equiv 0$, $p \equiv 0$.*

Proof. If Ω is unbounded then $\nabla \mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ by Proposition 11.3. Thus, we can suppose that $q \leq 2$. Corollary 11.2 and Proposition 11.3 give

$$\mathbf{u} = \mathcal{D}^a_\Omega \mathbf{u} - Z^a_\Omega(h\mathbf{u}), \quad p = \Pi^a_\Omega \mathbf{u} - Q_\Omega(h\mathbf{u}) \qquad \text{in } \Omega.$$

The limit at the boundary gives according to Theorem 9.1 and Theorem 10.1

$$\mathbf{u} = \left(\frac{1}{2}I + \tilde{K}^a_\Omega\right)\mathbf{u} - \mathcal{Z}^a_\Omega(h\mathbf{u}).$$

Define $S\mathbf{v} := (-\frac{1}{2}I + \tilde{K}^a_\Omega)\mathbf{v} - \mathcal{Z}^a_\Omega(h\mathbf{v})$. Then $S\mathbf{u} = 0$. The operator $(-\frac{1}{2}I + \tilde{K}^a_\Omega)$ is a Fredholm operator with index 0 in $W^{1,q}(\partial\Omega, \mathbb{R}^3)$ by Lemma 13.2. Since $\mathcal{Z}^a_\Omega : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is bounded by Theorem 9.1, the operator $\mathbf{v} \mapsto \mathcal{Z}^a_\Omega(h\mathbf{v})$ is compact on $W^{1,q}(\partial\Omega, \mathbb{R}^3)$ by [102, Lemma 18.4]. So, S is a Fredholm operator with index 0 in $W^{1,q}(\partial\Omega, \mathbb{R}^3)$. Since $S\mathbf{u} = 0$ and S is a Fredholm operator with index 0 in $W^{1,2}(\partial\Omega, \mathbb{R}^3)$, [78, Lemma 9] gives $\mathbf{u} \in W^{1,2}(\partial\Omega, \mathbb{R}^3)$. So, $(\mathbf{u}, p)$ is an L^2-solution of the problem (1.1), (1.3) by Lemma 13.1 and Theorem 9.1. Corollary 4.2 and Proposition 12.1 give

$$0 = \int_{\partial\Omega} \mathbf{u} \cdot \mathbf{f} \, \mathrm{d}\sigma = \int_{\partial\Omega} h|\mathbf{u}|^2 \, \mathrm{d}\sigma + 2\int_\Omega |\hat{\nabla}\mathbf{u}|^2 \, \mathrm{d}\mathbf{x}.$$

Thus $h\mathbf{u} = 0$ on $\partial\Omega$. Since $\hat{\nabla}\mathbf{u} \equiv 0$, [80, Lemma 3.1] gives that there exists an antisymmetric matrix A and a vector $\mathbf{b}$ such that $\mathbf{u}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. If Ω is unbounded then the condition $\mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ gives $\mathbf{u} \equiv 0$. Let now Ω be bounded. Since $h\mathbf{u} = 0$ on $\partial\Omega$, we deduce $\sigma(\{\mathbf{x} \in \partial\Omega; \mathbf{u}(\mathbf{x}) = 0\}) > 0$. [83, Lemma 5.1] gives that $\mathbf{u} \equiv 0$.

Since $\mathbf{u} \equiv 0$ the equation (1.1) gives $\nabla p \equiv 0$. Therefore there exists a constant c such that $p \equiv c$. Since $0 = \mathbf{f} = -c\mathbf{n}^\Omega$, we deduce $c = 0$. □

We look for a particular solution of the Robin problem (1.1), (1.3) in the form of the modified single layer potential with a density $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$, i.e. $(\mathbf{u}, p)$ is given by (12.3) for Ω bounded with connected boundary and by (12.4), (12.5) for Ω with holes. Then $(\mathbf{u}, p)$ is an L^q-solution of the problem (1.1), (1.3) if $S_h\mathbf{\Phi} = \mathbf{f}$, where

$$S_h^a\mathbf{\Phi} := \frac{1}{2}\mathbf{\Phi} + \tilde{K}_\Omega^a\mathbf{\Phi} + h\mathcal{Z}_\Omega^a\mathbf{\Phi} + \sum_{j=1}^k \mathbf{\Psi}_j^h \int_{\partial G(j)} \mathbf{n}^\Omega \cdot \mathbf{\Phi} \, \mathrm{d}\sigma,$$

$$\mathbf{\Psi}_j^h := T(\mathcal{D}_{B(j)}^a\mathbf{n}^{B(j)}, \Pi_{B(j)}^a\mathbf{n}^{B(j)})\mathbf{n}^\Omega + \frac{1}{2}[(\omega \times \mathbf{x}) \cdot \mathbf{n}^\Omega]\mathcal{D}_{B(j)}^a\mathbf{n}^{B(j)} + h\mathcal{D}_{B(j)}^a\mathbf{n}^{B(j)}.$$

