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THE EXTENSION OF COCHAIN COMPLEXES OF MEROMORPHIC FUNCTIONS TO MULTIPLICATIONS

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Abstract. Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra and G be the algebraic completion of its module. Using a geometric interpretation in terms of sewing two Riemann spheres with a number of marked points, we introduce a multiplication between elements of two spaces $\mathcal{M}^k_m(\mathfrak{g}, G)$ and $\mathcal{M}^n_{m'}(\mathfrak{g}, G)$ of meromorphic functions depending on a number of formal complex parameters $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ with specific analytic and symmetry properties, and associated to $\mathfrak{g}$-valued series. These spaces form a chain-cochain complex with respect to a boundary-coboundary operator. The main result of the paper shows that the multiplication is defined by an absolutely convergent series and takes values in the space $\mathcal{M}^{k+n}_{m+m'}(\mathfrak{g}, G)$.

1. Introduction: Motivation and geometrical interpretation

For purposes of construction of non-trivial cohomology classes for an infinite-dimensional Lie algebra $\mathfrak{g}$ [5] on manifolds it is important to define a multiplication of elements of cochain complex spaces of meromorphic functions with predetermined analytic properties and depending on $\mathfrak{g}$-series. Predetermined functions can be parameterized by formal complex parameters associated to local coordinates of marked points on Riemann spheres. In this paper we introduce the multiplication of elements of double cochain complex $(\mathcal{M}^n_m(\mathfrak{g}, G), \delta^n_m)$-spaces by involving the geometrical procedure [7] of sewing two Riemann spheres. Such multiplication is then parameterized by a nonzero complex number ϵ which is the complex parameter of the Riemann spheres sewing. For two chain complex spaces $\mathcal{M}^k_m(\mathfrak{g}, G)$ and $\mathcal{M}^n_{m'}(\mathfrak{g}, G)$, we associate formal complex parameters $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ to local coordinates vanishing on k and n marked points on two Riemann spheres. The sewing brings about another Riemann sphere, and formal complex parameters of predetermined meromorphic functions of the space $\mathcal{M}^{k+n}_{m+m'}$ are identified with parameters of the resulting sphere. The problem of defining a multiplication of elements of cochain complex spaces is very important for cohomological problems in conformal field theory [1, 6], infinite-dimensional Lie algebras [5], the theory of integrable models, as well as for further applications to cohomologies of smooth manifolds [2]. The plan of the paper is the following. In Section 2, for an infinite-dimensional Lie algebra, we describe the axiomatics of meromoprhic functions with predetermined analytic properties. The cochain complex spaces $\mathcal{M}^{\cdot}_{\cdot}$ of predetermined meromorphic functions

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are then defined. In Section 3 we describe first the geometric procedure of forming a Riemann sphere by sewing two initial Riemann spheres. A multiplication for elements of two $\mathcal{M}^{\cdot}_{\cdot}$-spaces is introduced. The main result of this paper, namely, Proposition 4 showing that the product of elements of spaces $\mathcal{M}^k_m$ and $\mathcal{M}^n_{m'}$ belongs to $\mathcal{M}^{k+n}_{m+m'}$ is proven. In particular, we show that functions obtained as a result of multiplication of elements of spaces of predetermined meromorphic functions are absolutely convergent meromorphic functions with predetermined analytic properties.

2. The cochain complex spaces $\mathcal{M}^n_m(\mathfrak{g}, G)$

Let $\mathfrak{g}$ be an infinite-dimensional Lie algebra, and W its module. Denote by $F_n\mathbb{C}$ the configuration space of $n \geq 1$ ordered formal complex parameters in $\mathbb{C}^n$, $F_n\mathbb{C} = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$. Denote by $G = G_{(z_1,\ldots,z_n)}$ the graded (with respect to a grading operator K) algebraic completion of W. It is assumed that on G there exists a non-degenerate bilinear pairing $(.,.)$. For $z = z_j$, $1 \leq j \leq n$, the element $T = -\partial_z$ acts as the translation operator, and $K = -z\partial_z$ acts as a grading operator. Let $G' = \coprod_{l\in\mathbb{Z}} G^*_l$ denotes the graded dual for $G = \bigoplus_{\lambda\in\mathbb{Z}} G_{(\lambda)}$ with respect to $(.,.)$. Assume that G is equipped with a map $\gamma_g : G_z \to G[[z, z^{-1}]]$, as a formal series $g \mapsto \gamma_g(z) \equiv \sum_{l\in\mathbb{Z}} g_l\ z^l$ for $z \in \mathbb{C}$. We assume that $G_{(l)} = \{w \in G | Kw = lw,\ l = \text{wt}\ (w)\}$. Moreover we require that $\dim G_{(l)} < \infty$, i.e., it is finite, and for fixed l, $G_{(n+l)} = 0$, for all small enough integers n. For g, $w \in G$, $n \in \mathbb{Z}$, $g_n w = 0$, $n \gg 0$, $\gamma_{\mathbf{1}}(z) = \text{Id}$. For $g \in G$, $\gamma_g(z)w$ contains only finitely many negative power terms, that is, $\gamma_g(z)w \in G$. By $\mathbf{1}_G$ we denote the highest weight element in G. We normalize the pairing by the condition $(\overline{\mathbf{1}}_G, \mathbf{1}_G) = 1$.

We now define the space of meromorphic functions $\mathcal{G}(g_1, z_1; \ldots; g_n, z_n)$ depending on n G-elements and n formal complex parameters. For $\mathcal{G}$ we allow poles only at $z_i = z_j$, $i \neq j$. We define left action of the permutation group S_n on $\mathcal{G}(g_1, z_1; \ldots; g_n, z_n)$ by $\sigma(\mathcal{G})(g_1, z_1; \ldots; g_n, z_n) = \mathcal{G}\left(g_1, z_{\sigma(1)}; \ldots; g_n, z_{\sigma(n)}\right)$. For g_1, $g_2 \in G$, $w \in G$, we require for $\mathcal{G}$ that the functions $\mathcal{G}(\gamma_{g_1}(z_1)\ \gamma_{g_2}(z_2)w)$, $\mathcal{G}(\gamma_{g_2}(z_2)\ \gamma_{g_1}(z_1)w)$, and $\mathcal{G}(\gamma_{\gamma_{g_1}(z_1-z_2)g_2}(z_2)w)$, are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common function in z_1 and z_2. The poles are only allowed at $z_1 = 0 = z_2$, $z_1 = z_2$. If g is homogeneous then $g_m G_{(n)} \subset G_{(\text{wt}\ u-m-1+n)}$. For a subgroup $\mathfrak{G} \subset \text{Aut}\ G$, $\mathfrak{G}$ acts on G as automorphisms if $g\ \gamma_h(z)\ g^{-1} = \gamma_{gh}(z)$, for all g, $h \in \mathfrak{G}$. The operator K satisfies the derivation property $\gamma_{Kg}(z) = \frac{d}{dz}\gamma_g(z)$. Denote by $T_i.$ the operator acting on the i-th entry. We then define the action of partial derivatives on an element $\mathcal{G}(g_1, z_1; \ldots; g_n, z_n)$

$$\begin{aligned} \partial_{z_i}\mathcal{G}(g_1, z_1; \ldots; g_n, z_n) &= \mathcal{G}(T_i.\ (g_1, z_1; \ldots; g_n, z_n)), \\ \sum_{i\geq 1} \partial_{z_i}\mathcal{G}(g_1, z_1; \ldots; g_n, z_n) &= T\mathcal{G}(g_1, z_1; \ldots; g_n, z_n). \end{aligned} \tag{2.1}$$

For $z \in \mathbb{C}$, let

$$e^{zT}\mathcal{G}(g_1, z_1; \ldots; g_n, z_n) = \mathcal{G}(g_1, z_1 + z; \ldots; g_n, z_n + z). \tag{2.2}$$

