INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

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Chris Lambie-Hanson
Šárka Stejskalová

Preprint No. 29-2023

PRAHA 2023

GUESSING MODELS, TREES, AND CARDINAL ARITHMETIC

CHRIS LAMBIE-HANSON AND ŠÁRKA STEJSKALOVÁ

Abstract. Since being isolated by Viale and Weiß in 2009, the Guessing Model Property has emerged as a particularly prominent and powerful consequence of the Proper Forcing Axiom. In this paper, we investigate connections between variations of the Guessing Model Property and cardinal arithmetic, broadly construed. We improve upon results of Viale and Krueger by proving that a weakening of the Guessing Model Property implies Shelah's Strong Hypothesis. We also prove that, though the Guessing Model Property is known not to put an upper bound on the size of the continuum, it does imply that 2^{ω_1} is as small as possible relative to the value of 2^ω. Building on work of Laver, we prove that, in the extension of any model of PFA by a measure algebra, every tree of height and size ω_1 is B-special (a generalization of specialness introduced by Baumgartner that can also hold of trees with uncountable branches). Finally, we investigate the impact of forcing axioms for Suslin and almost Suslin trees on guessing model properties. In particular, we prove that if S is a Suslin tree, then the axioms $\mathsf{PFA(S)}$ and $\mathsf{PFA(S)[S]}$ imply the Guessing Model Property and the Indestructible Guessing Model Property, respectively, and, if T^* is an almost Suslin Aronszajn tree, then the axiom $\mathsf{PFA(T^*)}$ implies the Indestructible Guessing Model Property. This answers a number of questions of Cox and Krueger.

Contents

2010 *Mathematics Subject Classification.* 03E55, 03E35.

Key words and phrases. guessing models, cardinal arithmetic, weak Kurepa hypothesis, meeting numbers, Singular Cardinals Hypothesis, Shelah's Strong Hypothesis, special trees, random reals, PFA(S)[S], coherent Suslin tree.

Both authors were supported by the Czech Academy of Sciences (RVO 67985840) and the GAČR project 23-04683S.

1. Introduction

A long line of research in set theory, dating back at least to the 1970s, concerns the study of the extent to which certain combinatorial properties of large cardinals can consistently hold at smaller cardinals and of the consequences and implications of such properties. Classical examples of such properties include stationary reflection principles and the tree property; particularly prominent in recent years have been two-cardinal tree properties, which, in their strongest forms, can usefully be reformulated as guessing model principles.

The study of two-cardinal tree properties began with work of Jech [7] and Magidor [16], who used them to give combinatorial characterizations of strongly compact and supercompact cardinals, respectively. Versions of two-cardinal tree properties consistent at small cardinals were first isolated by Weiß [32] in the late 2000s, and since then much work has been done to understand these principles. In [31], Viale and Weiß show that the strongest of these two-cardinal tree properties, ISP, can be reformulated in terms of the existence of *guessing models*; this reformulation has proven to be quite fruitful in deriving consequences of two-cardinal tree properties. Viale and Weiß also prove that the guessing model property at ω_2, which is equivalent to ISP_{ω_2} and often simply denoted GMP, follows from the Proper Forcing Axiom (PFA).

In this article, we continue the study of two-cardinal tree properties and guessing model principles, focusing in particular on their relationship to cardinal arithmetic, broadly construed. There has already been considerable work done in this direction. Notably, results of Viale [30] and Krueger [9] show that GMP implies the Singular Cardinals Hypothesis (SCH), and, more generally, that for any regular cardinal $\kappa \geq \omega_2$, ISP_κ implies SCH above κ. On the other hand Cox and Krueger prove in [2] that, unlike PFA, GMP does not place any restrictions on the value of the continuum other than implying that $2^\omega > \omega_1$.

In the first part of the paper, we continue these lines of investigation. In particular, we show that, for a regular cardinal $\kappa \geq \omega_2$, a certain weakening of ISP_κ implies Shelah's Strong Hypothesis (SSH) above κ. This improves upon Viale and Krueger's aforementioned result by weakening the hypotheses and strengthening the conclusion. (The terminology and notation in the statement of Theorem A will be introduced in later sections; for now, we note that the hypothesis of the theorem is a weakening of ISP_κ.)

Theorem A. *Let $\kappa \geq \omega_2$ be a regular cardinal and, for every singular cardinal $\lambda > \kappa$ of countable cofinality, let*

$$\mathcal{Y}_\lambda := \{M \in \mathscr{P}_\kappa H(\lambda^{++}) \mid M \textit{ is } (\omega_1, \lambda)\textit{-internally unbounded}\}.$$

If $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\kappa, \kappa, \lambda^{++})$ holds for all such λ, then SSH holds above κ.

In the process of proving Theorem A, we sharpen some results of Viale [29] regarding covering properties. In particular, in [29] and [30], Viale shows that various consequences of PFA, including GMP, imply certain covering properties at all singular cardinals of countable cofinality above 2^ω. Here, we improve upon this by removing the assumption that the singular cardinal is greater than the continuum. See Theorem 4.12 below for a more precise statement; the main new technical lemma leading to this improvement is Lemma 4.6.

By the work of Cox and Krueger [2], GMP is compatible with $\mathrm{cf}(2^\omega) = \omega_1$ and therefore, unlike PFA, or even MA_{ω_1}, it does not imply that $2^{\omega_1} = 2^\omega$. However, as a consequence of the proof Theorem A, we are able to prove that a weakening of GMP implies that 2^{ω_1} is as small as possible relative to the value of 2^ω. We additionally show that the negation of the weak Kurepa Hypothesis ($\neg\mathsf{wKH}$), which is a consequence of the same weakening of GMP, yields the same conclusion if $2^\omega < \aleph_{\omega_1}$ but can consistently fail to do so if $2^\omega \geq \aleph_{\omega_1}$.

Theorem B. *(1) If $\mathsf{wAGP}(\omega_2)$ holds and $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\lambda^{++})$ holds for all singular $\lambda < 2^\omega$ of countable cofinality, where*

$$\mathcal{Y}_\lambda := \{M \in \mathscr{P}_{\omega_2} H(\lambda^{++}) \mid M \text{ is } (\omega_1, \lambda)\text{-internally unbounded}\},$$

then

$$2^{\omega_1} = \begin{cases} 2^\omega & \text{if } \mathrm{cf}(2^\omega) \neq \omega_1 \\ (2^\omega)^+ & \text{if } \mathrm{cf}(2^\omega) = \omega_1. \end{cases}$$

(2) If $\neg\mathsf{wKH}$ holds and $2^\omega < \aleph_{\omega_1}$, then $2^{\omega_1} = 2^\omega$.

(3) Relative to the consistency of the existence of a supercompact cardinal, it is consistent that $\neg\mathsf{wKH}$ holds, $2^\omega = \aleph_{\omega_1}$, but $2^{\omega_1} > \aleph_{\omega_1+1}$.

In the second half of the paper, we prove some results motivated in part by the indestructible guessing model property (IGMP) introduced by Cox and Krueger in [3], and which we also feel are of independent interest. We first prove a result about special trees. Recall that a tree of height ω_1 is *special* if it is the union of countably many antichains. Special trees therefore have no uncountable branches. In [1], Baumgartner introduces a generalization of specialness, which we call here *B-specialness*, which can also hold of trees with uncountable branches. By results of Cox and Krueger [3] and Krueger [10], IGMP follows from the conjunction of GMP and the assertion that all trees of height and size ω_1 are B-special. Also, Cox and Krueger prove [3] that IGMP is compatible with any possible value of the continuum with cofinality at least ω_2. Motivated by the question of whether IGMP is compatible with $\mathrm{cf}(2^\omega) = \omega_1$, we prove the following variation on a theorem of Laver [15], who proved that forcing with a measure algebra over any model of Martin's Axiom preserves the fact that all trees of height and size ω_1 with no uncountable branches are special.

Theorem C. *If PFA holds and $\mathbb{B}$ is a measure algebra, then in $V^{\mathbb{B}}$ every tree of height and size ω_1 is B-special.*

Theorem C implies that, in the extension of any model of PFA by a measure algebra, an indestructible version of $\neg\mathsf{wKH}$ holds (see Corollary 5.8 for a precise statement). In particular, this indestructible version of $\neg\mathsf{wKH}$ is compatible with $\mathrm{cf}(2^\omega) = \omega_1$.

In the last section of the paper, we investigate the effect of forcing axioms for Suslin and almost Suslin trees on guessing model principles. The axiom $\mathsf{PFA(S)}$ (introduced by Todorcevic in [25] and defined more precisely in Section 6) is the assertion that S is a Suslin tree and the conclusion of PFA holds when restricted to proper forcings that preserve the fact that S is a Suslin tree.[1] $\mathsf{PFA(S)[S]}$ is the assertion that the universe is obtained from forcing over a model of $\mathsf{PFA(S)}$ with the Suslin tree S. The axiom $\mathsf{PFA(T^*)}$ (introduced by Krueger in [11] and again defined more precisely in Section 6), is the assertion that T^* is an almost Suslin

[1]Most other sources require S to be a *coherent* Suslin tree in the statement of $\mathsf{PFA(S)}$. Since we will not need coherence here, we state the axiom in a more general form.

Aronszajn tree and the conclusion of PFA holds when restricted to proper forcings preserving this fact. Our main theorem in this section is the following:

Theorem D. *(1) Let S denote a Suslin tree. Then* PFA(S) *implies* GMP*, and* PFA(S)[S] *implies* IGMP*.*
(2) Let T^* denote an almost Suslin Aronszajn tree. Then PFA(T^*) *implies* IGMP*.*

This theorem answers a number of questions of Cox and Krueger, which we record here and explicate more thoroughly in Section 6.

- In [3], Cox and Krueger ask whether IGMP implies that the pseudointersection number $\mathfrak{p}$ is greater than ω_1. Theorem D answers this negatively, since, in any model of PFA(S)[S], we have $\mathfrak{p} = \omega_1$.
- In [3], Cox and Krueger ask whether IGMP implies that every tree of height and size ω_1 with no cofinal branches is special. Theorem D answers this negatively, since, in any model of PFA(T^*), IGMP holds and T^* is a nonspecial Aronszajn tree.
- In [11], Krueger asks whether PFA(T^*) implies ¬wKH. Theorem D answers this positively, since GMP, and hence IGMP, implies ¬wKH.

The structure of the remainder of the paper is as follows. In Section 2, we review some background on $\mathscr{P}_\kappa\lambda$ combinatorics and guessing models. In Section 3, we investigate the effect of ¬wKH on cardinal arithmetic, proving clauses (2) and (3) of Theorem B. In Section 4, we investigate the effect of (weakenings of) GMP on cardinal arithmetic. Among the central technical results of this section is Lemma 4.6, a new lemma about covering matrices. This is then used to prove Theorem A and clause (1) of Theorem B. Section 5 contains the proof of Theorem C, and Section 6 contains the proof of Theorem D.

1.1. Notation and terminology.

Our terminology and notation is for the most part standard. We use [8] as our standard background reference for set theory and refer the reader there for any undefined notions or notations. We record a few notational conventions here at the outset. If $\kappa < \nu$ are infinite cardinals, with κ regular, then $S^\nu_\kappa := \{\alpha < \nu \mid \mathrm{cf}(\alpha) = \kappa\}$. If κ is an infinite cardinal and X is a set with $|X| \geq \kappa$, then $\mathscr{P}_\kappa X := \{x \subseteq X \mid |x| < \kappa\}$. If $\mathbb{P}$ is a forcing poset with greatest lower bounds and p, q are compatible conditions in $\mathbb{P}$, then $p \wedge q$ denotes their greatest common lower bound.

2. Background on two-cardinal combinatorics and guessing models

Though much of the previous work motivating this article concerns two-cardinal tree properties, we will be working here exclusively with the formulations of these properties in terms of guessing models. Since the definitions of the relevant thin and μ-slender (κ, λ)-lists and (κ, λ)-trees and the ensuing tree properties (I)TP(κ, λ) and (I)SP(μ, κ, λ) are somewhat involved and will not directly be used in this paper, we refer the reader to the companion paper [13] for their precise definition and their connection with the guessing model properties studied here and defined below. Before introducing these guessing model properties, though, we need some background on two-cardinal combinatorics.

2.1. **Two-cardinal combinatorics.** Temporarily fix a regular uncountable cardinal κ and a set X with $|X| \geq \kappa$.

Definition 2.1. Suppose that $\mathcal{C} \subseteq \mathscr{P}_\kappa X$.

(1) C is *closed* if whenever $D \subseteq C$ is such that $|D| < \kappa$ and D is linearly ordered by $\subseteq$, we have $\bigcup D \in C$;
(2) C is *strongly closed* if whenever $D \subseteq C$ and $|D| < \kappa$, we have $\bigcup D \in C$;
(3) C is *cofinal* if for all $x \in \mathscr{P}_\kappa \lambda$, there is $y \in C$ such that $x \subseteq y$;
(4) C is a *club* in $\mathscr{P}_\kappa X$ if it is closed and cofinal;
(5) C is a *strong club* in $\mathscr{P}_\kappa X$ if it is strongly closed and cofinal.

A set $S \subseteq \mathscr{P}_\kappa X$ is *(weakly) stationary* in $\mathscr{P}_\kappa X$ if $S \cap C \neq \emptyset$ for every (strong) club $C \subseteq \mathscr{P}_\kappa X$.

Given a set $x \subseteq X$ and a function $f : X \to \mathscr{P}_\kappa X$, we say that x is *closed under* f if $f(a) \subseteq x$ for all $a \in x$. Similarly, if $g : [X]^2 \to \mathscr{P}_\kappa X$, then x is closed under g if $g(a) \subseteq x$ for all $a \in [x]^2$. The following proposition is immediate.

Proposition 2.2. *Suppose that $f : X \to \mathscr{P}_\kappa X$ is a function. Then the set $\{x \in \mathscr{P}_\kappa X \mid x \text{ is closed under } f\}$ is a strong club in $\mathscr{P}_\kappa X$. In particular, if $\mathcal{Y} \subseteq \mathscr{P}_\kappa X$ is weakly stationary, then there is $x \in \mathcal{Y}$ such that x is closed under f.*

The following characterization of the club filter on $\mathscr{P}_\kappa X$ is due to Menas [19].

Proposition 2.3. *If $g : [X]^2 \to \mathscr{P}_\kappa X$ is a function, then the set*

$$C_g := \{x \in \mathscr{P}_\kappa X \mid x \text{ is infinite and closed under } g\}$$

is a club in $\mathscr{P}_\kappa X$. Moreover, for any club C in $\mathscr{P}_\kappa X$, there is $g : [X]^2 \to \mathscr{P}_\kappa X$ such that $C_g \subseteq C$.

2.2. **Guessing models.** We now review the notion of a *guessing model* and the subsequently defined *guessing model properties*, which provide an alternative formulation of the two-cardinal tree properties of the form $\mathsf{ISP}(\ldots)$.

Definition 2.4. Suppose that θ is a regular uncountable cardinal and $M \subseteq H(\theta)$.

(1) Given a set $x \in M$, a subset $d \subseteq x$, and a cardinal μ, we say that
   (a) d is (μ, M)*-approximated* if, for every $z \in M \cap \mathscr{P}_\mu(x)$, there is $e \in M$ such that $d \cap z = e \cap z$;[2]
   (b) d is M*-guessed* if there is $e \in M$ such that $d \cap M = e \cap M$.
(2) M is a μ*-guessing model for* x if every (μ, M)-approximated subset of x is M-guessed.
(3) M is a μ*-guessing model* if, for every $x \in M$, it is a μ-guessing model for x.
(4) Suppose that $\mu \leq \kappa \leq \theta$ are regular uncountable cardinals and $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$ is stationary. Then $\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ is the assertion that the set of $M \in \mathcal{Y}$ such that M is a μ-guessing models is stationary in $\mathscr{P}_\kappa H(\theta)$.

