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Abstract

We show that a strong solution of the compressible MHD system driven by inhomogeneous Dirichlet boundary conditions remains smooth as long as its L^∞-norm is controlled.

Keywords: Compressible MHD system, strong solution, conditional regularity, blow–up criterion

Contents

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1 Introduction

The present paper is a continuation of our work concerning conditional regularity of solutions to open systems arising in fluid dynamics, see [2]. The results can be seen as another evidence supporting the celebrated Nash's conjecture, see [16]:

> *Probably one should first try to prove a conditional existence and uniqueness theorem for flow equations. This should give existence, smoothness, and unique continuation (in time) of flows, conditional on the non-appearance of certain gross types of singularity, such as infinities of temperature or density.*

This principle can be stated in a more concise way:

$$\textit{boundedness} \ \Rightarrow \ \textit{regularity.}$$

The fact that bounded solutions of *semilinear parabolic* problems are smooth is well known, see e.g. the monograph of Ladyzhenskaya, Solonnikov, Uralceva [13]. A similar statement in the context of systems of equations in fluid dynamics is less obvious. The problem is relatively well understood in the context of incompressible fluids described by the standard Navier–Stokes system, where regularity is guaranteed by the Prodi–Serrin type conditions. The compressible case, where the Navier–Stokes system is of mixed type, the situation is less obvious. In the pioneering work of Sun, Wang, and Zhang [17], [18], regularity of solutions to the compressible Navier–Stokes system is conditioned by boundedness of the density. In view of the recent results of Buckmaster et al. [4], and Merle et al. [14], [15], this criterion seems almost optimal. Besides, there are several results based on boundedness of solutions available in the literature. Fan, Jiang, and Ou [5] consider the compressible Navier–Stokes–Fourier system in a bounded fluid domain with the conservative boundary conditions. The same problem is considered by by Huang, Li, Wang [9]. There are results for the Cauchy problem Huang and Li [8], and Jiu, Wang and Ye [10]. In [7], the blow up criterion for both the Cauchy problem and conservative boundary conditions is formulated in terms of the maximum of the density and a Serrin type regularity for the temperature. We refer to Wen and Zhu [22], [23] for previous results in this direction.

1.1 Compressible magnetohydrodynamics

Basically all results mentioned above apply either to the Cauchy problem or the boundary value problem with conservative boundary conditions. Much less seems to be known for physically relevant open systems, with a non–zero energy and possibly mass flux through the kinematic boundary. Following our work on the Navier–Stokes–Fourier system [2] we consider the *compressible magnetohydrodynamics (MHD) system* describing the time evolution of the density $\varrho = \varrho(t,x)$, the velocity $\mathbf{u} = \mathbf{u}(t,x)$, the (absolute) temperature $\vartheta = \vartheta(t,x)$, and the magnetic field $\mathbf{B} = \mathbf{B}(t,x)$ of a compressible, viscous, electrically and heat conducting fluid, see e.g. Weiss and Proctor [21]:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0; \tag{1.1}$$

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x\mathbf{u}) + \mathbf{curl}_x\mathbf{B}\times\mathbf{B} + \varrho\mathbf{g}; \tag{1.2}$$

$$\partial_t\mathbf{B} + \mathbf{curl}_x(\mathbf{B}\times\mathbf{u}) + \mathbf{curl}_x(\zeta\mathbf{curl}_x\mathbf{B}) = 0,\ \mathrm{div}_x\mathbf{B} = 0; \tag{1.3}$$

$$\partial_t(\varrho e(\varrho,\vartheta)) + \mathrm{div}_x(\varrho e(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x\mathbf{q}(\nabla_x\vartheta) = \mathbb{S}(\mathbb{D}_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} + \zeta|\mathbf{curl}_x\mathbf{B}|^2 - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u}. \tag{1.4}$$

The viscous stress $\mathbb{S}$ is given by Newton's rheological law

$$\mathbb{S}(\mathbb{D}_x\mathbf{u}) = 2\mu\left(\mathbb{D}_x\mathbf{u} - \frac{1}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta\mathrm{div}_x\mathbf{u}\mathbb{I},\ \mathbb{D}_x\mathbf{u} \equiv \frac{1}{2}\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u}\right),\ \mu > 0,\ \eta \geq 0. \tag{1.5}$$

The heat flux obeys Fourier's law

$$\mathbf{q}(\nabla_x\vartheta) = -\kappa\nabla_x\vartheta,\ \kappa > 0, \tag{1.6}$$

where the transport coefficients μ, η, κ as well as the the magnetic diffusion coefficient $\zeta > 0$ are constant. The equations of state for the pressure p and the internal energy e are given by the standard Boyle–Mariotte law of polytropic perfect gas,

$$p(\varrho,\vartheta) = \varrho\vartheta,\ e(\varrho,\vartheta) = c_v\vartheta,\ c_v > 0 \text{ constant}. \tag{1.7}$$

1.2 Boundary conditions

We suppose the fluid occupies a bounded domain $\Omega \subset R^3$ of class at least C^3. The boundary $\partial\Omega$ admits a decomposition into two components:

$$\partial\Omega = \Gamma_D \cup \Gamma_N,\ \Gamma_D, \Gamma_N \text{ compact},\ \Gamma_D \cap \Gamma_N = \emptyset. \tag{1.8}$$

We consider the following boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = 0; \tag{1.9}$$

$$\vartheta|_{\partial\Omega} = \vartheta_B; \tag{1.10}$$

$$\mathbf{B} \times \mathbf{n}|_{\Gamma_D} = \mathbf{b}_\tau, \ \mathbf{B} \cdot \mathbf{n}|_{\Gamma_N} = b_\nu, \ \mathbf{curl}_x \mathbf{B} \times \mathbf{n}|_{\Gamma_N} = 0. \tag{1.11}$$

Although the compressible MHD system shares many properties with the simpler Navier–Stokes–Fourier system studied in [2], incorporating the inhomogeneous boundary conditions (1.11) is not straightforward. One of the technical difficulties is the absence of a proper (local in time) existence theory in the class of strong solutions. Tang and Gao [19] adapted the elegant L^p-framework of Kotschote [11] to the compressible MHD equations replacing (1.11) by

$$\mathbf{B}|_{\Gamma_D} = \mathbf{l}_d, \ \mathbf{curl}_x \mathbf{B} \times \mathbf{n}|_{\Gamma_N} = 0.$$

Apparently, the problem is overdetermined with the former while underdetermined with the latter condition. We propose a remedy incorporating the (correct) boundary conditions (1.11) in a proper parabolic setting introduced by Kozono and Yanagisawa [12].

The paper consists of two parts. The first one concerns a conditional regularity criterion for the compressible MHD system accompanied with the boundary conditions (1.9), (1.10) in the framework of regular solutions introduced in the spirit of Valli and Zajaczkowski [20]. The second one presents a blow up criterion in the same class adapting the L^p-framework of Kotschote [11] and Tang, Gao [19].

