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Abstract The paper presets a coordinate-free analysis of deformation measures for shells modeled as 2D surfaces. These measures are represented by second-order tensors. As is well-known, two types are needed in general: the surface strain measure (deformations in tangential directions), and the bending strain measure (warping). Our approach first determines the 3D strain tensor E of a shear deformation of a 3D shell-like body and then linearizes E in two smallness parameters: the displacement and the distance of a point from the middle surface. The linearized expression is an affine function of the signed distance from the middle surface: the absolute term is the surface strain measure and the coefficient of the linear term is the bending strain measure. The main result of the paper determines these two tensors explicitly. It turns out that the derived surface strain measure coincides with Naghdi's surface strain measure, but *the bending strain measure is different from Naghdi's bending strain measure* (actually it seems to be new). In the particular case of a Kirchhoff-Love deformation our bending strain measure reduces to a tensor introduced earlier by Anicic & Léger in [2] (rather than to the Koiter bending strain measure frequently used in this context).

AMS 2010 classification 74B99

Keywords Naghdi's shell theory, shear deformation, Kirchhoff-Love deformation, surface strain tensor, bending strain tensor

Contents

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1 Introduction

The paper presents an analysis of deformations of thin shells. The ultimate goal is the determination of appropriate deformation measures for linear shells modeled as 2D surfaces. It is well-known that the literature contains multiple suggestions in this case, with different, often not entirely clear, motivations. Our motivation is based on a 3D-2D approximation of the geometry of deformation. The main novel features in our analysis are

– strict adherence to an index-free notation for curved surfaces and

– simultaneous linearization in small magnitude of deformation and small thickness of the 3D shell.

Christoffel's symbols and covariant derivatives do not occur in our analysis. It is important to realize that the linearization of, say, the second fundamental form of the deformed surface taken as a second-order tensor is different from the linearization of the components in a given coordinate system on the undeformed surface. The point is that the use of the same coordinate system on the deformed surface involves (implicitly) a change of the basis vectors a_α, $\alpha = 1, 2$, in the undeformed configuration to the *convected coordinate vectors* $\bar{a}_\alpha$ in the deformed configuration. The first way of linearization takes the change of the basis vectors into account but the second not. The details are presented in Section 5; the results in Theorems 5.3 and 5.5, respectively. The first linearization leads to a linearized curvature tensor ρ^{C} apparently not mentioned hitherto, while the second gives, standardly, Koiter's bending strain tensor ρ^{K}. When tested on a radial expansion of a sphere, the trace of ρ^{C} gives the correct linearized value of the scalar curvature, while the trace of ρ^{K} gives the *opposite value.* See Example 5.7. However, Sections 4 and 5 on the changes of geometric quantities under deformations of a surface are included only for completeness, because it is not apriori clear, why the change of curvature should be related to the elastic response of the shell.

The main line of our argument motivates deformation measures for linear shells by the 3D linear elasticity, which is governed by the 3D small strain tensor. We consider a 3D shell-like body with the reference configuration

$$\Omega := \{x = \xi + tn(\xi) : \xi \in \mathscr{S}, -h < t < h\},$$

where $\mathscr{S}$ is the middle surface with the normal n, and where $h > 0$ is the thickness of the shell. If h is sufficiently small, a general point $x \in \Omega$ can be written uniquely as

$$x = \xi + tn(\xi) \quad \text{where} \quad \xi \in \mathscr{S} \ \text{and} \ -h < t < h. \tag{1.1}$$

The pair (ξ, t) is called normal coordinates of x. A shear deformation is defined by the 3D displacement u of Ω of the format

$$u\big(\xi + tn(\xi)\big) = \omega(\xi) + t\delta(\xi) \tag{1.2}$$

where ω and δ some vector-valued functions defined on $\mathscr{S}$. Clearly, ω is the displacement on the middle surface. The quantity δ is called the change of the director; the deformation moves the normal fiber of direction $n(\xi)$ in the reference configuration into a fiber whose direction is $n(\xi) + \delta(\xi)$. Generally, $n(\xi) + \delta(\xi)$ is different from the normal $\bar{n}(\xi)$ to the deformed middle surface.

Next, we calculate the 3D small strain tensor of the displacement u in (1.2), i.e.,

$$E = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}})$$

where ∇ is the 3D gradient with respect to the variable $x \in \Omega$. We use (1.1) to express E as a function of ξ and t and of ω and δ and their surface derivatives, and linearize E simultaneously when t and ω and δ and their surface derivatives are small. The linearization is defined explicitly (but on a formal level); no ambiguities occur, see Subsection 7.2. The result is

$$E\big(\xi + tn(\xi)\big) \approx \varepsilon^{\mathrm{N}}(\xi) + t\rho^{\mathrm{D}}(\xi), \tag{1.3}$$

where

$$\varepsilon^{\mathrm{N}} = \frac{1}{2}\Big(P(\nabla\omega + \nabla\omega^{\mathrm{T}})P + (\delta + \nabla\omega^{\mathrm{T}}n) \otimes n + n \otimes \big(\delta + \nabla\omega^{\mathrm{T}}n\big)\Big),$$

$$\rho^{\mathrm{D}} = \frac{1}{2}(\nabla\delta + \nabla\delta^{\mathrm{T}} - \nabla\omega L - L\nabla\omega^{\mathrm{T}}).$$

Here ∇ denotes the surface gradient relative to $\mathscr{S}$ and $L = \nabla n$ the curvature of $\mathscr{S}$. The tensor ε^{N} is Naghdi's surface strain tensor [12; Equation (7.69)]; the tensor ρ^{D} seems to be new.

A linearization procedure similar to ours is undertaken by Naghdi in [12; Section 7], but he arrives at different results [12; Equation (7.69)], viz.,

$$E\big(\xi + tn(\xi)\big) \approx \varepsilon^{\mathrm{N}}(\xi) + t\rho^{\mathrm{N}}(\xi), \tag{1.4}$$

where ε^{N} is as before, but

$$\rho^{\mathrm{N}} = \frac{1}{2}\big(\nabla\delta + \nabla\delta^{\mathrm{T}} + L(\nabla\omega + \delta \otimes n) + (\nabla\omega + \delta \otimes n)^{\mathrm{T}}L\big),$$

which is called Naghdi's bending strain tensor in the literature. I am not able to reproduce Naghdi's result. Example 7.5 tests this discrepancy on a radial deformation of a spherical shell, where E can be evaluated directly. The calculation confirms (1.3), while (1.4) gives the *opposite sign* of the term linear in t.

The Kirchoff-Love deformations are the special case of shear deformations when the normal fiber in the reference configuration is deformed into the normal fiber to the deformed middle surface. A Kirchoff-Love deformation is completely determined by the displacement ω of the middle surface since $\delta = -\nabla\omega^{\mathrm{T}}n$, as will be shown below. The resulting asymptotics is

$$E\big(\xi + tn(\xi)\big) \approx \varepsilon(\xi) + t\rho^{\mathrm{AL}}(\xi) \tag{1.5}$$

where

$$\varepsilon = \frac{1}{2} P (\nabla \omega + \nabla \omega^{\mathrm{T}}) P,$$
$$\rho^{\mathrm{AL}} = -n \cdot \nabla^2 \omega - \varepsilon L - L \varepsilon$$

where P is the projection onto the tangent space to the middle surface in the reference configuration. The tensor ε is used standardly for the Kirchoff-Love deformations. The tensor ρ^{AL}, called the Anicic-Léger bending strain tensor in this paper, was introduced by Anicic & Léger in [2], but their motivation is different. Most of the authors use Koiter's or Budiansky-Sanders bending strain tensors, but Example 7.5 confirms (1.5).

Outline of the paper. Section 2 summarizes the notation and gives a synoptic view on the surface strain and bending strain tensors to be encountered. Section 3 presents a coordinate-free geometry of surfaces suitable for our purposes. Sections 4 and 5 determine the changes of geometric quantities under the deformation of surfaces (most importantly of the curvature); Sections 4 exact and Section 5 linearized. Section 6 introduces a shell-like body Ω as mentioned above and calculates the important formulas for the 3D gradients of the normal coordinates. Finally, Section 7 introduces the shear and Kirchhoff-Love's deformations of Ω, evaluates the exact value of E for them and applies the linearization to obtain formulas outlined above.

2 Glossary of notation

General:

$\mathscr{S}$, ξ	$\nu - 1$ dimensional oriented surface in $\mathbb{R}^\nu$ and its typical point (in applications, $\nu = 3$ or 2)
n, P, L	normal, projection onto the tangent space, and curvature tensor for $\mathscr{S}$
∇, ∇^2	surface gradient and second surface gradient for $\mathscr{S}$
$\eta : \mathscr{S} \to \bar{\mathscr{S}}$	deformation mapping $\mathscr{S}$ onto the deformed surface $\bar{\mathscr{S}}$
$\bar{n}$, $\bar{P}$, $\bar{L}$	normal, projection onto the tangent space, and curvature tensor for $\bar{\mathscr{S}}$
$F = \nabla \eta$	surface deformation gradient of η
$\bar{K}$	convected curvature of $\bar{\mathscr{S}}$
F^{-1}	generalized inverse of a possibly noninvertible tensor
$\omega = \eta - \xi$	displacement
$\mathfrak{k}(G) = \mathfrak{k}_0(G) + \mathfrak{k}_1(G)$	linearization of a differential function $G = G(\xi, \omega^{[k]})$
Ω, x, (ξ, t)	solid (bulk) 3D shell with middle surface $\mathscr{S} \subset \Omega$, typical point of Ω and its normal coordinates $(\xi, t) \in \mathscr{S} \times (-h, h)$
$y : \Omega \to \mathbb{R}^\nu$	shear deformation or Kirchhoff-Love deformation of Ω
$u : \Omega \to \mathbb{R}^\nu$	displacement for y
E	small strain tensor of y
d	the director of a shear deformation
$\delta = d - n$	change of the director
$\mathfrak{l} = \mathfrak{l}(G)$	linearization of a differential function $G = G(\xi, \varphi^{[k]}, t)$