Proposition 13.4. *Let $\Omega \subset \mathbb{R}^3$ be a domain with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $1 < q < \infty$. Suppose that $q \le 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$. If Ω is bounded suppose moreover that $\int_{\partial\Omega} h \, \mathrm{d}\sigma > 0$. Then S_h^a is an isomorphism on $L^q(\partial\Omega, \mathbb{R}^3)$.*

Proof. $\frac{1}{2}I + \tilde{K}_\Omega^a$ is a Fredholm operator with index 0 in $L^q(\partial\Omega, \mathbb{R}^3)$ by Lemma 13.2. The operator $\mathcal{Z}_\Omega^a : L^q(\partial\Omega, \mathbb{R}^3) \to W^{1,q}(\partial\Omega, \mathbb{R}^3)$ is bounded by Theorem 9.1. Thus $h\mathcal{Z}_\Omega^a$ is a compact operator on $L^q(\partial\Omega, \mathbb{R}^3)$ (compare [102, Lemma 18.4]). Hence S_h^a is a Fredholm operator with index 0 in $L^q(\partial\Omega, \mathbb{R}^3)$.

Let $\mathbf{\Phi} \in L^q(\partial\Omega, \mathbb{R}^3)$, $S_h^a\mathbf{\Phi} = 0$. Let $(\mathbf{u}, p)$ be the modified single layer potential with the density $\mathbf{\Phi}$. (See (12.4), (12.5).) Then $(\mathbf{u}, p)$ is an L^q-solution of the problem (1.1), (1.3) with $\mathbf{f} \equiv 0$. Proposition 13.3 gives that $\mathbf{u} = 0$, $p = 0$ in Ω. Thus $\mathcal{Z}_{mod}\mathbf{\Phi} \equiv 0$. According to Lemma 12.6 we have $\mathbf{\Phi} \equiv 0$ for Ω unbounded, and $\mathbf{\Phi} = c\mathbf{n}^\Omega\chi_{\partial G(0)}$ for Ω bounded. (Remember that $G(0)$ denotes the unbounded component of $\mathbb{R}^3 \setminus \overline{\Omega}$.) Corollary 11.5 gives $0 = p = -c$. Thus $\mathbf{\Phi} \equiv 0$.

$S_h^a : L^q(\partial\Omega, \mathbb{R}^3) \to L^q(\partial\Omega, \mathbb{R}^3)$ is a Fredholm operator with index 0 and trivial kernel. Hence S_h^a is an isomorphism. □

Theorem 13.5. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$. Suppose that $q \le 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$, $\int_{\partial\Omega} h \, \mathrm{d}\sigma > 0$. Let $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^3)$. Put $\mathbf{\Phi} = (S_h^a)^{-1}\mathbf{f}$. Let $(\mathbf{u}, p)$ be the modified single layer potential with the density $\mathbf{\Phi}$ (see (12.4), (12.5)). Then $(\mathbf{u}, p)$ is a unique L^q-solution of the Robin problem (1.1), (1.3). Moreover, $\mathbf{u} \in B_{1+1/q}^{q,\max(q,2)}(\Omega, \mathbb{R}^3)$, $p \in B_{1/q}^{q,\max(q,2)}(\Omega)$, and*

$$\|M_\beta(\mathbf{u})\|_{L^q(\partial\Omega)} + \|M_\beta(\nabla\mathbf{u})\|_{L^q(\partial\Omega)} + \|\mathbf{u}\|_{B_{1+1/q}^{q,\max(q,2)}(\Omega} \le C\|\mathbf{f}\|_{L^q(\partial\Omega)},$$

$$\|M_\beta(p)\|_{L^q(\partial\Omega)} + \|p\|_{B_{1/q}^{q,\max(q,2)}(\Omega} \le C\|\mathbf{f}\|_{L^q(\partial\Omega)},$$

where C does not depend on $\mathbf{f}$.