2 Conditional regularity criterion

We adopt the class of strong solutions introduced by Valli and Zajaczkowski [20, Theorem 2.5] in the context of the Navier–Stokes–Fourier system. Specifically, we say that $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ is a strong solution of the compressible MHD system in $[0, T]$ if

$$\begin{aligned}
&\varrho \in C([0,T]; W^{2,2}(\Omega), \ \partial_t \varrho \in C([0,T]; W^{1,2}(\Omega)), \ \inf_{(0,T)\times\Omega} \varrho > 0,\\
&\vartheta \in L^2(0,T; W^{3,2}(\Omega)), \ \partial_t \vartheta \in L^2(0,T; W^{1,2}(\Omega)), \ \inf_{(0,T)\times\Omega} \vartheta > 0,\\
&\mathbf{u} \in L^2(0,T; W^{3,2}(\Omega; R^3)), \ \partial_t \mathbf{u} \in L^2(0,T; W^{1,2}(\Omega; R^3)),\\
&\mathbf{B} \in L^2(0,T; W^{3,2}(\Omega; R^3)), \ \partial_t \mathbf{B} \in L^2(0,T; W^{1,2}(\Omega; R^3)).
\end{aligned} \tag{2.1}$$

In particular,

$$(\varrho, \vartheta, \mathbf{u}, \mathbf{B}) \in C([0,T]; W^{2,2}(\Omega; R^8)).$$

Accordingly, the initial/boundary data belong to the adequate trace spaces

$$\varrho(0,\cdot) = \varrho_0 \in W^{2,2}(\Omega), \ \inf_{x\in\Omega} \varrho_0 > 0, \tag{2.2}$$

$$\vartheta(0,\cdot) = \vartheta_0 \in W^{2,2}(\Omega),\ \inf_{x\in\Omega} \vartheta_0(x) > 0,\ \vartheta_B \in W^{\frac{5}{2},2}(\partial\Omega),\ \inf_{x\in\partial\Omega} \vartheta_B > 0, \tag{2.3}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0 \in W^{2,2}(\Omega; R^3), \tag{2.4}$$

$$\mathbf{B}(0,\cdot) = \mathbf{B}_0 \in W^{2,2}(\Omega; R^3),\ \mathrm{div}_x \mathbf{B}_0 = 0,\ \mathbf{b}_\tau \in W^{\frac{5}{2},2}(\Gamma_D; R^3),\ b_\nu \in W^{\frac{5}{2},2}(\Gamma_N). \tag{2.5}$$

Strong solutions automatically satisfy the relevant set of *compatibility conditions* specified in the second part of the paper.

Finally, we introduce a quantity reflecting the size of the data:

$$\|\mathrm{data}\| =$$
$$\max \left\{ (\inf_{\Omega} \varrho_0)^{-1}, (\inf_{\Omega} \vartheta_0)^{-1},\ (\inf_{\partial\Omega} \vartheta_B)^{-1}, \|\varrho_0, \vartheta_0, \mathbf{u}_0, \mathbf{B}_0\|_{W^{2,2}(\Omega)}, \|\vartheta_B, \mathbf{b}_\tau, b_\nu\|_{W^{\frac{5}{2},2}(\partial\Omega)}, \|\mathbf{g}\|_{W^{1,2}(\Omega)} \right\}.$$

The first result of this paper is a conditional regularity criterion in terms of the amplitude of solutions.

Theorem 2.1 (Conditional regularity.). *Let $\Omega \subset R^3$ be a bounded domains with boundary $\partial\Omega$ of class at least C^3. Suppose $(\varrho, \vartheta, \mathbf{u}, \mathbf{B})$ is a strong solution of the compressible MHD system* (1.1)–(1.6) *with the boundary conditions* (1.9)–(1.11) *on a time interval* $[0,T]$.

Then

$$\sup_{t\in[0,T]} \|\varrho(t,\cdot)\|_{W^{2,2}(\Omega)} + \sup_{t\in[0,T]} \|\partial_t \varrho(t,\cdot)\|_{W^{1,2}(\Omega)} + \int_0^T \|(\vartheta, \mathbf{u}, \mathbf{B})\|^2_{W^{3,2}(\Omega;R^7)}\,\mathrm{d}t$$
$$+ \int_0^T \|(\partial_t\vartheta, \partial_t\mathbf{u}, \partial_t\mathbf{B})\|^2_{W^{1,2}(\Omega;R^7)}\,\mathrm{d}t$$
$$\leq \Lambda \Big(T, \|\mathrm{data}\|, \|(\varrho, \vartheta, \mathbf{u}, \mathbf{B})\|_{L^\infty((0,T)\times\Omega;R^8)}\Big), \tag{2.6}$$

where $\Lambda : R^3 \to [0,\infty)$ is bounded for bounded arguments.

We point out that we do not assume *existence* of a strong solution. Theorem 2.1 should be seen as a kind of *a priori* bound in the class of strong solutions. If the L^∞ norm of a strong solution is controlled on a time interval $[0,T]$, then all smooth norms are controlled only in terms of the data. The solution remains regular and can be continued beyond the time T. The rest of this section is devoted to the proof of Theorem 2.1.

2.1 Extension of the boundary data

It is convenient to extend the boundary data inside Ω.

$$\Delta_x \tilde{\vartheta} = 0 \text{ in } \Omega,\ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B,$$

$$\mathbf{curl}_x\mathbf{curl}_x\widetilde{\mathbf{B}} = 0,\ \mathrm{div}_x\widetilde{\mathbf{B}} = 0 \text{ in } \Omega,\ \widetilde{\mathbf{B}} \times \mathbf{n}|_{\Gamma_D} = \mathbf{b}_\tau,\ \widetilde{\mathbf{B}} \cdot \mathbf{n}|_{\Gamma_N} = b_\nu,\ \mathbf{curl}_x\widetilde{\mathbf{B}} \times \mathbf{n}|_{\Gamma_N} = 0. \tag{2.7}$$

In what follows, we use the same symbol ϑ_B, $\mathbf{B}_B$ to denote both the boundary data and their extension inside Ω.

2.2 Induction equation

As observed by Kozono and Yanagisawa [12], the Maxwell system (1.3) with the boundary conditions (1.11) can be conveniently written as a standard (unconstrained) parabolic system

$$\begin{aligned} &\partial_t \mathbf{B} - \zeta \Delta_x \mathbf{B} = \mathbf{curl}_x(\mathbf{u} \times \mathbf{B}) \text{ in } (0,T) \times \Omega, \\ &\quad \mathbf{B} \times \mathbf{n}|_{\Gamma_D} = \mathbf{b}_\tau,\ \mathrm{div}_x \mathbf{B}|_{\Gamma_D} = 0, \mathbf{B} \cdot \mathbf{n}|_{\Gamma_N} = b_\nu,\ \mathbf{curl}_x \mathbf{B} \times \mathbf{n}|_{\partial\Omega} = 0. \end{aligned} \tag{2.8}$$

If a solution of (2.8) is regular, its divergence $U = \mathrm{div}_x \mathbf{B}$ satisfies the conventional parabolic equation

$$\begin{aligned} \partial_t U - \zeta \Delta_x \mathbf{U} &= 0 \text{ in } (0,T) \times \Omega, \\ U|_{\Gamma_D} &= 0,\ \nabla_x U \cdot \mathbf{n}|_{\Gamma_N} = 0, \\ U(0, \cdot) &= \mathrm{div}_x \mathbf{B}_0. \end{aligned} \tag{2.9}$$

Indeed the homogeneous Dirichlet boundary condition for U on Γ_D follows directly from (2.8). Moreover,

$$\nabla_x U \cdot \mathbf{n} = \nabla_x \mathrm{div}_x \mathbf{B} \cdot \mathbf{n} = \mathbf{curl}_x(\mathbf{curl}_x \mathbf{B}) \cdot \mathbf{n} + \Delta_x \mathbf{B} \cdot \mathbf{n}$$

on Γ_N, where

$$\mathbf{curl}_x \mathbf{B} \times \mathbf{n} = 0 \ \Rightarrow\ \mathbf{curl}_x(\mathbf{curl}_x \mathbf{B}) \cdot \mathbf{n} = 0 \text{ on } \Gamma_N.$$

while

$$\zeta \Delta_x \mathbf{B} \cdot \mathbf{n} = \mathbf{curl}_x(\mathbf{B} \times \mathbf{u}) \cdot \mathbf{n} = \Big((\mathbf{B} \cdot \nabla_x) \mathbf{u} - \mathrm{div}_x \mathbf{u} \mathbf{B} \Big) \cdot \mathbf{n} = 0 \text{ on } \Gamma_N.$$

Consequently, the required solenoidality $\mathrm{div}_x \mathbf{B} = 0$ is inherited from the initial data $\mathrm{div}_x \mathbf{B}_0 = 0$.