Review of the surface strain and bending strain tensors:

Shear deformations

Present analysis: Naghdi's surface strain ε^{N} and the a new bending strain tensor ρ^{D}:

$$\varepsilon^{\mathrm{N}} = \frac{1}{2}\left(P(\nabla\omega + \nabla\omega^{\mathrm{T}})P + (\delta + \nabla\omega^{\mathrm{T}}n) \otimes n + n \otimes (\delta + \nabla\omega^{\mathrm{T}}n)\right)$$

$$\rho^{\mathrm{D}} = \frac{1}{2}(\nabla\delta + \nabla\delta^{\mathrm{T}} - \nabla\omega L - L\nabla\omega^{\mathrm{T}})$$

Naghdi: Naghdi's surface strain ε^{N} and Naghdi's bending strain tensor ρ^{N}:

$$\rho^{\mathrm{N}} = \frac{1}{2}\left(\nabla\delta + \nabla\delta^{\mathrm{T}} + L(\nabla\omega + \delta \otimes n) + (\nabla\omega + \delta \otimes n)^{\mathrm{T}}L\right)$$

Kirchhoff-Love deformations

Present analysis: classical surface strain ε and the Anicic-Léger bending strain tensor ρ^{AL}:

$$\varepsilon = \frac{1}{2}P(\nabla\omega + \nabla\omega^{\mathrm{T}})P$$

$$\rho^{\mathrm{AL}} = -n \cdot \nabla^2\omega - \varepsilon L - L\varepsilon$$

Koiter, Ciarlet: classical surface strain ε and Koiter's bending strain tensor ρ^{K}:

$$\rho^{\mathrm{AL}} = -n \cdot \nabla^2\omega$$

Budiansky-Sanders: classical surface strain ε and Budiansky-Sanders bending strain tensor ρ^{BS}:

$$\rho^{\mathrm{BS}} = -n \cdot \nabla^2\omega - \frac{1}{2}(\varepsilon L + L\varepsilon)$$

Invariant change of curvature

Present analysis: a new bending strain tensor ρ^{C}:

$$\rho^{\mathrm{C}} = -n \cdot \nabla^2\omega - L(\nabla\omega - \nabla\omega^{\mathrm{T}}n \otimes n) - (\nabla\omega^{\mathrm{T}} - n \otimes \nabla\omega^{\mathrm{T}}n)L$$

3 Geometry of surfaces, surface gradients of orders 1 and 2

This section presents the geometry of $\nu - 1$ dimensional surfaces in $\mathbb{R}^\nu$. Central to the approach are the first and second surface gradients as defined in [14, 13]. Our approach has many features in common with [11, 8, 6, 3–4].

3.1 Surface and its tangent space By a surface we mean an oriented C^2 manifold of dimension $\nu - 1$ (without boundary) embedded in $\mathbb{R}^\nu$. Thus a surface is a pair $(\mathscr{S}, n)$ where $\mathscr{S} \subset \mathbb{R}^\nu$ is the set of points and $n : \mathscr{S} \to \mathbb{S}^{\nu-1}$ is the unit normal giving the orientation, to be specified below. By definition, $\mathscr{S}$ can be locally parametrized by a C^2 parametrizations $\pi : \operatorname{dom}\pi \to \mathscr{S}$. Specifically, we require that (i) π maps homeomorphically an open subset $\operatorname{dom}\pi$ of $\mathbb{R}^{\nu-1}$ onto a relatively open subset $\operatorname{ran}\pi$ of $\mathscr{S}$ and (ii) $\nabla\pi(\xi)$ is an injective linear transformation from

$\mathbb{R}^{\nu-1}$ into $\mathbb{R}^{\nu}$ for every $\xi \in \operatorname{dom} \pi$. The tangent space to $\mathscr{S}$ at $\xi \in \operatorname{cl} \mathscr{S}$ is the linear subspace of $\mathbb{R}^{\nu}$ of dimension $\nu - 1$ given by

$$\operatorname{Tan}(\mathscr{S}, \xi) = \operatorname{ran} \nabla \pi(\xi) \equiv \{\nabla \pi(\xi) a : a \in \mathbb{R}^{\nu-1}\}$$

where π is any local parametrization of $\mathscr{S}$ such that $\xi = \pi(\xi)$ for some $\xi \in \operatorname{dom} \pi$. The normal in the pair $(\mathscr{S}, n)$ is any continuous function $n : \mathscr{S} \to \mathbb{S}^{\nu-1}$ with values in the unit sphere such that $n(\xi)$ is perpendicular to $\operatorname{Tan}(\mathscr{S}, \xi)$ for every $\xi \in \mathscr{S}$. The class C^2 smoothness of $\mathscr{S}$ implies that n is of class C^1. We denote by $P(\xi)$ the orthogonal projection from $\mathbb{R}^{\nu}$ onto $\operatorname{Tan}(\mathscr{S}, \xi)$, given by

$$P = 1 - n \otimes n. \tag{3.1}$$

We often denote by $\mathscr{S}$ the oriented surface $(\mathscr{S}, n)$.

3.2 Definitions Let V be a finite dimensional vector space and $f : \mathscr{S} \to$ V.

(i) We say that f is differentiable at $\xi \in \mathscr{S}$ if f has an extension $\tilde{f} : U \to$ V to an n-dimensional neighborhood U of ξ that is differentiable at ξ. The surface gradient (= derivative) $\nabla f(\xi) \in \operatorname{Lin}(\mathbb{R}^{\nu}, \mathrm{V})$ of f at ξ is defined by

$$\nabla f(\xi)[a] = \nabla \tilde{f}(\xi)[P(\xi)a] \tag{3.2}$$

for any $a \in \mathbb{R}^{\nu}$. This definition is independent of the choice of the extension. The element of V defined by

$$\nabla_a f(\xi) := \nabla f(\xi)[a]$$

is called the directional derivative of f at $\xi \in \mathscr{S}$ in the direction $a \in \mathbb{R}^{\nu}$.

(ii) We say that f is twice differentiable at $\xi \in \mathscr{S}$ if f is differentiable in some neighborhood of ξ in $\mathscr{S}$, and ∇f has the derivative $\nabla(\nabla f)(\xi)$ at ξ. We identify $\nabla(\nabla f)(\xi)$ with an equally denoted bilinear form given by

$$\nabla(\nabla f)(\xi)[a, b] = \nabla_b(\nabla_a f)(\xi)$$

and define the second surface gradient of f at ξ as a bilinear form $\nabla^2 f(\xi)$ given by

$$\nabla^2 f(\xi)[a, b] = \nabla(\nabla f)(\xi)[Pa, b] \tag{3.3}$$

for any $a, b \in \mathbb{R}^{\nu}$.

We interpret the second-order tensors as linear transformations from $\mathbb{R}^{\nu}$ into $\mathbb{R}^{\nu}$. We denote the set of all second-order tensors by Ten^2. Define the functions $L : \mathscr{S} \to \mathrm{Ten}^2$ and $\kappa : \mathscr{S} \to \mathbb{R}$ by

$$L = \nabla n, \quad \kappa = \operatorname{tr} L$$

where tr denotes the trace of a second-order tensor. We call L the curvature of $\mathscr{S}$ and κ the scalar curvature of $\mathscr{S}$. For the boundary $(\mathscr{S}, n)$ of the unit ball oriented by the *exterior* normal n our choice of signs of L and κ provides L a *positive-semidefinite tensor and κ a positive number.* See Example 3.7.

3.3 Proposition *The curvature is symmetric and superficial, i.e.,*

$$L = L^{\mathrm{T}} \quad \text{and} \quad Ln = 0. \tag{3.4}$$

Moreover, we have

$$\nabla_b Pa = -(n \cdot a) Lb - n(La \cdot b) \tag{3.5}$$

for any $a, b \in \mathbb{R}^{\nu}$; equivalently,

$$\nabla P[b] = -Lb \otimes n - n \otimes Lb \tag{3.6}$$

for any $b \in \mathbb{R}^{\nu}$.

3.4 Proposition *If $f : \mathscr{S} \to \mathrm{V}$ is twice continuously differentiable then $\nabla^2 f$ is symmetric, i.e.,*

$$\nabla^2 f[a,b] = \nabla^2 f[b,a] \tag{3.7}$$

and we have

$$\nabla(\nabla f)[a,b] = \nabla^2 f[a,b] - (n \cdot a)\nabla f[Lb]. \tag{3.8}$$

3.5 Lemma (Normal extensions) *If $f : \mathscr{S} \to \mathrm{V}$ is a class C^2 function on $\mathscr{S}$ then for each $x \in \mathscr{S}$ there exists a class C^2 extension of f $\tilde{f} : U \to \mathrm{V}$ defined on some neighborhood U of x in $\mathbb{R}^\nu$ such that*

$$\nabla \tilde{f}(\xi)[n(\xi)] = 0 \quad \text{for every} \quad \xi \in \mathscr{S} \cap U. \tag{3.9}$$

Proof Let Z be the linear subspace of $\mathbb{R}^\nu$ given by

$$Z = \{(x_1, \dots, x_\nu) \in \mathbb{R}^\nu : x_1 = x_2 = \dots = x_{\nu-1} = 0\}.$$

By [7; Subsection 3.1.19], each point of $\mathscr{S}$ has a neighjorhood U in $\mathbb{R}^\nu$, a diffeomorphism $\sigma : U \to \mathbb{R}^\nu$ of class C^2, and a relatively open subset $\mathscr{G}$ of Z such that

$$\sigma(\mathscr{S} \cap U) = \mathscr{G}.$$

If $f : \mathscr{S} \to \mathrm{V}$ is a class C^2 function on $\mathscr{S}$ then $g := f \circ \sigma^{-1}$ is a class C^2 function on $\mathscr{G} \subset Z$. Let $\tilde{g} : \mathscr{G} \times \mathbb{R} \to \mathrm{V}$ be given by

$$\tilde{g}(x_1, \dots, x_\nu) = g(x_1, \dots, x_{\nu-1})$$

for any $(x_1, \dots, x_\nu) \in \mathscr{G} \times \mathbb{R}$. Then $\tilde{f} := \tilde{g} \circ \sigma$ has the required properties. □

***Proof of Propositions* 3.3 & 3.4** The properties (3.4) are well-known and their proof is omitted. To prove (3.5), we note that it is easy to verify that the directional surface gradient satisfies the product rule

$$\nabla_a(\beta \otimes \gamma) = (\nabla_a \beta) \otimes \gamma + \beta \otimes \nabla_a \gamma$$

for any $a \in \mathbb{R}^\nu$ and any two vector fields $\beta, \gamma : \mathscr{S} \to \mathbb{R}^\nu$. Let $a, b \in \mathbb{R}^\nu$ be fixed. From (3.1) we obtain

$$Pb = b - n(n \cdot b);$$

we then take the directional surface gradient ∇_a, use the product rule and the definition of L to obtain (3.5).