Remark 2.5. In order to cut down on the number of parameters in use and make statements of our results cleaner, we introduce some conventions, all of which are standard in the literature. In the notation $\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta)$, if $\mathcal{Y}$ is omitted, then it

[2]In other works, the conclusion of this definition is simply "$d \cap z \in M$". This is clearly equivalent to what is written here if $M \prec H(\theta)$ (and hence M is closed under intersections). We will want to apply this definition to more general situations, though, and for our purposes this seems like the most appropriate formulation.

should be understood to be $\mathscr{P}_\kappa H(\theta)$. $\mathsf{GMP}(\mu, \kappa, {\geq}\kappa)$ denotes the assertion that $\mathsf{GMP}(\mu, \kappa, \theta)$ holds for all regular $\theta \geq \kappa$. Since the most common first two parameters in $\mathsf{GMP}(\ldots)$ are ω_1 and ω_2, respectively, if $\mathcal{Y} \subseteq \mathscr{P}_{\omega_2} H(\theta)$, we let $\mathsf{GMP}_{\mathcal{Y}}(\theta)$ denote $\mathsf{GMP}_{\mathcal{Y}}(\omega_1, \omega_2, \theta)$ and let GMP denote the assertion that $\mathsf{GMP}(\theta)$ holds for all regular $\theta \geq \omega_2$. By [31, Propositions 3.2 and 3.3], for a regular cardinal $\kappa \geq \omega_2$, $\mathsf{GMP}(\omega_1, \kappa, {\geq}\kappa)$ is equivalent to $\mathsf{ISP}(\omega_1, \kappa, {\geq}\kappa)$, which is typically denoted in the literature as simply $\mathsf{ISP}(\kappa)$ or ISP_κ (we will use the latter in this paper). We note that GMP follows from the Proper Forcing Axiom [31, Theorem 4.8] and also holds in the extension by the Mitchell forcing $\mathbb{M}(\omega, \kappa)$ if κ is supercompact in the ground model [32, Theorem 5.4].

We next recall weakenings of $\mathsf{GMP}(\ldots)$ introduced in [13] that provide alternative formulations of two-cardinal tree properties of the form $\mathsf{SP}(\ldots)$.

Definition 2.6. Suppose that $\mu \leq \kappa \leq \theta$ are regular uncountable cardinals, $x \in H(\theta)$, $S \subseteq \mathscr{P}_\kappa H(\theta)$, and $M \subseteq H(\theta)$. We say that (M, x) is *almost guessed by S* if $x \in M$ and, for every (μ, M)-approximated subset $d \subseteq x$, there is $N \in S$ such that

- $x \in N \subseteq M$;
- d is N-guessed.

Suppose that $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$.

- $\mathsf{AGP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ is the assertion that, for every $\subseteq$-cofinal $S \subseteq \mathscr{P}_\kappa H(\theta)$ and every $x \in H(\theta)$, the set of $M \in \mathcal{Y}$ such that (M, x) is almost guessed by S is stationary in $\mathscr{P}_\kappa H(\theta)$.
- $\mathsf{wAGP}_{\mathcal{Y}}(\mu, \kappa, \theta)$ is defined in the same way, except that the set of M as in the conclusion is only assumed to be *weakly* stationary in $\mathscr{P}_\kappa H(\theta)$.

Remark 2.7. $\mathsf{(w)AGP}$ stands for "(weak) almost guessing principle". In the notation $\mathsf{(w)AGP}_{\mathcal{Y}}(\mu, \kappa, H(\theta))$, we again suppress mention of $\mathcal{Y}$ if it is equal to $\mathscr{P}_\kappa H(\theta)$. As with GMP, if $\mathcal{Y} \subseteq \mathscr{P}_{\omega_2} H(\theta)$, we let $\mathsf{(w)AGP}_{\mathcal{Y}}(\theta)$ denote $\mathsf{(w)AGP}_{\mathcal{Y}}(\omega_1, \omega_2, \theta)$.

It is immediate that, for a fixed trio of regular uncountable cardinals $\mu \leq \kappa \leq \theta$ and a stationary $\mathcal{Y} \subseteq \mathscr{P}_\kappa H(\theta)$, we have

$$\mathsf{GMP}_{\mathcal{Y}}(\mu, \kappa, \theta) \Rightarrow \mathsf{AGP}_{\mathcal{Y}}(\mu, \kappa, \theta) \Rightarrow \mathsf{wAGP}_{\mathcal{Y}}(\mu, \kappa, \theta).$$

To verify the first implication, the key observation is the fact that, for a fixed cofinal $S \subseteq \mathscr{P}_\kappa H(\theta)$, the set

$$\{M \in \mathscr{P}_\kappa H(\theta) \mid \forall a \in M \exists N \in S \; [a \in N \subseteq M]\}$$

is a club in $\mathscr{P}_\kappa H(\theta)$.

3. The weak Kurepa hypothesis and cardinal arithmetic

Recall that, for a regular uncountable cardinal μ, a weak μ-Kurepa tree is a tree of height and size μ with more than μ-many cofinal branches. We use $\mathsf{wKH}(\mu)$ to denote the *weak Kurepa hypothesis* at μ, i.e., the assertion that there is a weak μ-Kurepa tree. Then $\neg\mathsf{wKH}(\mu)$ is the assertion that every tree of height and size μ has at most μ-many cofinal branches. Note that $\neg\mathsf{wKH}$ entails $2^{<\mu} > \mu$, since otherwise ${}^{<\mu}2$ is a weak μ-Kurepa tree. We omit mention of μ and write simply wKH if $\mu = \omega_1$.

In [13, Theorem 9.3], we show that, if μ is a regular uncountable cardinal, then $\neg\mathsf{wKH}(\mu)$ follows from $\mathsf{wAGP}(\mu, \mu^+, \mu^+)$. In particular, $\neg\mathsf{wKH}$ follows from GMP (this conclusion had been proven earlier in [3, Theorem 2.8]).

In this section, we address the influence of $\neg\mathsf{wKH}(\mu)$ on the value of 2^μ. In particular, we will show that $\neg\mathsf{wKH}(\mu)$ forces 2^μ to be as small as possible relative to $2^{<\mu}$ *if* $2^{<\mu} < \mu^{+\mu}$ but does not have the same influence in general. In particular, $\neg\mathsf{wKH}$ implies that $2^{\omega_1} = 2^\omega$ if $2^\omega < \aleph_{\omega_1}$, but not necessarily otherwise. This should be contrasted with (a weakening of) GMP, which implies $\neg\mathsf{wKH}$ and, as we will see in Section 4, forces 2^{ω_1} to be as small as possible relative to 2^ω regardless of the value of 2^ω.

These arguments make use of the notion of *meeting numbers*, which will also be used in the arguments of Section 4.

Definition 3.1 ([18])**.** Suppose that $\kappa \leq \lambda$ are infinite cardinals. Then the *meeting number* $m(\kappa, \lambda)$ is the minimal cardinality of a collection $\mathcal{Y} \subseteq [\lambda]^\kappa$ such that, for all $x \in [\lambda]^\kappa$, there is $y \in \mathcal{Y}$ such that $|x \cap y| = \kappa$.

Remark 3.2. By a standard diagonalization argument, if $\kappa < \lambda$ and $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$, then $m(\kappa, \lambda) > \lambda$. Therefore, for general $\kappa < \lambda$, the smallest possible value $m(\kappa, \lambda)$ can take is λ if $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$ and λ^+ if $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$. As we will see in Section 4, a weakening of GMP forces $m(\kappa, \lambda)$ to always attain this minimum possible value.

One reason for interest in meeting numbers is the fact that they provide a simple alternate formulation of Shelah's Strong Hypothesis (SSH), which is the assertion that, for every singular cardinal λ, the pseudopower $\mathrm{pp}(\lambda)$ is equal to λ^+.

Theorem 3.3 (Matet, [17])**.** *The following are equivalent:*

(1) Shelah's Strong Hypothesis;
(2) for every singular cardinal λ of countable cofinality, we have $m(\omega, \lambda) = \lambda^+$;
(3) for all infinite cardinals $\kappa < \lambda$, we have $m(\kappa, \lambda) = \lambda^+$ if $\mathrm{cf}(\kappa) = \mathrm{cf}(\lambda)$ and $m(\kappa, \lambda) = \lambda$ if $\mathrm{cf}(\kappa) \neq \mathrm{cf}(\lambda)$.

Note that Theorem 3.3 makes it immediately evident that SSH is indeed a strengthening of SCH, since if λ is a singular cardinal of countable cofinality, then a straightforward calculation yields $\lambda^\omega = 2^\omega \cdot m(\omega, \lambda)$.

The following basic fact about $m(\kappa, \lambda)$ will be useful. For a proof, see [17, Corollary 2.5].

Proposition 3.4. *Suppose that $\kappa < \lambda < \kappa^{+\mathrm{cf}(\kappa)}$. Then $m(\kappa, \lambda) = \lambda$.*

Lemma 3.5. *Suppose that μ is a regular uncountable cardinal, and assume $\neg\mathsf{wKH}(\mu)$. Then $2^\mu = m(\mu, 2^{<\mu})$.*

Proof. First note that $m(\mu, 2^{<\mu}) \leq |[2^{<\mu}]^\mu| = 2^\mu$. It thus remains to show that $2^\mu \leq m(\mu, 2^{<\mu})$.

Fix $\mathcal{Y} \subseteq [{}^{<\mu}2]^\mu$ such that

- $|\mathcal{Y}| = m(\mu, 2^{<\mu})$;
- for every $x \in [{}^{<\mu}2]^\mu$, there is $y \in \mathcal{Y}$ such that $|x \cap y| = \mu$.

For each $y \in \mathcal{Y}$, let $T_y := \{f \in {}^{<\mu}2 \mid \exists g \in y\ [f \subseteq g]\}$, i.e., T_y is the downward closure of y in the tree ${}^{<\mu}2$. Note that T_y is a tree of cardinality μ and therefore, by $\neg\mathsf{wKH}(\mu)$, T_y has at most μ-many branches of size μ. We naturally identify branches through ${}^{<\mu}2$ of size μ with elements of ${}^\mu 2$.

Claim 3.6. *For every $h \in {}^\mu 2$, there is $y \in \mathcal{Y}$ such that h is a branch through T_y.*

Proof. Fix $h \in {}^{\mu}2$, and let $x := \{h \restriction \alpha \mid \alpha < \mu\}$ be the set of proper initial segments of h. Then $x \in [{}^{<\mu}2]^{\mu}$, so we can find $y \in \mathcal{Y}$ such that $|x \cap y| = \mu$. Then there are unboundedly many $\alpha < \mu$ such that $h \restriction \alpha \in y$; it follows that h is a branch through T_y. □

Since, for each $y \in \mathcal{Y}$, T_y has at most μ-many branches of size μ, and since each element of ${}^{\mu}2$ is a branch through T_y for some $y \in \mathcal{Y}$, it follows that $2^{\mu} \leq \mu \cdot m(\mu, 2^{<\mu})$. Since $\neg\mathsf{wKH}$ holds, we have $2^{<\mu} > \mu$, and thus $m(\mu, 2^{<\mu}) > \mu$. Therefore, $2^{\mu} = m(\mu, 2^{<\mu})$. □

The following corollary yields clause (2) of Theorem B as a special case.

Corollary 3.7. *Suppose that μ is a regular uncountable cardinal, $\neg\mathsf{wKH}(\mu)$ holds, and $2^{<\mu} < \mu^{+\mu}$. Then $2^{\mu} = 2^{<\mu}$.*

Proof. Since $\neg\mathsf{wKH}(\mu)$ holds, Lemma 3.5 implies that $2^{\mu} = m(\mu, 2^{<\mu})$. By $\neg\mathsf{wKH}(\mu)$, we have $2^{<\mu} > \mu$, and by assumption we have $2^{<\mu} < \mu^{+\mu}$. Therefore, Proposition 3.4 implies that $m(\mu, 2^{<\mu}) = 2^{<\mu}$. Altogether, this implies that $2^{\mu} = 2^{<\mu}$. □

We now prove clause (3) of Theorem B, showing that $\neg\mathsf{wKH}(\mu)$ no longer forces 2^{μ} to be as small as possible relative to $2^{<\mu}$ if we allow $2^{<\mu} \geq \mu^{+\mu}$. For concreteness, we focus on the case $\mu = \omega_1$ and produce a model in which $\neg\mathsf{wKH}$ holds, $2^{\omega} = \aleph_{\omega_1}$, and $2^{\omega_1} = \aleph_{\omega_1+2}$, but it will be evident how to modify the construction to produce other configurations.

Theorem 3.8. *If the existence of a supercompact cardinal is consistent, then it is consistent that $\neg\mathsf{wKH}$ holds, $2^{\omega} = \aleph_{\omega_1}$, and $2^{\omega_1} > \aleph_{\omega_1+1}$.*

Proof. Let λ be a supercompact cardinal. By forcing with the Laver preparation followed by the forcing to add λ^{++}-many Cohen subsets to λ, we may assume that $2^{\lambda} = \lambda^{++}$. Let $\kappa < \lambda$ be weakly compact. We can now force with a Radin-type forcing $\mathbb{R}$ with interleaved collapses such that:

- in $V^{\mathbb{R}}$, we have $\lambda = \kappa^{+\omega_1}$, λ is strong limit, and $2^{\lambda} = \lambda^{\omega_1} = \lambda^{++}$;
- $(V_{\kappa+1})^{V^{\mathbb{R}}} = (V_{\kappa+1})^{V}$.

For details about how to define such a forcing notion, see, for example, [4]. The forcing defined there produces an extension in which $\lambda = \aleph_{\omega_1}$, but by ensuring that all of the points of the Radin club are chosen above κ and by using an initial interleaved Lévy collapse of the form $\mathrm{Coll}(\kappa^{+3}, < \lambda_0)$, where λ_0 is the first point in the Radin club, we obtain a forcing notion with the desired properties.

Since $(V_{\kappa+1})^{V^{\mathbb{R}}} = (V_{\kappa+1})^{V}$, κ is still weakly compact in $V^{\mathbb{R}}$. Let $\dot{\mathbb{M}}$ be an $\mathbb{R}$-name for the standard Mitchell forcing $\mathbb{M}(\omega, \kappa)$. Then, in $V^{\mathbb{R}*\dot{\mathbb{M}}}$, we have:

- $\kappa = \omega_2 = 2^{\omega}$;
- $\lambda = \aleph_{\omega_1}$ is strong limit, and $2^{\lambda} = \lambda^{\omega_1} = \lambda^{++}$;
- $\neg\mathsf{wKH}$ (cf. [26]).

Finally, let $\dot{\mathbb{C}}$ be an $\mathbb{R} * \dot{\mathbb{M}}$-name for the forcing to add λ-many Cohen reals. Then, in $V^{\mathbb{R}*\dot{\mathbb{M}}*\dot{\mathbb{C}}}$, we have $2^{\omega} = \lambda = \aleph_{\omega_1}$ and $2^{\omega_1} = (\aleph_{\omega_1})^{\aleph_1} = \aleph_{\omega_1+2}$. It remains to verify that $\neg\mathsf{wKH}$ holds in $V^{\mathbb{R}*\dot{\mathbb{M}}*\dot{\mathbb{C}}}$.