Kozono and Yanagisawa [12, Lemma 4.4] showed that the Laplace operator Δ_x endowed with the boundary conditions (2.8) is uniformly elliptic satisfying the Lopatinski–Shapiro (complementing) boundary conditions. In particular, the classical Agmon–Douglis–Nirenberg theory [1] applies yielding the following elliptic estimates:

$$\begin{aligned} &\|\mathbf{B}\|_{W^{s,q}(\Omega;R^3)} \stackrel{<}{\sim} \Big(\|\mathbf{f}\|_{W^{s-2,q}(\Omega;R^3)} + \|\mathbf{b}_\tau\|_{W^{s-\frac{1}{q},q}(\partial\Omega;R^3)} + \|b_\nu\|_{W^{s-\frac{1}{q},q}(\partial\Omega)} + \|\mathbf{B}\|_{L^q(\Omega;R^3)} \Big) \\ &\qquad \text{whenever} \\ &\qquad \Delta_x \mathbf{B} = \mathbf{f} \text{ in } \Omega,\ \mathbf{B} \times \mathbf{n}|_{\Gamma_D} = 0,\ \mathbf{B} \cdot \mathbf{n}|_{\Gamma_N} = 0,\ \mathbf{curl}_x \mathbf{B} \times \mathbf{n}|_{\Gamma_N} = 0, \\ &\qquad s \geq 2 \text{ an integer},\ 1 < q < \infty. \end{aligned} \tag{2.10}$$

Accordingly, we recover the maximal regularity estimates for the parabolic system (2.8):

$$\|\partial_t \mathbf{B}\|_{L^p(0,T;L^q(\Omega;R^3))} + \|\mathbf{B}\|_{L^p(0,T;W^{2,q}(\Omega;R^3))}$$

$$\leq C\left(\|\mathbf{B}_0\|_{W^{2-\frac{2}{p},q}(\Omega;R^3)}+\|\mathbf{b}_\tau\|_{W^{1-\frac{1}{q},q}(\partial\Omega;R^3)}+\|b_\nu\|_{W^{1-\frac{1}{q},q}(\partial\Omega)}+\|\mathbf{curl}_x(\mathbf{u}\times\mathbf{B})\|_{L^p(0,T;L^q(\Omega;R^2))}\right) \tag{2.11}$$

for any $1<p,q<\infty$. In addition, in the class of bounded solutions,

$$\begin{aligned}&\|\mathbf{curl}_x(\mathbf{u}\times\mathbf{B})\|_{L^p(0,T;L^q(\Omega;R^3))}\\&\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\Big(\|\nabla_x\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}+\|\nabla_x\mathbf{B}\|_{L^p(0,T;L^q(\Omega))}\Big).\end{aligned} \tag{2.12}$$

As $\|\mathbf{B}\|_{L^\infty}$ is *a priori* bounded, we may combine (2.11) with (2.12) and a simple interpolation argument to conclude

$$\begin{aligned}&\|\partial_t\mathbf{B}\|_{L^p(0,T;L^q(\Omega;R^3))}+\|\mathbf{B}\|_{L^p(0,T;W^{2,q}(\Omega;R^3))}\\&\qquad\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\Big(1+\|\nabla_x\mathbf{u}\|_{L^p(0,T;L^q(\Omega))}\Big)\end{aligned} \tag{2.13}$$

for any $1<p\leq 2$, $1<q\leq 6$.

Remark 2.2. The restriction on the exponents p,q in (2.13) is due to the assumed regularity of the initial data $\mathbf{B}_0$.

2.3 Energy estimates

The following identity termed *ballistic energy balance* was derived in [3, Section 3.2]:

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}\varrho|\mathbf{u}|^2+c_v\varrho\vartheta+\frac{1}{2}|\mathbf{B}|^2-\vartheta_B\varrho s(\varrho,\vartheta)-\mathbf{B}_B\cdot\mathbf{B}\right)\ \mathrm{d}x\\&+\int_\Omega\frac{\vartheta_B}{\vartheta}\left(\mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}+\kappa\frac{|\nabla_x\vartheta|^2}{\vartheta}+\zeta|\mathbf{curl}_x\mathbf{B}|^2\right)\ \mathrm{d}x\\&\quad=-\int_\Omega\left(\varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\vartheta_B-\kappa\frac{\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\vartheta_B\right)\ \mathrm{d}x\\&\quad+\int_\Omega\Big((\mathbf{B}\times\mathbf{u})\cdot\mathbf{curl}_x\mathbf{B}_B+\zeta\mathbf{curl}_x\ \mathbf{B}\cdot\mathbf{curl}_x\mathbf{B}_B\Big)\ \mathrm{d}x+\int_\Omega\varrho\mathbf{g}\cdot\mathbf{u}\ \mathrm{d}x,\end{aligned} \tag{2.14}$$

where $s=s(\varrho,\vartheta)$ is the specific entropy,

$$s(\varrho,\vartheta)=\log\left(\frac{\vartheta^{c_v}}{\varrho}\right).$$

Next, observe that

$$\left|\int_\Omega\frac{\nabla_x\vartheta}{\vartheta}\cdot\nabla_x\vartheta_B\ \mathrm{d}x\right|=\left|\int_{\partial\Omega}\log(\vartheta_B)\nabla_x\vartheta_B\cdot\mathbf{n}\ \mathrm{d}\sigma_x\right|\leq c(\|\mathrm{data}\|),$$

and

$$\left|\int_\Omega \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\vartheta_B \ \mathrm{d}x\right| \leq c(\|\varrho,\mathbf{u}\|_{L^\infty})\left(1+\int_\Omega |\log(\vartheta)| \ \mathrm{d}x\right).$$

We point out that boundedness of the velocity is crucial to control the second integral.

It is a routine matter to apply the Gronwall argument along with the standard Poincaré and Korn–Poincaré inequalities to deduce the energy bounds:

$$\sup_{t\in[0,T]}\int_\Omega \varrho|\log(\vartheta)|(t,\cdot) \ \mathrm{d}x \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right), \tag{2.15}$$

$$\int_0^T\int_\Omega |\mathbb{D}_x\mathbf{u}-\frac{1}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}|^2 \ \mathrm{d}x\,\mathrm{d}t \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)$$

$$\Rightarrow \int_0^T \|\mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right), \tag{2.16}$$

$$\int_0^T\int_\Omega \left(|\nabla_x\vartheta|^2+|\nabla_x\log(\vartheta)|^2\right) \ \mathrm{d}x\,\mathrm{d}t \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right),$$

$$\Rightarrow \int_0^T \|\vartheta\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t+\int_0^T \|\log(\vartheta)\|^2_{W^{1,2}(\Omega)}\,\mathrm{d}t \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right), \tag{2.17}$$

$$\int_0^T\int_\Omega \|\mathbf{curl}_x\mathbf{B}\|^2_{L^2(\Omega;R^3)} \ \mathrm{d}x \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right). \tag{2.18}$$

where we have used Poincaré inequality to obtain (2.16). In addition, it follows from (2.16) and the estimate (2.13) that

$$\begin{aligned}\|\partial_t\mathbf{B}\|_{L^2(0,T;L^2(\Omega;R^3))} &\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right),\\ \|\mathbf{B}\|_{L^2(0,T;W^{2,2}(\Omega;R^3))} &\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right),\end{aligned} \tag{2.19}$$

and, consequently,

$$\sup_{t\in[0,T]}\|\mathbf{B}(t,\cdot)\|_{W^{1,2}(\Omega;R^3)} \leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right). \tag{2.20}$$

2.4 Estimates of the velocity gradient

This part follows to large extent the arguments of [2, Section 4] motivated by Fang, Zi, and Zhang [6, Section 3]. We therefore focus only on the estimates of the extra terms involving the magnetic field.