Let us denote by $\mathrm{Ins}_i(A)$ the operator of multiplication by $A \in \mathbb{C}$ at the i-th position. Then we assume that both sides of the expression

$$\mathcal{G}\left((g_1, \ldots, g_n)\ , \mathrm{Ins}_i(z_1, \ldots, z_n)\ \ (z_1, \ldots, z_n)\right) = \mathcal{G}\left(\mathrm{Ins}_i(e^{zT})\ \ (g_1, z_1; \ldots ; g_n, z_n)\right),$$

are absolutely convergent on the open disk $|z| < \min_{i \neq j}\{|z_i - z_j|\}$, and equal as power series expansions in z. For $z \in \mathbb{C}^\times$, $(zz_1, \ldots, zz_n) \in F_n\mathbb{C}$, we require for functions

$$z^K \mathcal{G}(g_1, z_1; \ldots ; g_n, z_n) = \mathcal{G}\left(z^K (g_1, zz_1; \ldots ; g_n, zz_n)\right). \tag{2.3}$$

For an arbitrary fixed $\theta \in G'$, a map linear in $(g_1, \ldots, g_n)$ and $(z_1, \ldots, z_n)$, $\mathcal{G}$: $(z_1; \ldots ; z_n) \mapsto (\theta, f(g_1, z_1; \ldots ; g_n, z_n))$, delivers a particular example of a meromorphic function in $(z_1, \ldots, z_n)$ which depends on $(g_1, \ldots, g_n)$.

Now we recall further conditions on meromorphic functions associated to a number of γ_G-series. We call such functions predetermined combined with a number of G-series on a domain. By this we mean functions with specific analytical behavior taking into account of Lie-algebra series. We denote by $\mathcal{P}_k : G \to G_{(k)}$, $k \in \mathbb{Z}$, the projection of G on $G_{(k)}$. Following [3], we formulate the following definitions and propositions. For $i = 1, \ldots, (l + k)n$, $k \geq 0$, $1 \leq l', l'' \leq n$, let $(l_1, \ldots, l_n)$ be a partition of $(l + k)n = \sum\limits_{i \geq 1} l_i$, and $k_i = \sum_{j=1}^{i-1} l_j$. For $\zeta_i \in \mathbb{C}$, define $H_i = \mathcal{G}(\gamma_{g_{k_1+1}}(z_{k_1+1} - \zeta_i) \ldots \gamma_{g_{k_i+l_i}}(z_{k_i+l_i} - \zeta_i)\mathbf{1}_G)$, for $i = 1, \ldots, n$. It is assumed that the function $\sum\limits_{(r_1, \ldots, r_l) \in \mathbb{Z}^l} \mathcal{G}(\mathcal{P}_{r_1} H_1, \zeta_1; \ldots ; \mathcal{P}_{r_l} H_l, \zeta_l)$, is absolutely convergent to an analytically extension in $(z_1, \ldots, z_n)_{l+k}$ in the domains $|z_{k_i+p} - \zeta_i| + |z_{k_j+q} - \zeta_j| < |\zeta_i - \zeta_j|$, for $i, j = 1, \ldots, k$, $i \neq j$, and for $p = 1, \ldots, l_i$, $q = 1, \ldots, l_j$. The convergence and analytic extension do not depend on complex parameters ζ_l. On the diagonal of $(z_1, \ldots, z_n)_{l+k}$ the order of poles is bounded from above by described positive numbers $\beta(g_{l',i}, g_{l'',j})$. For $(g_1, \ldots, g_{l+k}) \in G$, $z_i \neq z_j$, $i \neq j$ $|z_i| > |z_s| > 0$, for $i = 1, \ldots, k$, $s = k+1, \ldots, l+k$ the sum $\sum_{q \in \mathbb{Z}} \mathcal{G}(\gamma_{g_1}(x_1) \ldots \gamma_{g_k}(x_k) \mathcal{P}_q(\gamma_{g_{1+k}}(z_{1+k}) \ldots \gamma_{g_{l+k}}(x_{l+k}))\mathbf{1}_G)$, is absolutely convergent and analytically extendable to a function in variables $(z_1, \ldots, z_n)_{l+k}$. The order of pole that is allowed at $z_i = z_j$ is bounded from above by the numbers $\beta(g_{l',i}, g_{l'',j})$.

Let S_l be the permutation group. For $l \in \mathbb{N}$ and $1 \leq s \leq l-1$, let $J_{l;s}$ be the set of elements of S_l which preserve the order of the first s numbers and the order of the last $l - s$ numbers, that is, $J_{l,s} = \{\sigma \in S_l \mid \sigma(1) < \ldots < \sigma(s),\ \sigma(s+1) < \ldots < \sigma(l)\}$. The elements of $J_{l;s}$ are called shuffles, and we use the notation $J^{-1}_{l;s} = \{\sigma \mid \sigma \in J_{l;s}\}$. For any set of G-elements g_i, $g_j \in G$, and formal complex parameters z_i, z_j, $1 \leq i \leq n$, $1 \leq j \leq m$, $n \geq 0$, $m \geq 0$, we denote by $\mathcal{M}^n_m(\mathfrak{g}, G)$, the space of all predetermined meromorphic functions combined with m γ-series $(\mathcal{G}(z_1, \ldots, z_n), \gamma_{g_j}(z'_j))$ satisfying (2.2), (2.3), the conditions of the previous paragraph, and the symmetry property with respect to action of the symmetric group S_n:

$$\sum_{\sigma \in J^{-1}_{n;s}} (-1)^{|\sigma|} \mathcal{G}\left(g_{\sigma(1)}, z_{\sigma(1)}; \ldots ; g_{\sigma(n)}, z_{\sigma(n)}\right) = 0. \tag{2.4}$$

For fixed $\mathfrak{g}$ and G we will use the notation $\mathcal{M}^n_m$.

One defines $\mathcal{G}*((f(g_1, z_1; \ldots; g_{l_1}, z_{l_1}; \mathbf{1}_G, z); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z)) : G \to \mathbb{C}$:

$$\begin{aligned}&\mathcal{G} * (f(g_1, z_1; \ldots; g_{l_1}, z_{l_1}; \mathbf{1}_G, z); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z))\\ &\quad = \mathcal{G}(f(g_{l_1}, z_{l_1}; \mathbf{1}_G, z); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z)).\end{aligned}$$

The action $\mathcal{G}(g_1, z_1; \ldots; g_m, z_m) *_0 f'(g_{m+1}, z_{m+1}; \ldots; g_{m+n}, z_{m+n}) : G \to G_{z_1,\ldots,z_{m+n-1}}$, is given by

$$\begin{aligned}&\mathcal{G}(g_1, z_1; \ldots; g_m, z_m) *_0 f'(g_{m+1}, z_{m+1}; \ldots; g_{m+n}, z_{m+n})\\ &\quad = \mathcal{G}(f(g_1, z_1; \ldots; g_m, z_m); f'(g_{m+1}, z_{m+1}; \ldots; g_{m+n}, z_{m+n})).\end{aligned}$$

We introduce also $\mathcal{G}(g_1, z_1; \ldots; g_m, z_m) *_{m+1} f'(g_{n+1}, z_{n+1}; \ldots; g_{n+m}, z_{n+m}) : G \to G_{z_1,\ldots,z_{m+n-1}}$, defined by

$$\begin{aligned}&\mathcal{G}(g_1, z_1; \ldots; g_m, z_m) *_{m+1} (g_{n+1}, z_{n+1}; \ldots; g_{n+m}, z_{n+m}))\\ &\quad = \mathcal{G}(f(g_1', z_1'; \ldots; g_n', z_n'); g_{n+1}, z_{n+1}; \ldots; g_{n+m}, z_{n+m})).\end{aligned}$$