Proof. The theorem is a consequence of Proposition 13.4, Proposition 13.3, Theorem 9.1, Proposition 9.2 and Corollary 8.3. □

Theorem 13.6. *Let $\Omega \subset \mathbb{R}^3$ be an unbounded domain with compact Lipschitz boundary, $a \in \mathbb{R}^1$, $\omega = (0, 0, a)$, $1 < q < \infty$. Suppose that $q \le 2$ or $\partial\Omega$ is of class $\mathcal{C}^1$. Let $h \in L^\infty(\partial\Omega)$, $h \ge 0$. Let $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^3)$.*

- *Put $\mathbf{\Phi} = (S_h^a)^{-1}\mathbf{f}$. Let $(\mathbf{u}, p)$ be the modified single layer potential with the density $\mathbf{\Phi}$ (see (12.4), (12.5)). Then $(\mathbf{u}, p)$ is a unique L^q-solution of the*

Robin problem (1.1), (1.3) *such that* $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ *as* $|\mathbf{x}| \to \infty$. *Moreover,* $u_j \in B^{q,2}_{1+1/q}(\Omega, \rho_{-3})$, $p \in B^{q,2}_{1/q}(\Omega, \rho_{-3})$ *and*

$$M_\beta(\mathbf{u}) + M_\beta(\nabla \mathbf{u}) + \sum_{j=1}^{3} \|u_j\|_{B^{q,2}_{1+1/q}(\Omega,\rho_{-3})} \le C\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)}, \tag{13.4}$$

$$\|M_\beta(p)\|_{L^q(\partial\Omega)} + \|p\|_{B^{q,2}_{1/q}(\Omega,\rho_{-3})} \le C\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)} \tag{13.5}$$

where C *does not depend on* $\mathbf{f}$.

- *If* $(\mathbf{u}, p)$ *is an* L^q*-solution of the Robin problem* (1.1), (1.3) *then there exist constants* b *and* c *such that* $\mathbf{u}(\mathbf{x}) \to (0,0,b)$, $p(\mathbf{x}) \to c$ *as* $|\mathbf{x}| \to \infty$.
- *Let constants* b *and* c *be given. Then there exists a unique* L^q*-solution* $(\mathbf{u}, p)$ *of the Robin problem* (1.1), (1.3) *such that* $\mathbf{u}(\mathbf{x}) \to (0,0,b)$, $p(\mathbf{x}) \to c$ *as* $|\mathbf{x}| \to \infty$. *Moreover,*

$$M_\beta(\mathbf{u}) + M_\beta(\nabla \mathbf{u}) + \|M_\beta(p)\|_{L^q(\partial\Omega)} \le C\left(\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)} + |b| + |c|\right).$$

Proof. S^a_h is an isomorphism on $L^q(\partial\Omega, \mathbb{R}^3)$ by Proposition 13.4. Put $\mathbf{\Phi} = (S^a_h)^{-1}\mathbf{f}$. Let $(\mathbf{u}, p)$ be the modified single layer potential with the density $\mathbf{\Phi}$ (see (12.4), (12.5)). Then $(\mathbf{u}, p)$ is an L^q-solution of the Robin problem (1.1), (1.3) such that $\mathbf{u}(\mathbf{x}) \to 0$, $p(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ by Theorem 9.1 and Theorem 10.1. The uniqueness follows from Proposition 13.3. According to Theorem 9.1, Theorem 10.1 and Corollary 8.3 there exists a constant C_1 such that

$$M_\beta(\mathbf{u}) + M_\beta(\nabla \mathbf{u}) + \|M_\beta(p)\|_{L^q(\partial\Omega)} \le C_1\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)}.$$

Fix $r > 0$ such that $\partial\Omega \subset B(0; r)$. Proposition 9.2 and Theorem 10.1 give $u_j \in B^{q,\max(q,2)}_{1+1/q}(B(0; 2r) \cap \Omega, \rho_{-3})$, $p \in B^{q,\max(q,2)}_{1/q}(\Omega \cap B(0; 2r); \rho_{-3})$ and

$$\sum_{j=1}^{3} \|u_j\|_{B^{q,\max(q,2)}_{1+1/q}(\Omega\cap B(0;2r),\rho_{-3})} + \|p\|_{B^{q,\max(q,2)}_{1/q}(\Omega\cap B(0;2r),\rho_{-3})} \le C_2\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)}.$$

Theorem 9.1, Theorem 10.1 and Lemma 8.1 give $\rho_{-3}u_j, \rho_{-3}p \in W^{2,q}(\mathbb{R}^3 \setminus B(0; r))$ and

$$\|\mathbf{u}\|_{W^{2,q}(\mathbb{R}^3\setminus B(0;r))} + \|p\|_{W^{2,q}(\mathbb{R}^3\setminus B(0;r))} \le C_3\|\mathbf{f}\|_{L^q(\partial\Omega,\mathbb{R}^3)}.$$

Since

$$W^{2,q}(\mathbb{R}^3 \setminus B(0; r)) \hookrightarrow B^{q,\max(q,2)}_{1+1/q}(\mathbb{R}^3 \setminus B(0; r)) \hookrightarrow B^{q,\max(q,2)}_{1/q}(\mathbb{R}^3 \setminus B(0; r))$$

by [104, §2.3.3, Remark 4], we obtain (13.4), (13.5).