It is proven in [26, §1] that, after forcing over a ground model W with Mitchell forcing $\mathbb{M}(\omega, \mu)$ with μ inaccessible, $\neg\mathsf{wKH}$ is preserved by further forcing with a finite-support iteration of length at most ω_2 of c.c.c. posets of size ω_1 such that each iterand does not add a new uncountable branch to any tree of height ω_1. In particular, it follows that, in $V^{\mathbb{R}*\dot{\mathbb{M}}}$, $\neg\mathsf{wKH}$ is preserved after adding ω_2-many or fewer Cohen reals.

In $V^{\mathbb{R}*\dot{\mathbb{M}}}$, let $\dot{T}$ be a $\mathbb{C}$-name for a tree of height and size ω_1. We think of conditions in $\mathbb{C}$ as being finite partial functions from λ to 2. For each $A \subseteq \lambda$, let $\mathbb{C}_A$ be the complete suborder consisting of all elements of $\mathbb{C}$ whose domains are subsets of A. Since $\mathbb{C}$ has the c.c.c., we can find $A \in [\lambda]^{\aleph_1}$ such that there is a $\mathbb{C}_A$-name $\dot{T}'$ for which $\Vdash_{\mathbb{C}}$ "$\dot{T} = \dot{T}'$". Let T be the realization of $\dot{T}'$ in $V^{\mathbb{R}*\dot{M}*\dot{\mathbb{C}}_{\dot{A}}}$. By the previous paragraph, $\neg\mathsf{wKH}$ holds in that model, so T has at most ω_1-many branches there. But $V^{\mathbb{R}*\dot{M}*\dot{\mathbb{C}}}$ is an extension of $V^{\mathbb{R}*\dot{M}*\dot{\mathbb{C}}_{\dot{A}}}$ by the forcing to add λ-many Cohen reals, which cannot add new cofinal branches to a tree of height ω_1. Therefore, T still has at most ω_1-many branches in $V^{\mathbb{R}*\dot{M}*\dot{\mathbb{C}}}$, so $\neg\mathsf{wKH}$ holds there. □

4. Guessing models and cardinal arithmetic

In this section, we analyze the effect of guessing model principles on cardinal arithmetic, focusing on meeting numbers and pseudopowers. In the process, we will prove, for instance, that a weakening of GMP implies that 2^{ω_1} is as small as possible relative to 2^ω and that a weakening of $\mathsf{GMP}(\kappa, \kappa, \geq \kappa)$ implies SSH above κ.

We will need to introduce a bit of machinery to obtain these results. We first recall the notion of a *covering matrix*, introduced by Viale in his proof that PFA implies SCH [28].

4.1. Covering matrices.

The terminology associated with covering matrices is slightly inconsistent across sources; we will follow the terminology of [21] and [12].

Definition 4.1. Let $\theta < \lambda$ be regular cardinals. A *θ-covering matrix for λ* is a matrix $\mathcal{D} = \langle D(i,\beta) \mid i < \theta,\ \beta < \lambda\rangle$ such that:

(1) for all $\beta < \lambda$, $\langle D(i,\beta) \mid i < \theta\rangle$ is a $\subseteq$-increasing sequence and $\bigcup_{i<\theta} D(i,\beta) = \beta$;
(2) for all $\beta < \gamma < \lambda$ and $i < \theta$, there is $j < \theta$ such that $D(i,\beta) \subseteq D(j,\gamma)$.

For such a matrix $\mathcal{D}$, let $\beta_{\mathcal{D}}$ denote the least ordinal β such that $\mathrm{otp}(D(i,\gamma)) < \beta$ for all $\gamma < \lambda$ and $i < \theta$.

We will be especially interested in covering matrices satisfying certain additional properties.

Definition 4.2. Suppose that $\theta < \lambda$ are regular cardinals and $\mathcal{D}$ is a θ-covering matrix for λ.

(1) $\mathcal{D}$ is *transitive* if, for all $\beta < \gamma < \lambda$ and all $i < \theta$, if $\beta \in D(i,\gamma)$, then $D(i,\beta) \subseteq D(i,\gamma)$.
(2) $\mathcal{D}$ is *uniform* if, for every limit ordinal $\beta < \lambda$, there is $i < \theta$ such that $D(i,\beta)$ contains a club in β.
(3) $\mathcal{D}$ is *strongly locally downward coherent* if, for all $X \in [\lambda]^{\leq\theta}$, there is $\gamma_X < \lambda$ such that, for all $\beta \in [\gamma_X, \lambda)$, there is $i < \theta$ such that, for all $j \in [i, \theta)$, $X \cap D(j,\beta) = X \cap D(j,\gamma_X)$.
(4) $\mathsf{CP}(\mathcal{D})$ holds if there is an unbounded $A \subseteq \lambda$ such that $[A]^\theta$ is covered by $\mathcal{D}$, i.e., for all $X \in [A]^\theta$, there are $\beta < \lambda$ and $i < \theta$ for which $X \subseteq D(i,\beta)$.

The key ingredient of Viale's proof that PFA implies SCH is the fact that, for every singular cardinal μ of countable cofinality, PFA implies $\mathsf{CP}(\mathcal{D})$ for every strongly locally downward coherent ω-covering matrix $\mathcal{D}$ for μ^+, together with a lemma asserting that, for every singular cardinal $\mu > 2^\omega$ of countable cofinality, there is a strongly locally downward coherent ω-covering matrix $\mathcal{D}$ for μ^+ such that $\beta_{\mathcal{D}} = \mu$. The proof of this latter lemma can essentially be split into two steps. In the first, it is proven that, for every singular cardinal μ, there is a transitive $\mathrm{cf}(\mu)$-covering matrix $\mathcal{D}$ for μ^+ such that $\beta_{\mathcal{D}} = \mu$. In the second, it is shown that, if $\mu > 2^{\mathrm{cf}(\mu)}$, then every such covering matrix $\mathcal{D}$ is strongly locally downward coherent. We show here that, in this second step, the requirement of $\mu > 2^{\mathrm{cf}(\mu)}$ can be dropped if we additionally assume that our covering matrix is uniform. We first note that the proof of the first step already guarantees the existence of such covering matrices. In fact, we will be able to arrange the following strengthening of $\beta_{\mathcal{D}} = \mu$ that will be useful later in this section, when we address the influence of guessing model principles on pseudopowers.

Definition 4.3. Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals converging to μ. If $\mathcal{D} = \langle D(i, \beta) \mid i < \theta,\ \beta < \mu^+ \rangle$ is a θ-covering matrix for μ^+, we say that $\mathcal{D}$ *respects* $\vec{\mu}$ if $|D(i, \beta)| < \mu_i$ for all $i < \theta$.

The following lemma is essentially [21, Lemma 2.4]; our formulation is slightly stronger than the cited lemma, but the proof there easily yields our desired conclusion.

Lemma 4.4 ([21, Lemma 2.4])**.** *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals converging to μ. Then there is a uniform, transitive θ-covering matrix for μ^+ that respects $\vec{\mu}$.*

We now show that covering matrices as in Lemma 4.4 actually satisfy a strengthening of strong local downward coherence. Our proof is a modification of an argument of Shelah from his development of PCF theory. We first recall the following club-guessing theorem.

Theorem 4.5 ([22, Ch. III, §2])**.** *Suppose that κ and ν are regular cardinals and $\kappa^+ < \nu$. Then there is a sequence $\langle C_\alpha \mid \alpha \in S^\nu_\kappa \rangle$ such that:*

(1) for all $\alpha \in S^\nu_\kappa$, C_α is club in α;
(2) for every club C in ν, the set $\{\alpha \in S^\nu_\kappa \mid C_\alpha \subseteq C\}$ is stationary in ν.

Lemma 4.6. *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\mathcal{D}$ is a uniform, transitive θ-covering matrix for μ^+. Then, for every $X \in [\mu^+]^{<\mu}$, there is $\gamma_X < \mu^+$ such that, for all $\beta \in [\gamma_X, \mu^+)$, there is $i < \theta$ such that, for all $j \in [i, \theta)$, $X \cap D(j, \beta) = X \cap D(j, \gamma_X)$.*

Proof. Fix a set $X \in [\mu^+]^{<\mu}$, and suppose for sake of contradiction that there is no γ_X as in the conclusion of the lemma. Since $\mathcal{D}$ is transitive, it follows that, for all $\beta < \gamma < \mu^+$ and all $i < \theta$, if $\beta \in D(i, \gamma)$, then $X \cap D(i, \beta) \subseteq X \cap D(i, \gamma)$. Therefore, by the nonexistence of a γ_X as in the conclusion of the lemma, for every $\beta < \mu^+$ we can find a $\beta' > \beta$ and an unbounded set $I_\beta \subseteq \theta$ such that, for all $j \in I_\beta$, we have $X \cap D(j, \beta) \subsetneq X \cap D(j, \beta')$. Let E be a club in μ^+ such that, for all $\gamma \in E$ and all $\beta < \gamma$, we have $\beta' < \gamma$. Then, by the transitivity of $\mathcal{D}$, for all $\beta < \gamma$, both in E, and for all but boundedly many $j \in I_\beta$, we have $X \cap D(j, \beta) \subsetneq X \cap D(j, \gamma)$.

Let $\kappa := \max\{|X|^+, \theta^+\}$, and let $\nu := \kappa^{++}$. Fix a sequence $\langle C_\alpha \mid \alpha \in S^\nu_\kappa \rangle$ satisfying the conclusion of Theorem 4.5. Assume without loss of generality that each C_α contains only limit ordinals.

We now construct a strictly increasing, continuous sequence $\langle \beta_\xi \mid \xi < \nu \rangle$ of elements of E as follows. Begin by setting $\beta_0 := \min(E)$. If $\xi < \nu$ is a limit ordinal and we have defined $\langle \beta_\eta \mid \eta < \xi \rangle$, then we are obliged to set $\beta_\xi := \sup\{\beta_\eta \mid \eta < \xi\}$. Finally, suppose that $\xi < \nu$ and we have defined $\langle \beta_\eta \mid \eta \leq \xi \rangle$. Let B_ξ be the set of $\alpha \in S^\nu_\kappa$ for which there is $\beta < \mu^+$ such that, for some $i < \theta$, $\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq D(i, \beta)$. For each $\alpha \in B_\xi$, choose a $\beta_{\xi,\alpha}$ witnessing this. Finally, choose $\beta_{\xi+1} \in E \setminus (\beta_\xi + 1)$ large enough so that $\beta_{\xi+1} > \beta_{\xi,\alpha}$ for all $\alpha \in B_\xi$. This completes the construction of $\langle \beta_\xi \mid \xi < \nu \rangle$. We first note the following simple claim.

Claim 4.7. *For all $\xi < \nu$ and all $\alpha \in B_\xi$, there is $j < \theta$ such that $\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq D(j, \beta_{\xi+1})$.*

Proof. Fix $\xi < \nu$ and $\alpha \in B_\xi$. By construction, we can fix an $i < \theta$ such that $\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq D(i, \beta_{\xi,\alpha})$. Since $\beta_{\alpha,\xi} < \beta_{\xi+1}$, we can fix $j \in [i, \theta)$ such that $\beta_{\alpha,\xi} \in D(j, \beta_{\xi+1})$. By the definition of covering matrix and the transitivity of $\mathcal{D}$, we have

$$D(i, \beta_{\xi,\alpha}) \subseteq D(j, \beta_{\xi,\alpha}) \subseteq D(j, \beta_{\xi+1}),$$

so $\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq D(j, \beta_{\xi+1})$, as desired. □

Let $\gamma := \sup\{\beta_\xi \mid \xi < \nu\}$. Since $\mathcal{D}$ is uniform, there is a club $C \subseteq \gamma$ and an $i^* < \theta$ such that $C \subseteq D(i^*, \gamma)$. Let $\bar{C} := \{\xi < \nu \mid \beta_\xi \in C\}$. Since $\langle \beta_\xi \mid \xi < \nu \rangle$ enumerates a club in γ, it follows that $\bar{C}$ is a club in ν. We can therefore fix $\alpha \in S^\nu_\kappa$ such that $C_\alpha \subseteq \bar{C}$.

Note that, for all $\xi < \nu$, we have $\alpha \in B_\xi$, since

$$\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq C \subseteq D(i^*, \gamma).$$

Therefore, for each $\xi \in C_\alpha$, we can fix an $i_\xi < \theta$ such that $\{\beta_\eta \mid \eta \in C_\alpha \cap (\xi + 1)\} \subseteq D(i_\xi, \beta_{\xi+1})$. Also for each $\xi \in C_\alpha$, let $\xi^\dagger$ denote $\min(C_\alpha \setminus (\xi + 1))$, and fix $j_\xi < \theta$ such that $\beta_{\xi+1} \in D(j_\xi, \beta_{\xi^\dagger})$. Let $k_\xi := \max\{i_\xi, j_\xi\}$. Since κ is a regular cardinal and $\kappa > \theta$, we can find a fixed $k < \theta$ and an unbounded set $A \subseteq C_\alpha$ such that $k_\xi = k$ for all $\xi \in A$.

Claim 4.8. *Suppose that $j \in [k, \theta)$ and that $\eta < \xi$ are both in A. Then $D(j, \beta_{\eta^\dagger}) \subseteq D(j, \beta_{\xi^\dagger})$.*

Proof. By the choice of i_ξ and j_ξ, we have $\beta_{\eta^\dagger} \in D(i_\xi, \beta_{\xi+1})$ and $\beta_{\xi+1} \in D(j_\xi, \beta_{\xi^\dagger})$. Since $j \geq k = \max\{i_\xi, j_\xi\}$, the transitivity of $\mathcal{D}$ implies that $\beta_{\eta^\dagger} \in D(j, \beta_{\xi^\dagger})$ and then, through another application, that $D(j, \beta_{\eta^\dagger}) \subseteq D(j, \beta_{\xi^\dagger})$, as desired. □

To ease the notation, let $A^\dagger$ denote $\{\eta^\dagger \mid \eta \in A\}$. Note that $A^\dagger$ is an unbounded subset of C_α. The previous claim then can be reworded to assert that, for all $\eta < \xi$, both in $A^\dagger$, and for all $j \in [k, \theta)$, we have $D(j, \beta_\eta) \subseteq D(j, \beta_\xi)$. For each $\eta \in A^\dagger$, let $\hat{\eta} = \min(A^\dagger \setminus (\eta + 1))$. Since both β_η and $\beta_{\hat{\eta}}$ are in E, we can find $\ell_\eta \in [k, \theta)$ such that $X \cap D(\ell_\eta, \beta_\eta) \subsetneq X \cap D(\ell_\eta, \beta_{\hat{\eta}})$. Again, since κ is a regular cardinal greater than θ, we can find a fixed ℓ and an unbounded $A^* \subseteq A^\dagger$ such that $\ell_\eta = \ell$ for all $\eta \in A^*$. Now, for all $\eta < \xi$, both in A^*, we have

$$X \cap D(\ell, \beta_\eta) \subsetneq X \cap D(\ell, \beta_{\hat{\eta}}) \subseteq X \cap D(\ell, \beta_\xi),$$

so $\langle X \cap D(\ell, \beta_\eta) \mid \eta \in A^* \rangle$ is a strictly $\subsetneq$-increasing sequence of subsets of X, contradicting the fact that $\mathrm{otp}(A^*) = \kappa$ and $\kappa > |X|$. $\square$

4.2. **Guessing models and meeting numbers.** We next show that $\mathsf{GMP}(\kappa, \kappa, \geq \kappa)$ implies that $\mathsf{CP}(\mathcal{D})$ holds whenever $\lambda > \kappa$ is a singular cardinal of countable cofinality and $\mathcal{D}$ is a uniform, transitive ω-covering matrix for λ^+. We will in fact prove something a bit stronger; to state the result precisely, we will need the following notion.