First, let us recall the concept of material derivative of a function g,

$$D_t g = \partial_t g + \mathbf{u}\cdot\nabla_x g.$$

The momentum equation (1.2) reads

$$\varrho D_t\mathbf{u}+\nabla_x p = \mathrm{div}_x\mathbb{S}+\mathbf{curl}_x\mathbf{B}\times\mathbf{B}+\varrho\mathbf{g}. \tag{2.21}$$

The scalar product of (2.21) with $D_t\mathbf{u}$ yields

$$\varrho|D_t\mathbf{u}|^2 + \nabla_x p\cdot D_t\mathbf{u} = \mathrm{div}_x\mathbb{S}(\mathbb{D}_x\mathbf{u})\cdot D_t\mathbf{u} + \varrho\mathbf{g}\cdot D_t\mathbf{u} + \mathbf{curl}_x\mathbf{B}\times\mathbf{B}\cdot D_t\mathbf{u}. \tag{2.22}$$

Repeating step by step the arguments of [2, Section 4.1], we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}\ \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega p\mathrm{div}_x\mathbf{u}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x\\
&\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\left(1+\int_\Omega \varrho|D_t\vartheta||\nabla_x\mathbf{u}|\ \mathrm{d}x + \int_\Omega|\nabla_x\mathbf{u}|^3\ \mathrm{d}x\right.\\
&\qquad\left.+\left|\int_\Omega(\mathbf{curl}_x\mathbf{B}\times\mathbf{B})\cdot D_t\mathbf{u}\ \mathrm{d}x\right|\right).
\end{aligned} \tag{2.23}$$

Finally, estimating the last integral in (2.23) by the available bounds (2.20) we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}\ \mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega p\mathrm{div}_x\mathbf{u}\ \mathrm{d}x + \frac{1}{2}\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x\\
&\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\left(1+\int_\Omega \varrho|D_t\vartheta||\nabla_x\mathbf{u}|\ \mathrm{d}x + \int_\Omega|\nabla_x\mathbf{u}|^3\ \mathrm{d}x + \|D_t\mathbf{u}\|_{L^2(\Omega;R^3)}\right),
\end{aligned} \tag{2.24}$$

where we have used (2.17). Integrating the above inequality in time and using standard Hölder type estimates we conclude

$$\begin{aligned}
&\|\mathbf{u}(\tau,\cdot)\|^2_{W^{1,2}(\Omega;R^3)} + \int_0^\tau\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x\\
&\leq \Lambda\left(T,\|\mathrm{data}\|,\|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\left(1+\int_0^\tau\int_\Omega \varrho|D_t\vartheta|^2\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \|D_t\mathbf{u}\|_{L^2(\Omega;R^3)}\,\mathrm{d}t\right.\\
&\left.+\int_0^\tau\int_\Omega|\nabla_x\mathbf{u}|^4\ \mathrm{d}x\,\mathrm{d}t\right).
\end{aligned} \tag{2.25}$$

2.4.1 Higher order velocity material derivative estimates

Applying material derivative to the momentum equation (2.21) we get

$$\begin{aligned}
&\varrho D_t^2\mathbf{u} + \nabla_x\partial_t p + \mathrm{div}_x(\nabla_x p\otimes\mathbf{u})\\
&= \mu\Big(\Delta_x\partial_t\mathbf{u} + \mathrm{div}_x(\Delta_x\mathbf{u}\otimes\mathbf{u})\Big) + \left(\eta+\frac{\mu}{3}\right)\Big(\nabla_x\mathrm{div}_x\partial_t\mathbf{u} + \mathrm{div}_x\left((\nabla_x\mathrm{div}_x\mathbf{u})\otimes\mathbf{u}\right)\Big) + \varrho\mathbf{u}\cdot\nabla_x\mathbf{g}\\
&+\partial_t(\mathbf{curl}_x\mathbf{B}\times\mathbf{B}) + \mathbf{u}\cdot\nabla_x(\mathbf{curl}_x\mathbf{B}\times\mathbf{B}).
\end{aligned} \tag{2.26}$$

Thus, exactly as in [2, Section 4.1], we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|D_t\mathbf{u}|^2\ \mathrm{d}x + \mu\int_\Omega|\nabla_x D_t\mathbf{u}|^2\ \mathrm{d}x + \left(\eta+\frac{\mu}{3}\right)\int_\Omega|\mathrm{div}_x D_t\mathbf{u})|^2\ \mathrm{d}x$$

$$\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right) \left(1 + \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x + \int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x + \int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x \right.$$

$$\left. + \int_\Omega \Big[\partial_t (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) + \mathbf{u} \cdot \nabla_x (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \Big] \cdot D_t \mathbf{u} \ \mathrm{d}x \right). \tag{2.27}$$

Next, using the relation

$$\mathbf{curl}_x \mathbf{B} \times \mathbf{B} = \mathrm{div}_x \Big(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} |\mathbf{B}|^2 \mathbb{I} \Big)$$

we may rewrite the integral

$$\int_\Omega \partial_t (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot D_t \mathbf{u} \ \mathrm{d}x = - \int_\Omega \partial_t \Big(\mathbf{B} \otimes \mathbf{B} - \frac{1}{2} |\mathbf{B}|^2 \mathbb{I} \Big) \cdot \nabla_x D_t \mathbf{u} \ \mathrm{d}x. \tag{2.28}$$

As $\partial_t \mathbf{B}$ is already controlled by (2.19), the integral (2.28) can be absorbed by the left–hand side of (2.27). Accordingly, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x + \mu \int_\Omega |\nabla_x D_t \mathbf{u}|^2 \ \mathrm{d}x + \Big(\eta + \frac{\mu}{3} \Big) \int_\Omega |\mathrm{div}_x D_t \mathbf{u})|^2 \ \mathrm{d}x$$

$$\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right) \left(1 + \int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x + \int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x + \int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x \right.$$

$$\left. + \left| \int_\Omega \mathbf{u} \cdot \nabla_x (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot D_t \mathbf{u} \ \mathrm{d}x \right| \right). \tag{2.29}$$

Finally, by parts integration yields

$$\int_\Omega \mathbf{u} \cdot \nabla_x (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot D_t \mathbf{u} \ \mathrm{d}x = - \int_\Omega \mathbf{u} \cdot (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot \nabla_x D_t \mathbf{u} \ \mathrm{d}x$$

$$- \int_\Omega \nabla_x \mathbf{u} \cdot (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot D_t \mathbf{u} \ \mathrm{d}x,$$

where, similarly to the above, the first integral on the right–hand side can be absorbed because of (2.20). Next, by Hölder's inequality,

$$\left| \int_\Omega \nabla_x \mathbf{u} \cdot (\mathbf{curl}_x \mathbf{B} \times \mathbf{B}) \cdot D_t \mathbf{u} \ \mathrm{d}x \right|$$

$$\leq \|\nabla_x \mathbf{u}\|_{L^4(\Omega;R^3)} \|\mathbf{curl}_x \mathbf{B} \times \mathbf{B}\|_{L^4(\Omega;R^3)} \|D_t \mathbf{u}\|_{L^2(\Omega;R^3)}$$

$$\leq \delta \|D_t \mathbf{u}\|^2_{L^2(\Omega;R^3)} + c(\delta) \left(\int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x + \|\mathbf{curl}_x \mathbf{B} \times \mathbf{B}\|^4_{L^4(\Omega;R^3)} \right),$$

where, by virtue of (2.19) and Gagliardo-Nirenberg inequality,

$$\sup_{t \in (0,T)} \|\mathbf{curl}_x \mathbf{B} \times \mathbf{B}\|^4_{L^4(\Omega;R^3)} \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.30}$$