The following result holds

Proposition 1. *For $(g_1, \ldots, g_n) \in G$, $\mathcal{G}(z_1, \ldots, z_n)$, is absolutely convergent in the region $|z_1| > \ldots > |z_n| > 0$, to a function in $(z_1, \ldots, z_n)$ with the only possible poles at $z_i = z_j$, $i \neq j$, and $z_i = 0$. The $\mathcal{G}$ is invariant with respect to the action of $\sigma \in S_n$ on all entries $(z_1, \ldots, z_n)$.*

The following useful proposition holds:

Proposition 2. *Let $\mathcal{G} : G \to \mathbb{C}$ is a predetermined meromorphic function combined with m series $\gamma_{g_j}(y_j)$, $1 \leq j \leq m$, $m \geq 0$. Then*

(1) *For $p \leq m$, $\mathcal{G}$ is a predetermined function combined with p $\gamma_{g_p}(x_p)$-series, and for p, $q \in \mathbb{Z}_+$ such that $p + q \leq m$ and $(l_1, \ldots, l_n) \in \mathbb{Z}_+$ such that $l_1 + \ldots + l_n = p + n$, $\mathcal{G}*(f(g_1', z_1'; \ldots; g_{l_1}', z_{l_1}'; \mathbf{1}_G, z'); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z^{'\cdots'}))$ and $\mathcal{G}(g_1, z_1; \ldots; g_n, z_n) *_{p+1} \mathcal{G}$ are combined with q series.*

(2) *For p, $q \in \mathbb{Z}_+$ such that $p + q \leq m$, $(l_1, \ldots, l_n) \in \mathbb{Z}_+$ such that $l_1 + \ldots + l_n = p + n$ and $k_1, \ldots, k_{p+n} \in \mathbb{Z}_+$ such that $k_1 + \ldots + k_{p+n} = q + p + n$, we have*

$$\begin{aligned}&(\mathcal{G} * f(g_1', z_1'; \ldots; g_{l_1}', z_{l_1}'; \mathbf{1}_G, z'); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z^{'\cdots'}))\\ &\quad * f(g_1^*, z_1^*; \ldots; g_{k_1}^*, z_{k_1}^*; \mathbf{1}_G, z^*); \ldots f(g_1^{*\ldots*}, z_1^{*\ldots*}; \ldots; g_{k_{p+n}}^{*\ldots*}, z_{k_{p+n}}^{*\ldots*}; \mathbf{1}_G, z^{*\ldots*}))\\ &\quad = \mathcal{G} * f(g_1^{*\ldots*}, z_1^{*\ldots*}; \ldots; g_{k_1+\ldots+k_{l_1}}^{*\ldots*}, z_{k_1+\ldots+k_{l_1}}^{*\ldots*}; \mathbf{1}_G, z^{*\ldots*});\\ &\quad \ldots f(g_1^{*\ldots*}, z_1^{*\ldots*}; \ldots; g_{k_{l_1+\ldots+l_{n-1}+1}+\ldots+k_{p+n})}^{*\ldots*}, z_{k_{l_1+\ldots+l_{n-1}+1}+\ldots+k_{p+n})}^{*\ldots*}; \mathbf{1}_G, z^{*\ldots*}).\end{aligned}$$

(3) *For p, $q \in \mathbb{Z}_+$ such that $p + q \leq m$ and $(l_1, \ldots, l_n) \in \mathbb{Z}_+$ such that $\sum_{i=1}^n l_i = p + n$, we have*

$$\begin{aligned}&\mathcal{G}(g_1, z_1; \ldots; g_q, z_q) *_{q+1} (\mathcal{G} * (f(g_1', z_1'; \ldots; g_{l_1}', z_{l_1}'; \mathbf{1}_G, z'); \ldots;\\ &\quad f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots; g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z^{'\cdots'}))\\ &\quad = (\mathcal{G}(g_1, z_1; \ldots; g_q, z_q) *_{q+1} \mathcal{G}) * (f(g_1', z_1'; \ldots; g_{l_1}', z_{l_1}'; \mathbf{1}_G, z'); \ldots; f(g_1^{'\cdots'}, z_1^{'\cdots'}; \ldots;\\ &\quad g_{l_n}^{'\cdots'}, z_{l_n}^{'\cdots'}; \mathbf{1}_G, z^{'\cdots'})).\end{aligned}$$

(4) *For* p, $q \in \mathbb{Z}_+$ *such that* $p+q \leq m$, *we have*

$$\mathcal{G}(g_1, z_1; \dots; g_p, z_p) *_{p+1} (\mathcal{G}(g'_1, z'_1; \dots; g'_q, z'_q) *_{q+1} \mathcal{G}')$$
$$= \mathcal{G}(g_1, z_1; \dots; g_p, z_p; g'_1, z'_1; \dots; g'_q, z'_q; g, z) *_{p+q+1} \mathcal{G}'.$$

Proposition 3. *For* k, $(l_1, \dots, l_{n+1}) \in \mathbb{Z}_+$ *and* $(g_1^{(1)}, \dots, g_{l_1}^{(1)}, \dots, g_1^{(n+1)}, \dots, g_{l_{n+1}}^{(n+1)}) \in G$, *the series*

$$\sum_{(r_1,\dots,r_n)\in\mathbb{Z},\ r_{n+1}\in\mathbb{Z}} \mathcal{G}(\mathcal{P}_{r_1}(f(g_1^{(1)}, z_1^{(1)}; \dots; g_{l_1}^{(1)}, z_{l_1}^{(1)}; \mathbf{1}_G, z_1^{(0)})); \dots;$$
$$\mathcal{P}_{r_n}(f(g_1^{(n)}, z_1^{(n)}; \dots; g_{l_n}^{(n)}, z_{l_n}^{(n)}; \mathbf{1}_G, z_n^{(0)}))$$
$$\mathcal{P}_{r_{n+1}}(f(g_1^{(n+1)}, z_1^{(n+1)}; \dots; g_{l_{n+1}}^{(n+1)}, z_{l_{n+1}}^{(n+1)}; g, z_{n+1}^{(0)})),$$

converges absolutely to

$$\mathcal{G}(g_1^{(1)}, z_1^{(1)} + z_1^{(0)}; \dots; g_{l_1}^{(1)}, z_{l_1}^{(1)} + z_1^{(0)}; \dots; g_1^{(n+1)}, z_1^{(n+1)} + z_{n+1}^{(0)}; g_{l_{n+1}}^{(n+1)}, z_{l_{n+1}}^{(n+1)} + z_{n+1}^{(0)}),$$

when $0 < |z_p^{(i)}| + |z_q^{(j)}| < |z_i^{(0)} - z_j^{(0)}|$ *for* i, $j = 1, \dots, n+1$, $i \neq j$, $p = 1, \dots, l_i$, $q = 1, \dots, l_j$.

