The Rayleigh–Bénard problem for compressible fluid flows

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Rayleigh–Bénard problem

Navier–Stokes–Fourier system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$
$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \nabla_x G$$
$$\partial_t(\varrho e(\varrho, \vartheta)) + \mathrm{div}_x(\varrho e(\varrho, \vartheta)\mathbf{u}) + \nabla_x \mathbf{q}(\nabla_x \vartheta) = \mathbb{S} : \mathbb{D}_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}$$

Boundary conditions

$$\Omega = \mathbb{T}^2 \times (0,1)$$

$$\mathbf{u}|_{x_3=0} = \mathbf{u}|_{x_3=1} = 0,$$
$$\vartheta|_{x_3=0} = \Theta_B, \ \vartheta|_{x_3=1} = \Theta_U.$$

$$\mathbb{S}(\vartheta, \mathbb{D}_x \mathbf{u}) = \mu(\vartheta) \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta(\vartheta) \mathrm{div}_x \mathbf{u} \mathbb{I}$$
$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta) \nabla_x \vartheta$$

Long–time behavior

- **Levinson dissipativity or bounded absorbing set.** Any global–in–time weak solution to the Navier–Stokes–Fourier system in a domain with impermeable boundary endowed with the Dirichlet boundary conditions for the temperature enters eventually a bounded absorbing set.
- **Asymptotic compactness.** Any bounded family of global solutions is precompact in a suitable topology of the trajectory space, whereas any of its accumulation points represents a weak solution of the same problem.

Weak solutions, I

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0,$$

$$\partial_t b(\varrho) + \mathrm{div}_x(b(\varrho)\mathbf{u}) + \Big(b'(\varrho)\varrho - b(\varrho) \Big) \mathrm{div}_x \mathbf{u} = 0$$

for any $b \in C^1(R)$, $b' \in C_c(R)$

Momentum equation

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \nabla_x G$$

Entropy inequality

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta} \right) \geq \frac{1}{\vartheta} \left[\mathbb{S}(\mathbb{D}_x \mathbf{u}) : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right]$$

Gibbs' law

$$\vartheta D s = D e + p D \left(\frac{1}{\varrho} \right)$$

Weak solutions, II

Total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \mathbf{u} \right] + \mathrm{div}_x (p\mathbf{u}) + \mathrm{div}_x \mathbf{q}$$
$$= \mathrm{div}_x (\mathbb{S} \cdot \mathbf{u}) + \varrho \mathbf{G} \cdot \mathbf{u}$$

Ballistic energy

$$E_{\widetilde{\vartheta}} = \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \widetilde{\vartheta} \varrho s \right]$$

$$\widetilde{\vartheta} > 0, \ \widetilde{\vartheta}|_{x_3=0} = \Theta_B, \ \widetilde{\vartheta}|_{x_3=1} = \Theta_U.$$

Ballistic energy boundary flux

$$\mathbf{q} \cdot \mathbf{n} - \frac{\widetilde{\vartheta}}{\vartheta} \mathbf{q} \cdot \mathbf{n}|_{\partial \Omega} = 0$$

Weak solutions, III

Ballistic energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e - \widetilde{\vartheta} \varrho s \right] \ \mathrm{d}x$$
$$+ \int_\Omega \frac{\widetilde{\vartheta}}{\vartheta} \left[\mathbb{S} : \mathbb{D}_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right] \ \mathrm{d}x$$
$$\leq \int_\Omega \left[\varrho \mathbf{u} \cdot \nabla_x G - \varrho s \mathbf{u} \cdot \nabla_x \widetilde{\vartheta} - \frac{\mathbf{q}}{\vartheta} \cdot \nabla_x \widetilde{\vartheta} \right] \ \mathrm{d}x$$

for any $\widetilde{\vartheta} \in C^1([0, T] \times \overline{\Omega})$, $\widetilde{\vartheta} > 0$ satisfying the boundary conditions

- **Compatibility [Chaudhuri–EF 2021].** Smooth weak solutions are classical solutions
- **Weak–strong uniqueness [Chaudhuri–EF 2021].** A weak solution coincides with the strong solution as long as the latter exists

Bounded absorbing set

Bounded absorbing set [EF - A. Świerczewska-Gwiazda]
For any global–in–time weak solution $(\varrho, \vartheta, \mathbf{u})$ defined on a time interval (T, ∞), there exists a constant $\mathcal{E}_\infty$ that depends only on the boundary data and the total mass of the fluid

$$M = \int_\Omega \varrho \ \mathrm{d}x,$$

such that

$$\mathrm{ess} \limsup_{t \to \infty} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty, \ \ E(\varrho, \vartheta, \mathbf{u}) \equiv \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta)$$

If, moreover,

$$\mathrm{ess} \limsup_{t \to T+} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0 < \infty,$$

then the convergence is uniform in $\mathcal{E}_0$. Specifically, for any $\varepsilon > 0$, there exists a time $T(\varepsilon, \mathcal{E}_0)$ such that

$$\mathrm{ess} \sup_{t > T(\varepsilon, \mathcal{E}_0)} \int_\Omega E(\varrho, \vartheta, \mathbf{u})(t, \cdot) \ \mathrm{d}x \leq \mathcal{E}_\infty + \varepsilon.$$

Asymptotic compactness, attractor

Attractor.