Definition 4.9. Given a set M, a subset $x \subseteq M$ is said to be *bounded* in M if there is $z \in M$ such that $x \subseteq z$. Given uncountable cardinals $\nu \leq \mu$, with ν regular, we say that M is *internally* (ν, μ)*-unbounded* if, for all $x \in \mathscr{P}_\nu M$ such that x is bounded in M, there is $y \in M$ such that $|y| < \mu$ and $x \subseteq y$.

Remark 4.10. Notice that the property of being internally (ν, μ)-unbounded becomes stronger as ν increases (for fixed μ) and weaker as μ increases (for fixed ν). In the extant literature, *internally unbounded* typically means (ω_1, ω_1)-unbounded. Also, for a set M and uncountable cardinals $\nu \leq \mu$ with ν regular, the following weak form of internal approachability is readily seen to imply that M is (ν, μ)-unbounded: $M = \bigcup_{\alpha < \nu} M_\alpha$ for some $\subseteq$-increasing sequence $\langle M_\alpha \mid \alpha < \nu \rangle$ such that $M_\alpha \in M$ and $|M_\alpha| < \mu$ for all $\alpha < \nu$. It then immediately follows that, for all regular uncountable cardinals $\nu < \kappa \leq \theta$, the set

$$\{M \in \mathscr{P}_\kappa H(\theta) \mid M \text{ is } (\nu, \kappa)\text{-internally unbounded}\}$$

is stationary in $\mathscr{P}_\kappa H(\theta)$.

Proposition 4.11. *Suppose that $\nu < \mu$ are infinite cardinals, θ is a sufficiently large regular cardinal, and $M \prec H(\theta)$ is a μ-guessing model such that ${}^{<\nu}M \subseteq M$ and $\nu \in M$. Then M is internally (ν^+, μ)-unbounded. In particular, every μ-guessing model is internally (ω_1, μ)-unbounded.*

Proof. The proof is quite similar to that of [10, Theorem 1.4]. Suppose for sake of contradiction that M is not internally (ν^+, μ)-unbounded, and fix a set $z \in M$ and an $x \in \mathscr{P}_{\nu^+}(z \cap M)$ such that there is no $y \in M$ such that $|y| < \mu$ and $x \subseteq y$. Since ${}^{<\nu}M \subseteq M$, we know that $|x| = \nu$; enumerate it as $\langle a_\beta \mid \beta < \nu \rangle$. For each $\gamma < \nu$, let $b_\gamma := \{a_\beta \mid \beta < \gamma\}$. By the closure of M, we have $b_\gamma \in M$. Let $B := \{b_\gamma \mid \gamma < \nu\}$, and note that $B \subseteq \mathscr{P}_\nu z \in M$ and $B \subseteq M$.

We claim that B is (μ, M)-approximated. To this end, fix an arbitrary $w \in M \cap \mathscr{P}_\mu(\mathscr{P}_\nu z)$, and note that $\bigcup w \in \mathscr{P}_\mu z \cap M$. If $w \cap B$ had cardinality ν, then we would have $b_\gamma \in w$ for unboundedly many $\gamma < \nu$, which would then imply that $x \subseteq \bigcup w$, contradicting our choice of x. Therefore, $w \cap B$ has cardinality less than ν, so, by the closure of M, we have $w \cap B \in M$.

Since w was arbitrary, it follows that B is (μ, M)-approximated. Therefore, B is M-guessed, so we can fix $d \in M$ such that $d \cap M = B$. By elementarity, we have $d \subseteq \mathscr{P}_\nu z$. Moreover, since $|B| = \nu$ and $\nu + 1 \subseteq M$, it follows again from elementarity that $|d| = \nu$, and therefore that $d = B$. But then $\bigcup d \in M$ and $\bigcup d = \bigcup B = x$, again contradicting our choice of x. $\square$

By Proposition 4.11 and Remark 2.7, the assumption of $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^{++})$ in the following theorem follows immediately from $\mathsf{GMP}(\kappa, \kappa, \lambda^{++})$ and hence also from the stronger $\mathsf{GMP}(\omega_1, \kappa, \geq\kappa)$, which is equivalent to ISP_κ (cf. Remark 2.5).

Theorem 4.12. *Suppose that $\kappa < \lambda$ are uncountable cardinals, with κ regular and λ singular of countable cofinality. Let*

$$\mathcal{Y} := \{M \in \mathscr{P}_\kappa H(\lambda^{++}) \mid M \text{ is } (\omega_1, \lambda)\text{-internally unbounded}\},$$

and suppose that $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^{++})$ holds. Then $\mathsf{CP}(\mathcal{D})$ holds for every uniform, transitive ω-covering matrix $\mathcal{D}$ for λ^+. Moreover, if $m(\omega, \mu) \leq \lambda^+$ for all $\mu < \lambda$, then $m(\omega, \lambda) = \lambda^+$.

Proof. Fix a uniform, transitive ω-covering matrix $\mathcal{D}$ for λ^+. For all $\alpha < \beta < \lambda^+$, let $j_{\alpha\beta}$ be the least $j < \omega$ such that $\alpha \in D(j, \beta)$. For each $\beta < \lambda^+$ and each $i < \omega$, define a function $g_{\beta,i} : \beta \to \omega$ by letting $g_{\beta,i}(\alpha) := \max\{j_{\alpha\beta}, i\}$ for all $\alpha < \beta$.

By Lemma 4.6, for each $X \in [\lambda^+]^{<\lambda}$, we can fix $\gamma_X < \lambda^+$ such that, for all $\beta \in [\gamma_X, \lambda^+)$, there is $i < \omega$ such that, for all $j \in [i, \theta)$, we have $X \cap D(j, \beta) = X \cap D(j, \gamma_X)$. For each $X \in [\lambda^+]^{<\lambda}$ and each $i < \omega$, define a function $h_{X,i} : X \to \omega$ by letting $h_{X,i} := g_{\gamma_X, i} \restriction X$. Also, let $\pi_0 : \lambda^+ \times \omega \to \lambda^+$ and $\pi_1 : \lambda^+ \times \omega \to \omega$ be the projection maps. We view functions of the form $g_{\beta,i}$ and $h_{X,i}$ as subsets of $\lambda^+ \times \omega$ in the natural way.

Let $S := \{N \in \mathscr{P}_\kappa H(\lambda^{++}) \mid N \prec H(\lambda^{++}) \text{ and } \mathrm{cf}(\sup(N \cap \lambda^+)) > \omega\}$. Since $\mathsf{wAGP}_{\mathcal{Y}}(\kappa, \kappa, \lambda^{++})$ holds, we can find $M \in \mathcal{Y}$ such that

- $\lambda^+ \times \omega \in M$;
- for all $y \in M$, we have $y \cap (\lambda^+ \times \omega), \sup(y \cap \lambda^+) \in M$;
- for all $y \in M \cap \mathscr{P}_\lambda(\lambda^+ \times \omega)$ and every $i < \omega$, we have $\gamma_{\pi_0[y]}, h_{\pi_0[y],i} \in M$ and, if $|y| < \kappa$, then $y \subseteq M$;
- $(M, \lambda^+ \times \omega)$ is almost κ-guessed by S.

Let $\delta := \sup(M \cap \lambda^+)$.

Claim 4.13. *There is $i < \omega$ such that $g_{\delta,i}$ is (κ, M)-approximated.*

Proof. Suppose not. Then, for every $i < \omega$, we can fix $y_i \in M \cap \mathscr{P}_\kappa(\lambda^+ \times \omega)$ such that there is no $z \in M$ for which $g_{\delta,i} \cap y_i = z \cap y_i$. Let $y := \bigcup_{i<\omega} y_i$. Since $M \in \mathcal{Y}$, we can find $w \in M$ such that $y \subseteq w$ and $|w| < \lambda$. By our choice of M, we can assume that $w \subseteq \lambda^+ \times \omega$. Let $X := \pi_0[w]$. Again by our choice of M, we have $\gamma_X \in M$ and $h_{X,i} \in M$ for all $i < \omega$. In particular, $\gamma_X < \delta$. We can therefore find $i < \omega$ such that, for all $j \in [i, \omega)$, we have $X \cap D(j, \delta) = X \cap D(j, \gamma_X)$. Unraveling the definitions, this implies that $g_{\delta,i} \restriction X = h_{X,i}$. But then $g_{\delta,i} \cap y_i = h_{X,i} \cap y_i$, and we have $h_{X,i} \in M$, contradicting our choice of y_i. □

Since $(M, \lambda^+ \times \omega)$ is almost κ-guessed by S, we can find $N \in S$ such that $\lambda^+ \times \omega \in N \subseteq M$ and an $e \in N$ such that $e \cap N = g_{\delta,i} \cap N$. By elementarity, e is a function from λ^+ to ω. Let $\gamma := \sup(N \cap \lambda^+)$. Since $\mathrm{cf}(\gamma) > \omega$, we can find $j \geq i$ such that $\{\alpha \in N \cap \lambda^+ \mid e(\alpha) = j\}$ is unbounded in γ. Let $H := \{\alpha < \lambda^+ \mid e(\alpha) = j\}$. By elementarity, H is unbounded in λ^+. Moreover, for every $x \in [H]^{\aleph_0} \cap N$, we have $x \subseteq D(j, \delta)$, so, by elementarity,

$$N \models \exists \beta < \lambda^+ \ (x \subseteq D(j, \beta))$$

But then, by another application of elementarity, we have

$$H(\lambda^{++}) \models \forall x \in [H]^{\aleph_0} \ \exists \beta < \lambda^+ \ (x \subseteq D(j, \beta)).$$

In particular, H witnesses $\mathsf{CP}(\mathcal{D})$.

For the "moreover" clause, suppose that $m(\omega, \mu) \leq \lambda^+$ for all $\mu < \lambda$. We will show that $m(\omega, \lambda^+) = \lambda^+$. Fix a uniform, transitive ω-covering matrix $\mathcal{D}$ for λ^+

such that $\beta_{\mathcal{D}} = \lambda$. By hypothesis, for all $i < \omega$ and $\beta < \lambda^+$, we can fix a set $\mathcal{Y}(i,\beta) \subseteq [D(i,\beta)]^\omega$ such that $|\mathcal{Y}(i,\beta)| \leq \lambda^+$ and, for all $x \in [D(i,\beta)]^\omega$, there is $y \in \mathcal{Y}(i,\beta)$ such that $|x \cap y| = \omega$. Let $H \in [\lambda^+]^{\lambda^+}$ witness $\mathsf{CP}(\mathcal{D})$, let $\pi : H \to \lambda^+$ be the unique order-preserving bijection, and let

$$\mathcal{Y} := \{\pi[x \cap H] \mid i < \omega,\ \beta < \lambda^+,\ x \in \mathcal{Y}(i,\beta)\}.$$

Clearly, we have $\mathcal{Y} \subseteq [\lambda^+]^\omega$ and $|\mathcal{Y}| = \lambda^+$. To see that $\mathcal{Y}$ witnesses $m(\omega,\lambda^+) = \lambda^+$, fix an arbitrary $x \in [\lambda^+]^\omega$. Let $\bar{x} = \pi^{-1}[x]$. Then $\bar{x} \in [H]^\omega$, so we can fix $i < \omega$ and $\beta < \lambda^+$ such that $\bar{x} \subseteq D(i,\beta)$. By our choice of $\mathcal{Y}(i,\beta)$, there is $\bar{y} \in \mathcal{Y}(i,\beta)$ such that $|\bar{x} \cap \bar{y}| = \omega$. Then $y := \pi[\bar{y} \cap H]$ is in $\mathcal{Y}$ and $|x \cap y| = \omega$, as desired. □

In [2], Cox and Krueger prove that GMP is consistent with arbitrarily large values of the continuum. In particular, their methods allow for the construction of a model in which GMP holds and the continuum is a singular cardinal of cofinality ω_1. In such a model, we necessarily have $2^\omega < 2^{\omega_1}$, so GMP, in contrast with, for instance, MA_{ω_1}, does not imply that $2^{\omega_1} = 2^\omega$. However, as we now show, it turns out that GMP does have a significant effect on 2^{ω_1}, in fact forcing it to be as small as possible relative to the value of 2^ω.

The following corollary, together with the ensuing remark, yields clause (1) of Theorem B. Similar corollaries can readily be obtained about the influence of principles of the form $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\mu,\mu^+,\lambda^{++})$ on the relationship between 2^μ and $2^{<\mu}$ under appropriate hypotheses about the values of $m(\nu,\mu^+)$ for $\nu \leq \mu$, where μ is an arbitrary regular uncountable cardinal.

Corollary 4.14. *Suppose that $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\lambda^{++})$ holds for all singular λ of countable cofinality, where*

$$\mathcal{Y}_\lambda := \{M \in \mathscr{P}_{\omega_2} H(\lambda^{++}) \mid M \textit{ is } (\omega_1,\lambda)\textit{-internally unbounded}\}.$$

Then, for every uncountable cardinal μ, we have

$$m(\omega,\mu) = \begin{cases} \mu & \textit{if } \mathrm{cf}(\mu) \neq \omega \\ \mu^+ & \textit{if } \mathrm{cf}(\mu) = \omega. \end{cases}$$

In particular,

$$2^{\omega_1} = \begin{cases} 2^\omega & \textit{if } \mathrm{cf}(2^\omega) \neq \omega_1 \\ (2^\omega)^+ & \textit{if } \mathrm{cf}(2^\omega) = \omega_1. \end{cases}$$

Proof. The proof is by induction on μ. The base case of $\mu = \omega_1$ is trivial. Suppose that $\mu = \nu^+$ and we know that $m(\omega,\nu) \leq \mu$. Then, by [17, Proposition 2.4(v)], we have $m(\omega,\mu) = \max\{\mu, m(\omega,\nu)\} = \mu$. Suppose next that μ is a limit cardinal of uncountable cofinality and $m(\omega,\nu) < \mu$ for all $\nu < \mu$. Then [17, Proposition 2.4(vi)] implies that $m(\omega,\mu) = \sup\{m(\omega,\nu) \mid \nu < \mu\} = \mu$. Finally, suppose that μ is a singular cardinal of countable cofinality and $m(\omega,\nu) < \mu$ for all $\nu < \mu$. Then the "moreover" clause of Theorem 4.12 implies that $m(\omega,\mu) = \mu^+$.

Now [17, Theorem 1.1] implies that, for all infinite cardinals $\sigma < \mu$, we have

$$m(\sigma,\mu) = \begin{cases} \mu & \text{if } \mathrm{cf}(\mu) \neq \mathrm{cf}(\sigma) \\ \mu^+ & \text{if } \mathrm{cf}(\mu) = \mathrm{cf}(\sigma), \end{cases}$$

as desired.

To see the "in particular" clause, first note that, as proven in [13, Theorem 9.3], $\mathsf{wAGP}(\omega_2)$ implies $\neg\mathsf{wKH}$, so, *a fortiori*, the hypothesis of the corollary also implies

$\neg\mathsf{wKH}$. By Lemma 3.5, we therefore have $2^{\omega_1} = m(\omega_1, 2^\omega)$, and the conclusion follows. □

Remark 4.15. As the proof of Corollary 4.14 makes clear, to obtain the "in particular" clause, we only require $\neg\mathsf{wKH}$ together with the conclusion of the corollary applied to $m(\omega_1, 2^\omega)$. It therefore suffices to assume $\mathsf{wAGP}(\omega_2)$ together with $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\lambda^{++})$ for all singular $\lambda < 2^\omega$ of countable cofinality, as reflected in the statement of Theorem B.