Let $(\mathbf{u}, p)$ be an L^q-solution of the Robin problem (1.1), (1.3). According to Proposition 11.3 there exist constants b and c such that $\mathbf{u}(\mathbf{x}) \to (0,0,b)$, $p(\mathbf{x}) \to c$ as $|\mathbf{x}| \to \infty$.

Let b, c be given. We have proved the existence of an L^q-solution $(\mathbf{v}, \pi)$ of the Robin problem (1.1), $\mathbf{v} = \mathbf{f} + c\mathbf{n}^\Omega - [(\omega \times x) \cdot \mathbf{n}^\Omega/2 + h](0,0,b)$ on $\partial\Omega$ such that $\mathbf{v}(\mathbf{x}) \to 0$, $\pi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Put $\mathbf{u} = \mathbf{v} + (0,0,b)$, $p = \pi + c$. Then $(\mathbf{u}, p)$ is a unique L^q-solution of the Robin problem (1.1), (1.3) such that $\mathbf{u}(\mathbf{x}) \to (0,0,b)$, $p(\mathbf{x}) \to c$ as $|\mathbf{x}| \to \infty$. $\square$

14. Appendix

Lemma 14.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded open set with Lipschitz boundary. Then there is a sequence of open sets Ω_j with boundaries of class C^∞ such that*

- $\overline{\Omega}_j \subset \Omega$.
- *There are $a > 0$ and homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for each j and each $y \in \partial\Omega$ and $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})|; \mathbf{y} \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that $\omega_j \to 1$ point-wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$.*
- *The normal vectors to Ω_j, $\mathbf{n}(\Lambda_j(\mathbf{y}))$, converge point-wise a.e. and in every $L^s(\partial\Omega)$, $1 \le s < \infty$, to $\mathbf{n}(\mathbf{y})$.*

(See [111, Theorem 1.12].)

References

[1] N. Abada, T. Z. Boulmezaoud, N. Kerdid: The Stokes flow around a rotating body in the whole space. J. Math. Soc. Japan 65 (2013), No. 2, 607–632

[2] J. Bemelmans, G. P. Galdi, M. Kyed: Fluid flow around floating bodies, I: The hydrostatic case. J. Math. Fluid. Mech. 14 (2012), 751–770

[3] R. Brown, I. Mitrea, M. Mitrea, M.Wright: Mixed boundary value problems for the Stokes system. Trans. Amer. Math. Soc. 362 (2010), No. 3, 1211–1230

[4] T. Chang: Boundary integral operators over Lipschitz surfaces for a Stokes equation in R^n. Potential Anal. 29 (2008), 105–117

[5] T. K. Chang, H. J. Choe: Estimates of the Green's functions for the elasto-static equations and Stokes equations in a three dimensional Lipschitz domain. Potential Anal. 30 (2009), 85–99

[6] T. Chang, K. Lee: Spectral properties of the layer potentials on Lipschitz domains. Illinois J. of Math., vol. 52, No. 2 (2008), 463–472

[7] T. K. Chang, D. H. Pahk: Spectral properties for layer potentials associated to the Stokes equation in Lipschitz domains. Manuscripta Math. 130 (2009), 359–373

[8] P. Deuring, *L^p-theory for the Stokes system in 3D domains with conical boundary points.* Indiana Univ. Math. J. 47 (1998), 11–47.

[9] P. Deuring: The resolvent problem for the Stokes system in exterior domains. An elementery approach. Math. Methods Appl. Sci. 13 (1990), 335–349

[10] R. A. Devore, R. C. Sharpley: Besov spaces on domains in R^d. Transactions of the Mathematical Society 335 (1993), No. 2, 843–864

[11] M. Dindoš, M. Mitrea: The stationary Navier-Stokes system in nonsmooth manifolds: The Poisson problem in Lipschitz and C^1 domains. Arch. Rational Mech. Anal. 174 (2004), 1–37.

[12] M. Dobrowolski: Angewandte Functionanalysis. Functionanalysis, Sobolev-Räume und elliptische Differentialgleichungen, Springer-Verlag, Berlin Heidelberg, 2006

[13] N. Dunford, J. T. Schwartz: Linear Operators I, General Theory. Interscience Publishers, New York - London 1958.

[14] D. E. Edmunds, H. Triebel: Function spaces, entropy numbers, differentiable operators. Cambridge Univ. Press 1996

[15] E. B. Fabes, C. E. Kenig, G. C. Verchota: The Dirichlet problem for the Stokes system on Lipschitz domains, Duke Math. J. **57** (1988), 769–793.