Thus we rewrite (2.29) in the final form

$$
\begin{aligned}
&\int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x(\tau,\cdot) + \int_0^\tau \|D_t \mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)} \,\mathrm{d}t \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(1 + \int_0^\tau \left[\int_\Omega \varrho |D_t \vartheta|^2 \ \mathrm{d}x + \int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x + \int_\Omega \varrho |D_t \mathbf{u}|^2 \ \mathrm{d}x \right] \mathrm{d}t \right).
\end{aligned}
\tag{2.31}
$$

2.4.2 Velocity decomposition

Following Sun, Wang, and Zhang [17], we decompose the velocity field in the form:

$$
\mathbf{u} = \mathbf{v} + \mathbf{w}, \tag{2.32}
$$

$$
\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{v}) = \nabla_x p(\varrho,\vartheta) \text{ in } (0,T)\times\Omega, \ \mathbf{v}|_{\partial\Omega} = 0, \tag{2.33}
$$

$$
\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{w}) = \varrho D_t \mathbf{u} - \varrho \mathbf{g} - \mathbf{curl}_x \mathbf{B} \times \mathbf{B} \text{ in } (0,T)\times\Omega, \ \mathbf{w}|_{\partial\Omega} = 0. \tag{2.34}
$$

Since

$$
\mathrm{div}_x \mathbb{S}(\mathbb{D}_x \partial_t \mathbf{v}) = \nabla_x \partial_t p \text{ in } (0,T)\times\Omega, \ \mathbf{v}|_{\partial\Omega} = 0,
$$

we get

$$
\int_\Omega \partial_t p \ \mathrm{div}_x \mathbf{v} \ \mathrm{d}x = - \int_\Omega \nabla_x \partial_t p \cdot \mathbf{v} \ \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \mathbb{S}(\mathbb{D}_x \mathbf{v}) : \mathbb{D}_x \mathbf{v} \ \mathrm{d}x. \tag{2.35}
$$

Moreover, the standard elliptic estimates for the Lamé operator yield:

$$
\|\mathbf{v}\|_{W^{1,q}(\Omega;R^3)} \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \text{ for all } 1 \leq q < \infty, \tag{2.36}
$$

$$
\|\mathbf{v}\|_{W^{2,q}(\Omega;R^3)} \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(\|\nabla_x \varrho\|_{L^q(\Omega;R^3)} + \|\nabla_x \vartheta\|_{L^q(\Omega;R^3)} \right), \ 1 < q < \infty. \tag{2.37}
$$

Similarly,

$$
\begin{aligned}
&\|\mathbf{w}\|_{W^{2,2}(\Omega;R^3)} \\
\leq &\Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(1 + \|\sqrt{\varrho}\partial_t \mathbf{u}\|_{L^2(\Omega;R^3)} + \|\nabla_x \mathbf{u}\|_{L^2(\Omega;R^{3\times 3})} + \|\nabla_x \mathbf{B}\|^2_{L^2(\Omega;R^{3\times 3})} \right) \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(1 + \|\sqrt{\varrho}\partial_t \mathbf{u}\|_{L^2(\Omega;R^3)} + \|\nabla_x \mathbf{u}\|_{L^2(\Omega;R^{3\times 3})} \right),
\end{aligned}
\tag{2.38}
$$

where the last estimate follows from boundedness of the magnetic field stated in (2.20).

In view of (2.36)–(2.38), we deduce a Gagliardo–Nirenberg type estimates for the velocity field

$$
\begin{aligned}
&\int_\Omega |\nabla_x \mathbf{u}|^4 \ \mathrm{d}x = \|\nabla_x \mathbf{u}\|^4_{L^4(\Omega;R^{3\times 3})} \lesssim \|\nabla_x \mathbf{v}\|^4_{L^4(\Omega;R^{3\times 3})} + \|\nabla_x \mathbf{w}\|^4_{L^4(\Omega;R^{3\times 3})} \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(1 + \|\mathbf{w}\|^2_{L^\infty(\Omega;R^3)} \|\Delta_x \mathbf{w}\|^2_{L^2(\Omega;R^3)} \right) \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right) \left(1 + \|\sqrt{\varrho}\partial_t \mathbf{u}\|^2_{L^2(\Omega;R^3)} + \|\nabla_x \mathbf{u}\|^2_{L^2(\Omega;R^3)} \right),
\end{aligned}
\tag{2.39}
$$

where $\lesssim$ means there is a constant $c > 0$ such that $f \leq cg$. Note that $\mathbf{u}$ and $\mathbf{v}$ are bounded in L^∞ in terms of the data and so is $\mathbf{w}$.

2.4.3 Temperature estimates

Here again, we anticipate the estimates obtained in [2, Section 4.3] with the exception of the $\mathbf{B}-$dependent source term. Accordingly, multiplying the internal energy equation on $\partial_t\vartheta$ and integrating by parts, we report the following estimate:

$$\begin{aligned}\int_\Omega |\nabla_x\vartheta|^2(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau\int_\Omega \varrho|D_t\vartheta|^2\ \mathrm{d}x\,\mathrm{d}t - \int_\Omega \vartheta\mathbb{S}(\mathbb{D}_x\mathbf{u}):\mathbb{D}_x\mathbf{u}(\tau,\cdot)\ \mathrm{d}x - \zeta\int_\Omega \vartheta|\mathbf{curl}_x\mathbf{B}|^2(\tau,\cdot)\ \mathrm{d}x \\ \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\left(1 + \int_0^\tau\int_\Omega \left(|\nabla_x\vartheta|^2 + |\nabla_x\mathbf{u}|^2\right)\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \|\sqrt{\varrho}\partial_t\mathbf{u}\|^2_{L^2(\Omega;R^3)}\,\mathrm{d}t \right. \\ \left. - \zeta\int_0^\tau\int_\Omega \vartheta\partial_t|\mathbf{curl}_x\mathbf{B}|^2\ \mathrm{d}x\,\mathrm{d}t\right), \end{aligned} \tag{2.40}$$

where, by virtue of (2.20),

$$\int_\Omega \vartheta|\mathbf{curl}_x\mathbf{B}|^2(\tau,\cdot)\ \mathrm{d}x \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right).$$

The last integral in (2.40) can be rewritten in the form

$$\int_\Omega \partial_t|\mathbf{curl}_x\mathbf{B}|^2\vartheta\ \mathrm{d}x = 2\int_\Omega \vartheta\mathbf{curl}_x\mathbf{B}\cdot\mathbf{curl}_x\partial_t\mathbf{B}\ \mathrm{d}x = 2\int_\Omega \partial_t\mathbf{B}\cdot\mathbf{curl}_x\Big(\vartheta\mathbf{curl}_x\mathbf{B}\Big)\ \mathrm{d}x.$$

Indeed

$$\vartheta\mathbf{curl}_x\mathbf{B}\cdot\mathbf{curl}_x\partial_t\mathbf{B} = \mathbf{curl}_x\Big(\vartheta\mathbf{curl}_x\mathbf{B}\Big)\cdot\partial_t\mathbf{B} - \mathrm{div}_x\Big(\vartheta(\partial_t\mathbf{B}\times\mathbf{curl}_x\mathbf{B})\Big).$$