Finally, we note

Lemma 1.

$$\sum_{q\in\mathbb{Z}} \mathcal{G}(g''_1, z_1; \dots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \dots; g''_{m+m'+k+n}, z_{m+m'+k+n}))$$
$$= \sum_{l\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g_{k+1}, x_{k+1}; \dots; g_{k+m}, x_{k+m}; \mathcal{P}_q(f(g_1, x_1; \dots; g_k, x_k); g, \zeta_1)))$$
$$\mathcal{G}(g'_{n+1}, y_{n+1}; \dots; g'_{n+m'}, y_{n+m'}; \mathcal{P}_q(f(g'_1, y_1; \dots; g'_n, y_n; \overline{g}, \zeta_2))).$$

Proof. Consider

$$\sum_{l\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g''_1, z_1; \dots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \dots; g''_{m+m'+k}, z_{m+m'+k}); g, \zeta_1))$$
$$\mathcal{G}(g''_1, z_1; \dots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; g''_{m+m'+k+n}, z_{m+m'+k+n}); \overline{g}, \zeta_2))$$
$$= \sum_{l\in\mathbb{Z},\ q\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g''_1, z_1; \dots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \dots; g''_{m+m'+k}, z_{m+m'+k})); g, \zeta_1)$$
$$\mathcal{G}(g''_1, z_1; \dots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \dots; g''_{m+m'+k+n}, z_{m+m'+k+n})); \overline{g}, \zeta_2). \quad (2.5)$$

Thus, we can rewrite the last expression as

$$\sum_{l\in\mathbb{Z},\ q\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k}, z_{m+m'+k})); g, \zeta_1)$$
$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n})); \overline{g}, \zeta_2)$$

$$= \sum_{l\in\mathbb{Z},\ q\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k}, z_{m+m'+k})); g, \zeta_1)$$
$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n}; \overline{g}, -\zeta_2))$$

$$= \sum_{l\in\mathbb{Z},\ q\in\mathbb{Z},\ \widetilde{g}'\in G'_{(l)}} \epsilon^l \mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'}; \widetilde{g}')$$
$$\sum_{l\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l\ \mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k}, z_{m+m'+k}; g, -\zeta_1)))$$
$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'}; \widetilde{g}')$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n}; \overline{g}, -\zeta_2)))$$

$$= \sum_{q\in\mathbb{Z}} \mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k}, z_{m+m'+k};$$
$$g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n})).$$

Now note that as an element of $\mathcal{M}^{k+n+m+m'}_{\cdot}$,

$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'}; \mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots;$$
$$g''_{m+m'+k}, z_{m+m'+k}; g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n})),$$

is invariant with respect to the action of $\sigma \in S_{k+n+m+m'}$. Thus we are able to use this invariance to show that the last expression is reduced to

$$\mathcal{G}(g''_{k+1}, z_{k+1}; \ldots; g''_{k+1+m}, z_{k+1+m}; g''_{n+1}, z_{n+1}; \ldots; g''_{n+1+m'}, z_{n+1+m'};$$
$$\mathcal{P}_q(f(g''_1, z_1; \ldots; g''_k, z_k; g''_{k+1}, z_{k+1}; \ldots; g''_{k+n}, z_{k+n})))$$
$$= \mathcal{G}(g_{k+1}, x_{k+1}; \ldots; g_{k+1+m}, x_{k+1+m}; g'_{n+1}, y_{n+1}; \ldots; g'_{n+1+m'}, y_{n+1+m'};$$
$$\mathcal{P}_q(f(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n)).$$

Similarly, since

$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k}, z_{m+m'+k}; g, \zeta_1))),$$

$$\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'};$$
$$\mathcal{P}_q(f(g''_{m+m'+k+1}, z_{m+m'+k+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n}); \overline{g}, \zeta_2))),$$

correspond to elements of $\mathcal{M}^{m+m'+k}_{\cdot}$ and $\mathcal{M}^{m+m'+k+n}_{\cdot}$, we then obtain $\mathcal{G}(g_{k+1}, x_{k+1}; \ldots; g_{k+m}, x_{k+m}; \mathcal{P}_q(f(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)))$ and $\mathcal{G}(g'_{n+1}, y_{n+1}; \ldots; g'_{n+m'}, y_{n+m'}; \mathcal{P}_q(f(g'_1, y_1; \ldots; g'_n, y_n); \overline{g}, \zeta_2)))$, respectively. Thus, Lemma follows. □

3. Multiplication of $\mathcal{M}^n_m$-elements

In this Section we define the multiplication of the spaces $\mathcal{M}^n_m$, $n \geq 0$, $m \geq 0$, and coboundary operators δ^n_m for chain-cochain double complexes $(\mathcal{M}^n_m, \delta^n_m)$, and study their properties. The matrix element for a number of Lie algebra-valued series represents usually [6] a character associated to a Riemann sphere. We extrapolate this notion to the case of $\mathcal{M}^n_m$ spaces. A space $\mathcal{M}^n_m$ can be associated with a Riemann sphere with n marked points, while the multiplication of two such spaces is then associated with a sewing of such two spheres with a number of marked points and extra points with local coordinates identified with formal parameters of $\mathcal{M}^k_m$ and $\mathcal{M}^n_{m'}$. In order to supply an appropriate geometric construction for the multiplication, we use the ϵ-sewing procedure for two initial spheres to obtain a matrix element associated with the multiplication of $\mathcal{M}^n_m$ spaces.

In our specific case of functions obtained by multiplying elements of $\mathcal{M}^{\cdot}_{\cdot}$-spaces, we take Riemann spheres $\mathcal{S}^{(0)}_a$, $a = 1, 2$, as two initial auxiliary spaces. The resulting space is formed by the sphere $\mathcal{S}^{(0)}$ obtained by the procedure of sewing $\mathcal{S}^{(0)}_a$. The formal parameters $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ are identified with local coordinates of k and n points on two initial spheres $\mathcal{S}^{(0)}_a$, $a = 1, 2$ correspondingly. In the ϵ sewing procedure, some r points among $(p_1, \ldots, p_k)$ may coincide with points among $(p'_1, \ldots, p'_n)$ when we identify the annuli. This corresponds to the singular case of coincidence of r formal parameters.

Consider the sphere formed by sewing together two initial spheres in the sewing scheme referred to as the ϵ-formalism in [7]. Let $\mathcal{S}^{(0)}_a$, $a = 1, 2$ be to initial spheres. Introduce a complex sewing parameter ϵ where $|\epsilon| \leq \rho_1\rho_2$, Consider k distinct points on $p_i \in \mathcal{S}^{(0)}_1$, $i = 1, \ldots, k$, with local coordinates $(x_1, \ldots, x_k) \in F_k\mathbb{C}$, and distinct points $p_j \in \mathcal{S}^{(0)}_2$, $j = 1, \ldots, n$, with local coordinates $(y_1, \ldots, y_n) \in F_n\mathbb{C}$, with $|x_i| \geq |\epsilon|/\rho_2$, $|y_j| \geq |\epsilon|/\rho_1$. Choose a local coordinate $z_a \in \mathbb{C}$ on $\mathcal{S}^{(0)}_a$ in the neighborhood of points $p_a \in \mathcal{S}^{(0)}_a$, $a = 1, 2$. Consider the closed disks $|\zeta_a| \leq \rho_a$, and excise the disk $D_a = \{\zeta_a,\ |\zeta_a| \leq |\epsilon|/\rho_{\overline{a}}\} \subset \mathcal{S}^{(0)}_a$, to form a punctured sphere $\widehat{\mathcal{S}}^{(0)}_a = \mathcal{S}^{(0)}_a \backslash \{\zeta_a, |\zeta_a| \leq |\epsilon|/\rho_{\overline{a}}\}$. We use the convention $\overline{1} = 2$, $\overline{2} = 1$. Define the annulus $\mathcal{A}_a = \{\zeta_a, |\epsilon|/\rho_{\overline{a}} \leq |\zeta_a| \leq \rho_a\} \subset \widehat{\mathcal{S}}^{(0)}_a$, and identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as a single region $\mathcal{A} = \mathcal{A}_1 \simeq \mathcal{A}_2$ via the sewing relation $\zeta_1\zeta_2 = \epsilon$. In this way we obtain a genus zero compact Riemann surface $\mathcal{S}^{(0)} = \left\{\widehat{\mathcal{S}}^{(0)}_1 \backslash \mathcal{A}_1\right\} \cup \left\{\widehat{\mathcal{S}}^{(0)}_2 \backslash \mathcal{A}_2\right\} \cup \mathcal{A}$. We introduce the multiplication of two double complex spaces with the image in another double complex space coherent with respect to the original coboundary operator (3.15), and the symmetry property (2.4). For G-elements $(g_1, \ldots, g_n)$, $(g'_1, \ldots, g'_n) \in G$, $\mathcal{G}(g_1, x_1;\ \ldots\ ; g_k, x_k) \in \mathcal{M}^k_m$, and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n)$, $\in \mathcal{M}^n_{m'}$, are combined with