4.3. **Pseudopowers.** We end this section by investigating the effect of guessing model principles on pseudopowers, leading up to the proof of Theorem A. We first recall some necessary definitions; for efficiency, we opt to give these definitions in the specific settings in which we will need them rather than in full generality.

Definition 4.16. Suppose that θ is an infinite regular cardinal and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is a sequence of regular cardinals. We let $\prod \vec{\mu}$ denote $\prod_{i<\theta} \mu_i$. If $f, g \in \prod \vec{\mu}$, then we write $f <^* g$ to indicate that $|\{i < \theta \mid g(i) \leq f(i)\}| < \theta$.

Recall that, if μ is a singular cardinal, then $\mathrm{pp}(\mu)$ denotes the *pseudopower* of μ. Since we will not need it here, we do not give the precise definition of $\mathrm{pp}(\mu)$, referring the reader to [17] for all necessary definitions or to [22] for a more encyclopedic treatment. Instead, we simply note here that we always have $\mathrm{pp}(\mu) \geq \mu^+$, and that the following consequence of $\mathrm{pp}(\mu) > \mu^+$ follows immediately from [17, Observation 4.4].

Lemma 4.17. *Suppose that μ is a singular cardinal and $\mathrm{pp}(\mu) > \mu^+$. Then there is an increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \mathrm{cf}(\mu) \rangle$ converging to μ such that $\mathrm{cf}\left(\prod \vec{\mu}, <^*\right) = \mu^{++}$.*

We also recall that Shelah's Strong Hypothesis (SSH) is the assertion that $\mathrm{pp}(\mu) = \mu^+$ for every singular cardinal μ. Given a cardinal κ, we say that SSH *holds above* κ if $\mathrm{pp}(\mu) = \mu^+$ for every singular cardinal $\mu > \kappa$. Recall also that the Singular Cardinals Hypothesis (SCH) is the assertion that, for every singular strong limit cardinal μ, we have $2^\mu = \mu^+$. As with SSH, we say that SCH holds above some cardinal κ if $2^\mu = \mu^+$ for every singular strong limit cardinal $\mu > \kappa$. As noted after Theorem 3.3, the characterization of SSH in terms of meeting numbers makes it clear that SSH implies SCH. It is not as immediately obvious that, for an arbitrary cardinal κ, SSH above κ implies SCH above κ, since the characterization of SSH above κ in terms of meeting numbers is not as clean as that of Theorem 3.3. Nonetheless, it is true, and follows from related results in [17] and [22]:

Proposition 4.18. *Let κ be an infinite cardinal such that SSH holds above κ. Then SCH holds above κ.*

Proof. Suppose not, and let $\mu > \kappa$ be the least witness to the failure of SCH above κ. In particular, μ is strong limit, $2^\mu = \mu^{\mathrm{cf}(\mu)} > \mu^+$, and, by Silver's Theorem [24], we must have $\mathrm{cf}(\mu) = \omega$. By the discussion after Theorem 3.3 above, we have $\mu^\omega = 2^\omega \cdot m(\omega, \mu) = m(\omega, \mu)$, so we have $m(\omega, \mu) > \mu^+$.

Recall that $\mathrm{cov}(\mu, \mu, \omega_1, 2)$ is the least cardinality of a set $\mathcal{X} \subseteq \mathscr{P}_\mu \mu$ such that, for every $a \in \mathscr{P}_{\omega_1} \mu$, there is $b \in \mathcal{X}$ such that $a \subseteq b$. By a straightforward argument (cf. [17, Proposition 2.4(viii)]), we have

$$m(\omega, \mu) \leq \max\{\mathrm{cov}(\mu, \mu, \omega_1, 2), \sup\{m(\omega, \chi) \mid \chi < \mu\}\}.$$

Since μ is strong limit, we have $m(\omega, \chi) < \mu$ for all $\chi < \mu$, and hence $m(\omega, \mu) \leq \mathrm{cov}(\mu, \mu, \omega_1, 2)$. Now note that

- $\mathrm{pp}(\chi) < \mu$ for every singular $\chi < \mu$ (because μ is strong limit and $\mathrm{pp}(\mu) \leq 2^\mu$ for all μ); and
- for all $\chi \in (\kappa, \mu)$, if χ is a singular cardinal of cofinality ω_1, then $\mathrm{pp}(\chi) = \chi^+$ (because SSH holds above κ).

Therefore, [22, §IX, Conclusion 1.8] implies that $\mathrm{pp}(\mu) = \mathrm{cov}(\mu, \mu, \omega_1, 2)$. Since $\mathrm{cov}(\mu, \mu, \omega_1, 2) \geq m(\omega, \mu) > \mu^+$, this contradicts the assumption that SSH holds above κ. □

We now prove the main result of this subsection, indicating the impact of instances of $\mathsf{CP}(\mathcal{D})$ on values of the form $\mathrm{cf}\,(\prod \vec{\mu}, <^*)$.

Theorem 4.19. *Suppose that μ is a singular cardinal, $\theta = \mathrm{cf}(\mu)$, and $\vec{\mu} = \langle \mu_i \mid i < \theta \rangle$ is an increasing sequence of regular cardinals converging to μ. Suppose moreover that there is a θ-covering matrix for μ^+, $\mathcal{D} = \langle D(i, \beta) \mid i < \theta,\ \beta < \mu^+ \rangle$, such that $\mathcal{D}$ respects $\vec{\mu}$ and $\mathsf{CP}(\mathcal{D})$ holds. Then $\mathrm{cf}\,(\prod \vec{\mu}, <^*) = \mu^+$.*

Proof. Let $\mathcal{D}$ be as in the statement of the proposition, and let $A \subseteq \mu^+$ witness $\mathsf{CP}(\mathcal{D})$. Let $\langle \gamma_\eta \mid \eta < \mu \rangle$ enumerate the first μ-many elements of A in increasing order and, for each $i < \theta$, let $\delta_i := \sup\{\gamma_\eta \mid \eta < \mu_i\}$. Note that $\mathrm{cf}(\delta_i) = \mu_i$. For each $\beta < \mu^+$, define a function $g_\beta \in \prod \vec{\mu}$ as follows: for each $i < \theta$, let $g_\beta(i)$ be the least $\eta < \mu_i$ such that $\sup(D(i, \beta) \cap \delta_i) < \gamma_\eta$. This is well-defined, since $\mathcal{D}$ respects $\vec{\mu}$ and therefore $|D(i, \beta)| < \mu_i = \mathrm{cf}(\delta_i)$.

We claim that $\{g_\beta \mid \beta < \mu^+\}$ is cofinal in $(\prod \vec{\mu}, <^*)$. To this end, fix $f \in \prod \vec{\mu}$. Let $x = \{\gamma_{f(i)} \mid i < \theta\}$. Since $x \in [A]^{\leq \theta}$ and A witnesses $\mathsf{CP}(\mathcal{D})$, we can fix $i < \theta$ and $\beta < \mu^+$ such that $x \subseteq D(i, \beta)$, and hence $x \subseteq D(j, \beta)$ for all $j \in [i, \theta)$. In particular, for every $j \in [i, \theta)$, we have $\gamma_{f(j)} \in D(j, \beta) \cap \delta_j$, and hence $f(j) < g_\beta(j)$. Therefore, $f <^* g_\beta$, as desired. □

We obtain Theorem A as a corollary.

Proof of Theorem A. Suppose that $\mathsf{wAGP}_{\mathcal{Y}_\lambda}(\kappa, \kappa, \lambda^{++})$ holds for all singular $\lambda > \kappa$ of countable cofinality. By [22, §2, Claim 2.4], if μ is a singular cardinal of uncountable cardinality and the set of singular cardinals $\nu < \mu$ for which $\mathrm{pp}(\nu) = \nu^+$ is stationary in μ, then $\mathrm{pp}(\mu) = \mu^+$. Therefore, to establish SSH above κ, it suffices to show that $\mathrm{pp}(\mu) = \mu^+$ for all singular $\mu > \kappa$ of countable cofinality.

Fix such a μ. By Lemma 4.17, it suffices to show that $\mathrm{cf}\,(\prod \vec{\mu}, <^*) = \mu^+$ for every increasing sequence of regular cardinals $\vec{\mu} = \langle \mu_i \mid i < \omega \rangle$ converging to μ. Fix such a sequence $\vec{\mu}$. By Lemma 4.4, we can fix a uniform, transitive ω-covering matrix $\mathcal{D}$ for μ^+ that respects $\vec{\mu}$. By Theorem 4.12, $\mathsf{CP}(\mathcal{D})$ holds, and then, by Theorem 4.19, we have $\mathrm{cf}\,(\prod \vec{\mu}, <^*) = \mu^+$, as desired. □

5. Special trees and random reals

In this section, we take a slight detour to prove a variation of a theorem of Laver [15] concerning special trees in random extensions of models of forcing axioms. At the end of the section, we will rejoin our main narrative path by connecting this result with Cox and Krueger's *indestructible guessing model principle* [3], a strengthening of GMP that will also provide part of the motivation for the results in Section 6.

Recall that a tree T of height ω_1 is *special* if there is a function $f : T \to \omega$ such that, for all $s, t \in T$, if $s <_T t$, then $f(s) \neq f(t)$. It is immediate that a special tree cannot have an uncountable branch. Baumgartner introduced a generalization of this notion of specialness that can also be satisfied by trees of height ω_1 that have uncountable branches; this notion was used to prove that PFA implies $\neg\mathsf{wKH}$. In order to avoid confusion with the more familiar notion of specialness, we will call Baumgartner's generalization *B-specialness*.

Definition 5.1 ([1, §7])**.** Suppose that T is a tree of height ω_1. We say that T is *B-special* if there is a function $f : T \to \omega$ such that, for all $s, t, u \in T$, if $f(s) = f(t) = f(u)$ and $s <_T t, u$, then t and u are $<_T$-comparable.

This can indeed be seen as a generalization of the notion of specialness, since, if T is a tree of height ω_1 with no uncountable branches, then T is special if and only if T is B-special. It is well-known that MA_{ω_1} implies that every tree of height and size ω_1 with no uncountable branch is special. An elaboration of this argument, also due to Baumgartner [1, Theorem 7.10] shows that PFA implies that every tree of height and size ω_1 is B-special.

In [15], Laver proves that, if one forces with a measure algebra over any model of MA_{ω_1}, then, in the resulting forcing extension, it remains true that every tree of height and size ω_1 with no uncountable branch is special. In particular, this resolved positively the question of whether Suslin's Hypothesis is consistent with $\mathrm{cf}(2^\omega) = \omega_1$. In this section, we modify Laver's argument to prove an analogous result indicating that, if one forces with a measure algebra over any model of PFA, then, in the resulting forcing extension, every tree of height and size ω_1 is B-special.

The following proposition will be useful.

Proposition 5.2 ([3, Proposition 4.3])**.** *Suppose that T is a B-special tree of height ω_1, and suppose that W is an outer model of V with $(\omega_1)^W = (\omega_1)^V$. Then every uncountable branch of T that is in W is also in V.*

We will also need the following definition and lemma.

Definition 5.3. Suppose that T is a tree and B is a set of cofinal branches of T. A function $g : B \to T$ is called a *Baumgartner function* if g is injective and

(1) for all $b \in B$, we have $g(b) \in b$;
(2) for all $b, b' \in B$, if $g(b) < g(b')$, then $g(b') \notin b$.

If B is small, then a Baumgartner function with domain B always exists:

Lemma 5.4 ([1, Lemma 7.6])**.** *Suppose that κ is a regular cardinal, T is a tree of height κ, B is a set of cofinal branches of T, and $|B| \leq \kappa$. Then there is a Baumgartner function $g : B \to T$.*

Theorem 5.5. *Suppose that* PFA *holds, κ is an infinite cardinal, and $\mathbb{B}$ is the measure algebra on 2^κ, with associated measure μ. Then, in $V^{\mathbb{B}}$, every tree of height and size ω_1 is B-special.*

Proof. For notational simplicity, assume that $\kappa \geq \omega_2$; the same ideas will work if $\kappa \in \{\omega, \omega_1\}$. Fix a $\mathbb{B}$-name $\dot{T}$ for a tree of height and size ω_1. Note that, since $\Vdash_{\mathbb{B}}$ "$2^\omega = 2^{\omega_1} = \kappa$", we have $\Vdash_{\mathbb{B}}$ "$\dot{T}$ has at most κ-many uncountable branches". Let $\mathbb{C} = \mathrm{Coll}(\omega_1, \kappa)$ (as defined in V). Since $\mathbb{B}$ has the c.c.c. and $\mathbb{C}$ is ω_1-closed, [27, Lemma 6] implies that every uncountable branch of $\dot{T}$ in the extension by $\mathbb{B} \times \mathbb{C}$ is already in the extension by $\mathbb{B}$. In particular, in $V^{\mathbb{C}}$, we have

$\Vdash_{\mathbb{B}}$ "$\dot{T}$ has at most ω_1-many uncountable branches". Therefore, working in $V^{\mathbb{C}}$ and letting $\dot{B}$ be a $\mathbb{B}$-name for the set of all uncountable branches through $\dot{T}$, Lemma 5.4 implies that we can find a $\mathbb{B}$-name $\dot{g} : \dot{B} \to \dot{T}$ that is forced to be a Baumgartner function.

Still working in $V^{\mathbb{C}}$, let $\dot{T}_0$ be a $\mathbb{B}$-name for the set

$$\{t \in \dot{T} \mid \exists b \in \dot{B}\ \dot{g}(b) <_{\dot{T}} t \in \dot{b}\},$$

and let $\dot{T}_1$ be a $\mathbb{B}$-name for $\dot{T} \setminus \dot{T}_0$. Then $\dot{T}_1$ is forced to be a subtree of $\dot{T}$ (with the induced ordering), and, since a tail of every element of $\dot{B}$ is forced to be in $\dot{T}_0$, we have $\Vdash_{\mathbb{B}}$ "$\dot{T}_1$ has no uncountable branch".

In V, $\mathbb{C}$ is ω_1-closed and therefore does not add any new codes for Borel subsets of 2^κ. As a result, in $V^{\mathbb{C}}$, the measure algebra on 2^κ is isomorphic to $\mathbb{B}$, so, when working in $V^{\mathbb{C}}$, we can still think of $\mathbb{B}$ as being the measure algebra on 2^κ. Therefore, the proof of the main result of [15] implies that, in $V^{\mathbb{C}}$, there is a c.c.c. forcing poset $\mathbb{P}$ that adds a $\mathbb{B}$-name for a specializing function for $\dot{T}_1$. Let us recall how this poset is defined.

Work in $V^{\mathbb{C}}$. Without loss of generality, we may assume that the underlying set of $\dot{T}$ is forced to be ω_1, and therefore the underlying sets of $\dot{T}_0$ and $\dot{T}_1$ are forced to be subsets of ω_1. We can also assume that $\Vdash_{\mathbb{B}}$ "$\forall \alpha, \beta < \omega_1 (\alpha <_{\dot{T}} \beta \Rightarrow \alpha < \beta)$".