$$\mathcal{A} = \left\{ (\varrho, S, \mathbf{m}) \ \Big| \ (\varrho, S, \mathbf{m}) \text{ a weak solution of the Navier–Stokes–Fourier system} \right.$$
$$\left. \text{for any } t \in R, \ \sup_{t \in R} \int_{\Omega} E(\varrho, S, \mathbf{u})(t, \cdot) \ \mathrm{d}x < \mathcal{E}_{\infty} \right\},$$

Trajectory attractor [EF - A. Świerczewska-Gwiazda 2021]
Let $M > 0$, $\mathcal{E}_0$ be given. Let $\mathcal{F}[M, \mathcal{E}_0]$ be a family of weak solutions to the Rayleigh–Bénard problem for the Navier–Stokes–Fourier system on the time interval $(0, \infty)$ satisfying

$$\int_{\Omega} \varrho \ \mathrm{d}x = M, \ \mathrm{ess} \limsup_{\tau \to 0+} \int_{\Omega} E(\varrho, S, \mathbf{m})(\tau, \cdot) \ \mathrm{d}x \leq \mathcal{E}_0.$$

We identify the set $\mathcal{F}[M, \mathcal{E}_0]$ with a subset of the entire trajectories space extending them by constant values for $\tau < 0$.

Then for any $\varepsilon > 0$, there exists a time $T(\varepsilon)$ such that

$d_{\mathcal{T}}[(\varrho, S, \mathbf{m})(\cdot + T); \mathcal{A}] < \varepsilon$ for any $(\varrho, S, \mathbf{m}) \in \mathcal{F}[M, \mathcal{E}_0]$ and any $T > T(\varepsilon)$.

Stationary statistical solutions

Stationary statistical solutions [EF - A. Świerczewska-Gwiazda 2021]
Let $\mathcal{U} \subset \mathcal{A}$ be a non-empty time–shift invariant set, meaning

$$(\varrho, S, \mathbf{m}) \in \mathcal{U} \ \Rightarrow \ (\varrho, S, \mathbf{m})(\cdot + T) \in \mathcal{U} \text{ for any } T \in R.$$

Then there exists a stationary statistical solution $\mathcal{V}$ supported by $\overline{\mathcal{U}}$:

- $\mathcal{V}$ is a Borel probability measure, $\mathcal{V} \in \mathfrak{P}(\overline{\mathcal{U}})$;
- $\mathrm{supp} \mathcal{V} \subset \overline{\mathcal{U}}$, where the closure of a $\mathcal{U}$ is a compact invariant set;
- $\mathcal{V}$ is shift invariant, i.e., $\mathcal{V}[\mathfrak{B}] = \mathcal{V}[\mathfrak{B}(\cdot + T)]$ for any Borel set $\mathfrak{B} \subset \mathcal{T}$ and any $T \in R$.

Ergodic means

Phase space

$$H = W^{-k,2}(\Omega) \times W^{-k,2}(\Omega) \times W^{-k,2}(\Omega; R^3).$$

Convergence of ergodic means [application of Birkhoff–Khinchin ergodic theorem]
Let $\mathcal{V}$ be a stationary statistical solution and $(\varrho, S, \mathbf{m})$ the associated stationary process. Let $F : H \to R$ be a Borel measurable function such that

$$\int_{\mathcal{T}} |F(\varrho(0,\cdot), S(0,\cdot), \mathbf{m}(0,\cdot)| \ \mathrm{d}\mathcal{V} < \infty.$$

Then there exists a measurable function $\overline{F}$,

$$\overline{F} : (\mathcal{T}, \mathcal{V}) \to R$$

such that

$$\frac{1}{T} \int_0^T F(\varrho(t,\cdot), S(t,\cdot), \mathbf{m}(t,\cdot)) \mathrm{d}t \to \overline{F} \text{ as } T \to \infty$$

$\mathcal{V}-$a.s. and in $L^1(\mathcal{T}, \mathcal{V})$.

Scaled system

Mass conservation:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance:

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \boxed{\frac{1}{\varepsilon} \varrho \nabla_x G}$$

Boundary conditions:

$$\Theta_B - \Theta_U \approx \varepsilon \overline{\theta}$$

Limit system – Oberbeck–Boussinesq approximation

Incompressibility:

$$\mathrm{div}_x \mathbf{U} = 0$$

Momentum balance:

$$\overline{\varrho}\Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) + \nabla_x \Pi = \mu(\overline{\vartheta}) \Delta_x \mathbf{U} + r \nabla_x G$$

Heat equation:

$$\overline{\varrho} c_p(\overline{\varrho}, \overline{\vartheta}) \Big(\partial_t \Theta + \mathbf{U} \cdot \nabla_x \Theta \Big) - \overline{\varrho}\ \overline{\vartheta} \alpha(\overline{\varrho}, \overline{\vartheta}) \mathbf{U} \cdot \nabla_x G = \kappa(\overline{\vartheta}) \Delta_x \Theta$$

Boussinesq relation:

$$\frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \varrho} \nabla_x r + \frac{\partial p(\overline{\varrho}, \overline{\vartheta})}{\partial \vartheta} \nabla_x \Theta = \overline{\varrho} \nabla_x G.$$

Nonlocal boundary conditions

$$\Theta|_{\partial\Omega} = \overline{\theta} - \frac{\lambda(\overline{\varrho}, \overline{\vartheta})}{1 - \lambda(\overline{\varrho}, \overline{\vartheta})} \frac{1}{|\Omega|} \int_\Omega \Theta \ \mathrm{d}x, \ \ 0 < \lambda(\overline{\varrho}, \overline{\vartheta}) < 1$$