m and m' γ-series correspondingly, we introduce the multiplication for $\epsilon = \zeta_1\zeta_2$, $\cdot_\epsilon : \mathcal{M}^k_{\cdot} \times \mathcal{M}^n_{\cdot} \to \mathcal{M}^{k+n}_{\cdot}$, for $(x_1, \ldots, x_k; y_1, \ldots, y_n) \in F_{k+n}\mathbb{C}$. Let us assume that for any $g \in G$, $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1) \in \mathcal{M}^{k+1}_m$, and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2)$, $\in \mathcal{M}^{n+1}_{m'}$, with $\zeta_1\zeta_2 = \epsilon$, and $\overline{g}$ is dual to g with respect to $(.,.)$. The most natural choice of the multiplication supported by the geometrical consideration above has the following form

$$\begin{aligned} &\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon) \\ &= \sum_{l\in\mathbb{Z},\ g\in G_{(l)}} \epsilon^l \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)\ \mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2), \end{aligned} \tag{3.1}$$

parameterized by ζ_1, $\zeta_2 \in \mathbb{C}$. The sum is taken over any $G_{(l)}$-basis, where $\overline{g}$ is the dual to g with respect to the canonical pairing $(.,.)$, with the dual space to G. By the standard reasoning [8], (3.1) does not depend on the choice of a basis of $g \in G_{(l)}$, $l \in \mathbb{Z}$. The definition of a multiplication is also supported by Proposition (3). In what follows, we will see that, since $g \in G$ and $\overline{g} \in G'$ are connected by the sewing condition, ζ_1 and ζ_2 appear in a relation to each other. The form of the multiplication defined above is natural in terms of the theory of characters in conformal field theory [1,6].

Let t be the number of common series the mappings $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k) \in \mathcal{M}^k_m$ and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n) \in \mathcal{M}^n_{m'}$, are combined with m and m' series. Similar to the case of common formal parameters, this case is separately treated with a decrease to $m+m'-t$ of number of combined series. Since we assume that $(x_1, \ldots, x_k; y_1, \ldots, y_n) \in F_{k+n}\mathbb{C}$, i.e., coincidences of x_i and y_j are excluded by the definition of $F_{k+n}\mathbb{C}$. In what follows, we exclude this case from considerations. We define the action of $\partial_l = \partial_{z_l} = \partial/\partial_{z_l}$, $1 \le l \le k+n$, the differentiation of $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k;\ g'_1, y_1; \ldots; g'_n, y_n; \epsilon)$ with respect to the l-th entry of $(x_1, \ldots, x_k; y_1, \ldots, y_n)$ as follows

$$\begin{aligned} &\partial_l \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon) \\ &= \sum_{m\in\mathbb{Z}, g\in G_{(m)}} \epsilon^m \partial^{\delta_{l,i}}_{x_i} \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)\ \partial^{\delta_{l,j}}_{y_j} \mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2). \end{aligned} \tag{3.2}$$

We define the action of the operator z^K on (3.1) as

$$\begin{aligned} &z^K \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon) \\ &= \sum_{m\in\mathbb{Z}, g\in G_{(m)}} \epsilon^m \mathcal{G}(g_1, zx_1; \ldots; g_k, zx_k; g, \zeta_1)\ \mathcal{G}(g'_1, zy_1; \ldots; g'_n, zy_n; \overline{g}, \zeta_2). \end{aligned} \tag{3.3}$$

We define the action of an element $\sigma \in S_{k+n}$ on the multiplication of $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k) \in \mathcal{M}^k_{\cdot}$, and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n) \in \mathcal{M}^n_{\cdot}$, as

$$\begin{aligned} &\sigma(\mathcal{G})(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon) \\ &= \mathcal{G}(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \epsilon) \\ &= \sum_{g\in G_{(m)}} \mathcal{G}(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; g, \zeta_1)\ \mathcal{G}(g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \overline{g}, \zeta_2). \end{aligned} \tag{3.4}$$

Now we formulate the main result of this paper

Proposition 4. *For $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k) \in \mathcal{M}^k_m$ and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n) \in \mathcal{M}^n_{m'}$, the multiplication $\mathcal{G}\left(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon\right)$ (3.4) belongs to the space $\mathcal{M}^{k+n}_{m+m'}$, i.e., $\cdot_\epsilon : \mathcal{M}^k_m \times \mathcal{M}^n_{m'} \to \mathcal{M}^{k+n}_{m+m'}$.*

We start from the proof of the convergence of the multiplication of two elements of double complexes to an predetermined meromorphic function defining their multiplication. In order to prove convergence we have to use a geometrical interpretation [4,7]. For an infinite-dimensional Lie algebra $\mathfrak{g}$, the definition of predetermined meromorphic functions combined with a number of G-series with formal parameters taken as local coordinates on a Riemann sphere. Geometrically, each space $\mathcal{M}^n_m$ is associated to a Riemann sphere with a few marked points, and local coordinates vanishing at these points [4]. Two extra points can be associated to centers of annuli used in order to sew two spheres to form another sphere. The multiplication (3.1) has then a transparent geometric interpretation and associated to a Riemann sphere formed as a result of sewing procedure. Let us identify (as in [8]) two sets $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_n)$ of complex formal parameters, with local coordinates of two sets of points on the first and the second Riemann spheres correspondingly. Identify complex parameters ζ_1, ζ_2 of (3.1) with coordinates as in D_a of the annuluses $\mathcal{A}_a$. After identification of annuluses $\mathcal{A}_a$ and $\mathcal{A}_{\overline{a}}$, r coinciding coordinates may occur. This takes into account case of coinciding formal parameters. The multiplication is defined by a sum of multiplications of matrix elements associated to each of two spheres. Such sum is supposed to describe a predetermined meromorphic function defined on a sphere formed as a result of geometrical sewing [7] of two initial spheres. Since two initial spaces $\mathcal{M}^k_\cdot$ and $\mathcal{M}^n_\cdot$ consists of predetermined meromorphic functions, it is then proved (Proposition 4), that the multiplication results in a predetermined meromorphic function, i.e., an element of the space $\mathcal{M}^{k+n}_{m+m'}$ form by means of an absolute convergent matrix element on the resulting sphere. The complex sewing parameter, parameterizing the module space of sewing spheres, parameterizes also the multiplication of $\mathcal{M}_\cdot^\cdot$-spaces.

Proof. We would like to show that the multiplication (3.1) of elements of the spaces $\mathcal{M}^k_\cdot$ and $\mathcal{M}^n_\cdot$ corresponds to an absolutely converging in ϵ meromorphic function with only possible poles at $x_i = x_j$, $y_{i'} = y_{j'}$, and $x_i = y_{j'}$, $1 \le i, i' \le k$, $1 \le j, j' \le n$. In order to prove this we use the geometrical interpretation of the multiplication (3.1) in terms of Riemann spheres with marked points. We consider two sets of G-elements $(g_1, \ldots, g_k)$ and $(g'_1, \ldots, g'_k)$, and two sets of formal complex parameters $(x_1, \ldots, x_k)$, $(y_1, \ldots, y_n)$. Formal parameters are identified with local coordinates of k points on the Riemann sphere $\widehat{\mathcal{S}}^{(0)}_1$, and n points on $\widehat{\mathcal{S}}^{(0)}_2$, with excised annuli $\mathcal{A}_a$. Recall the sewing parameter condition $\zeta_1\zeta_2 = \epsilon$ of the sewing procedure. Recall from definition of the disks D_a that in two sphere ϵ-sewing formulation, the complex parameters ζ_a, $a = 1, 2$ are coordinates inside identified annuluses $\mathcal{A}_a$, and $|\zeta_a| \le \rho_a$. Therefore, due to Proposition 1 the m-th coefficients of the expansions in ζ_1 and ζ_2, $\mathcal{R}_m(x_1, \ldots, x_k) =$ coeff $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)$ and $\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n) =$ coeff $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2)$, are absolutely convergent in powers of ϵ with some radia of convergence $R_a \le \rho_a$, with $|\zeta_a| \le R_a$. The dependence of the above coefficients on ϵ is expressed via ζ_a,