Moreover, since $\mathbf{b}_\tau$ is independent, it follows that

$$\begin{aligned}\int_\Omega \mathrm{div}_x\Big(\vartheta(\partial_t\mathbf{B}\times\mathbf{curl}_x\mathbf{B})\Big)\ \mathrm{d}x = \int_{\Gamma_D} \vartheta(\partial_t\mathbf{B}\times\mathbf{curl}_x\mathbf{B})\cdot\mathbf{n}\ \mathrm{d}\sigma \\ + \int_{\Gamma_N} \vartheta(\partial_t\mathbf{B}\times\mathbf{curl}_x\mathbf{B})\cdot\mathbf{n}\ \mathrm{d}\sigma = 0.\end{aligned}$$

Next, we write

$$\int_\Omega \partial_t\mathbf{B}\cdot\mathbf{curl}_x\Big(\vartheta\mathbf{curl}_x\mathbf{B}\Big)\ \mathrm{d}x = \int_\Omega \vartheta\partial_t\mathbf{B}\cdot\mathbf{curl}_x\mathbf{curl}_x\mathbf{B}\ \mathrm{d}x + \int_\Omega \partial_t\mathbf{B}\cdot(\mathbf{curl}_x\mathbf{B}\times\nabla_x\vartheta)\ \mathrm{d}x, \tag{2.41}$$

where

$$\left|\int_\Omega \partial_t\mathbf{B}\cdot\mathbf{curl}_x\Big(\vartheta\mathbf{curl}_x\mathbf{B}\Big)\ \mathrm{d}x\right| \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho,\vartheta,\mathbf{u},\mathbf{B}\|_{L^\infty}\right)\|\mathbf{B}\|_{W^{2,2}(\Omega;R^3)}\|\partial_t\mathbf{B}\|_{L^2(\Omega;R^3)}. \tag{2.42}$$

Finally, the last integral in (2.41) can be handled as follows:

$$\left|\int_\Omega \partial_t\mathbf{B}\cdot(\mathbf{curl}_x\mathbf{B}\times\nabla_x\vartheta)\ \mathrm{d}x\right| \leq \|\nabla_x\vartheta\|_{L^2(\Omega)}\|\mathbf{B}\|_{W^{1,4}(\Omega;R^3)}\|\partial_t\mathbf{B}\|_{L^4(\Omega;R^3)}$$

$$\leq \|\nabla_x \vartheta\|^2_{L^2(\Omega;R^3)} + \|\mathbf{B}\|^2_{W^{1,4}(\Omega;R^3)} \|\partial_t \mathbf{B}\|^2_{L^4(\Omega;R^3)}.$$

Moreover, the maximal regularity estimates (2.13) for the induction equation yield

$$\sup_{t\in(0,\tau)} \|\mathbf{B}\|_{W^{1,4}(\Omega;R^3)} + \|\partial_t \mathbf{B}\|_{L^2(0,T;L^4(\Omega;R^3))} \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right)\left(1 + \|\nabla_x \mathbf{u}\|_{L^4(\Omega;R^{3\times 3})}\right). \tag{2.43}$$

Thus we may infer that

$$\begin{aligned}
&\left|\int_0^\tau \int_\Omega \partial_t \mathbf{B} \cdot (\mathbf{curl}_x \mathbf{B} \times \nabla_x \vartheta)\ \mathrm{d}x\,\mathrm{d}t\right| \leq \int_0^\tau \int_\Omega |\nabla_x \vartheta|^2\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \|\mathbf{B}\|^2_{W^{1,4}(\Omega;R^3)} \|\partial_t \mathbf{B}\|^2_{L^4(\Omega;R^3)}\,\mathrm{d}t \\
&\leq \int_0^\tau \int_\Omega |\nabla_x \vartheta|^2\ \mathrm{d}x\,\mathrm{d}t + \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right)\left[1 + \left(\int_0^\tau \|\nabla_x \mathbf{u}\|^2_{L^4(\Omega;R^{3\times 3})}\mathrm{d}t\right)^2\right] \\
&\leq \int_0^\tau \int_\Omega |\nabla_x \vartheta|^2\ \mathrm{d}x\,\mathrm{d}t + \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right)\left(1 + \int_0^\tau \int_\Omega |\nabla_x \mathbf{u}|^4\ \mathrm{d}x\,\mathrm{d}t\right). \qquad (2.44)
\end{aligned}$$

Consequently, in view of the previous estimates, we may rewrite (2.40) in the form

$$\begin{aligned}
&\int_\Omega |\nabla_x \vartheta|^2(\tau,\cdot)\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho |D_t \vartheta|^2\ \mathrm{d}x\,\mathrm{d}t - \int_\Omega \vartheta \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u}(\tau,\cdot)\ \mathrm{d}x \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right)\left(1 + \int_0^\tau \int_\Omega \left(|\nabla_x \vartheta|^2 + |\nabla_x \mathbf{u}|^2\right)\ \mathrm{d}x\,\mathrm{d}t + \int_0^\tau \|\sqrt{\varrho}\partial_t \mathbf{u}\|^2_{L^2(\Omega;R^3)}\,\mathrm{d}t\right. \\
&\qquad\qquad \left. + \int_0^\tau \int_\Omega |\nabla_x \mathbf{u}|^4\ \mathrm{d}x\,\mathrm{d}t\right). \qquad (2.45)
\end{aligned}$$

2.4.4 Estimates of the velocity gradient - conclusion

Controlling $\int_\Omega |\nabla_x \mathbf{u}|^4\ \mathrm{d}x$ by means of (2.39) we can put together (2.25), (2.31), and (2.45) to conclude

$$\begin{aligned}
&\int_\Omega \left[\varrho |D_t \mathbf{u}|^2 + |\nabla_x \mathbf{u}|^2 + \delta \varrho |D_t \vartheta|^2 - \delta \vartheta \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u}\right](\tau,\cdot)\ \mathrm{d}x \\
&+ \int_0^\tau \|D_t \mathbf{u}\|^2_{W^{1,2}(\Omega;R^3)}\,\mathrm{d}t + \int_0^\tau \int_\Omega \varrho |D_t \mathbf{u}|^2\ \mathrm{d}x\,\mathrm{d}t \\
&\leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right)\left(1 + \int_0^\tau \left[\int_\Omega \varrho |D_t \vartheta|^2\ \mathrm{d}x + \int_0^\tau \int_\Omega \varrho |D_t \mathbf{u}|^2\ \mathrm{d}x\right]\,\mathrm{d}t\right) \qquad (2.46)
\end{aligned}$$

for any $\delta > 0$. As ϑ is bounded, we may choose $\delta > 0$ small enough and use the standard Gronwall argument to close the estimates.