For all $\alpha < \omega_1$, let $A_\alpha \subseteq \mathbb{B}$ be a maximal antichain of conditions deciding the truth value of the statement "$\alpha \in \dot{T}_1$". For all $\alpha < \beta < \omega_1$, let $A'_{\alpha\beta}$ be a maximal antichain of conditions $b \in \mathbb{B}$ such that one of the following holds:

- b forces at least one of α and β to be in $\dot{T}_0$; or
- b forces both α and β to be in $\dot{T}_1$ and b decides the truth value of the statement "$\alpha <_{\dot{T}_1} \beta$".

For all $\gamma < \omega_1$, let $\mathbb{B}_\gamma$ be the complete subalgebra of $\mathbb{B}$ generated by $\bigcup_{\alpha \leq \gamma} A_\alpha \cup \bigcup_{\alpha < \beta \leq \gamma} A'_{\alpha\beta}$. Since $\mathbb{B}$ has the c.c.c., each $\mathbb{B}_\gamma$ is countably generated and is thus isomorphic to the measure algebra on 2^ω. We can therefore fix a countable subset $\mathbb{B}^*_\gamma \subseteq \mathbb{B}_\gamma$ such that, for all $b \in \mathbb{B}_\gamma$ and all $\varepsilon > 0$, there is $b^* \in \mathbb{B}^*_\gamma$ such that $\mu(b^* - b) < \varepsilon \cdot \mu(b^*)$.

We now define our poset $\mathbb{P}$. The conditions in $\mathbb{P}$ are all functions p with domains of the form $E_p \times W_p$, where $E_p \in [\omega_1]^{<\omega}$ and $W_p \in [\omega]^{<\omega}$, such that

- for all $(\alpha, n) \in E_p \times W_p$, either $p(\alpha, n) = 0$ or $p(\alpha, n) \in \mathbb{B}_\alpha$ and $\mu(p(\alpha, n)) > \frac{1}{2}$;
- for all $\alpha < \beta$ in E_p and all $n \in W_p$, either $p(\alpha, n) \wedge p(\beta, n) = 0$ or $p(\alpha, n) \wedge p(\beta, n) \Vdash_{\mathbb{B}}$ "$\alpha \not<_{\dot{T}_1} \beta$".

Note that, for a condition $b \in \mathbb{B}$ and ordinals $\alpha < \beta < \omega_1$, if we write $b \Vdash_{\mathbb{B}}$ "$\alpha <_{\dot{T}_1} \beta$", then implicit in this assertion is the fact that b forces both α and β to be in $\dot{T}_1$. Therefore, one of the ways in which we could have $p(\alpha, n) \wedge p(\beta, n) \Vdash_{\mathbb{B}}$ "$\alpha \not<_{\dot{T}_1} \beta$" in the above bullet point is for $p(\alpha, n) \wedge p(\beta, n)$ to force either α or β to be in $\dot{T}_0$.

If $p, q \in \mathbb{P}$, then we let $q \leq_{\mathbb{P}} p$ if and only if $E_q \supseteq E_p$, $W_q \supseteq W_p$, $q(\alpha, n) \leq_{\mathbb{B}} p(\alpha, n)$ for all $(\alpha, n) \in E_p \times W_p$, and, for all such (α, n), if $p(\alpha, n) > 0$, then $q(\alpha, n) > 0$.

By [15, Lemma 3], $\mathbb{P}$ has the c.c.c. in $V^{\mathbb{C}}$. (Our poset $\mathbb{P}$ has some minor cosmetic differences from the poset considered in [15] due to the fact that we are adding a name for a specializing function for a *subtree* of $\dot{T}$ rather than the entire tree, but all of the arguments of [15] go through without change to our setting.) In V, let $\dot{\mathbb{P}}$ be a $\mathbb{C}$-name for $\mathbb{P}$. Then $\mathbb{C} * \dot{\mathbb{P}}$ is proper. Note that, in $V^{\mathbb{C}}$, every condition of

$\dot{\mathbb{P}}$ is in fact in V, since it is a function from a finite set of pairs of ordinals into $\mathbb{B}$. Therefore, in V, the set of conditions $(c, \dot{p})$ such that there is a function $p \in V$ for which $c \Vdash_{\mathbb{C}}$ "$\dot{p} = p$" is dense in $\mathbb{C} * \dot{\mathbb{P}}$, so we will assume that we are working exclusively with such conditions and will write (c, p) instead of $(c, \dot{p})$.

We now isolate a collection of ω_1-many dense subsets of $\mathbb{C} * \dot{\mathbb{P}}$ to which we will apply PFA. First note that, in $V^{\mathbb{C} \times \mathbb{B}}$, for all $\alpha \in T_0$, the fact that g is a Baumgartner function implies that there is a unique uncountable branch b of T such that $g(b) <_T \alpha \in b$. Denote the value $g(b)$ for this unique branch b by η_α, and note that $\eta_\alpha \in T_1$. If $\alpha \in T_1$, then let $\eta_\alpha = 0$. In V, let $\dot{\eta}_\alpha$ be a $\mathbb{C} \times \mathbb{B}$-name for η_α.

For each $\alpha < \omega_1$, let D_α be the set of $c \in \mathbb{C}$ for which there exists a maximal antichain $A(c, \alpha)$ of $\mathbb{B}$ such that, for every $b \in A(c, \alpha)$, $(c, b) \in \mathbb{C} \times \mathbb{B}$ decides the truth value of the statement "$\alpha \in \dot{T}_1$" and decides the value of $\dot{\eta}_\alpha$. Since $\mathbb{C}$ is ω_1-closed and $\mathbb{B}$ has the c.c.c., D_α is dense in $\mathbb{C}$. Let D^*_α be the set of $(c, p) \in \mathbb{C} * \dot{\mathbb{P}}$ such that $c \in D_\alpha$. Then D^*_α is dense in $\mathbb{C} * \dot{\mathbb{P}}$.

Recall that, in $V^{\mathbb{C}}$, for every $\gamma < \omega_1$, we constructed $\mathbb{B}_\gamma$ to be a countably generated complete subalgebra of $\mathbb{B}$ and also specified a countable subset $\mathbb{B}^*_\gamma \subseteq \mathbb{B}_\gamma$ such that, for all $b \in \mathbb{B}_\gamma$ and every $\varepsilon > 0$, there is $b^* \in \mathbb{B}^*_\gamma$ such that $\mu(b^* - b) < \varepsilon \cdot \mu(b^*)$. In V, let $\dot{h}_\gamma$ be a $\mathbb{C}$-name for a bijection from ω to $\mathbb{B}^*_\gamma$, and let $\dot{\mathbb{A}}_\gamma$ be a $\mathbb{C}$-name for a countable generating set for $\mathbb{B}_\gamma$. Since $\mathbb{C}$ is ω_1-closed, $\dot{h}_\gamma$ and $\dot{\mathbb{A}}_\gamma$ are forced to be elements of the ground model. For all $\gamma < \omega_1$ and $k < \omega$, let $E^*_{\gamma,k}$ be the set of $(c, p) \in \mathbb{C} * \dot{\mathbb{P}}$ such that

- c decides the value of $\dot{h}_\gamma$, say as some function $h_\gamma : \omega \to \mathbb{B}$;
- c decides the value of $\dot{\mathbb{A}}_\gamma$;
- there is an $n < \omega$ such that $(\gamma, n) \in \mathrm{dom}(p)$, $\mu(p(\gamma, n)) > \frac{1}{2}$, and $\mu(p(\gamma, n) - h_\gamma(k)) < \frac{1}{2} - \frac{1}{2}\mu(h_\gamma(k)))$.

We claim that $E^*_{\gamma,k}$ is dense in $\mathbb{C} * \dot{\mathbb{P}}$. To see this fix an arbitrary $(c_0, p_0) \in \mathbb{C} * \dot{\mathbb{P}}$. First find a condition $c_1 \leq_{\mathbb{C}} c_0$, a function $h_\gamma : \omega \to \mathbb{B}$, and a set $\mathbb{A}_\gamma$ such that $c_1 \Vdash_{\mathbb{C}}$ "$\dot{h}_\gamma = h_\gamma$ and $\dot{\mathbb{A}}_\gamma = \mathbb{A}_\gamma$". Then, fix an $n \in \omega \setminus W_{p_0}$. Let $\dot{p}_1$ be a $\mathbb{C}$-name for a condition in $\dot{\mathbb{P}}$ that is forced by c_1 to have the following properties:

- $E_{\dot{p}_1} = E_{p_0} \cup \{\gamma\}$;
- $W_{\dot{p}_1} = W_{p_0} \cup \{n\}$;
- for all $(\alpha, m) \in E_{p_0} \times W_{p_0}$, $\dot{p}_1(\alpha, m) = p_0(\alpha, m)$;
- for all $(\alpha, m) \in (E_{\dot{p}_1} \times W_{\dot{p}_1}) \setminus (E_{p_0} \times W_{p_0})$, if $(\alpha, m) \neq (\gamma, n)$, then $\dot{p}_1(\alpha, m) = 0$;
- $\dot{p}_1(\gamma, n)$ is an element b of $\mathbb{B}_\gamma$ such that
  - $h_\gamma(k) \leq b$;
  - $\frac{1}{2} < \mu(b) < \frac{1}{2} + \frac{1}{2}\mu(h_\gamma(k))$.

It is possible to satisfy the final requirement above due to the fact that $\mathbb{B}_\gamma$ is forced to be atomless. Finally, find $c_2 \leq_{\mathbb{C}} c_1$ deciding the value of $\dot{p}_1$ as some function $p_1 \in V$. Then $(c_2, p_1) \in E^*_{\gamma,k}$ and $(c_2, p_1) \leq_{\mathbb{C} * \dot{\mathbb{P}}} (c_0, p_0)$.

Now apply PFA to find, in V, a filter $G \subseteq \mathbb{C} * \dot{\mathbb{P}}$ that meets D^*_α for all $\alpha < \omega_1$ and $E^*_{\gamma,k}$ for all $\gamma < \omega_1$ and $k < \omega$. Also, let $H \subseteq \mathbb{B}$ be a V-generic filter. Working in $V[H]$, let T denote the realization of $\dot{T}$. We will use G and H to construct a function $f : T \to \omega$ witnessing that T is B-special.

We first define subtrees $T^*_0, T^*_1 \subseteq T$ as follows. For every $\alpha < \omega_1$, find a condition $(c^*_\alpha, p^*_\alpha) \in G \cap D^*_\alpha$. Then, in V, $A(c^*_\alpha, \alpha)$ was a maximal antichain in $\mathbb{B}$, so there is a unique $b^*_\alpha \in H \cap A(c^*_\alpha, \alpha)$. Moreover, we know that (c^*_α, b^*_α) decides the truth value of the statement "$\alpha \in \dot{T}_1$". If (c^*_α, b^*_α) forces α to be in $\dot{T}_1$, then put α into T^*_1; otherwise, put α into T^*_0. In addition, (c^*_α, b^*_α) decides the value of $\dot{\eta}_\alpha$; let η^*_α be this value. In V, let $\dot{T}^*_0$ and $\dot{T}^*_1$ be $\mathbb{B}$-names for T^*_0 and T^*_1, respectively.

Claim 5.6. *For all $\alpha < \beta \in T^*_0$, if $\alpha \not<_T \beta$, then either $\eta^*_\beta \not<_T \alpha$ or $\eta^*_\alpha \not<_T \beta$.*

Proof. Fix $\alpha < \beta \in T^*_0$, and suppose for sake of contradiction that $\alpha \not<_T \beta$ but $\eta^*_\beta <_T \alpha$ and $\eta^*_\alpha <_T \beta$. Let c^* be the greatest lower bound of c^*_α and c^*_β in $\mathbb{C}$, and let $b^* \leq b^*_\alpha \wedge b^*_\beta$ be a condition in H forcing that $\alpha \not<_T \beta$, $\eta^*_\beta <_T \alpha$, and $\eta^*_\alpha <_T \beta$. Then, in V, $(c^*, b^*) \Vdash_{\mathbb{C}\times\mathbb{B}}$ "$\dot{\eta}_\alpha = \eta^*_\alpha$ and $\dot{\eta}_\beta = \eta^*_\alpha$. In particular, there are $\mathbb{B}$-names $\dot{d}_\alpha, \dot{d}_\beta$ for uncountable branches through $\dot{T}$ such that

$$(c^*, b^*) \Vdash_{\mathbb{C}\times\mathbb{B}} \text{“}\alpha \in \dot{d}_\alpha,\ \beta \in \dot{d}_\beta,\ \dot{g}(\dot{d}_\alpha) = \eta^*_\alpha,\ \dot{g}(\dot{d}_\beta) = \eta^*_\beta\text{”}.$$

Since b^* forces $\alpha \not<_T \beta$, it must be the case that (c^*, b^*) forces $\dot{d}_\alpha$ and $\dot{d}_\beta$ to be distinct branches of T, and hence $\eta^*_\alpha \neq \eta^*_\beta$. Since (c^*, b^*) forces $\eta^*_\alpha, \eta^*_\beta <_T \alpha$, it forces η^*_α and η^*_β to be $<_T$-comparable. Suppose that $\eta^*_\alpha < \eta^*_\beta$, in which case $(c^*, b^*) \Vdash_{\mathbb{C}\times\mathbb{B}}$ "$\eta^*_\alpha <_T \eta^*_\beta$". Since $\dot{g}$ is forced to be a Baumgartner function, it must be the case that $(c^*, b^*) \Vdash_{\mathbb{C}\times\mathbb{B}}$ "$\eta^*_\beta \notin \dot{d}_\alpha$". However, $(c^*, b^*) \Vdash_{\mathbb{C}\times\mathbb{B}}$ "$\eta^*_\beta <_{\dot{T}} \alpha \in \dot{d}_\alpha$", which is a contradiction. A symmetric argument yields a contradiction if $\eta^*_\beta < \eta^*_\alpha$, thus completing the proof of the claim. □

For each $\gamma < \omega_1$, since $G \cap E^*_{\gamma,0} \neq \emptyset$, we can find a condition $(c, p) \in G$ that decides the value of $\dot{h}_\gamma$ and $\dot{\mathbb{A}}_\gamma$, say as h_γ and $\mathbb{A}_\gamma$. Let $\mathbb{B}^*_\gamma$ be the subalgebra of $\mathbb{B}$ generated by $\mathbb{A}_\gamma$ (in V).

For each $\alpha \in T^*_1$ and each $n < \omega$, let

$$b_{\alpha,n} = \bigwedge\{p(\alpha, n) \mid (\alpha, n) \in \mathrm{dom}(p) \text{ and } \exists c \in \mathbb{C}\ (c, p) \in G\}.$$

Note that $b_{\alpha,n}$ is in $\mathbb{B}^*_\alpha$ (possibly equal to 0) and, if $b_{\alpha,n} > 0$, then $\mu(b_{\alpha,n}) \geq \frac{1}{2}$. In V, let $\dot{H}$ be the canonical $\mathbb{B}$-name for the generic filter.

Claim 5.7. *For every $\alpha \in T^*_1$, there is $n < \omega$ such that $b_{\alpha,n} > 0$ and $b_{\alpha,n} \in H$.*

Proof. Fix $\alpha \in T^*_1$, and suppose for sake of contradiction that there is $b \in H$ such that, in V, we have

- $b \Vdash_{\mathbb{B}}$ "$\alpha \in \dot{T}^*_1$";
- $b \Vdash_{\mathbb{B}}$ "there is no $n < \omega$ such that $b_{\alpha,n} > 0$ and $b_{\alpha,n} \in \dot{H}$".