$a = 1, 2$. Let us rewrite the multiplication (3.1) as

$$\begin{aligned}&\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon)\\ &= \sum_{l\in\mathbb{Z},\ g\in G_{(l)},\ m\in\mathbb{Z}} \epsilon^{l-m-1}\ \mathcal{R}_m(x_1, \ldots, x_k)\ \widetilde{\mathcal{R}}_m(y_1, \ldots, y_n),\end{aligned} \tag{3.5}$$

as a formal series in ϵ for $|\zeta_a| \le R_a$, where and $|\epsilon| \le r$ for $r < \rho_1\rho_2$. Then we apply Cauchy's inequality to coefficients in ζ_1, ζ_2 above to find

$$|\mathcal{R}_m(x_1, \ldots, x_k)| \le M_1 R_1^{-m}, \tag{3.6}$$

with $M_1 = \sup_{|\zeta_1|\le R_1, |\epsilon|\le r} |\mathcal{R}(x_1, \ldots, x_k)|$. Similarly,

$$\left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n)\right| \le M_2 R_2^{-m}, \tag{3.7}$$

for $M_2 = \sup_{|\zeta_2|\le R_2, |\epsilon|\le r} \left|\widetilde{\mathcal{R}}(y_1, \ldots, y_n)\right|$. Using (3.6) and (3.7) we obtain for (3.5)

$$\begin{aligned}|(\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon)_l| &\le |\mathcal{R}_m(x_1, \ldots, x_k)|\ \left|\widetilde{\mathcal{R}}_m(y_1, \ldots, y_n)\right|\\ &\le M_1\ M_2\ (R_1 R_2)^{-m}.\end{aligned}$$

Thus, for $M = \min\{M_1, M_2\}$ and $R = \max\{R_1, R_2\}$,

$$|\mathcal{R}_l(x_1; \ldots, x_k; y_1, \ldots, y_n)| \le M R^{-l+m+1}. \tag{3.8}$$

Due to completeness of C^{k+n} and density of the space of meromorphic functions, we see that (3.1) is absolute convergent to a meromorphic function $\mathcal{G}(x_1, \ldots, x_k; y_1, \ldots, y_n; \epsilon)$ as a formal series in ϵ for $|\zeta_a| \le \rho_a$, and $|\epsilon| \le r$ for $r < \rho_1\rho_2$, with extra poles only at $x_i = y_j$, $1 \le i \le k$, $1 \le j \le n$.

Now we prove that multiplication (3.1) satisfies the (2.2), (2.3). By using (2.2) for $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k)$ and $\mathcal{G}(g'_1,\ y_1;\ \ldots;\ g'_n,\ y_n)$, we consider

$$\begin{aligned}&\partial_l \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon)\\ &= \sum_{m\in\mathbb{Z}\ g\in G_{(m)}} \epsilon^m \partial_{x_i}^{\delta_{l,i}} \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)\ \partial_{y_j}^{\delta_{l,j}} \mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2)\\ &= \sum_{m\in\mathbb{Z},\ g\in G_{(m)}} \epsilon^m \mathcal{G}(g_1, x_1; \ldots; T^{\delta_{l,i}} g_i, x_i; \ldots; g_k, x_k; g, \zeta_1)\\ &\qquad \mathcal{G}(g'_1, y_1; \ldots; T^{\delta_{l,j}} g'_j, y_j; \ldots; g'_n, y_n; \overline{g}, \zeta_2)\\ &= \mathcal{G}(g_1, x_1; \ldots; T_l .., .; \ldots; g'_n, y_n; \epsilon),\end{aligned} \tag{3.9}$$

where T_l. acts on the l-th entry of $(g_1, \ldots; g_k; g'_1, \ldots, g'_n)$. Summing over l we obtain

$$\begin{aligned}\sum_{l=1}^{k+n} \partial_l \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon) &= \sum_{l=1}^{k+n} \mathcal{G}(g_1, x_1; \ldots; T_l .., .; \ldots; g'_n, y_n; \epsilon)\\ &= T.\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon).\end{aligned}$$

Due to (2.3) and (3.3), consider

$$\begin{aligned}
&\mathcal{G}(z^K.(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon))\\
&= \sum_{m\in\mathbb{Z}\ g\in G_{(m)}} \epsilon^m \mathcal{G}(z^K g_1, z\, x_1; \ldots; z^K g_k, z\, x_k; z^K g, z\, \zeta_1)\\
&\qquad \mathcal{G}(z^K g'_1, z\, y_1; \ldots; z^K g'_n, z\, y_n; z^K \overline{g}, z\, \zeta_2)\\
&= \sum_{m\in\mathbb{Z},\ g\in G_{(m)}} \epsilon^m z^K.\mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g, \zeta_1)\\
&\qquad z^K.\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n; \overline{g}, \zeta_2)\\
&= z^K \mathcal{G}(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon)
\end{aligned}$$

Now we prove compatibility of the multiplication with extra Lie algebra series. We will show that $\mathcal{G}\,(g_1, x_1; \ldots; g_k, x_k; g'_1, y_1; \ldots; g'_n, y_n; \epsilon)$ (3.4) is combined with $m + m'$ series. Recall that $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k)$ is combined with m series, and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n)$ is combined with m' series. For $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k)$ we have the following. Let $(l_1, \ldots, l_k) \in \mathbb{Z}_+$ such that $\sum_{i=1}^{k} l_i = k + m$, and $(g_1, \ldots, g_{k+m}) \in G$. Set $h_i = \mathcal{G}(g_{k_1}, x_{k_1} - \zeta_i; \ldots; g_{k_i}, x_{k_i} - \zeta_i; \mathbf{1}_G)$, where $k_1 = \sum_{j=1}^{i-1} l_j + 1$, …, $k_i = \sum_{j=1}^{i-1} l_j + l_i$, for $i = 1, \ldots, k$. Then the series

$$\mathcal{C}^k_m(\mathcal{G}) = \sum_{(r_1,\ldots,r_k)\in\mathbb{Z}} \mathcal{G}(\mathcal{P}_{r_1} h_1, \zeta_1; \ldots; \mathcal{P}_{r_k} h_k, \zeta_k), \tag{3.10}$$

is absolutely convergent when

$$|x_{l_1+\ldots+l_{i-1}+p} - \zeta_i| + |x_{l_1+\ldots+l_{j-1}+q} - \zeta_j| < |\zeta_i - \zeta_j|, \tag{3.11}$$

for $i,\ j = 1, \ldots, k,\ i \neq j$ and for $p = 1, \ldots, l_i$ and $q = 1, \ldots, l_j$. There exist positive integers $\beta^k_m(g_i, g_j)$, depending only on g_i and g_j for $i, j = 1, \ldots, k,\ i \neq j$, such that the sum is analytically extended to a meromorphic function in $(x_1, \ldots, x_{k+m})$, independent of $(\zeta_1, \ldots, \zeta_k)$, with the only possible poles at $x_i = x_j$, of order less than or equal to $\beta^k_m(g_i, g_j)$, for $i,\ j = 1, \ldots, k,\ i \neq j$. For $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n)$, let $(l'_1, \ldots, l'_n) \in \mathbb{Z}_+$ such that $\sum_{i}^{n} l'_i = n + m'$, $(g'_1, \ldots, g'_{n+m'}) \in G$. Set $h'_{i'} = f(g'_{k'_1}, y_{k'_1} - \zeta'_{i'}; \ldots; g'_{k'_{i'}}, y_{k'_{i'}} - \zeta'_{i'})$, where $k'_1 = \sum_{j=1}^{i'-1} l'_j + 1$, …, $k'_{i'} = \sum_{j=1}^{i'-1} l'_j + l'_{i'}$, for $i' = 1, \ldots, n$. Then the series