[16] E. B. Fabes, J. E. Lewis, N. M. Riviere: Singular integrals and hydrodynamic potentials. American Journal of Mathematics 99 (1977), No. 3, 601–625

[17] R. Farwig: Estimates of lower order derivatives of viscous fluid past a rotating obstacle. Regularity and other aspects of the Navier-Stokes equations. Banach Center Publications, vol. 70, Institute of Mathematics Polish Academy of Sciences, Warszawa 2005

[18] R. Farwig: An L^q-analysis of viscous fluid flow past a rotating obstacle. Tohoku Math. J. 58 (2005), 129–147.

[19] R. Farwig, G. P. Galdi, M. Kyed: Asymptotic structure of a Leray solution to the Navier-Stokes flow around a rotating body. Pacific J. Math. 253 (2011), 367–382
[20] R. Farwig, T. Hishida: Stationary Navier-Stokes flow around a rotating obstacle. Functialaj Ekvacioj 50 (2007), 371–403
[21] R. Farwig, T. Hishida: Asymptotic profiles of steady Stokes and Navier-Stokes flows around a rotating obstacle. Ann. Univ. Ferrara 55 (2009), 263–277
[22] R. Farwig, T. Hishida: Asymptotic profile of steady Stokes flow around a rotating obstacle. Manuscripta Math. 136 (2011), 315–338
[23] R. Farwig, T. Hishida: Leading term at infinity of steady Navier-Stokes flow around a rotating obstacle. Math. Nachr. 284 (2011), No. 16, 2065-2077
[24] R. Farwig, T. Hishida, D. Müller: L^q-theory of a singular "winding" integral operator arising from fluid dynamics. Pacific J. Math. 215 (2004), No. 2, 297–313
[25] R. Farwig, S. Kračmar, M. Krbec, Š. Nečasová, P. Penel: Weighted L^2 and L^q approaches to fluid flow past a rotating body. in P. B. Mucha (ed.) et al., Nonlocal and abstract parabolic equations and their applications. Based on the conference, Bedlewo, Poland, 2007. Warsaw: Polish Academy of Sciences, Institute of Mathematics. Banach Center Publications 86 (2009), 59–81
[26] R. Farwig, M. Krbec, Š. Nečasová: A weighted L^q-approach to Oseen flow around a rotating body. Math. Meth. Appl. Sci. 31 (2008), 551–574
[27] L. E. Fraenkel: Introduction to Maximum Principles and Symmetry in Elliptic Problems. Cambridge Tracts in Mathematics 128. Cambridge University Press 2000
[28] G. P. Galdi: Steady flow of a Navier-Stokes fluid around a rotating obstacle. J. Elasticity 71 (2003), 1–31
[29] Galdi, G. P.: An introduction to the Mathematical Theory of the Navier-Stokes Equations. Steady-State Problems Springer Monographs in Mathematics. 2nd Edition, Springer Verlag 2011
[30] G. P. Galdi, M. Kyed: Steady-state Navier-Stokes flows past a rotating body: Leray solutions are physically reasonable. Arch. Rational Mech. Anal. 200 (2011), 21–58
[31] G. P. Galdi, M. Kyed: A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part I: Strong solutions. Proc. Am. Math. Soc. 141 (2013), No. 2, 573–583
[32] G. P. Galdi, M. Kyed: A simple proof of L^q-estimates for the steady-state Oseen and Stokes equations in a rotating frame. Part II: Weak solutions. Proc. Am. Math. Soc. 141 (2013), No. 4, 1313–1322
[33] G. P. Galdi, A. L. Silvestre: The steady motion of a Navier-Stokes liquid around a rigid body. Arch. Rational Mech. Anal. 184 (2007), 371–400
[34] G. P. Galdi, Ch. G. Simader: Existence, uniqueness and L^q-estimates for the Stokes problem in an exterior domain. Arch. Rational Mech. Anal. 112 (1990), 291–318
[35] G.P. Galdi, G. Simader, H. Sohr: On the Stokes problem in Lipschitz domains. Annali di Matematica pura ed applicata CLXVII (1994), IV, 147–163
[36] T. Hishida: L^q estimates of weak solutions to the stationary Stokes equations around a rotating body. J. Math. Soc. Japan 58 (2006), No. 3, 743–767
[37] R. Hunt, R. L. Wheeden: On the boundary value of harmonic functions. Trans. Amer. Math. Soc. 132 (1968), 307–322
[38] D. S. Jerison, C. E. Kenig: An identity with applications to harmonic measure. Bull. Amer. Math. Soc. 2 (1980), No. 3, 447–451
[39] C. E. Kenig: Weighted H^p spaces on Lipschitz domains, Amer. J. Math. 102 (1980), 129–163
[40] C. E. Kenig: Boundary value problems of linear elastostatics and hydrostatics on Lipschitz domains. Seminaire Goulaovic - Meyer - Schwartz 1983–1984. Équat. dériv. part., Exposé No. 21 (1984), 1–12
[41] C. E. Kenig: Harmonic Analysis Techniques for Second Order Elliptic Boundary Value Problems. American Mathematical Society, Providence, Rhode Island 1994
[42] J. Kilty: The L^p Dirichlet problem for the Stokes system on Lipschitz domain. Indiana Univ. Math. J. 58 (2009), 1219–1234
[43] M. Kohr: Boundary value problems for a compressible Stokes system in bounded domains in R^n. J. Comput. Appl. Math., vol. 201 (2007), Issue 1, 128–145
[44] M. Kohr: The Dirichlet problems for the Stokes resolvent equations in bounded and exterior domains in R^n. Math. Nachr. 280 (2007), No. 5–6, 534–559