In particular, for each $n < \omega$ such that $b_{\alpha,n} > 0$, it must be the case that $b_{\alpha,n}$ and b are incompatible. Since each $b_{\alpha,n}$ is in $\mathbb{B}^*_\alpha$, we can assume that b is in $\mathbb{B}^*_\alpha$ as well. We can therefore find $k < \omega$ such that $\mu(h_\alpha(k) - b) < \frac{1}{4}\mu(h_\alpha(k))$, and thus $\mu(h_\alpha(k) \wedge b) > \frac{3}{4}\mu(h_\alpha(k))$. Now find $(c, p) \in G \cap E^*_{\alpha,k}$. Then we can find $n < \omega$ such that $(\alpha, n) \in \mathrm{dom}(p)$, $\mu(p(\alpha, n)) > \frac{1}{2}$, and $\mu(p(\alpha, n) - h_\alpha(k)) < \frac{1}{2} - \frac{1}{2}\mu(h_\alpha(k))$. In particular, since $b_{\alpha,n} \leq p(\alpha, n)$ and $\mu(b_{\alpha,n}) \geq \frac{1}{2}$, we must have $\mu(b_{\alpha,n} \wedge h_\alpha(k)) > \frac{1}{2}\mu(h_\alpha(k))$. Altogether, this implies that $b_{\alpha,n}$ and b are compatible in $\mathbb{B}$, contradicting our choice of b. □

For each $\alpha \in T^*_1$, let $f(\alpha)$ be the least n such that $b_{\alpha,n} > 0$ and $b_{\alpha,n} \in H$. For each $\alpha \in T^*_0$, recall that we defined an ordinal η^*_α such that $\eta^*_\alpha <_T \alpha$ and $\eta^*_\alpha \in T^*_1$. For such α, let $f(\alpha) = f(\eta^*_\alpha)$.

It remains to show that f witnesses that T is B-special. We first claim that $f \restriction T_1^*$ witnesses that T_1^* is special. To this end, fix $\alpha, \beta \in T_1^*$ and suppose that $f(\alpha) = f(\beta) = n$. Then we have $b_{\alpha,n} \wedge b_{\beta,n} \in H$. Find $(c, p) \in G$ such that (α, n) and (β, n) are both in $\mathrm{dom}(p)$. Then, by the definition of $\mathbb{P}$ in $V^{\mathbb{C}}$, we must have

$$c \Vdash_{\mathbb{C}} \text{“}p(\alpha, n) \wedge p(\beta, n) \Vdash_{\mathbb{B}} \text{“}\alpha \not<_{\dot{T}_1} \beta\text{””}.$$

Since $\alpha, \beta \in T_1^*$, we can find an extension c^* of c such that $(c^*, p) \in G$ and $(c^*, b_\alpha^* \wedge b_\beta^*) \Vdash$ “$\alpha, \beta \in \dot{T}_1$”. Then $p(\alpha, n) \wedge p(\beta, n) \wedge b_\alpha^* \wedge b_\beta^*$ is in H and forces that $\alpha \not<_{\dot{T}} \beta$. Therefore, α and β are incomparable in T, so $f \restriction T_1^*$ really does witness that T_1^* is special.

Now suppose that $\alpha < \beta < \gamma < \omega_1$, $f(\alpha) = f(\beta) = f(\gamma) = n$, and $\alpha <_T \beta, \gamma$. Suppose first that $\beta \in T_1^*$. Since $f \restriction T_1^*$ witnesses that T_1^* is special, we must have $\alpha \in T_0^*$. But then we have $\eta_\alpha^* <_T \alpha <_T \beta$, $\eta_\alpha^* \in T_1^*$, and $f(\eta_\alpha^*) = f(\alpha) = f(\beta)$, which is again a contradiction. Thus, we must have $\beta \in T_0^*$; similarly, we also have $\gamma \in T_0^*$.

If $\alpha <_T \eta_\beta^*$, then we reach a contradiction exactly as in the previous paragraph, since $f(\alpha) = f(\beta) = f(\eta_\beta^*)$ and $\eta_\beta^* \in T_1^*$. Therefore, we must have $\eta_\beta^* \leq_T \alpha$. Similarly, $\eta_\gamma^* \leq_T \alpha$. But then $\eta_\beta^* <_T \gamma$ and $\eta_\gamma^* <_T \beta$, so, by Claim 5.6, it must be the case that $\beta <_T \gamma$. Therefore, f witnesses that T is B-special. $\square$

As a corollary, we can show that an “indestructible” version of the negation of the weak Kurepa Hypothesis is compatible with any possible value of the continuum except ω_1. More precisely:

Corollary 5.8. *Suppose that* PFA *holds, $\kappa \geq \omega_2$ is a cardinal of uncountable cofinality, and $\mathbb{B}$ is the measure algebra on 2^κ. Then, in $V^{\mathbb{B}}$, $2^\omega = \kappa$ and, for every tree T of size and height ω_1 and every outer model W of $V^{\mathbb{B}}$ such that $(\omega_1)^W = (\omega_1)^{V^{\mathbb{B}}}$, T has at most ω_1-many uncountable branches in W.*

Proof. The fact that $2^\omega = \kappa$ in $V^{\mathbb{B}}$ is a standard fact about random forcing. Let $T \in V^{\mathbb{B}}$ be a tree of size and height ω_1, and let W be an outer model of $V^{\mathbb{B}}$ with $(\omega_1)^W = (\omega_1)^{V^{\mathbb{B}}}$. By Theorem 5.5, T is B-special in $V^{\mathbb{B}}$. By [1, Theorem 7.4], it follows that T has at most ω_1-many uncountable branches in $V^{\mathbb{B}}$. By Proposition 5.2, every uncountable branch of T in W is already in V. Therefore, T has at most ω_1-many uncountable branches in W. $\square$

We now connect the results in this section back to the main subject of this paper. In [3], Cox and Krueger introduce the *indestructible guessing model property*:

Definition 5.9 ([3])**.** Let $\theta \geq \omega_2$ be a regular cardinal. $M \in \mathscr{P}_{\omega_2} H(\theta)$ is said to be an *indestructible ω_1-guessing model* if it is an ω_1-guessing model and remains an ω_1-guessing model in any forcing extension that preserves ω_1. $\mathsf{IGMP}(\theta)$ is the assertion that there are stationarily many indestructible ω_1-guessing models in $\mathscr{P}_{\omega_2} H(\theta)$. IGMP is the assertion that $\mathsf{IGMP}(\theta)$ holds for all regular $\theta \geq \omega_2$.

In [3], Cox and Krueger show that IGMP follows from the conjunction of the following two statements:

- for all regular $\theta \geq \omega_2$, there are stationarily many internally unbounded ω_1-guessing models in $\mathscr{P}_{\omega_2} H(\theta)$;
- every tree of size and height ω_1 with no cofinal branches is special.

In [10], Krueger proves that if $\theta \geq \omega_2$ is a regular cardinal and $N \prec H(\theta)$ is an ω_1-guessing model with $\omega_1 \subseteq N$, then N is internally unbounded. Therefore, IGMP follows from the conjunction of GMP and the assertion that all trees of size and height ω_1 with no cofinal branches are special; in particular, it follows from PFA.

IGMP has some consequences that GMP does not have. For example, it implies that there are no Suslin trees, whereas other work of Cox and Krueger [2] shows that GMP is compatible with the existence of a Suslin tree (we give another model for this conjunction in the next section by showing that GMP follows from the forcing axiom $\mathsf{PFA}(\mathsf{S})$).

By the argument of [3, Theorem 2.8], it is clear that $\mathsf{IGMP}(\omega_2)$ implies the indestructible version of $\neg\mathsf{wKH}$ isolated in Corollary 5.8 in the case in which W is a forcing extension of V. In Corollary 5.8, we saw that this indestructible version of $\neg\mathsf{wKH}$ is compatible with any possible value of the continuum, including values of cofinality ω_1. In [3], Cox and Krueger show that IGMP is compatible with any possible value of the continuum with cofinality at least ω_2. The combination of these two results naturally raises the following question:

Question 5.10. *Is* IGMP *compatible with* $\mathrm{cf}(2^\omega) = \omega_1$*? What about just* $\mathsf{IGMP}(\omega_2)$*?*

6. Forcing axioms for Suslin and almost Suslin trees

In this section, we continue investigations motivated by the study of IGMP, connecting it with the forcing axioms $\mathsf{PFA}(\mathsf{S})$ and $\mathsf{PFA}(\mathsf{T}^*)$, introduced by Todorcevic [25] and Krueger [11], respectively. In the process, we answer questions of Cox and Krueger [3] and Krueger [11]. Let us begin by recalling the relevant definitions.

Definition 6.1. Suppose that T is an ω_1-tree, i.e., a tree of height ω_1, all of whose levels are countable.

(1) T is an *Aronszajn tree* if it has no cofinal branches.
(2) T is a *Suslin tree* if it is an Aronszajn tree and has no uncountable antichains.
(3) T is an *almost Suslin tree* if it has no stationary antichains, i.e., no antichains $A \subseteq T$ for which the set $\{\mathrm{ht}(s) \mid s \in A\}$ is stationary in ω_1, where $\mathrm{ht}(s)$ denotes the level of s in T.

We note that an almost Suslin tree need not be an Aronszajn tree, though in this paper we will only be interested in almost Suslin trees that are Aronszajn. Throughout this section, in accordance with established notation, we will always use S to denote a Suslin tree and T^* to denote an almost Suslin Aronszajn tree. With this convention, there will be no confusion introduced by the following slight abuse of notation.

Definition 6.2. Let $\mathbb{P}$ be a forcing notion.

(1) For a Suslin tree S, we say that $\mathbb{P}$ is *S-preserving* if $\Vdash_{\mathbb{P}}$ "S is a Suslin tree".
(2) For an almost Suslin Aronszajn tree T^*, we say that $\mathbb{P}$ is *T^*-preserving* if $\Vdash_{\mathbb{P}}$ "T^* is an almost Suslin Aronszajn tree".

We now introduce the forcing axioms that form the subject of this section. We note that in most other works, the forcing axioms $\mathsf{MA}_{\omega_1}(\mathsf{S})$ and $\mathsf{PFA}(\mathsf{S})$ require that S be a *coherent* Suslin tree. Since coherence will not be necessary in any of our arguments, we choose to present the axioms in a more general form and not require coherence. We note that, when we interpret a Suslin tree as a forcing notion, then the forcing order is understood to be the reverse of the tree order.

Definition 6.3. If $\mathcal{C}$ is a class of forcing posets, then $\mathsf{FA}(\mathcal{C})$ is the assertion that, for every $\mathbb{P} \in \mathcal{C}$ and every collection $\mathcal{D} = \{D_\alpha \mid \alpha < \omega_1\}$ of ω_1-many dense subsets of $\mathbb{P}$, there is a filter $G \subseteq \mathbb{P}$ such that $G \cap D_\alpha \neq \emptyset$ for all $\alpha < \omega_1$.

(1) $\mathsf{MA}_{\omega_1}(\mathsf{S})$ is the assertion that S is a Suslin tree and $\mathsf{FA}(\mathcal{C})$ holds, where $\mathcal{C}$ is the class of c.c.c. S-preserving posets.
(2) $\mathsf{PFA}(\mathsf{S})$ is the assertion that S is a Suslin tree and $\mathsf{FA}(\mathcal{C})$ holds, where $\mathcal{C}$ is the class of proper S-preserving posets.
(3) If we start with a model satisfying $\mathsf{PFA}(\mathsf{S})$ and then force with the Suslin tree S, then we say that the resulting forcing extension satisfies $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$. Asserting that $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$ implies a statement φ should be understood as asserting that, in any model of ZFC satisfying $\mathsf{PFA}(\mathsf{S})$ for some coherent Suslin tree S, we have $\Vdash_S \varphi$. $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ is defined analogously, with $\mathsf{MA}_{\omega_1}(\mathsf{S})$ replacing $\mathsf{PFA}(\mathsf{S})$.
(4) $\mathsf{PFA}(\mathsf{T}^*)$ is the assertion that T^* is an almost Suslin Aronszajn tree and $\mathsf{FA}(\mathcal{C})$ holds, where $\mathcal{C}$ is the class of proper T^*-preserving posets.

Remark 6.4. The consistency of $\mathsf{PFA}(\mathsf{S})$ (and hence of $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$) and of $\mathsf{PFA}(\mathsf{T}^*)$ follows from the consistency of the existence of a supercompact cardinal in much the same way that the consistency of PFA follows from the same hypothesis. For sketches of the proof, see [25, Theorem 4.1] and [11, Theorem 2.6].

The following lemma, due to Woodin (cf. [33, Proof of Theorem 2.53]) will be useful. Recall that, if $\mathbb{P}$ is a forcing poset and $\mathbb{P} \in M \prec H(\theta)$ for some regular uncountable cardinal θ, then we say that a filter $G \subseteq \mathbb{P}$ is *M-generic* if $G \cap D \cap M \neq \emptyset$ for all dense subsets D of $\mathbb{P}$ that are elements of M.

Lemma 6.5. *Suppose that $\mathcal{C}$ is a class of forcing posets and $\mathsf{FA}(\mathcal{C})$ holds. Then, for every poset $\mathbb{P} \in \mathcal{C}$ and all sufficiently large regular cardinals θ, the set*

$$\{M \in \mathscr{P}_{\omega_2} H(\theta) \mid |M| = \omega_1 \subseteq M \wedge \exists G \subseteq \mathbb{P}\ [G \textit{ is an } M\textit{-generic filter}]\}$$

is stationary in $\mathscr{P}_{\omega_2} H(\theta)$.

6.1. **Suslin trees.** In [3], Cox and Krueger ask whether IGMP implies $\mathfrak{p} > \omega_1$. In this section, we prove that the axiom $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$ implies IGMP. Since $\mathfrak{p} = \omega_1$ in any forcing extension by a Suslin tree (cf. [5, Lemma 2]), and hence in any model of $\mathsf{PFA}(\mathsf{S})[\mathsf{S}]$, this answers Cox and Krueger's question negatively. We also show that $\mathsf{PFA}(\mathsf{S})$ implies GMP. Thus, models of $\mathsf{PFA}(\mathsf{S})$ provide examples of models in which GMP holds and there exists a Suslin tree, and therefore IGMP fails, but forcing with a particular Suslin tree yields a model of IGMP.

Given a tree T of height ω_1 and a set $C \subseteq \omega_1$, let T_C denote the set of $s \in T$ such that $\mathrm{ht}(s) \in C$, with the inherited ordering from T. Note that, if S is a Suslin tree and $C \subseteq \omega_1$ is unbounded, then S_C is itself a Suslin tree.

The following theorem is due to Raghavan and Yorioka [20] under the additional assumption that S is coherent, though they indicate that it was known beforehand in the case in which $\dot{T}$ is a name for an Aronszajn tree. Since we remove the requirement that S be coherent, and because we find our proof to be simpler than that presented in [20], we include a proof.

Theorem 6.6. *Suppose that S is a Suslin tree and $\dot{T}$ is an S-name for a tree of height and size ω_1 with no cofinal branches. Then there is a forcing notion $\mathbb{P}(\dot{T})$ such that $\mathbb{P}(\dot{T}) \times S$ is c.c.c. and $\Vdash_{\mathbb{P}(\dot{T})\times S}$ "$\dot{T}$ is special".*

Proof. We may assume that it is forced by S that:

- the underlying set of $\dot{T}$ is ω_1;
- for all $\alpha, \beta < \omega_1$, if $\alpha <_{\dot{T}} \beta$, then $\alpha < \beta$.