$$\mathcal{C}^n_{m'}(\mathcal{G}) = \sum_{(r'_1,\ldots,r'_n)\in\mathbb{Z}} \mathcal{G}(\mathcal{P}_{r'_1} h'_1, \zeta'_1; \ldots; \mathcal{P}_{r'_n} h'_n, \zeta'_n), \tag{3.12}$$

is absolutely convergent when

$$|y_{l'_1+\ldots+l'_{i'-1}+p'} - \zeta'_{i'}| + |y_{l'_1+\ldots+l'_{j'-1}+q'} - \zeta'_{j'}| < |\zeta'_{i'} - \zeta'_{j'}|, \tag{3.13}$$

for $i',\ j' = 1, \ldots, n,\ i' \neq j'$ and for $p' = 1, \ldots, l'_i$ and $q' = 1, \ldots, l'_j$. There exist positive integers $\beta^n_{m'}(g'_{i'}, g'_{j'})$, depending only on $g'_{i'}$ and $g'_{j'}$ for $i',\ j' = 1, \ldots, n,\ i' \neq j'$, such that the sum is analytically extended to a meromorphic function in $(y_1, \ldots, y_{n+m'})$, independent of $(\zeta'_1, \ldots, \zeta'_n)$, with the only possible poles at $y_{i'} = y_{j'}$, of order less than or equal to $\beta^n_{m'}(g'_{i'}, g'_{j'})$, for i', $j' = 1, \ldots, n$, $i' \neq j'$.

Now let us consider the conditions of compatibility for the multiplication (3.4) of $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k)$ and $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n)$ combined with a number of series. We redefine the notations $(g''_1, \ldots, g''_k; g''_{k+1}, \ldots, g''_{k+m}; g''_{k+m+1}, \ldots, g''_{k+n+m+m'}; g_{n+1}, \ldots, g'_{n+m'}) = (g_1, \ldots, g_k; g_{k+1}, \ldots, g_{k+m}; g'_1, \ldots, g'_n; g'_{n+1}, \ldots, g'_{n+m'})$, $(z_1, \ldots, z_k; z_{k+1}, \ldots, z_{k+n}) = (x_1, \ldots, x_k; y_1, \ldots, y_n)$, of G-elements. Introduce $(l''_1, \ldots, l''_{k+n}) \in \mathbb{Z}_+$, such that $\sum_{j=1}^{k+n} l''_j = k+n+m+m'$. Define $h''_i = f(g''_{k''_1}, z_{k''_1} - \zeta''_{i''}; \ldots; g''_{k''_{i''}}, z_{k''_{i''}} - \zeta''_{i''})$, where $k''_1 = \sum_{j=1}^{i''-1} l''_j + 1, \quad \ldots, \quad k''_{i''} = \sum_{j=1}^{i''-1} l''_j + l''_{i''}$, for $i'' = 1, \ldots, k+n$, and we take $(\zeta''_1, \ldots, \zeta''_{k+n}) = (\zeta_1, \ldots, \zeta_k; \zeta'_1, \ldots, \zeta'_n)$. Then we consider

$$\mathcal{C}^{k+n}_{m+m'}(\mathcal{G}) = \sum_{r''_1, \ldots, r''_{k+n} \in \mathbb{Z}} \mathcal{G}(\mathcal{P}_{r''_1} h''_1, \zeta''_1; \ldots; \mathcal{P}_{r''_{k+n}} h''_{k+n}, \zeta''_{k+n}), \tag{3.14}$$

and prove it is absolutely convergent with some conditions. The condition $|z_{l''_1+\ldots+l''_{i-1}+p''} - \zeta''_i| + |z_{l''_1+\ldots+l''_{j-1}+q''} - \zeta''_j| < |\zeta''_i - \zeta''_j|$, of absolute convergence for (3.14) for i'', $j'' = 1, \ldots, k+n$, $i'' \neq j''$ and for $p'' = 1, \ldots, l''_i$ and $q'' = 1, \ldots, l''_j$, follows from the conditions (3.11) and (3.13). We obtain

$$\left|\mathcal{C}^{k+n}_{m+m'}(\mathcal{G})\right| = \left| \sum_{r''_1, \ldots, r''_{k+n} \in \mathbb{Z}} \mathcal{G}(\mathcal{P}_{r''_1} h''_1, \zeta''_1; \ldots; \mathcal{P}_{r''_{k+n}} h''_{k+n}, \zeta''_{k+n}) \right|$$

$$= \left| \sum_{l \in \mathbb{Z},\ g \in G_{(l)},\ (r_1, \ldots, r_k) \in \mathbb{Z}} \mathcal{G}(\mathcal{P}_{r_1} h_1, \zeta_1; \ldots; \mathcal{P}_{r_k} h_k, \zeta_k; g, \zeta) \right.$$

$$\left. \sum_{(r'_1, \ldots, r'_n) \in \mathbb{Z}} \mathcal{G}(\mathcal{P}_{r'_1} h'_1, \zeta'_1; \ldots; \mathcal{P}_{r'_n} h'_n, \zeta'_n; \overline{g}, \widetilde{\zeta}) \right| \le \left|\mathcal{C}^k_m(\mathcal{G})\right| \ \left|\mathcal{C}^n_{m'}(\mathcal{G})\right|.$$

Thus, (3.14) is absolutely convergent. The maximal orders of possible poles of (3.14) are $\beta^k_m(g_i, g_j)$, $\beta^n_{m'}(g'_{i'}, g'_{j'})$ at $x_i = x_j$, $y_{i'} = y_{j'}$. In (3.1) we obtain an expansion in powers of x_i and y_j we see that new poles at $x_i = y_j$ may occur. From the last expression we infer that there exist positive integers $\beta^{k+n}_{m+m'}(g''_{i''}, g''_{j''})$, such that $\beta^k_m(g_i, g_j)\beta^n_{m'}(g'_{i'}, g'_{j'}) \le \beta^{k+n}_{m+m'}(g''_{i''}, g''_{j''})$, for i, $j = 1, \ldots, k$, $i \neq j$, i', $j' = 1, \ldots, n$, $i' \neq j'$, depending only on $g''_{i''}$ and $g''_{j''}$ for i'', $j'' = 1, \ldots, k+n$, $i'' \neq j''$ such that the series (3.14) can be analytically extended to a meromorphic function in $(x_1, \ldots, x_k; y_1, \ldots, y_n)$, independent of $(\zeta''_1, \ldots, \zeta''_{k+n})$, with extra possible poles at and $x_i = y_j$ of order less than or equal to $\beta^{k+n}_{m+m'}(g''_{i''}, g''_{j''})$, for i'', $j'' = 1, \ldots, k+n$, $i'' \neq j''$.