[45] M. Kohr, M. Lanza de Cristoforis, W. L. Wendland: Nonlinear Neumann–transmission problems for Stokes and Brinkman equations on Euclidean Lipschitz domains. Potential Anal. 38 (2013), 1123–1171

[46] M. Kohr, M. Lanza de Cristoforis, W. L. Wendland: Boundary value problems of Robin type for the Brinkman and Darcy-Forchheimer-Brinkman systems in Lipschitz domains. J. Math. Fluid Mech. 16 (2014), 595–630

[47] Kohr, M., Pintea, C., Wendland, W. L.: Brinkman-type operators on Riemannian manifolds: Transmission problems in Lipschitz and C^1 domains, Potential Anal. 32, 229–273 (2010)

[48] M. Kohr, C. Pintea, W. L. Wendland, *Stokes-Brinkman transmission problems on Lipschitz and C^1 domains in Riemannian manifolds.* Commun. Pure Appl. Anal. *9* (2010), 493–537.

[49] Kohr, M., Pintea, C., Wendland, W. L.: Dirichlet–transmission problems for general Brinkman operators in Lipschitz and C^1 domains in Riemannian manifolds. Discrete and continuous dynamical systems, ser. B, 15 (2011), 999–1018

[50] Kohr, M., Pintea, C., Wendland, W. L.: Dirichlet–transmission problems for pseudodifferential Brinkman operators on Sobolev and Besov spaces associated to Lipschitz domains in Riemannian manifolds. Z. Angew. Math. Mech. 93 (2013), No. 6–7, 446–458

[51] Kohr, M., Pintea, C., Wendland, W. L.: Potential analysis for pseudodifferential matrix operators in Lipschitz domains on Riemannian manifolds. Applications to Brinkman operators. Mathematica 54 (2012), 156–173

[52] M. Kohr, I. Pop: Viscous Incompressible Flow for Low Reynolds Numbers. Advances in Boundary Elements 16, WIT Press, Southampton 2004

[53] M. Kohr, G. P. Raja Sekhar, E. M. Ului, W. L. Wendland: Two-dimensional Stokes-Brinkmann cell model - a boundary integral formulation. Applicable Analysis 91 (2012), 251–275

[54] M. Kohr, G. P. Raja Sekhar, W. L. Wendland: Boundary integral equations for a three-dimensional Stokes-Brinkman cell model. Mathematical Models and Methods in Applied Sciences 18 (2008), No. 12, 2055–2085

[55] M. Kohr, G. P. Raja Sekhar, W. L. Wendland: Boundary integral method for Stokes flow past a porous body. Mathematical Methods in Applied Sciences 31 (2008), 1065–1097

[56] Kohr, M., Wendland W. L.: Boundary integral equations for a three-dimensional Brinkmann flow problem, Math. Nachr. 282, no. 9, 1305–1333 (2009)

[57] M. Kohr, W. L. Wendland, G. P. Raja Sekhar: Boundary integral equations for two-dimensional low Reynolds number flow past a porous body. Math. Meth. Appl. Sci. 32 (2009), 922–962, DOI: 10.1002/mma.1074

[58] V. A. Kozlov, V. G. Maz'ya: On "power-logarithmic" solutions to the Dirichlet problem for the Stokes system in a Dirichlet angle. Math. Methods Appl. Sci., vol. 20 (1997), 315–346

[59] S. Kračmar, D. Medková, Š. Nečasová, W. Varnhorn: A Maximum Modulus Theorem for the Oseen Problem. Annali di Matematica Pura ed Applicata, Volume 192, Issue 6 (2013), 1059–1076