To prove (3.8), we take an arbitrary point x of $\mathscr{S}$, a class C^2 extension $\tilde{f}$ of f to a neighborhood of x in $\mathbb{R}^\nu$ satisfying (3.9) and a class C^1 extension $\tilde{P}$ of P to a neighborhood of x in $\mathbb{R}^\nu$. We can assume that $\tilde{f}$ and $\tilde{P}$ are defined on the same neighborhoo U of x. Throughout the proof let a b be fixed vectors in $\mathbb{R}^\nu$. The definition gives

$$\nabla f(\xi)[a] = \nabla \tilde{f}(\xi)[P(\xi)a] \tag{3.10}$$

for any $\xi \in \mathscr{S} \cap U$. Let $g_a : U \to \mathrm{V}$ be defined by

$$g_a(\eta) = \nabla \tilde{f}(\eta)[\tilde{P}(\eta)a], \quad \eta \in U. \tag{3.11}$$

A differentiation in the direction $\tilde{P}b$ yields

$$\nabla g_a[\tilde{P}b] = \nabla^2 \tilde{f}[\tilde{P}a, \tilde{P}b] + \nabla \tilde{f}[\nabla(\tilde{P}a)[\tilde{P}b]] \tag{3.12}$$

throughout U. A comparison of the right-hand sides of (3.10) and (3.11) shows that g_a is a class C^1 extension of $\nabla f[a]$ to U and hence the restriction of (3.12) to $\mathscr{S} \cap U$ provides

$$\nabla(\nabla f[a])[b] = \nabla^2 \tilde{f}[Pa, Pb] + \nabla \tilde{f}[\nabla(Pa)[b]]. \tag{3.13}$$

The insertion of the formula

$$\nabla(Pa)[b] = -(n \cdot a)Lb - n(La \cdot b),$$

which is a consequence of (3.5), into the second term on the right-hand side of (3.13) and the use of (3.9) yieds

$$\nabla(\nabla f[a])[b] = \nabla^2 \tilde{f}[Pa, Pb] - (n \cdot a)\nabla f[Lb]. \tag{3.14}$$

By (3.3) then

$$\nabla^2 f[a, b] = \nabla^2 \tilde{f}[Pa, Pb],$$

which proves (3.7) and reduces (3.14) to (3.8). □

3.6 ***Example*** (Surface gradients of the identity map) Let id : $\mathscr{S} \to \mathbb{R}^\nu$ be the identity map on $\mathscr{S}$ given by

$$\mathrm{id}(\xi) = \xi, \quad \xi \in \mathscr{S}. \tag{3.15}$$

Then

$$\nabla \,\mathrm{id} = P, \quad \nabla^2 \,\mathrm{id} = -n \otimes L. \tag{3.16}$$

Proof The identity map $\mathrm{id}_{\mathbb{R}^\nu}$, given by $\mathrm{id}_{\mathbb{R}^\nu}(\eta) = \eta$ for every $\eta \in \mathbb{R}^\nu$, is a local extension of id. Then (3.2) provides $\nabla \,\mathrm{id} = \nabla \,\mathrm{id}_{\mathbb{R}^\nu} P = P$, which is $(3.16)_1$. From (3.6) we obtain

$$\nabla_b(\nabla_a \mathrm{id}) = \nabla_b(Pa) = \nabla_b\big(a - n(n \cdot a)\big) = -Lb(n \cdot a) - n(La \cdot b),$$

which in combination with (3.3) yields $(3.16)_2$. □

3.7 ***Example*** (The curvature of a sphere) Let $\mathscr{S} = \{\xi \in \mathbb{R}^\nu : |\xi| = r\}$ be the sphere of radius $r > 0$. This is a surface of dimension $\nu - 1$ of class ∞ and we have

$$\begin{aligned} n &= \xi/r, \\ P &= 1 - \xi \otimes \xi/r^2, \\ L &= P/r, \\ \nabla^2 n &= -n \otimes P/r^2 \end{aligned}$$

where the quantities on the left-hand sides are evaluated at a general point $\xi \in \mathscr{S}$.

Proof The expressions for n and P are immediate. To obtain the curvature, we use the local normal extension $\tilde{n}$ of n to $\mathbb{R}^\nu$ given by $\tilde{n}(\eta) = \eta/|\eta|$ for every $\eta \in \mathbb{R}^\nu \sim \{0\}$. Then $\nabla \tilde{n}(\eta) = (|\eta|^2 - \eta \otimes \eta)/|\eta|^3$ and (3.2) yields $L = \nabla n = \nabla \tilde{n} = P/r$. Next we note that for sphere $\mathscr{S}$ we have $n = \mathrm{id}/r$ and thus the expression for $\nabla^2 n$ follows from $(3.16)_2$. □

4 Deformation of a surface. Changes of normal and curvature under deformation

This and the next sections determine the changes of the normal, projection onto the tangent space and curvature under the deformation of a surface. Here we treat exact formulas, the next section linearizations. The central result are the transformation formulas for the curvature in Proposition 4.4. As a preparation, we introduce the surface deformation gradient F and its generalized inverse F^{-1} as a particular case of the generalized of a non-invertible linear transformation to be introduced below.

4.1 Deformation of surfaces Let $(\mathscr{S}, n)$ be an oriented surface. We say that a map $\eta : \mathscr{S} \to \mathbb{R}^\nu$ is a deformation of $\mathscr{S}$ if η is twice continuously differentiable, injective, and the surface deformation gradient of η,

$$F = \nabla \eta, \tag{4.1}$$

has the rank equal to $\nu - 1$ everywhere on $\mathscr{S}$. It follows that the pair $(\mathscr{S}, n)$, given by

$$\bar{\mathscr{S}} = \eta(\mathscr{S}), \quad \bar{n} \circ \eta = |\operatorname{cof} F|^{-1} \operatorname{cof} F n \tag{4.2}$$

is an oriented surface. Indeed, if $\pi : \operatorname{dom} \pi \to \mathscr{S}$ is a local parametrisation of $\mathscr{S}$ then $\eta \circ \pi$ is a local parametrisation of $\bar{\mathscr{S}}$; moreover, the tangent space to $\bar{\mathscr{S}}$ at $\bar{x} \in \bar{\mathscr{S}}$ is given by

$$\operatorname{Tan}(\bar{\mathscr{S}}, \bar{x}) = F(\xi) \operatorname{Tan}(\mathscr{S}, \xi) \quad \text{where} \quad \xi = \eta^{-1}(\bar{x}).$$

To prove $(4.2)_2$, we note that the tensor of cofactors is the unique continuous function $\operatorname{cof} : \mathrm{Ten}^2 \to \mathrm{Ten}^2$ satisfying $F^{\mathrm{T}} \operatorname{cof} F = (\det F) 1$ for every $F \in \mathrm{Ten}^2$. In the case (4.1), we have $F^{\mathrm{T}} \operatorname{cof} F = 0$ from which $\operatorname{cof} F n \cdot F t = 0$ for every $t \in \operatorname{Tan}(\mathscr{S}, \xi)$ and hence $\operatorname{cof} F n \cdot \bar{t} = 0$ for every $\bar{t} \in \operatorname{Tan}(\bar{\mathscr{S}}, \bar{x})$.

We call the pair $(\bar{\mathscr{S}}, \bar{n})$ the image of $(\mathscr{S}, n)$ under the deformation η. The general unit normal to $\bar{\mathscr{S}}$ is, of course, locally $\pm |\operatorname{cof} F|^{-1} \operatorname{cof} F n$, but our sign convention for the image is always as in $(4.2)_2$.

