On the motion of several small rigid bodies in a viscous incompressible fluid

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System of rigid bodies

Rigid bodies

$$\left(\mathcal{S}_\varepsilon^1, \dots, \mathcal{S}_\varepsilon^N\right),\ N \to \infty,\ \varepsilon \to 0$$

$$\mathcal{S}_\varepsilon^i \text{ compact subset of } R^d,\ d = 2, 3$$

Fluid equations

$$\mathrm{div}_x \mathbf{u} = 0$$

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\mathbb{D}_x \mathbf{u}) + \mathbf{g}$$

$$\mathbb{S}(\mathbb{D}_x \mathbf{u}) = \mu \mathbb{D}_x \mathbf{u},\ \mathbb{D}_x \mathbf{u} = \frac{\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}}{2},\ \mu > 0$$

Compatibility

- **No slip.** The fluid velocity coincides with the rigid bodies velocity on $\partial \mathcal{S}_\varepsilon^i$
- **Momentum continuity.** Momenta are continuous on $\partial \mathcal{S}_\varepsilon^i$

Main result

Hypotheses.

- $\left(\mathcal{S}_\varepsilon^1, \dots, \mathcal{S}_\varepsilon^N\right)$ compact subsets of R^d, $d = 2, 3$

$$N = N(\varepsilon) \approx -\alpha \log(\varepsilon),\ \ \alpha \in (0, \frac{5}{7})$$

- $$D_\varepsilon = \max_{i=1,\dots,N(\varepsilon)} \mathrm{diam}[\mathcal{S}_\varepsilon^i] \to 0 \text{ as } \varepsilon \to 0$$

$$0 < \lambda D_\varepsilon^\beta \leq |\mathcal{S}_\varepsilon^i|,\ \ d \leq \beta < \begin{cases} 15 - 21\alpha \text{ if } d = 3 \\ \text{arbitrary finite if } d = 2 \end{cases}$$

- $$0 < \varrho_{\mathcal{S}_\varepsilon^i} \leq \overline{\varrho} \text{ uniformly for } \varepsilon \to 0,\ \ i = 1, \dots, N(\varepsilon)$$

Conclusion.

$\mathbf{u}_\varepsilon \to \mathbf{u}$ weakly in $L^2(0, T; W^{1,2}(\Omega; R^d))$ and strongly in $L^2(0, T; L^2(\Omega; R^d))$

where $\mathbf{u}$ solves the Navier–Stokes system without obstacles

Previous results

M. Bravin and Š. Nečasová.
On the velocity of a small rigid body in a viscous incompressible fluid in dimension two and three
2022, arXiv preprint arXiv:2208.12351.

- **heavy single body**

J. He and D. Iftimie.
On the small rigid body limit in 3D incompressible flows.
J. Lond. Math. Soc. (2), **104**(2):668–687, 2021.

D. Iftimie, M. C. Lopes Filho, and H. J. Nussenzveig Lopes.
Two-dimensional incompressible viscous flow around a small obstacle.
Math. Ann., **336**(2):449–489, 2006.

- **single body, density** $\to \infty$

C. Lacave and T. Takahashi.
Small moving rigid body into a viscous incompressible fluid.
Arch. Ration. Mech. Anal., **223**(3):1307–1335, 2017.

- **single body with constant (bounded) density**

Principal difficulties

- **Contacts.** As the number of bodies tends to infinity $\Rightarrow$ Contacts must be allowed
- **Low density.** The result depends only on the shape of bodies, density is allowed to vanish $\Rightarrow$ Better energy estimates needed
- **Extension operator.** A new construction of extension operator for the multibody system is necessary

Weak solutions

Weak solution concept introduced by Judakov [1964]

Mass conservation.

$$\int_0^T \int_{R^d} \left[\rho \partial_t \varphi + \rho \mathbf{u} \cdot \nabla_x \varphi \right] \mathrm{d}t = - \int_{R^d} \rho_0 \varphi(0, \cdot)$$

for any $\varphi \in C^1_c([0,T) \times R^d)$

Momentum balance.

$$\int_0^T \int_{R^d} \left[\rho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \rho \mathbf{u} \otimes \mathbf{u} : \mathbb{D}_x \boldsymbol{\varphi} \right] \mathrm{d}t$$

$$= \int_0^T \int_{R^d} \left[\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \boldsymbol{\varphi} - \rho \mathbf{g} \cdot \boldsymbol{\varphi} \right] \mathrm{d}t - \int_{R^d} \rho_0 \mathbf{u}_0 \cdot \boldsymbol{\varphi}(0, \cdot)$$

for any function $\boldsymbol{\varphi} \in C^1_c([0,T) \times \Omega; R^d)$, $\mathrm{div}_x \boldsymbol{\varphi} = 0$,

$$\mathbb{D}_x \boldsymbol{\varphi}(t, \cdot) = 0 \text{ on an open neighbourhood of } \mathcal{S}^i_\varepsilon(t)$$