[60] M. A. Krasnosel'skii, P. P. Zabreiko, E. I. Pustyl'nik, P. E. Sobolevskii: Integral Operators in Spaces of Summable Functions. Izdat. Nauka, Moskva 1966 (Russian)

[61] P. A. Krutitskii: On properties of some integrals related to potentials for Stokes equations, Quarterly of Applied Math. LXV (2007), No. 3, 549–569

[62] M. Kyed: On a mapping property of the Oseen operator with rotation. Discrete and Continuous Dynamical Systems, ser. S, 6 (2013), No. 5, 1315–1322

[63] M. Kyed: Asymptotic profile of a linearized Navier-Stokes flow past a rotating body. Q. Appl. Math. 71 (2013), No. 3, 489–500

[64] M. Kyed: On the asymptotic structure of a Navier-Stokes flow past a rotating body. J. Math. Soc. Japan 66 (2014), No. 1, 1–16

[65] O. A. Ladyzenskaya: The mathematical theory of viscous incompressible flow. New York-London-Paris: Gordon and Breach 1969

[66] O. Lepsky: On Laplace and Stokes potentials. Math. Methods Appl. Sci. 21 (1998), 227–249

[67] J. E. Lewis: Layer potentials for elastostatics and hydrostatics in curvilinear domains. Trans. Amer. Math. Soc. 320 (1990), 53–76

[68] J. L. Lions, E. Magenes: Problèmes aux limites non homogènes et applications, Vol. 1, Dunod, Paris 1968

[69] M.-S. Lubuma: Classical solutions of two dimensional Stokes problems on non smooth domains I: the Radon integral operators. Math. Methods Appl. Sci. 16 (1993), No. 9, 643–664

[70] P. Maremonti: On the Stokes equations: The maximum modulus theorem. Math. Models Methods Appl. Sci. 10 (2000), 1047–1072.

[71] P. Maremonti, R. Russo: On the maximum modulus theorem for the Stokes system. Annali della scuola normale superiore di Pisa XXI (1994), 629–643.

[72] P. Maremonti, R. Russo, G. Starita: On the Stokes equations: the boundary value problem. In Advances in Fluid Dynamics. Eds. P. Maremonti. 1999, Dipartimento di Matematica Seconda Università di Napoli; pp. 69–140

[73] V. Maz'ya, M. Mitrea, T. Shaposhnikova: The inhomogenous Dirichlet problem for the Stokes system in Lipschitz domains with unit normal close to VMO^*. Functional Analysis and Its Applications 43 (2009), No. 3, 217–235

[74] V. G. Maz'ya, S. V. Poborchi: Differentiable Functions on Bad Domains. World Scientific, Singapore 1997

[75] V. Maz'ya, J. Rossmann: Pointwise estimates for Green's kernel of a mixed boundary value problem to the Stokes system in a polyhedral cone. Math. Nachr. 278 (2005), No. 15, 1766 – 1810

[76] V. Maz'ya, J. Rossmann: L_p estimates of solutions to mixed boundary value problems for the Stokes system in polyhedral domains. Math. Nachr. 280 (2007), No. 7, 751 – 793

[77] W. McLean: Strongly Elliptic Systems and Boundary Integral Equations. 2000, Cambridge University Press

[78] D. Medková: Regularity of solutions of the Neumann problem for the Laplace equation. Le Matematiche, LXI (2006), 287–300.

[79] D. Medková: Integral representation of a solution of the Neumann problem for the Stokes system. Numer. Algorithms 54 (2010), No. 4, 459–484

[80] Medková, D.: Convergence of the Neumann series in BEM for the Neumann problem of the Stokes system. Acta Appl. Math. 116, 281–304 (2011)

[81] D. Medková: The Neumann problem for the planar Stokes system. Annali dell' Universita' di Ferrara 58 (2012), Issue 2, 307–329

[82] D. Medková: Integral equation method for the first and second problems of the Stokes system. Potential Analysis 39 (2013), 389–409

[83] D. Medková: Transmission problem for the Brinkman system. Complex Variables and Elliptic Equations 59 (2014), No. 12, 1664–1678

[84] D. Medková: L^q-solution of the Robin problem for the Oseen system. Acta Applicanda Mathematicae 142 (2016), No. 1, 61–79, DOI: 10.1007/s10440-015-0014-5

[85] D. Medková: Bounded solutions of the Dirichlet problem for the Stokes resolvent system. Complex Variables and Elliptic Equations 61 (2016), Issue 12, 1689–1715,

[86] D. Mitrea, M. Mitrea, L. Yan: Boundary value problems for the Laplacian in convex and semiconvex domains. J. Funct. Anal. 258 (2010), 2507–2585