For $\mathcal{G}(g_1, x_1; \ldots; g_k, x_k) \in \mathcal{M}^k_m$, the series $\mathcal{D}^k_m(\mathcal{G}) = \sum_{q \in \mathbb{Z}} \mathcal{G}(g_1, x_1; \ldots; g_m, x_m; \mathcal{P}_q(f(g_{m+1}, x_{m+1}; \ldots; g_{m+k}, x_{m+k}\ )$, is absolutely convergent when $x_i \neq x_j$, $i \neq j$, $|x_i| > |x_{k'}| > 0$, for $i = 1, \ldots, m$, and $k' = m+1, \ldots, k+m$. The sum can be analytically extended to a meromorphic function in $(x_1, \ldots, x_{k+m})$ with the only possible poles at $x_i = x_j$, of orders less than or equal to $\beta^k_m(g_i, g_j)$, for $i, j =$

$1, \ldots, k$, $i \neq j$. For $\mathcal{G}(g'_1, y_1; \ldots; g'_n, y_n) \in \mathcal{M}^n_{m'}$, the series $\mathcal{D}^n_{m'}(\mathcal{G})= \sum_{q\in\mathbb{Z}} \mathcal{G}(g'_1$, y_1; …; $g'_{m'}$, $y_{m'}$; $\mathcal{P}_q(f($ $g'_{m'+1}$, $y_{m'+1}$; …; $g'_{m'+n}, y_{m'+n})$)), is absolutely convergent when $y_{i'} \neq y_{j'}$, $i' \neq j'$, $|y_{i'}| > |y_{k''}| > 0$, for $i' = 1, \ldots, m'$, and $k'' = m'+1, \ldots, n+m'$, and the sum can be analytically extended to a meromorphic function in $(y_1, \ldots, y_{n+m'})$ with the only possible poles at $y_{i'} = y_{j'}$, of orders less than or equal to $\beta^n_{m'}(g'_{i'}, g'_{j'})$, for i', $j' = 1, \ldots, n$, $i' \neq j'$. For the multiplication (3.4), $(g''_1, \ldots, g''_{k+n+m+m'}) \in G$, and $(z_1, \ldots, z_{k+n+m+m'}) \in \mathbb{C}$, and we find positive integers $\beta^{k+n}_{m+m'}(g'_i, g'_j)$, depending only on v'_i and v''_j, for i'', $j'' = 1, \ldots, k+n$, $i'' \neq j''$. Under conditions $z_{i''} \neq z_{j''}$, $i'' \neq j''$, $|z_{i''}| > |z_{k'''}| > 0$, for $i'' = 1, \ldots, m+m'$, and $k''' = m+m'+1, \ldots, m+m'+k+n$, let us introduce $\mathcal{D}^{k+n}_{m+m'}(\mathcal{G}) = \sum_{q\in\mathbb{Z}}$ $\mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'}; \mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n})); \epsilon)$. Using Lemma 1 we then obtain

$$\begin{aligned}
\left|\mathcal{D}^{k+n}_{m+m'}(\mathcal{G})\right| &= \Bigg|\sum_{q\in\mathbb{Z}} \mathcal{G}(g''_1, z_1; \ldots; g''_{m+m'}, z_{m+m'}; \\
&\quad \mathcal{P}_q(f(g''_{m+m'+1}, z_{m+m'+1}; \ldots; g''_{m+m'+k+n}, z_{m+m'+k+n})); \epsilon)\Bigg| \\
&= \Bigg|\sum_{q\in\mathbb{Z},\ g\in G} \mathcal{G}(g_{k+1}, x_{k+1}; \ldots; g_{k+m}, x_{k+m}; \mathcal{P}_q(f(g_1, x_1; \ldots; g_k, x_k); g, \zeta_1)) \\
&\quad \mathcal{G}(g'_{n+1}, y_{n+1}; \ldots; g'_{n+m'}, y_{n+m'}; \mathcal{P}_q(f(g'_1, y_1; \ldots; g'_n, y_n); \overline{g}, \zeta_2)))\Bigg| \\
&\leq \left|\mathcal{D}^k_m(\mathcal{G})\right| \left|\mathcal{D}^n_{m'}(\mathcal{G})\right|,
\end{aligned}$$

where we have used the invariance of (3.4) with respect to $\sigma \in S_{m+m'+k+n}$. In the last expression, according to Proposition 3 $\mathcal{D}^k_m(\mathcal{G})$ and $\mathcal{D}^n_{m'}(\mathcal{G})$ are absolute convergent. Thus, $\mathcal{D}^{k+n}_{m+m'}(\mathcal{G})$ is absolutely convergent, and (3.14) is analytically extendable to a meromorphic function in $(z_1, \ldots, z_{k+n+m+m'})$ with the only possible poles at $x_i = x_j$, $y_{i'} = y_{j'}$, and at $x_i = y_{j'}$, i.e., the only possible poles at $z_{i''} = z_{j''}$, of orders less than or equal to $\beta^{k+n}_{m+m'}(v''_{i''}, v''_{j''})$, for i'', $j'' = 1, \ldots, k'''$, $i'' \neq j''$.

Finally, for the action of $\sigma \in S_{k+n}$ on the product we have

$$\begin{aligned}
&\sum_{\sigma\in J^{-1}_{k+n;s}} (-1)^{|\sigma|} \mathcal{G}(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \epsilon) \\
&= \sum_{\sigma\in J^{-1}_{k+n;s},\ g\in G_{(l)}} (-1)^{|\sigma|} \epsilon^l \mathcal{G}(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; g, \zeta_1) \\
&\qquad \mathcal{G}(g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \overline{g}, \zeta_2) \\
&= \sum_{r\in\mathbb{Z},\ \sigma\in J^{-1}_{k;s}} \epsilon^r\ (-1)^{|\sigma|} \mathcal{G}_r(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; \zeta_1) \\
&\quad \sum_{r'\in\mathbb{Z},\ \sigma\in J^{-1}_{n;s}} \epsilon^{r'}\ (-1)^{|\sigma|} \mathcal{G}_{r'}(g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \zeta_2) = 0,
\end{aligned}$$

due to $J^{-1}_{k+n;s} = J^{-1}_{k;s} \times J^{-1}_{n;s}$, definition (3.4), and $\mathcal{G}_r(g_{\sigma(1)}, x_{\sigma(1)}; \ldots; g_{\sigma(k)}, x_{\sigma(k)}; \zeta_1) \in \mathcal{M}^k_m$, $\mathcal{G}_{r'}(g'_{\sigma(1)}, y_{\sigma(1)}; \ldots; g'_{\sigma(n)}, y_{\sigma(n)}; \zeta_2) \in \mathcal{M}^n_{m'}$, and, therefore, (2.4) is satisfied. This finishes the proof of the proposition. □

Let us now recall [3] the definition of the coboundary operator for the spaces $\mathcal{M}^n_m$,

$$\begin{aligned} \delta^n_m \mathcal{G}(g_1, z_1; \ldots; g_n, z_n) &= \sum_{i=1}^{n} (-1)^i \, \mathcal{G}\left(\ldots; \gamma_{g_i}(z_i - z_{i+1}) \, g_{i+1}; \ldots\right) \\ &+ \mathcal{G}(\gamma_{g_1}(z_1); g_2, z_2; \ldots; g_n, z_n) \\ &+ (-1)^{n+1} \mathcal{G}(\gamma_{g_{n+1}}(z_{n+1}); g_1, z_1; \ldots; g_n, z_n). \end{aligned} \tag{3.15}$$

The following lemma takes place:

Lemma 2. *The operator* (3.15) *obeis* $\delta^n_m : \mathcal{M}^n_m \to \mathcal{M}^{n+1}_{m-1}$, $\delta^{n+1}_{m-1} \circ \delta^n_m = 0$, $0 \longrightarrow \mathcal{M}^0_m \xrightarrow{\delta^0_m} \mathcal{M}^1_{m-1} \xrightarrow{\delta^1_{m-1}} \ldots \xrightarrow{\delta^{m-1}_1} \mathcal{M}^m_0 \longrightarrow 0$, *i.e., provides the double chain-cochain complex* $(\mathcal{M}^n_m, \delta^n_m)$. □

Then one has

Corollary 1. *The multiplication* (3.4) *extends the chain-cochain complex* $(\mathcal{M}^n_m, \delta^n_m)$ *to all multiplications* $\mathcal{M}^k_m \times \mathcal{M}^n_{m'}$, k, $n \geq 0$, m, $m' \geq 0$. □

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