The formula

$$\bar{P} = 1 - \bar{n} \otimes \bar{n}$$

gives the projection onto the tangent space of $\bar{\mathscr{S}}$. We further denote by $\bar{L} : \bar{\mathscr{S}} \to \mathrm{Ten}^2$ the curvature of $\bar{\mathscr{S}}$, given by

$$\bar{L} = \nabla \bar{n},$$

where, of course, ∇ denotes the surface differentiation with respect to $\bar{\mathscr{S}}$. Finally, we denote by $\bar{K} : \mathscr{S} \to \mathrm{Sym}$ the convected curvature, i.e., the pullback of $\bar{L}$ to $\mathscr{S}$, given by

$$\bar{K} = F^{\mathrm{T}} (\bar{L} \circ \eta) F.$$

4.2 Generalized inverse To state the formulas that follow in a notationally simple form, we now introduce the generalized inverse of any $F \in \mathrm{Ten}^2$, invertible or not. Namely, we shall prove that for each $F \in \mathrm{Ten}^2$ there exists a unique $F^{-1} \in \mathrm{Ten}^2$ such that

$$F^{-1} F = P, \quad F F^{-1} = \bar{P}. \tag{4.3}$$

where P is the orthogonal projection onto the orthogonal complement $(\ker F)^{\perp}$ of the kernel of F and $\bar{P}$ is the orthogonal projection onto the range $\operatorname{ran} F$ of F. It suffices to note that F maps bijectively $(\ker F)^{\perp}$ onto $\operatorname{ran} F$ and to put

$$F^{-1}a = \begin{cases} F_0^{-1}a & \text{if} \quad a \in \operatorname{ran} F, \\ 0 & \text{if} \quad a \in (\operatorname{ran} F)^{\perp}, \end{cases}$$

where $F_0^{-1} : \operatorname{ran} F \to (\ker F)^{\perp}$ is the inverse of the restriction F_0 of F to $(\ker F)^{\perp}$. Clearly,

$$F = FP = \bar{P}F, \quad F^{-1} = F^{-1}\bar{P} = PF^{-1},$$
$$(F^{-1})^{-1} = F.$$

If F is bijective, the generalized inverse coincides with the usual inverse of F. One has $0^{-1} = 0$ and $P^{-1} = P$ for any orthogonal projection P.

If D is a $\mathbb{R}^{\nu}$-valued bilinear form on $\mathbb{R}^{\nu} \times \mathbb{R}^{\nu}$ and $c \in \mathbb{R}^{\nu}$ a vector, we define the product $c \cdot \mathsf{D}$ as a second-order tensor satisfying

$$(c \cdot \mathsf{D})a \cdot b = c \cdot \mathsf{D}(a, b) \tag{4.4}$$

for every a, and $b \in \mathbb{R}^{\nu}$.

4.3 Lemma *If* $g : \mathscr{S} \to \mathbb{R}^{\nu}$ *is a* C^2 *function and* $a \in \mathbb{R}^{\nu}$ *a fixed vector then*

$$\nabla(\nabla g^{\mathrm{T}} a) = a \cdot \nabla^2 g - n \otimes L \nabla g^{\mathrm{T}} a \tag{4.5}$$

for ever throughout $\mathscr{S}$.

Proof Let $f : \mathscr{S} \to \mathbb{R}$ be defined by $f(\xi) = g(\xi) \cdot a$ for every $\xi \in \mathscr{S}$. Then $\nabla f[b] = \nabla g[b] \cdot a = \nabla g^{\mathrm{T}} a \cdot b$ for every $b \in \mathbb{R}^{\nu}$. Hence (3.8) yields, for every $c \in \mathbb{R}^{\nu}$,

$$\begin{aligned} \nabla_c(\nabla g^{\mathrm{T}} a \cdot b) &= \nabla_c(\nabla_b f) \\ &= \nabla^2 f[b, c] - (b \cdot n)\, \nabla f[Lc] \\ &= a \cdot \nabla^2 g[b, c] - (b \cdot n)\, (L \nabla g^{\mathrm{T}} a \cdot c) \end{aligned} \tag{4.6}$$

where we have used $\nabla^2 f = a \cdot \nabla^2 g$. Since b and c are arbitrary, we can restate (4.6) as (4.5). □

The following important proposition expresses the quantities $\bar{L}$ and $\bar{K}$ in terms of the second surface gradient of the deformation function η.

4.4 Proposition *We have*

$$\bar{L} \circ \eta = -F^{-\mathrm{T}}[(\bar{n} \circ \eta) \cdot \nabla^2 \eta] F^{-1}, \tag{4.7}$$

$$\bar{K} = -(\bar{n} \circ \eta) \cdot \nabla^2 \eta. \tag{4.8}$$

By (4.4), at each $\xi \in \mathscr{S}$, the right-hand sides of (4.7) and (4.8) are symmetric second-order tensors S, T satisfying

$$a \cdot Sb = -\bar{n}(\eta) \cdot \nabla^2 \eta[F^{-1}a, F^{-1}b],$$
$$a \cdot Tb = -\bar{n}(\eta) \cdot \nabla^2 \eta[a, b],$$

for every a, $b \in \mathbb{R}^{\nu}$, where the argument ξ has been omitted.

Proof Since the normal to the deformed surface is always in the orthogonal complement of F, we have the relation $F^{\mathrm{T}}(\bar{n} \circ \eta) = 0$ at every poin ξ of $\mathscr{S}$. The surface gradient of this relation provides

$$F^{\mathrm{T}} \nabla(\bar{n} \circ \eta) + (\nabla F^{\mathrm{T}})(\bar{n} \circ \eta) = 0.$$

We now apply (4.5) with $g = \eta$ to evaluate the second term to obtain

$$F^{\mathrm{T}}(\nabla \bar{n} \circ \eta) + (\bar{n} \circ \eta) \cdot \nabla^2 \eta - n \otimes L F^{\mathrm{T}}(\bar{n} \circ \eta) = 0.$$

We multiply this equation by F^{-1} from the right and combine with the chain rule to obtain

$$F^{\mathrm{T}} \bar{L} \circ \eta + (\bar{n} \circ \eta) \cdot (\nabla^2 \eta) F^{-1} - n \otimes F^{-\mathrm{T}} L F^{\mathrm{T}}(\bar{n} \circ \eta) = 0.$$

A multiplication from the right by $F^{-\mathrm{T}}$ and the use of $F^{-\mathrm{T}} n = 0$ gives (4.7). Equation (4.8) is a consequence. □

5 Linearized changes of normal and curvature under deformation of a surface

This section gives approximate formulas for the changes of the normal, projection onto the tangent space and curvature under the deformation $\eta : \mathscr{S} \to \bar{\mathscr{S}}$ provided the displacement

$$\omega(\xi) = \eta(\xi) - \xi, \quad \xi \in \mathscr{S},$$

and its surface gradients up to some order k are small. We recall that by the results of Section 4 the quantities $\bar{n}$, $\bar{P}$, $\bar{L}$, and $\bar{K}$ can be expressed as functions of η and its surface derivatives up to order 2. Hence they can be expressed as functions of ω and its surface derivatives up to order 2.

5.1 Linearization in small displacement If $\varphi : \mathscr{S} \to \mathrm{W}$ is a function on $\mathscr{S}$ with values in a finite dimensional vector space W and $k = 1$ or 2, we denote by

$$\varphi^{[k]}(\xi) = \begin{cases} (\varphi, \nabla \varphi) & \text{if} \quad k = 1, \\ (\varphi, \nabla \varphi, \nabla^2 \varphi) & \text{if} \quad k = 2 \end{cases} \tag{5.1}$$

the collection of surface derivatives of φ up to order k. By a differential function G of φ of order k with values in a finite dimensional vector space V we mean a function $G = G(\xi, \varphi^{[k]})$ where G is a continuously differentiable function on its domain and with values in V. We consider a field g on $\mathscr{S}$ with values in a finite dimensional vector space V given by a differential function of order k with values in V

$$g(\xi) = G(\xi, \omega^{[k]}(\xi)).$$

By the linearization★ of the field g with respect to ω we mean a quantity $\bar{g} : \mathscr{S} \to \mathrm{V}$ given by

★ The concept of linearization applies verbatim to differential functions arbitrary order k, with (5.1) replaced by

$$\varphi^{[k]}(\xi) = (\varphi(\xi), \dots, \nabla^k \varphi(\xi)). \tag{5.2}$$

$$\bar{g}(\xi) = \mathfrak{k}(G)(\xi, \omega^{[k]}(\xi))$$

where $\mathfrak{k}(G)$ is a differential function of ω of order k given by

$$\mathfrak{k}(G)(\xi, \omega^{[k]}(\xi)) = \mathfrak{k}_0(G)(\xi) + \mathfrak{k}_1(G)(\xi)[\omega^{[k]}],$$

with $\mathfrak{k}_0(G)$ and $\mathfrak{k}_1(G)$ the differential functions of ω of order k defined by

$$\mathfrak{k}_0(G)(\xi) = F(\xi, 0), \quad \mathfrak{k}_1(G)(\xi)[\omega^{[k]}] = \frac{dG(\xi, s\omega^{[k]})}{ds}\Big|_{s=0}.$$

We note that $\mathfrak{k}_1(G)(\xi)[\omega^{[k]}]$ depends on $\omega^{[k]}$ linearly. We then write

$$g \approx \mathfrak{k}(G) \quad \text{as} \quad \omega^{[k]} \to 0$$

or equivalently but more standardly

$$g = \mathfrak{k}(G) + o(|\omega^{[k]}|).$$

Let V, W and X finite-dimensional vector spaces and let $(a, b) \mapsto a \odot b$ be a bilinear operation from V $\times$ W into X. The operation $\odot$ can be, e.g., the product of two real numbers, scalar or tensor products, etc. Let G and H be differential function with values in V and W. Then we have Leibniz's rule

$$\mathfrak{k}_1(G \odot H) = \mathfrak{k}_1(G) \odot \mathfrak{k}_0(H) + \mathfrak{k}_0(G) \odot \mathfrak{k}_1(H). \tag{5.3}$$

5.2 Proposition ***(Linearized changes of normal and tangential projection)*** *We have*

$$\bar{n} \approx n - \nabla\omega^{\mathrm{T}} n, \tag{5.4}$$

$$\bar{P} \approx P + \nabla\omega^{\mathrm{T}} n \otimes n + n \otimes \nabla\omega^{\mathrm{T}} n \tag{5.5}$$

as $\omega^{[1]} \to 0$.