Summarizing we record the following bounds:

$$\sup_{t\in[0,T]} \|\mathbf{u}(t,\cdot)\|_{W^{1,2}(\Omega;R^3)} \leq \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right), \tag{2.47}$$

$$\sup_{t\in[0,T]} \|\sqrt{\varrho} D_t \mathbf{u}(t,\cdot)\|_{L^2(\Omega;R^3)} \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right), \tag{2.48}$$

$$\sup_{t\in[0,T]} \|\vartheta(t,\cdot)\|_{W^{1,2}(\Omega)} \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right), \tag{2.49}$$

$$\int_0^T \int_\Omega |\nabla_x D_t \mathbf{u}|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right), \tag{2.50}$$

$$\int_0^T \int_\Omega \varrho |D_t \vartheta|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.51}$$

Moreover, it follows from (2.36), (2.39), (2.48)

$$\sup_{t\in[0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^4(\Omega;R^{3\times 3})} \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.52}$$

In addition, in view of (2.20), (2.51), (2.52), the internal energy balance (1.3) yields

$$\int_0^T \|\vartheta\|^2_{W^{2,2}(\Omega)} \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.53}$$

2.5 Higher order estimates

2.5.1 Velocity

It follows from (2.38), (2.47), and (2.48) that

$$\sup_{t\in[0,T)} \|\mathbf{w}(t,\cdot)\|_{W^{2,2}(\Omega;R^3)} \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.54}$$

This bound, together with (2.36) and Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$, gives rise to

$$\sup_{t\in[0,T)} \|\nabla_x \mathbf{u}(t,\cdot)\|_{L^6(\Omega;R^{3\times 3})} \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.55}$$

Moreover, by virtue of (2.50), the material derivative $D_t \mathbf{u}$ is bounded in $L^2(L^6)$, which, combined with (2.55), yields

$$\int_0^T \|\partial_t \mathbf{u}\|^2_{L^6(\Omega;R^3)} \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.56}$$

Going back to (2.34) and using (2.20) with (2.55), (2.56), we get

$$\int_0^T \|\mathbf{w}\|^2_{W^{2,6}(\Omega;R^3)} \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right), \tag{2.57}$$

and, in accordance with (2.36),

$$\int_0^T \|\mathbf{u}\|^2_{W^{1,q}(\Omega;R^3)} \, \mathrm{d}t \leq \Lambda \left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right) \text{ for any } 1 \leq q < \infty. \tag{2.58}$$

Remark 2.3. Strictly speaking, the bound (2.58) depends also on q.

2.5.2 Density

Using (2.57), (2.58), we may proceed exactly as in [18, Section 5] to deduce the bounds on the density

$$\sup_{t\in[0,T)} \left(\|\partial_t \varrho(t,\cdot)\|_{L^6(\Omega)} + \|\varrho(t,\cdot)\|_{W^{1,6}(\Omega)} \right) \leq \Lambda\left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.59}$$

2.5.3 Momentum equation revisited

The momentum equation reads

$$\text{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) = \varrho D_t \mathbf{u} + \nabla_x p - \varrho \mathbf{g} - \mathbf{curl}_x \mathbf{B} \times \mathbf{B},$$

where, by virtue of (2.57), (2.58), (2.59), and (2.19), the right–hand side is bounded in $L^2(L^6)$. Thus the standard elliptic estimates for the Lamé operator yield

$$\int_0^T \|\mathbf{u}\|^2_{W^{2,6}(\Omega;R^3)} \, \text{d}t \leq \Lambda\left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.60}$$

In particular, as $W^{1,6} \hookrightarrow L^\infty$,

$$\int_0^T \|\nabla_x \mathbf{u}\|^2_{L^\infty(\Omega;R^{3\times 3})} \, \text{d}t \leq \Lambda\left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.61}$$

2.5.4 Lower bounds on the density and the temperature

In view of (2.61), the standard method of characteristics an be applied to the equation of continuity (1.1) to deduce a positive lower bound on the density,

$$\inf_{(t,x)\in[0,T)\times\Omega} \varrho(t,x) \geq \underline{\varrho}(T) > 0, \tag{2.62}$$

in other words,

$$\left\|\varrho^{-1}\right\|_{L^\infty((0,T)\times\Omega)} \leq \Lambda\left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.63}$$

Now, we may write the internal energy equation (1.3) in the form

$$c_v \left(\partial_t \vartheta + \mathbf{u} \cdot \nabla_x \vartheta\right) - \frac{\kappa}{\varrho} \Delta_x \vartheta = \frac{1}{\varrho} \left(\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} + \zeta |\mathbf{curl}_x \mathbf{B}|^2\right) - \vartheta \text{div}_x \mathbf{u}. \tag{2.64}$$

Applying the standard parabolic maximum/minimum principle we conclude

$$\left\|\vartheta^{-1}\right\|_{L^\infty((0,T)\times\Omega)} \leq \Lambda\left(T, \|\text{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.65}$$

2.5.5 Induction equation revisited

Using the bound (2.55) on the velocity gradient, we may apply the maximal regularity estimates (2.13) to obtain

$$\|\partial_t \mathbf{B}\|_{L^2(0,T;L^6(\Omega;R^3))} + \|\mathbf{B}\|_{L^2(0,T;W^{2,6}(\Omega;R^3))} \stackrel{<}{\sim} \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right) \tag{2.66}$$

for any $1 \le p < \infty$.

2.5.6 Parabolic estimates for the internal energy balance

Introducing a new variable $\Theta = \vartheta - \vartheta_B$ we may rewrite the internal energy equation as

$$c_v \partial_t \Theta - \frac{\kappa}{\varrho}\Delta_x \Theta = \frac{1}{\varrho}\left(\mathbb{S} : \mathbb{D}_x \mathbf{u} + \zeta |\mathbf{curl}_x \mathbf{B}|^2\right) - \vartheta \mathrm{div}_x \mathbf{u} - c_v \mathbf{u} \cdot \nabla_x \vartheta \tag{2.67}$$

with the *homogeneous* Dirichlet boundary conditions

$$\Theta|_{\partial\Omega} = 0. \tag{2.68}$$

In view of the bounds (2.55), (2.66), the right–hand side of the above equation is bounded in $L^p(L^3)$, $1 \le p < \infty$. Thus the maximal regularity parabolic estimates yield, in particular,

$$\|\partial_t \vartheta\|_{L^2(0,T;L^3(\Omega))} + \|\vartheta\|_{L^2(0,T;W^{2,3}(\Omega))} \stackrel{<}{\sim} \Lambda\left(T, \|\mathrm{data}\|, \|\varrho, \vartheta, \mathbf{u}, \mathbf{B}\|_{L^\infty}\right). \tag{2.69}$$

2.6 Final estimates

The final estimates are obtained by differentiating the internal and induction equation in time. Introducing new variables $\mathcal{T} = \partial_t \vartheta$ and $\mathcal{B} = \partial_t \mathbf{B}$, we obtain

$$\begin{aligned} c_v \partial_t \mathcal{T} + c_v \mathbf{u} \cdot \nabla_x \mathcal{T} - \frac{\kappa}{\varrho}\Delta_x \mathcal{T} &= -c_v \partial_t \mathbf{u} \cdot \nabla_x \vartheta - \frac{\partial_t \varrho}{\varrho^2}\Big(\kappa \Delta_x \vartheta + \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} + \zeta |\mathbf{curl}_x \mathbf{B}|^2\Big) \\ &+ \frac{2}{\varrho}\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \partial_t \mathbf{u} + \frac{1}{\varrho}\partial_t (\zeta |\mathbf{curl}_x \mathbf{B}|^2) - \partial_t \vartheta \mathrm{div}_x \mathbf{u} - \vartheta \mathrm{div}_x \partial_t \mathbf{u}, \\ \mathcal{T}|_{\partial\Omega} &= 0; \end{aligned} \tag{2.70}$$

and

$$\begin{aligned} \partial_t \mathcal{B} + \mathbf{curl}_x (\mathcal{B} \times \mathbf{u}) + \zeta \mathbf{curl}_x \mathbf{curl}_x \mathcal{B} &= \mathbf{curl}_x (\partial_t \mathbf{u} \otimes \mathbf{B}), \\ \mathcal{B} \times \mathbf{n}|_{\Gamma_D} = 0,\ \mathcal{B} \cdot \mathbf{n}|_{\Gamma_N} &= 0,\ \mathbf{curl}_x \mathcal{B} \times \mathbf{n}|_{\Gamma_N} = 0. \end{aligned} \tag{2.71}$$

Using the bounds previously obtained, we get

$$\partial_t \vartheta \in L^2(0,T;W^{1,2}(\Omega)),\ \partial_t \mathbf{B} \in L^2(0,T;W^{1,2}(\Omega;R^3)) \tag{2.72}$$

and

$$\vartheta \in L^2(0,T;W^{3,2}(\Omega)),\ \mathbf{B} \in L^2(0,T;W^{3,2}(\Omega;R^3)). \tag{2.73}$$

Finally, going back to the momentum equation and using the same argument, we deduce

$$\partial_t \mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathbf{u} \in L^2(0,T;W^{3,2}(\Omega;R^3)). \tag{2.74}$$

the required bounds for the density follow directly from the equation of continuity (1.1).