Energy dissipation.

$$\int_{R^d} \rho |\mathbf{u}|^2(\tau, \cdot) + \int_0^\tau \int_{R^d} \mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} \leq \int_{R^d} \rho_0 |\mathbf{u}_0|^2 + \int_0^\tau \int_{R^d} \rho \mathbf{g} \cdot \mathbf{u} \, \mathrm{d}t$$

Restriction operator

$$(\mathbf{h}_1, \dots, \mathbf{h}_N), \ \mathbf{h}_i \in R^d$$

- $$\varphi \in C^\infty(R^d; R^d) \ \Rightarrow \ \mathbf{R}_\varepsilon(\mathbf{h_1}, \dots, \mathbf{h_N})[\varphi] \in \mathbf{C}^\infty(\mathbf{R^d}; \mathbf{R^d})$$

- $$\mathrm{div}_x \mathbf{R}_\varepsilon(\mathbf{h}_1, \cdots, \mathbf{h}_N)[\varphi] = \mathrm{div}_x \varphi (= 0)$$

- $$\mathbf{R}_\varepsilon(\mathbf{h}_1, \cdots, \mathbf{h}_N)[\varphi] = \boxed{\Lambda_i - \text{a constant vector on}} \ B_\varepsilon(\mathbf{h}_i), \ i = 1, \cdots, N$$

- $$\mathbf{R}_\varepsilon(\mathbf{h}_1, \cdots, \mathbf{h}_N)[\varphi](x) = \varphi(x) \text{ whenever } x \in R^d \setminus \cup_{i=1}^N B_{2 \cdot 5^{i-1}\varepsilon}(\mathbf{h}_i)$$

- $$\|\mathbf{R}_\varepsilon(\mathbf{h}_1, \cdots, \mathbf{h}_N)[\varphi]\|_{L^p(R^d;R^d)} \leq c(p, N)\|\varphi\|_{L^p(R^d;R^d)}$$
  $$\|\nabla_x \mathbf{R}_\varepsilon(\mathbf{h}_1, \cdots, \mathbf{h}_N)[\varphi]\|_{L^p(R^d;R^{d\times d})} \leq c(p, N)\|\nabla_x \varphi\|_{L^p(R^d;R^{d\times d})}$$

Restriction operator, construction I

Step 1 - a single body.

$$H \in C^\infty(\mathbb{R}),\ 0 \le H(Z) \le 1,\ H'(Z) = H'(1-Z) \text{ for all } Z \in \mathbb{R}$$

$$H(Z) = 0 \text{ for } -\infty < Z \le \frac{1}{4},\ H(Z) = 1 \text{ for } \frac{3}{4} \le Z < \infty$$

$$E_r[\varphi](x) = \frac{1}{|B_r|} \int_{B_r} \varphi \ dz \ H\left(2 - \frac{|x|}{r}\right) + \varphi(x) H\left(\frac{|x|}{r} - 1\right)$$

for $\varphi \in L^1_{\rm loc}(R^d)$, $r > 0$

Step 2 - solenoidality.

$$\mathbf{R}_r[\varphi] = E_r[\varphi] - \mathcal{B}_{2r,r}\left[{\rm div}_x E_r[\varphi]|_{B_{2r} \setminus B_r} \right],$$

where $\mathcal{B}_{2r,r}$ is the so–called Bogovskii operator $\approx {\rm div}_x^{-1}$ on

$$B_{2r} \setminus B_r$$

Restriction operator, construction II

Step 3 - space shift

$$\mathbf{R}_r(\mathbf{h}) = S_{-h} \circ \mathbf{R}_r \circ S_h$$

$$S_h[\varphi](x) = \varphi(\mathbf{h} + x)$$

Step 4 - the 5-principle.

$$\mathbf{R}_\varepsilon(\mathbf{h}_1, \dots, \mathbf{h}_N)[\varphi] = \mathbf{R}_\varepsilon(\mathbf{h}_1) \circ \boxed{\mathbf{R}_{5\varepsilon}(\mathbf{h}_2)} \circ \cdots \circ \mathbf{R}_{5^{n-1}\varepsilon}(\mathbf{h}_N)[\varphi]$$