Proof Equation (5.4): We apply the operator $\mathfrak{k}_1$ to the equations

$$F^{\mathrm{T}}\bar{n} = 0 \quad \text{and} \quad \bar{n} \cdot \bar{n} = 1.$$

The use of Leibniz's rule (5.3) and of the relations

$$\mathfrak{k}_1(F) = \nabla\omega, \quad \mathfrak{k}_0(F) = P, \quad \mathfrak{k}_0(\bar{n}) = n,$$

then gives

$$\nabla\omega^{\mathrm{T}} n + P\mathfrak{k}_1(\bar{n}) \approx 0, \quad n \cdot \mathfrak{k}_1(\bar{n}) \approx 0.$$

The elimination of P from the first equation via $P = 1 - n \otimes n$ and the simplification of the result by the second equation provides

$$\nabla\omega^{\mathrm{T}} n + \mathfrak{k}_1(n) = 0$$

which is (5.4).

Equation (5.5) is obtained by the application of the operator $\mathfrak{k}_1$ to the equation $P = 1 - n \otimes n$ and the use of Leibniz's rule in combination with (5.4). $\square$

5.3 Theorem ***(Linearized changes of plain and convected curvatures)*** *We have*

$$\bar{L} \approx L + \rho^{\mathrm{C}}, \tag{5.6}$$

$$\bar{K} \approx L + \rho^{\mathrm{K}} \tag{5.7}$$

as $\omega^{[2]} \to 0$, *where* ρ^{C} *and* ρ^{K} *are second-order tensors given by*

$$\rho^{\mathrm{C}} = -n \cdot \nabla^2 \omega - L(\nabla\omega - \nabla\omega^{\mathrm{T}} n \otimes n) - (\nabla\omega^{\mathrm{T}} - n \otimes \nabla\omega^{\mathrm{T}} n) L,$$

$$\rho^{\mathrm{K}} = -n \cdot \nabla^2 \omega.$$

Proof Recall the identity map $\mathrm{id} : \mathscr{S} \to \mathbb{R}^\nu$ from (3.15).

Equation (5.6): The application of Leibniz's rule (5.3) to the right-hand side of (4.7) gives

$$\mathfrak{k}_1(\bar{L}) = -\mathfrak{k}_1(F^{-\mathrm{T}})(n \cdot \nabla^2 \mathrm{id}) - \mathfrak{k}_1(\bar{n}) \cdot \nabla^2 \mathrm{id} - n \cdot \nabla^2 \omega - (n \cdot \nabla^2 \mathrm{id}) \mathfrak{k}_1(F^{-1}) \tag{5.8}$$

where we have used

$$\mathfrak{k}_0(\nabla^2 \eta) = \nabla^2 \mathrm{id}, \quad \mathfrak{k}_1(\nabla^2 \eta) = \nabla^2 \omega \tag{5.9}$$

since $\eta = \mathrm{id} + \omega$.

We now linearize the individual terms occurring in (5.8). First, let us prove that

$$\mathfrak{k}_1(F^{-1}) = -P\nabla\omega + \nabla\omega^{\mathrm{T}} n \otimes n. \tag{5.10}$$

Indeed, the application of the operator $\mathfrak{k}_1$ to the equations $F^{-1}F = P$ and $F^{-1}\bar{n} = 0$ and Leibniz's rule give

$$\mathfrak{k}_1(F^{-1})P + P\nabla\omega = 0, \quad \mathfrak{k}_1(F^{-1})n - \nabla\omega^{\mathrm{T}} n = 0.$$

The elimination of P from the first equation via $P = 1 - n \otimes n$ and the simplification of the result by the second equation provides (5.10). Further, Equations $(3.16)_2$ and (5.4) provide

$$n \cdot \nabla^2 \mathrm{id} = -L, \quad \mathfrak{k}_1(\bar{n}) \cdot \nabla^2 \mathrm{id} = 0. \tag{5.11}$$

The insertion of (5.10) and (5.11) into (5.8) gives (5.6).

Equation (5.7): We linearize (4.8) by Leibniz's rule to obtain

$$\mathfrak{k}_1(\bar{K}) = -\mathfrak{k}_1(\bar{n}) \cdot \nabla^2 \mathrm{id} = -n \cdot \nabla^2 \omega. \tag{5.12}$$

Equations $(3.16)_2$ and (5.4) provide $\mathfrak{k}_1(\bar{n}) \cdot \nabla^2 \mathrm{id} = 0$ and thus (5.12) reduces to (5.7). □

5.4 Coordinates and convected coordinates Let $\pi : \mathrm{dom}\,\pi \to \mathscr{S}$ be a local parametrization of $\mathscr{S}$ defined on an open subset $\mathrm{dom}\,\pi$ of $\mathbb{R}^{\nu-1}$ (Section 3.1). Each $\xi \in \mathscr{S}$ that belong also to the range $\mathrm{ran}\,\pi$ of π can be written as $\xi = \pi(x^1, \dots, x^{\nu-1})$ where $(x^1, \dots, x^{\nu-1}) \in \mathrm{dom}\,\pi$ are called local coordinates of ξ. Let $a_\alpha : \mathrm{ran}\,\pi \to \mathbb{R}^\nu$ be the coordinate vectors, i.e.,

$$a_\alpha \circ \pi = \partial\pi/\partial x^i, \quad a_\alpha(\xi) \in \mathrm{Tan}(\mathscr{S}, \xi), \quad \alpha = 1, \dots, \nu - 1,$$

and let a^α be the dual basis (in the tangent space). If $\eta : \mathscr{S} \to \bar{\mathscr{S}}$ is a deformation, then each point $\bar{\xi} \in \bar{\mathscr{S}} \cap \eta(\mathrm{ran}\,\pi)$ can be written as $\bar{\xi} = \eta\big(\pi(x^1, \dots, x^{\nu-1})\big)$ where $(x^1, \dots, x^{\nu-1}) \in \mathrm{dom}\,\pi$ are called convected local coordinates of $\bar{\xi}$. We denote by

$\bar{a}_\alpha(\bar{\xi})$, $\bar{a}^\alpha(\bar{\xi}) \in \mathrm{Tan}(\bar{\mathscr{S}}, \bar{\xi})$ the coordinate vectors corresponding to the convected coordinates. Clearly,

$$\bar{a}_\alpha \circ \eta = F a_\alpha, \quad \bar{a}^\alpha \circ \eta = F^{-\mathrm{T}} a^\alpha \tag{5.13}$$

where F is the deformation gradient of η. Let $L_{\alpha\beta}$ be the components of L in the basis a^α and let $\bar{L}_{\alpha\beta}$ be the components of $\bar{L}$ in the basis $\bar{a}^\alpha$ of the convected coordinate system i.e.,

$$L = L_{\alpha\beta} a^\alpha \otimes a^\beta, \quad \bar{L} = \bar{L}_{\alpha\beta} \bar{a}^\alpha \otimes \bar{a}^\beta,$$

where we use the summation convention.

5.5 Theorem ***(Linearized changes of components of the curvature tensor)*** *We have*

$$\bar{L}_{\alpha\beta} \approx L_{\alpha\beta} + \rho^{\mathrm{K}}_{\alpha\beta} \quad \text{as} \quad \omega^{[2]} \to 0, \tag{5.14}$$

where $\rho^{\mathrm{K}}_{\alpha\beta}$ are the components of ρ^{K} in the basis a^α.

Since the linearization of the components in (5.14) does not take into account the change $a^\alpha \to \bar{a}^\alpha$, the result does not coincide the linearization of $\bar{L}$ given by (5.6). For the linearization of the components of $\bar{L}$, see [10], [5; p. 340].

Proof Let $\bar{K}_{\alpha\beta}$ be the components of $\bar{K}$ in the basis a^α, i.e.,

$$\bar{K} = \bar{K}_{\alpha\beta} a^\alpha \otimes a^\beta.$$

The definition of $\bar{L}$ and $\bar{K}$ and (5.13) show that $\bar{L}_{\alpha\beta} = \bar{K}_{\alpha\beta}$. Consequently, $\bar{K} = \bar{L}_{\alpha\beta} a^\alpha \otimes a^\beta$ and hence

$$\mathfrak{k}_1(\bar{K}) = \mathfrak{k}_1(\bar{L}_{\alpha\beta}) a^\alpha \otimes a^\beta = \rho^{\mathrm{K}}_{\alpha\beta} a^\alpha \otimes a^\beta$$

by (5.7). Equation (5.14) follows. □

The eigenvalues of the curvature tensor L are called the principal curvatures of $\mathscr{S}$. We recall that L is superficial, i.e., $Ln = 0$ and hence at least one eigenvalue of L is equal to 0. The remaining eigenvalues are denoted by $\kappa_1 \geq \ldots \geq \kappa_{\nu-1}$, where in this descending sequence we take into account multiplicities. We define the principal curvatures $\bar{\kappa}_1 \geq \ldots \geq \bar{\kappa}_{\nu-1}$ of $\bar{\mathscr{S}}$ analogously.