We have proved Theorem 2.1.

3 Blow up criterion

The main obstacle in converting the estimates presented in Theorem 2.1 into a blow up criterion is the absence of a local existence result in the function spaces class used in Theorem 2.1. Indeed the local existence proved by Tang and Gao [19], modulo the correct choice of the boundary conditions (1.11), is established in the L^p- setting. In particular, the magnetic field $\mathbf{B}$ belongs to the class

$$\partial_t\vartheta, \partial_t\mathbf{u}, \partial_t\mathbf{B} \in L^p(L^p),\ \vartheta, \mathbf{u}, \mathbf{B} \in L^p(W^{2,p}),\ \partial_t\varrho \in L^p(L^p),\ \varrho \in C(W^{1,p}),\ p > 3. \tag{3.1}$$

The regularity of the initial data in Theorem 2.1 is $W^{2,2}$. Seeing that

$$W^{2,2} \hookrightarrow W^{2-\frac{2}{p},p} \text{ for } 1 \leq p \leq \frac{10}{3}$$

we conclude that the initial data considered in Theorem 2.1 give rise to a (unique) local strong solution in the class (3.1) for any $3 < p \leq \frac{10}{3}$.

Consider the induction equation (1.3) written for the modified magnetic field $\mathbf{b} = \mathbf{B} - \mathbf{B}_B$:

$$\partial_t\mathbf{b} - \zeta\mathbf{curl}_x\mathbf{curl}_x\mathbf{b} = \mathbf{curl}_x(\mathbf{B}\times\mathbf{u}),\ \mathbf{b}\times\mathbf{n}|_{\Gamma_D} = 0,\ \mathbf{b}\cdot\mathbf{n}|_{\Gamma_N} = 0,\ \mathbf{curl}_x\mathbf{b}\times\mathbf{n}|_{\Gamma_N} = 0. \tag{3.2}$$

Applying the operator $\mathbf{curl}_x$ to (3.2), we get

$$\partial_t\mathbf{h} - \zeta\mathbf{curl}_x\mathbf{curl}_x\mathbf{h} = \mathbf{curl}_x\mathbf{curl}_x(\mathbf{B}\times\mathbf{u}),\ \mathbf{h}\cdot\mathbf{n}|_{\Gamma_N},\ \mathbf{h}\times\mathbf{n}|_{\Gamma_D} = 0, \tag{3.3}$$

where we have denote $\mathbf{h} = \mathbf{curl}_x\mathbf{b}$.

Next, in view of the bounds (3.1), we claim that

$$\mathbf{curl}_x\mathbf{curl}_x(\mathbf{B}\times\mathbf{u}) \in L^p(0,T;L^p(\Omega;R^3)).$$

Applying the maximal regularity estimates 2.11 we conclude

$$\begin{aligned}&\mathbf{curl}_x\mathbf{b}_t \in L^2(0,T;L^p(\Omega;R^3)),\ \mathbf{curl}_x\mathbf{b} \in L^2(0,T;W^{2,p}(\Omega;R^3)),\\ &\text{and } \mathbf{curl}_x\mathbf{b} \in C([0,T];W^{1,p}(\Omega;R^3)).\end{aligned} \tag{3.4}$$

It follows from (3.4) that the Lorentz force in the momentum equation (1.2) as well as the heat source in the internal energy equation (1.3) belong to the regularity class:

$$
\begin{aligned}
\mathbf{curl}_x\mathbf{B} \times \mathbf{B} \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \partial_t(\mathbf{curl}_x\mathbf{B} \times \mathbf{B}) \in L^2(0,T;L^2(\Omega;R^3)),\\
|\mathbf{curl}_x\mathbf{B}|^2 \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \partial_t(|\mathbf{curl}_x\mathbf{B}|^2) \in L^2(0,T;L^2(\Omega;R^3)).
\end{aligned}
\tag{3.5}
$$

In view of the estimates (3.5), we may interpret solutions of the compressible MHD system as solutions of the Navier–Stokes–Fourier system (1.1), (1.2), (1.4) driven by a source term on the right–hand side. By virtue of [20, Theorem 2.5], the solution $(\varrho, \vartheta, \mathbf{u})$ belongs to the class (2.1). Thus we have shown that the local solution guaranteed by [11], [19] emanating from the more regular data (2.2)–(2.5) belongs to the regularity class (2.1). This fact, combined with Theorem 2.1 yields the following blow–up criterion.

Theorem 3.1 (Blow–up criterion). *Let $\Omega \subset R^3$ be a domain of the class at least C^3. Let the initial and boundary data belong to the regularity class* (2.2)–(2.5) *and satisfy the compatibility conditions:*

$$
\begin{aligned}
\vartheta_0|_{\partial\Omega} = \vartheta_B,\ \mathbf{u}_0|_{\partial\Omega} = 0,\ \mathbf{B}_0 \times \mathbf{n}|_{\Gamma_D} = \mathbf{b}_\tau,\ \mathbf{B}_0 \cdot \mathbf{n}|_{\Gamma_N} = b_\nu,\ \mathbf{curl}_x\mathbf{B}_0 \times \mathbf{n}|_{\Gamma_N} = 0,\\
\mathrm{div}_x(\varrho_0\mathbf{u}_0 \otimes \mathbf{u}_0) + \nabla_x p(\varrho_0, \vartheta_0) = \mathrm{div}_x(\mathbb{S}(\mathbb{D}_x\mathbf{u}_0)) + \mathbf{curl}_x\mathbf{B}_0 \times \mathbf{B}_0 + \varrho_0\mathbf{g}|_{\partial\Omega} = 0,\\
\Big[\mathbf{curl}_x(\mathbf{B}_0 \times \mathbf{u}_0) + \zeta\mathbf{curl}_x\mathbf{curl}_x\mathbf{B}_0\Big] \times \mathbf{n}|_{\Gamma_D} = 0,\\
\Big[\mathbf{curl}_x(\mathbf{B}_0 \times \mathbf{u}_0) + \zeta\mathbf{curl}_x\mathbf{curl}_x\mathbf{B}_0\Big] \cdot \mathbf{n}|_{\Gamma_N} = 0,\\
\varrho_0 e(\varrho_0, \vartheta_0)\mathbf{u}_0 + \mathrm{div}_x\mathbf{q}(\nabla_x\vartheta_0) - \mathbb{S}(\mathbb{D}_x\mathbf{u}_0) : \mathbb{D}_x\mathbf{u}_0 - \zeta|\mathbf{curl}_x\mathbf{B}_0|^2 - p(\varrho_0, \vartheta_0)\mathrm{div}_x\mathbf{u}_0|_{\partial\Omega} = 0.
\end{aligned}
\tag{3.6}
$$

Then the compressible MHD system (1.1)–(1.11) *admits a strong solution in the class* (2.1) *defined on a maximal time interval* $[0, T_{\max})$*. If* $T_{\max} < \infty$*, then*

$$
\limsup_{t \to T_{\max}-} \|(\varrho, \mathbf{u}, \vartheta, \mathbf{B})(t, \cdot)\|_{L^\infty(\Omega;R^8)} = \infty.
$$

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