[87] I. Mitrea: Spectral radius properties for layer potentials associated with the elastic and hydrostatics equations in nonsmooth domains. J. Fourier Anal. Appl. 5 (1999), 385–408

[88] I. Mitrea: On the spectra of elastostatic and hydrostatic layer potentials on curvilinear polygons. J. Fourier Anal. Appl. 8 (2002), 443–487

[89] M. Mitrea, S. Monniaux, M. Wright: The Stokes operator with Neumann boundary conditions in Lipschitz domains. J. Math. Sci. 176 (2011), No. 3, 409 –457

[90] M. Mitrea, M. Taylor: Navier-Stokes equations on Lipschitz domains in Riemannian manifolds. Math. Ann. 321 (2001), 955–987

[91] M. Mitrea, M. Taylor: Potential theory on Lipschitz domains in Riemannian manifolds: the case of Dini metric tensors. Transactions of the American Mathematical Society, 355 (2002), No. 5, 1961–1985

[92] M. Mitrea, M. Wright: Boundary value problems for the Stokes system in arbitrary Lipschitz domains. Astérisque 344, Paris 2012

[93] V. Müller: Spectral Theorey of Linear Operators and Spectral Systems in Banach Algebras. Operator Theory Advances and Applications. Vol. 139, Birkhäuser, Basel 2007

[94] Pokorný, M.: Comportement asymptotique des solutions de quelques equations aux derivees partielles decrivant l'ecoulement de fluides dans les domaines non-bornes. These de doctorat. Universite de Toulon et Du Var, Universite Charles de Prague, (1999).

[95] C. Pozrikidis: Boundary integral and singularity methods for linearized viscous flow. Cambridge texts in Applied Math., Cambridge University Press, New York 1992

[96] A. Russo, A. Tartaglione: On the Oseen and Navier-Stokes systems with a slip boundary condition. Appl. Math. Letters 22 (2009), 674–678

[97] A. Russo, A. Tartaglione: On the Navier problem for the stationary Navier-Stokes equations. J. Differential Equations 251 (2011), 2387–2408

[98] Z. Shen: A note on the Dirichlet problem for the Stokes system in Lipschitz domain. Proc. Amer. Math. Soc. 123 (1995), No. 3, 801–811

[99] Z. Shen: Resolvent estimates in L^p for the Stokes operator in Lipschitz domains. Arch. Rational Mech. Anal. 205 (2012), 395–424

[100] A. L. Silvestre: On the existence of steady flows of a Navier-Stokes liquid around a moving rigid body. Math. Meth. Appl. Sci. 27 (2004), 1399–1409, DOI: 10.1002/mma.509

[101] E. M. Stein: Harmonic Analysis. Real-Variable Methods, Orthogonality, and Oscilatory Integrals, Princeton Univ. Press, Princeton 1993

[102] L. Tartar: An Introduction to Sobolev Spaces and Interpolation Spaces. Springer-Verlag, Berlin Heidelberg 2007

[103] E. A. Thomann, R. B. Guenther: The fundamental solution of the linearized Navier-Stokes equations for spinning bodies in three spatial dimensions - time dependent case. J. Math. Fluid Mech. 8 (2006), 77 –98

[104] H. Triebel: Interpolation Theory, Function Spaces, Differential Operators. VEB Deutscher Verlag der Wissenschaften, Berlin 1978

[105] H. Triebel: Theory of function spaces. Birkhäuser, Basel - Boston - Stuttgart 1983

[106] H. Triebel: Function spaces in Lipschitz domains and on Lipschitz manifolds. Characteristic functions as pointwise multipliers. Rev. Mat. Comput. 15 (2002), 475–524

[107] H. Triebel, Theory of function spaces III, Monographs in Mathematics 100, Birkhäuser, Basel, 2006.

[108] W. Varnhorn, The Stokes equations, Akademie Verlag, Berlin, 1994.

[109] W. Varnhorn: The boundary value problems of the Stokes resolvent equations in n dimensions. Math. Nachr. 269–270 (2004), 210–230

[110] W. Varnhorn: Boundary integral equations and maximum modulus estimates for the Stokes system. Proc. Appl. Math. Mech. 7 (2007), 1042603–1042604

[111] G. Verchota: Layer potentials and regularity for the Dirichlet problem for Laplace's equation in Lipschitz domains. Journal of Functional Analysis 59 (1984), 572–611

[112] W. Wei, Z. Zhang: L^p resolvent estimates for constant coefficient elliptic systems on Lipschitz domains. J. Funct. Anal. 267 (2014), 3262–3293

Institute of Mathematics of the Czech Academy of Sciences, Žitná 25, 115 67 Praha 1, Czech Republic

E-mail address: medkova@math.cas.cz