5.6 Theorem ***(Linearized changes of principal curvatures,*** Anicic [1]) *Let $\mathscr{S}$ have distinct principal curvature κ_α, $\alpha = 1, \ldots, \nu - 1$, with the corresponding normalized eigenvectors of L denoted by e_α. Then the linearizations of the principal curvatures of $\bar{\mathscr{S}}$ are*

$$\bar{\kappa}_\alpha \approx \kappa_\alpha + \rho^{\mathrm{C}} e_\alpha \cdot e_\alpha \equiv \kappa_\alpha + \rho^{\mathrm{AL}} e_\alpha \cdot e_\alpha \tag{5.15}$$

as $\omega^{[2]} \to 0$, where ρ^{AL} is a second-order tensor given by

$$\rho^{\mathrm{AL}} = -n \cdot \nabla^2 \omega - \varepsilon L - L\varepsilon. \tag{5.16}$$

Proof Since the principal curvatures $\bar{\kappa}_\alpha$ are the eigenvalues of $\bar{L}$, which we write as $\bar{\kappa}_\alpha = \hat{\kappa}_\alpha(\bar{L})$, the well-known continuity and differentiability properties of eigenvalues of a symmetric tensor (see, e.g., [9; Eq. (6.10), p. 125] or [15; Section 2, Eq. (2.1)]) show that $\hat{\kappa}_\alpha(\bar{L})$ are Lipschitz continuous functions of $\bar{L}$ and if $\hat{\kappa}_\alpha(L)$ is a simple eigenvalue of L, then $\hat{\kappa}_\alpha(\bar{L})$ is infinitely differentiable in some neighborhood of L and the derivative at L is given by

$$\mathrm{D}\hat{\kappa}_\alpha(L)[B] = Be_\alpha \cdot e_\alpha$$

for every symmetric $B \in \mathrm{Ten}^2$ where e_α is the corresponding unit eigenvector of L. The last formula in combination with the chain rule and (5.6) gives

$$\mathfrak{k}_1(\bar{\kappa}_\alpha) = \mathfrak{k}_1(\bar{L})e_\alpha \cdot e_\alpha = \rho^{\mathrm{AL}} e_\alpha \cdot e_\alpha.$$

Observing that $\rho^{\mathrm{AL}} e_\alpha \cdot e_\alpha = \rho^{\mathrm{AL}} e_\alpha \cdot e_\alpha$, we obtain (5.15). □

5.7 ***Example*** (Comparison of linearized curvature tensors) We here consider an isotropic expansion a sphere of radius $r > 0$ in $\mathbb{R}^\nu$ and the goal is to compare the changes of ρ^{K}, ρ^{BS}, ρ^{AL}, and ρ^{C}. Thus we denote by $\mathscr{S}$ the sphere of radius r as in Example 3.7 and by $\bar{\mathscr{S}}$ the sphere of radius $\bar{r} > r$. By the results of that example, the curvature tensors of $\mathscr{S}$ and $\bar{\mathscr{S}}$ are

$$L = P/r, \quad \bar{L} = \bar{P}/\bar{r},$$

where P and $\bar{P}$ are the projections onto the tangents spaces of $\mathscr{S}$ and $\bar{\mathscr{S}}$. Let $\eta : \mathscr{S} \to \bar{\mathscr{S}}$ be the deformation given by $\eta(\xi) = (\bar{r}/r)\xi$ for every $\xi \in \mathscr{S}$ so that the displacement is

$$\omega(\xi) = vn(\xi)$$

where $v = \bar{r} - r$. Easy calculations bases on the equation $\nabla^2 n = -n \otimes P/r^2$, proved in Example 3.7, gives that

$$\nabla^2\omega = -(v/r)n \otimes L = -(v/r^2)n \otimes P.$$

Based on that we deduce

$$\rho^{\mathrm{K}} = (v/r^2)P, \quad \rho^{\mathrm{BS}} = 0, \quad \rho^{\mathrm{C}} = \rho^{\mathrm{AL}} = -(v/r^2)P.$$

We now linearize the difference $\bar{L}(\bar{\xi}) - L(\xi)$ where $\bar{\xi} = \eta(\xi) = (r+v)\xi/r$ using calculations independent of the proof of Theorem 5.3. One finds that

$$\bar{L}(\bar{\xi}) - L(\xi) = \big(1/(r+v) - 1/r\big)\, P(\xi)$$

and after a linearization of the difference $1/(r+v) - 1/r \approx -v/r^2$, we obtain

$$\bar{L}(\bar{\xi}) - L(\xi) \approx -(v/r^2)\, P(\xi).$$

Thus only ρ^{AL} and ρ^{C} give the correct results.

5.8 ***Example*** (Normal expansion of a surface) The preceding example can be generalized. Let $\mathscr{S}$ be a general surface and consider a map $\eta : \mathscr{S} \to \mathbb{R}^\nu$ given by

$$\eta(\xi) = \xi + tn(\xi),$$

$\xi \in \mathscr{S}$, where $t \in \mathbb{R}$ is a given number. Using $(3.16)_1$ we find that the surface deformation gradient is

$$F(\xi) = P(\xi) + tL(\xi).$$

The normal to the deformed surface $\bar{\mathscr{S}} = \eta(\mathscr{S})$ is, obviously,

$$\bar{n}(\bar{\xi}) = n(\xi)$$

where $\bar{\xi} = \eta(\xi)$. The curvature of the deformed surface is the surface gradient of $\bar{\xi}$ with respect to $\bar{\xi}$. By the chain rule, $\bar{L} = \nabla \bar{n} = F^{-\mathrm{T}} \nabla n$ and one finds

$$\bar{L} = (P + tL)^{-1} L \equiv L(P + tL)^{-1}.$$

Thus the change of curvature under the passage from $\mathscr{S}$ to $\bar{\mathscr{S}}$ is

$$\bar{L}(\bar{\xi}) - L(\xi) = L(P + tL)^{-1} - L.$$

It is easy to verify that $\bar{L}(\bar{\xi}) - L(\xi)$ is negative semidefinite if $t \geq 0$. Thus a genuine normal expansion of $\mathscr{S}$ can only *decease* the curvature, an intuitively clear fact. The linearized change of curvature is

$$\bar{L}(\bar{\xi}) - L(\xi) \approx -tL^2.$$

6 Shells-like bodies

The following notion is central for this and the next sections.

A shell-like body is a subset Ω of $\mathbb{R}^\nu$ of the form

$$\Omega := \{x = \xi + tn(\xi) : \xi \in \mathscr{S}, \ -h < t < h\},$$

where $(\mathscr{S}, n)$ is an $\nu - 1$-dimensional oriented surface and h a positive number. For technical reasons we assume that $\mathscr{S}$ is bounded and that there exists a C^2 oriented surface $\tilde{\mathscr{S}}$ such that the closure $\operatorname{cl}\mathscr{S}$ of $\mathscr{S}$ in $\mathbb{R}^\nu$ satisfies $\operatorname{cl}\mathscr{S} \subset \tilde{\mathscr{S}}$. We call $\mathscr{S}$ the middle surface of Ω and $2h$ the thickness of Ω. As before, we denote by P, $L : \mathscr{S} \to \mathrm{Ten}^2$ the projection onto the tangent space of $\mathscr{S}$, and the curvature of $\mathscr{S}$, respectively.

6.1 Proposition (Normal coordinates) *If the thickness of a shell-like body Ω is sufficiently small then the map $x_\circ : \mathscr{S} \times (-h, h) \to \mathbb{R}^\nu$ defined by*

$$x_\circ(\xi, t) = \xi + tn(\xi), \quad (\xi, t) \in \mathscr{S} \times (-h, h), \tag{6.1}$$

is a diffeomorphism onto Ω. The inverse map associates with any $x \in \Omega$ an element of $\mathscr{S} \times (-h, h)$ which we denote by $(\xi_\circ(x), t_\circ(x))$. We have

$$\nabla \xi_\circ(x) = \big(P(\xi) + tL(\xi)\big)^{-1}, \quad \nabla t_\circ(x) = n(\xi) \tag{6.2}$$

for any $x \in \Omega$, where $\xi = \xi_\circ(x)$ and where the right-hand side of (6.1)$_1$ *is the generalized inverse of the noninvertible tensor $P + tL$, see Subsection* 4.2. *We call the parameters $\xi_\circ(x), t_\circ(x)$ the normal coordinates of x.*

Throughout this chapter, Ω denotes a shell-like body that admits normal coordinates $(\xi_\circ, t_\circ)$.

Proof The normal coordinates are solutions of the equation

$$x = \xi_\circ(x) + t_\circ(x) n\big(\xi_\circ(x)\big) \tag{6.3}$$

Let $\tilde{\mathscr{S}}$ be the extension of $\operatorname{cl}\mathscr{S}$ as in the definition of a shell-like body. Consider an extension $\tilde{x}_\circ : \tilde{\mathscr{S}} \times (-h, h) \to \mathbb{R}^\nu$ of $x_\circ$ given by the right-hand side of (6.1) for every $(\xi, t) \in \tilde{\mathscr{S}} \times (-h, h)$. The derivative of $\tilde{x}_\circ$ at $\xi \in \tilde{\mathscr{S}}$ and $t = 0$ is given by

$$\nabla \tilde{x}_\circ(\xi, 0)(a, b) = P(\xi)a + b\bar{n}(\xi)$$

for every $a \in \mathrm{Tan}(\tilde{\mathscr{S}}, \xi)$ and $b \in \mathbb{R}$, where we have used that the surface gradient of the identity map $\mathrm{id}_{\tilde{\mathscr{S}}}$ on $\tilde{\mathscr{S}}$ is $\nabla\, \mathrm{id}_{\tilde{\mathscr{S}}} = P$, see (3.16). From the definition of a surface one readily deduces that $\nabla \tilde{x}_\circ(\xi, 0)$ maps $\mathbb{R}^\nu$ bijectively onto $\mathbb{R}^\nu$ for any $\xi \in \tilde{\mathscr{S}}$. Thus by the inverse function theorem, for every point $\xi \in \tilde{\mathscr{S}}$, $\tilde{x}_\circ$ has a continuously differentiable inverse $\tilde{x}_\circ^{-1} : \mathscr{O}_\xi \to \tilde{\mathscr{S}} \times \mathbb{R}$ defined in some neighborhood of ξ. Since $\mathrm{cl}\, \mathscr{S}$ is compact, a standard compactness argument shows that $\tilde{x}_\circ$ is invertible in some neighborhood of $\mathrm{cl}\, \mathscr{S}$. Hence $x_\circ$ has a continuously differentiable inverse in Ω if h is sufficiently small. To prove (6.2), we differentiate (6.3) in the direction $c \in \mathbb{R}^\nu$. The definition of L provides

$$c = \nabla \xi_\circ[c] + \nabla t_\circ[c] n(\xi_\circ) + t_\circ L(\xi_\circ) \nabla \xi_\circ[c]$$

where we have omitted the argument x. Splitting the last equation into the tangential and normal components and solving for $\nabla \xi_\circ(x)$ and $\nabla t_\circ(x)$ we obtain (6.2). □

7 Linearized strain tensors: the shear and Kirchhoff-Love's cases

This section deals with shear deformations and Kirchhoff-Love's deformations of a shell-like body of Ω. The strain tensor E is determined for these two types of deformations as functions of the normal coordinates (ξ, t) of a point $x \in \Omega$. The objective is to linearize E under the assumption that t and the displacement are small.

7.1 Shear deformations A map $y : \Omega \to \mathbb{R}^\nu$ is said to be a shear deformation if

$$y(x) = \eta(\xi) + t d(\xi)$$

for any $x \in \Omega$ with the normal coordinates (ξ, t), where $\eta : \mathscr{S} \to \mathbb{R}^\nu$ is a deformation of $\mathscr{S}$ (see Subsection 4.1) and $d : \mathscr{S} \to \mathbb{R}^\nu$ is a map.

The function d gives the direction after the deformation of the material line that was normal to the middle surface before the deformation.

The bulk displacement u and the bulk strain tensor E of any deformation are given by

$$u(x) = y(x) - x, \quad x \in \Omega,$$

and

$$E = \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}).$$

One obtains

$$u(x) = \omega(\xi) + t\delta(\xi)$$

where (ξ, t) are the normal coordinates of $x \in \Omega$ and

$$\omega(\xi) = \eta(\xi) - \xi, \quad \delta(\xi) = d(\xi) - n(\xi) \tag{7.1}$$

are the displacement of the middle surface and the change of d, respectively. A differention using (6.2) provides

$$\nabla u = (\nabla \omega + t \nabla \delta)(P + tL)^{-1} + \delta \otimes n. \tag{7.2}$$

The derivative in (7.2), and hence also E, depends on $(\omega^{[1]}, \delta^{[1]}, t)$ (see the notation in (5.1)). In addition, there is a parametric dependence on ξ through n, P and L. The dependence on t is nonlinear. It will be seen below that in the Kirchhoff-Love case ∇u depends on $(\omega^{[2]}, t)$ (and the parametric dependence on ξ). For thin shells $|t|$ is small. In addition, we shall consider small displacements. Hence we linearize simultaneously in $(\omega^{[1]}, \delta^{[1]}, t)$ or in $(\omega^{[2]}, t)$. The following format covers these two cases (and more).

7.2 Linearization in $(\varphi^{[k]}, t)$ Let $\varphi : \mathscr{S} \to \mathrm{W}$ be a function on $\mathscr{S}$ with values in a finite dimensional vector space W and $k = 1$ or 2. By a differential function G of (φ, t) of order k with values in a finite dimensional vector space V we mean a function $G = G(\xi, \varphi^{[k]}, t)$ where G is a continuously differentiable function on its domain and with values in V. We consider a field g on $\mathscr{S}$ with values in a finite dimensional vector space V given by a differential function of (φ, t) of order k with values in V

$$g(\xi, t) = G(\xi, \varphi^{[k]}(\xi), t).$$

By the linearization[⋆] of the field g with respect to (φ, t) we mean a quantity $\bar{g} : \mathscr{S} \to \mathrm{V}$ given by

$$\bar{g}(\xi) = \mathrm{I}(G)(\xi, \varphi^{[k]}(\xi), t)$$

where $\mathrm{I}(G)$ is a differential function of $(\varphi^{[k]}, t)$ of the following form

$$\begin{aligned}
&\mathrm{I}(G)(\xi, \varphi^{[k]}, t) = \\
&\quad = \mathrm{I}_{00}(G)(\xi) + \mathrm{I}_{10}(G)(\xi)[\varphi^{[k]}] + t\mathrm{I}_{01}(G)(\xi) + t\mathrm{I}_{11}(G)(\xi)[\varphi^{[k]}].
\end{aligned} \tag{7.3}$$

Here

$$\begin{aligned}
\mathrm{I}_{00}(G)(\xi) &= G(\xi, 0, 0), \\
\mathrm{I}_{10}(G)(\xi)[\varphi^{[k]}] &= \frac{\partial G(\xi, s\varphi^{[k]}, 0)}{\partial s}, \\
t\mathrm{I}_{01}(G)(\xi) &= \frac{\partial G(\xi, 0, \tau t)}{\partial \tau}, \\
t\mathrm{I}_{11}(G)(\xi)[\varphi^{[k]}] &= \frac{\partial^2 G(\xi, s\varphi^{[k]}, \tau t)}{\partial s \partial \tau}
\end{aligned}$$

where the derivatives are evaluated at $s = \tau = 0$. We note that $\mathrm{I}_{10}(G)(\xi)[\varphi^{[k]}]$ and $\mathrm{I}_{11}(G)(\xi)[\varphi^{[k]}]$ depend on $\varphi^{[k]}$ linearly. We express the linearization by the relation

$$g \approx \mathrm{I}(G) \quad \text{as} \quad (\varphi^{[k]}, t) \to 0,$$

or equivalently but more standardly, by

$$g = \mathrm{I}(G) + o(|\varphi^{[k]}| + t).$$

We now apply this formalism to $\varphi = (\omega, \delta)$ and to a differential function G given by the right-hand side of (7.1) to obtain the following result.

[⋆] The concept of linearization applies verbatim to differential functions arbitrary order k, with (5.1) replaced by (5.2).

7.3 Proposition *We have the following approximate formula for the strain tensor E of a shear deformation at a point $x \in \Omega$ with normal coordinates (ξ, t):*

$$E \approx \varepsilon^{\mathrm{N}} + t\rho^{\mathrm{D}} \quad \text{as} \quad (\omega^{[1]}, \delta^{[1]}, t) \to 0, \tag{7.4}$$

where

$$\varepsilon^{\mathrm{N}} = \frac{1}{2}(\nabla\omega + \nabla\omega^{\mathrm{T}} + \delta \otimes n + n \otimes \delta), \tag{7.5}$$

$$\rho^{\mathrm{D}} = \frac{1}{2}(\nabla\delta + \nabla\delta^{\mathrm{T}} - \nabla\omega L - L\nabla\omega^{\mathrm{T}}) \tag{7.6}$$

where all quantities are evaluated at ξ.

Proof We first linearize the expression (7.2). Let us preliminarily prove that

$$\frac{d(P + tL)^{-1}}{dt}\bigg|_{t=0} = -L. \tag{7.7}$$

Indeed, if $t \in \mathbb{R}$ is sufficiently close to 0 then $\ker(P + tL) = n^{\perp}$ is the orthogonal complement of n and hence $\big(\ker(P + tL)\big)^{\perp} = \operatorname{ran} P$. Thus, invoking the definition of the generalized inverse, we see that $(4.3)_1$ reads

$$(P + tL)^{-1}(P + tL) = P.$$

The differentiation at $t = 0$ gives (7.7). To linearize the right-hand side of (7.2), let $\varphi = (\omega, \delta)$ and let G be the differential function given by

$$G(\xi, \varphi^{[1]}, t) = (\nabla\omega + t\nabla\delta)(P + tL)^{-1} + \delta \otimes n.$$

We now linearize G with respect to $(\varphi^{[1]}, t)$ by (7.3). Clearly,

$$\mathrm{l}_{00}(G)(\xi) = 0, \quad \mathrm{l}_{01}(G)(\xi) = 0$$

and one obtains directly from definition with the help of (7.7) that

$$\mathrm{l}_{10}(G)(\xi)[\varphi^{[k]}] = \nabla\omega + \delta \otimes n, \quad \mathrm{l}_{11}(G)(\xi)[\varphi^{[k]}] = \nabla\delta - \nabla\omega L.$$

Hence the linearized form the expression for ∇u is

$$\nabla u \approx \nabla\omega + \delta \otimes n + t(\nabla\delta - \nabla\omega L).$$

A symmetrization gives (7.4), (7.5) and (7.6). □

7.4 *Remark* In [12; Equation (7.69)] Naghdi derives a different approximate expression for the components γ^*_{ij} of E. In our notation, he derives

$$E \approx \varepsilon^{\mathrm{N}} + t\rho^{\mathrm{N}} \quad \text{as} \quad (\omega^{[1]}, \delta^{[1]}, t) \to 0, \tag{7.8}$$

where ε^{N} is as before, but

$$\rho^{\mathrm{N}} = \frac{1}{2}\big(\nabla\delta + \nabla\delta^{\mathrm{T}} + L(\nabla\omega + \delta \otimes n) + (\nabla\omega + \delta \otimes n)^{\mathrm{T}} L\big),$$

which is called Naghdi's bending strain tensor in the literature. It is often used with the additional realistic restriction $\delta \cdot n = 0$, in which case we have [5; p. 365], [4]

$$\rho^{\mathrm{N}} = \frac{1}{2}(P\nabla\delta + \nabla\delta^{\mathrm{T}} P + L\nabla\omega + \nabla\omega^{\mathrm{T}} L). \tag{7.9}$$

The following example is devoted to this discrepancy.

7.5 ***Example*** Let Ω be a solid spherical shell with the middle surface $\mathbb{S}^{\nu-1}$ and $2h$, $0 < h < 1$, i.e.,

$$\Omega = \{x \in \mathbb{R}^\nu : \big||x| - 1\big| < h\}.$$

Let $y : \Omega \to \mathbb{R}^\nu$ be a deformation given by

$$y(x) = c(r-1)x/|x| + x$$

where c is a constant. The corresponding displacement, strain tensor and the gradient of displacement are

$$u(x) = c(r-1)x/|x|, \quad E(x) = \nabla u(x) = \frac{c(r-1)\big(|x|^2 1 - x \otimes x\big)}{|x|^3}.$$

In terms of the normal coordinates (ξ, t), $\xi = x/|x|$, $t = |x| - 1$ we have

$$E(x) = c(r-1)P/(t+1) \tag{7.10}$$

where $P = 1 - \xi \otimes \xi$ is the projection onto the tangent space of $\mathbb{S}^{\nu-1}$. It is easy to see that y is a shear deformation of Ω with

$$\omega = c(r-1)\xi = c, \quad \delta = 0.$$

The linearization of (7.10) in (ω, t) is

$$E(x) = c\big((r-1)P - t(r-1)P\big). \tag{7.11}$$

Since $\nabla\omega = (r-1)P$ and since the curvature tensor of $\mathbb{S}^{\nu-1}$ is P (see Example 3.7) we find that

$$\varepsilon = (r-1)P, \quad \rho^{\mathrm{AL}} = -(r-1)P, \quad \rho^{\mathrm{K}} = (r-1)P.$$

Equation (7.11) then shows that Equation (7.4) holds, but (7.8) does not.

7.6 Kirchhoff-Love's deformations A map $y : \Omega \to \mathbb{R}^\nu$ is said to be a Kirchhoff-Love's deformation if

$$y(x) = \eta(\xi) + t\bar{n}(\eta(\xi))$$

for any $x \in \Omega$ with the normal coordinates (ξ, t), where $\eta : \mathscr{S} \to \mathbb{R}^\nu$ is a deformation of $\mathscr{S}$ and $\bar{n}$ is the normal to the deformed surface $\bar{\mathscr{S}} = \eta(\mathscr{S})$.

The Kirchhoff-Love's deformation is a special case in which the normal line before the deformation is mapped into a normal line to the deformed middle surface, i.e., $d = \bar{n}$. The Kirchhoff-Love hypothesis is usually formulated for small deformations; in the verbal statements this restriction is missing. The reader is referred to e.g., [16; Assumption 4, Section 3.1], [5; p. 336 and p. 372] for essentially the same assumptions.

The bulk displacement of a Kirchhoff-Love deformation is

$$u(x) = \omega(\xi) + t\big(\bar{n}(\eta(\xi)) - n(\xi)\big) \tag{7.12}$$

where ω is given by $(7.1)_1$. We now differentiate (7.12) with the help of (6.2); the subsequent use of the chain rule and the definition of the curvature of $\bar{\mathscr{S}}$ provides

$$\nabla u = \Big(\nabla\omega + t\big((\bar{L} \circ \eta)\nabla\eta - L\big)\Big)(P + tL)^{-1} + (\bar{n} \circ \eta - n) \otimes n. \tag{7.13}$$

Equations $(4.2)_2$ and (4.7) expresses $\bar{n}$ and $\bar{L}$ as a functions of η and its derivatives up to order 2 and hence also of ω and its derivatives up to order 2. Thus the right-hand side of (7.13) can be interpreted as an implicitly defined differential function of $(\omega^{[2]}, t)$.

Thus we can apply the linearization of Subsection 7.2 to $\varphi := \omega$ and G given by the right-hand side of (7.13) to obtain the following result.

7.7 Proposition *We have the following approximate formula for the strain tensor E of a Kirchhoff-Love deformation at a point $x \in \Omega$ with normal coordinates (ξ, t):*

$$E(u) \approx \varepsilon + t\rho^{\mathrm{AL}} + t\rho^{\mathrm{D}} \quad \text{as} \quad (\omega^{[2]}, t) \to 0,$$

where

$$\varepsilon = \frac{1}{2} P(\nabla\omega + \nabla\omega^{\mathrm{T}})P,$$

$$\rho^{\mathrm{AL}} = -n \cdot \nabla^2\omega - \varepsilon L - L\varepsilon$$

where all quantities are evaluated at ξ. We note that ρ^{AL} has already been encountered in (5.16).

Proof Let $G = G(\xi, \omega^{[2]}(\xi), t)$ be the differential function implicitly defined by the right-hand side of (7.13) as explained above. One immediately obtains

$$\mathrm{l}_{00}(G)(\xi) = 0, \quad \mathrm{l}_{01}(G)(\xi) = 0. \tag{7.14}$$

To calculate $\mathrm{l}_{10}(G)(\xi)[\omega^{[2]}]$, we observe that

$$G(\xi, \omega^{[2]}, 0) = \nabla\omega + (\bar{n} \circ \eta - n) \otimes n$$

and that

$$\mathrm{l}_{10}(G)(\xi)[\omega^{[2]}] = \frac{\partial G(\xi, s\omega^{[2]}, 0)}{\partial s}\bigg|_{s=0} = \mathfrak{l}\big(\nabla\omega + (\bar{n} \circ \eta - n) \otimes n\big)$$

where $\mathfrak{l}(\cdot)$ is the linearization in ω defined in Subsection 5.1. Using (5.4), we find from the last expression, that

$$\mathrm{l}_{10}(G)(\xi)[\omega^{[2]}] = \nabla\omega - \nabla\omega^{\mathrm{T}} n \otimes n. \tag{7.15}$$

We shall now calculate $\mathrm{l}_{11}(G)(\xi)[\omega^{[2]}]$. A calculation based on (7.13) in combination with (7.7) yields

$$\frac{\partial G(\xi, \omega^{[k]}, \tau t)}{\partial \tau}\bigg|_{\tau=0} = t\big(-\nabla\omega L + (\bar{L} \circ \eta)\nabla\eta - L\big).$$

Next we observe that

$$\frac{\partial^2 G(\xi, s\omega^{[2]}, \tau t)}{\partial s \partial \tau}\bigg|_{\tau=s=0} = t\,\mathfrak{l}\big(-\nabla\omega L + (\bar{L} \circ \eta)\nabla\eta - L\big).$$

We have

$$\mathfrak{l}\big(-\nabla\omega L + (\bar{L} \circ \eta)\nabla\eta - L\big) = -\nabla\omega L + \mathfrak{l}(H) \tag{7.16}$$

where

$$H = (\bar{L} \circ \eta)\nabla\eta - L.$$

We calculate $\mathfrak{k}(H)$ by a method similar to that in the proof of Theorem 5.3. Namely, from (4.7) we find that

$$H = -F^{-\mathrm{T}}\big((\bar{n} \circ \eta) \cdot \nabla^2 \eta\big).$$

We linearize the products on the right-hand side of the last relation by Leibniz's rule (5.3) in combination with (5.9) to obtain

$$\mathfrak{k}_1(H) = -\mathfrak{k}_1(F^{-\mathrm{T}})(n \cdot \nabla^2 \mathrm{id}) - \mathfrak{k}_1(\bar{n}) \cdot \nabla^2 \mathrm{id} - n \cdot \nabla^2 \omega \tag{7.17}$$

We now insert $\mathfrak{k}_1(F^{-\mathrm{T}}) = \mathfrak{k}_1(F^{-1})^{\mathrm{T}}$ from (5.10), $\nabla^2 \mathrm{id}$ and $n \cdot \nabla^2 \mathrm{id}$ from $(3.16)_2$, and $\mathfrak{k}_1(\bar{n})$ from (5.4) into (7.17). We obtain

$$\mathfrak{k}_1(H) = (n \otimes \nabla\omega^{\mathrm{T}} n - \nabla\omega^{\mathrm{T}} P)L - n \cdot \nabla^2 \omega.$$

By (7.16) then

$$\mathfrak{k}\big(-\nabla\omega L + (\bar{L} \circ \eta)\nabla\eta - L\big) = -P(\nabla\omega + \nabla\omega^{\mathrm{T}})L - n \cdot \nabla^2 \omega$$

and consequently

$$t\mathfrak{l}_{11}(\xi)[\varphi^{[k]}] = -t\big(n \cdot \nabla^2\omega + P(\nabla\omega^{\mathrm{T}} + \nabla\omega)L\big). \tag{7.18}$$

Thus (7.14), (7.15) and (7.18) yield

$$\nabla u \approx \nabla\omega - \nabla\omega^{\mathrm{T}} n \otimes n - t\big(n \cdot \nabla^2\omega + P(\nabla\omega^{\mathrm{T}} + \nabla\omega)L\big)$$

A symmetrization provides the result. □

7.8 *Remark* (Consistency) The linearization of the right-hand side of (7.12) in ω using (5.4) gives

$$u \approx \omega - t\nabla\omega^{\mathrm{T}} n.$$

In this approximation, a Kirchhoff-Love deformation is a shear deformation with

$$\delta = -\nabla\omega^{\mathrm{T}} n. \tag{7.19}$$

Let us show that with the choice (7.19), the bulk stain tensor E of a shear deformation from Proposition 7.3 reduces to that of a Kirchhoff-Love deformation from Proposition 7.7. That is, let us prove that

$$\varepsilon^{\mathrm{N}} = \varepsilon, \quad \rho^{\mathrm{D}} = \rho^{\mathrm{AL}}. \tag{7.20}$$

Equation $(7.20)_1$ is immediate. To obtain $(7.20)_2$, we have to determine $\nabla\delta$. By the product rule,

$$\nabla\delta = -\nabla(\nabla\omega^{\mathrm{T}})n - \nabla\omega^{\mathrm{T}} L.$$

The evaluation of the first term on the right-hand side by (4.5) with $\psi = \omega$ provides

$$\nabla\delta = -n \cdot \nabla^2\omega + n \otimes L\nabla\omega^{\mathrm{T}} n - \nabla\omega^{\mathrm{T}} L. \tag{7.21}$$

The insertion into (7.6) provides $(7.20)_2$.

Next, let us show that with the choice (7.19), we have

$$\rho^{\mathrm{N}} = -n \cdot \nabla^2\omega. \tag{7.22}$$

Indeed, from (7.21) we obtain

$$P\nabla\delta = -n \cdot \nabla^2\omega - \nabla\omega^{\mathrm{T}} L$$

and the insertion into (7.9) provides (7.22).

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