We can therefore think of $\dot{T}$ as a name for a subset $\dot{X} \subseteq [\omega_1]^2$, where, for all $\alpha < \beta < \omega_1$, we have

$$\Vdash_S \text{“}\alpha <_{\dot{T}} \beta \Leftrightarrow \{\alpha, \beta\} \in \dot{X}\text{”}.$$

By the argument immediately after the statement of Theorem 4.2 of [14], we can find an unbounded $C \subseteq \omega_1$ and a subset $K \subseteq [S_C]^2$ such that for all $t \in S_C$ and all $\alpha < \beta < \mathrm{ht}_{S_C}(t)$, we have $t \Vdash_S$ "$\{\alpha, \beta\} \in \dot{X}$" if and only if $\{s, r\} \in K$, where s and r are the predecessors of t at levels α and β of S_C, respectively. Since forcing with S_C is equivalent to forcing with S, for ease of notation we will assume that $C = \omega_1$ and write S instead of S_C. Also, since the preceding sentences only mention pairs $\{s, r\} \in K$ such that $s <_S r$ or $r <_S s$, we can assume that K only consists of pairs that are comparable via $<_S$.

Let $\mathbb{P}(\dot{T}) = \mathbb{P}$ be the forcing notion whose conditions are all finite partial functions $p : S \to \omega$ with the property that, for all $s, t \in \mathrm{dom}(p)$, if $p(s) = p(t)$, then $\{s, t\} \notin K$. Order $\mathbb{P}$ by reverse inclusion. We can think of $\mathbb{P}$ as the natural forcing to add an S-name for a specializing function for $\dot{T}$.

Claim 6.7. *$\mathbb{P} \times S$ is c.c.c.*

Proof. Let $\langle (p_\eta, s_\eta) \mid \eta < \omega_1 \rangle$ be a sequence of conditions in $\mathbb{P} \times S$. We will prove that there are $\eta_0 < \eta_1$ such that (p_{η_0}, s_{η_0}) and (p_{η_1}, s_{η_1}) are compatible. Since S is Suslin, and hence c.c.c., there is $s^* \in S$ such that $s^* \Vdash_S$ "$\{\eta < \omega_1 \mid s_\eta \in \dot{G}\}$ is uncountable," where $\dot{G}$ is the canonical name for the S-generic filter. Let $G \subseteq S$ be a V-generic filter with $s^* \in G$, and move to $V[G]$. Let $A := \{\eta < \omega_1 \mid s_\eta \in G\}$. By our choice of G, A is uncountable. Using the Δ-system lemma, find an unbounded $A' \subseteq A$ such that $\{\mathrm{dom}(p_\eta) \mid \eta \in A'\}$ forms a Δ-system, with root R. Since each level of S is countable and the codomain of each p_η is ω, by thinning out A' further if necessary, we may also assume the following.

- There is $n < \omega$ such that, for all $\eta \in A'$, $R_\eta := \mathrm{dom}(p_\eta) \setminus R$ has size n. Enumerate each R_η as $\{r^\eta_m \mid m < n\}$.
- For all $\eta_0, \eta_1 \in A'$, we have $p_{\eta_0} \restriction R = p_{\eta_1} \restriction R$.
- For all $\eta_0 < \eta_1$, both in A', for all $m_0, m_1 < n$, and for all $r \in R$, we have $\mathrm{ht}(r) < \mathrm{ht}(r^{\eta_0}_{m_0}) < \mathrm{ht}(r^{\eta_1}_{m_1})$.

For all $\eta_0 < \eta_1$ in A', we know that s_{η_0} and s_{η_1} are both in G and are therefore compatible. Therefore, if we can find such η_0 and η_1 for which p_{η_0} and p_{η_1} are compatible, then we will be finished. Suppose that this is not the case. We proceed as in Baumgartner's classical proof that the forcing to specialize an ω_1-tree with no cofinal branch is c.c.c. Namely, for each pair $\eta < \xi$, both from A', our uniformization of A' implies that the only obstacle to the compatibility of p_η and p_ξ would be the existence of $m_{\eta,\xi,0}, m_{\eta,\xi,1} < n$ such that $p_\eta(r^\eta_{m_{\eta,\xi,0}}) = p_\xi(r^\xi_{m_{\eta,\xi,1}})$ and $\{r^\eta_{m_{\eta,\xi,0}}, r^\xi_{m_{\eta,\xi,1}}\} \in K$. It follows from our assumptions on K that $r^\eta_{m_{\eta,\xi,0}} <_S r^\xi_{m_{\eta,\xi,1}}$.

Let U be a uniform ultrafilter over A'. For each $\eta \in A'$, we can fix a set $Y_\eta \in U$ and numbers $m_{\eta,0}, m_{\eta,1} < n$ such that, for all $\xi \in Y_\eta$, we have $\eta < \xi$, $m_{\eta,\xi,0} = m_{\eta,0}$, and $m_{\eta,\xi,1} = m_{\eta,1}$. Now find an unbounded $A'' \subseteq A'$ and numbers $m_0, m_1 < n$ such that $m_{\eta,0} = m_0$ and $m_{\eta,1} = m_1$ for all $\eta \in A''$. Given $\eta_0 < \eta_1$, both in A'', find $\xi \in Y_{\eta_0} \cap Y_{\eta_1}$. Then $r^{\eta_i}_{m_0} <_S r^{\xi}_{m_1}$ for each $i < 2$, so we have $r^{\eta_0}_{m_0} <_S r^{\eta_1}_{m_0}$. Also, $\{r^{\eta_i}_{m_0}, r^{\xi}_{m_1}\} \in K$ for each $i < 2$, so, by our assumptions on $\dot{T}$, we must have $\{r^{\eta_0}_{m_0}, r^{\eta_1}_{m_0}\} \in K$, as well.

By the previous paragraph, the downward closure of $\{r^{\eta}_{m_0} \mid \eta \in A''\}$ is a cofinal branch through S; let b' denote this cofinal branch. Since S is Suslin in V, b' generates a V-generic filter G' over S. Let T' be the interpretation of $\dot{T}$ using G'. For all $\eta_0 < \eta_1$ in A'', we have $\{r^{\eta_0}_{m_0}, r^{\eta_1}_{m_0}\} \in K$, so the $<_{T'}$-downward closure of $\{\mathrm{ht}(r^{\eta}_{m_0}) \mid \eta \in A''\}$ generates a cofinal branch in T', contradicting the fact that, in V, we have $\Vdash_S$ "$\dot{T}$ has no cofinal branches". □

For each $s \in S$, let $D_s := \{p \in \mathbb{P} \mid s \in \mathrm{dom}(p)\}$, and note that D_s is a dense open subset of $\mathbb{P}$. Therefore, the generic object for $\mathbb{P}$ can be seen as a function from S to ω. Let $\dot{g}$ be a canonical $\mathbb{P}$-name for this function, and let $\dot{c}$ be a $\mathbb{P} \times S$-name for a function from ω_1 to ω such that, for all $\alpha < \omega$ and all $t \in S$ with $\mathrm{ht}(t) > \alpha$, we have

$$(\emptyset, t) \Vdash_{\mathbb{P}\times S} \text{“}\dot{c}(\alpha) = \dot{g}(s)\text{”},$$

where s is the predecessor of t on level α of S.

We will be done if we show that $\dot{c}$ is forced to be a specializing function for $\dot{T}$. Suppose for sake of contradiction that there are $(p, t) \in \mathbb{P} \times S$ and $\alpha < \beta < \omega_1$ such that $(p, t) \Vdash_{\mathbb{P}\times S}$ "$\alpha <_{\dot{T}} \beta$ and $\dot{c}(\alpha) = \dot{c}(\beta)$". We can assume that $\mathrm{ht}(t) > \beta$. Let s and r be the predecessors of t on levels α and β of S, respectively. We can also assume that $\{s, r\} \subseteq \mathrm{dom}(p)$. Then we have $\{s, r\} \in K$ and $p(s) = p(r)$, contradicting the fact that p is a condition in $\mathbb{P}$. □

Corollary 6.8. $\mathsf{MA}_{\omega_1}(\mathsf{S})[\mathsf{S}]$ *implies that every tree of height and size* ω_1 *with no cofinal branches is special.*

We now prove that $\mathsf{PFA(S)[S]}$ implies GMP. Since $\mathsf{PFA(S)[S]}$ also implies that all trees of height and size ω_1 with no cofinal branches are special, it will follow from the discussion at the end of Section 5 that $\mathsf{PFA(S)[S]}$ implies IGMP. Our proof is a modification of the proof from [31] of the fact that PFA implies GMP.

We first recall the *covering* and *approximation* properties, introduced by Hamkins (cf. [6]).

Definition 6.9. Suppose that $V \subseteq W$ are transitive models of ZFC and μ is a regular uncountable cardinal.

(1) (V, W) satisfies the *μ-covering property* if, for every $x \in W$ such that $x \subseteq V$ and $|x|^W < \mu$, there is $y \in V$ such that $|y|^V < \mu$ and $x \subseteq y$.
(2) (V, W) satisfies the *μ-approximation property* if, for all $x \in W$ such that $x \subseteq V$ and $x \cap z \in V$ for all $z \in V$ with $|z| < \mu$, we in fact have $x \in V$.

A poset $\mathbb{P}$ has the *μ-covering property* (resp. *μ-approximation property*) if, for every V-generic filter $G \subseteq \mathbb{P}$, the pair $(V, V[G])$ has the μ-covering property (resp. μ-approximation property).

For the rest of this subsection, let S denote a Suslin tree. If $\dot{T}$ is an S-name for a tree of height and size ω_1 with no cofinal branch, then $\mathbb{P}(\dot{T})$ denotes the forcing from Theorem 6.6 that adds an S-name for a specializing function for $\dot{T}$. If $\mathbb{P}$ is an S-preserving forcing notion and $\dot{T}$ is a $\mathbb{P} \times S$-name for a tree of height and size

ω_1 with no cofinal branch, then we will typically let $\dot{\mathbb{P}}(\dot{T})$ be a $\mathbb{P}$-name for $\mathbb{P}(\dot{T})$ (where, in $V^{\mathbb{P}}$, $\dot{T}$ is reinterpreted as an S-name). The following is analogous to, and largely follows the proof of, [31, Lemma 4.6].

Lemma 6.10. *Suppose that $\lambda \geq 2^\omega$ is an infinite cardinal, and let θ be a sufficiently large regular cardinal. Assume that $\mathbb{P}$ preserves S, collapses 2^λ to have cardinality ω_1, satisfies the ω_1-covering and ω_1-approximation properties, and continues to satisfy the ω_1-approximation property in V^S. Then there is a $\mathbb{P} \times S$-name $\dot{T}_1$ for a tree of height and size ω_1 with no cofinal branches and a $w \in H(\theta)$ such that, for every $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$, if there is $G \subseteq \mathbb{P} * \dot{\mathbb{P}}(\dot{T}_1)$ that is M-generic, then, in V^S, M^S is an ω_1-guessing model for λ.*

Proof. Work for now in V, and let $\dot{B}$ be an S-name for $({}^\lambda 2)^{V^S}$. Using the fact that $\mathbb{P}$ collapses 2^λ to have cardinality ω_1 and satisfies the ω_1-covering property, let $\dot{c}$ be a $\mathbb{P}$-name for a $\subseteq$-increasing, continuous, and cofinal function from ω_1 to $(\mathscr{P}_{\omega_1}\lambda)^{V^{\mathbb{P}}}$. Since $\mathbb{P}$ satisfies the ω_1-covering property, we can assume that, for all $\alpha < \omega_1$, we have $\Vdash_{\mathbb{P}}$ "$\dot{c}(\alpha+1) \in V$". Let $\dot{\ell}$ be a $\mathbb{P} \times S$-name for a bijection from ω_1 to B, and let $\dot{T}$ be a $\mathbb{P} \times S$-name for $\{\dot{\ell}(\eta) \restriction \dot{c}(\alpha) \mid \eta, \alpha < \omega_1\}$. Note that $\dot{T}$ is forced to be a tree of height and size ω_1. Moreover, since $\mathbb{P}$ has the ω_1-approximation property in V^S, $\dot{B}$ is forced to be precisely the set of cofinal branches through $\dot{T}$. Since $\dot{B}$ is forced to have cardinality ω_1 in $V^{\mathbb{P} \times S}$, we can apply Lemma 5.4 to find a $\mathbb{P} \times S$-name $\dot{g}$ for a Baumgartner function from $\dot{B}$ to $\dot{T}$.

Let $\dot{T}_0$ be a $\mathbb{P} \times S$-name for the set $\{t \in \dot{T} \mid \exists b \in \dot{B}\ \dot{g}(b) <_{\dot{T}} t \in b\}$, and let $\dot{T}_1$ be a $\mathbb{P} \times S$-name for $\dot{T} \setminus \dot{T}_0$. Since $\dot{T}_0$ is forced to contain a tail of every $b \in \dot{B}$, it follows that $\dot{T}_1$ is forced to have no uncountable branches in $V^{\mathbb{P} \times S}$. We can therefore let $\dot{\mathbb{Q}}$ be a $\mathbb{P}$-name for $\mathbb{P}(\dot{T}_1)$, where $\dot{T}_1$ is reinterpreted as an S-name in $V^{\mathbb{P}}$. Since $\dot{\mathbb{Q}}$ is forced to add an S-name for a specializing function for $\dot{T}_1$, we can fix a $(\mathbb{P} * \dot{\mathbb{Q}}) \times S$-name $\dot{f}$ for a specializing function from $\dot{T}_1$ to ω.

We claim that $\dot{T}_1$ is as desired. Let w be a set containing all relevant information, including $S, \dot{c}, \dot{T}, \dot{T}_0, \dot{T}_1, \dot{f}, \dot{g}$, and $\dot{\ell}$. Now fix $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$, and suppose that $G = G_0 * G_1 \subseteq (\mathbb{P} * \dot{\mathbb{Q}}) \cap M$ is M-generic. We will show that, in V^S, M^S is an ω_1-guessing model for λ.

Let H be an S-generic filter over V and move to $V[H]$. Let $c := \dot{c}^{G_0} : \omega_1 \to \mathscr{P}_{\omega_1}(\lambda \cap M)$, and note that c is $\subseteq$-increasing and continuous, and $c(\alpha+1) \in M$ for all $\alpha < \omega_1$. Also, c is cofinal in $(\mathscr{P}_{\omega_1}\lambda)^M$, and, since S is ω_1-distributive, it is also cofinal in $(\mathscr{P}_{\omega_1}\lambda)^{M[H]}$. Let $g := \dot{g}^{G_0 \times H}$, $T := \dot{T}^{G_0 \times H}$ (and analogously for T_0 and T_1), $f := \dot{f}^{(G_0 * G_1) \times H}$, $B = \dot{B}^H$, and $\ell := \dot{\ell}^{G_0 \times H}$. Let $B \restriction M[H] := \{b \restriction M[H] \mid b \in B \cap M[H]\}$.

By elementarity and the fact that $G_0 * G_1$ is M-generic, we have the following facts:

- $\ell : \omega_1 \to B \cap M[H]$ is a bijection.
- $T = \{b \restriction c(\alpha) \mid b \in B \cap M[H],\ \alpha < \omega_1\}$.
- $B \restriction M[H]$ is a set of uncountable branches through T.
- $\mathrm{dom}(g) = B \cap M[H]$; we will slightly abuse notation and identify $\mathrm{dom}(g)$ with $B \restriction M[H]$.
- $g : B \restriction M[H] \to T$ is a Baumgartner function.
- $T = T_0 \cup T_1$.
- $f : T_1 \to \omega$ is a specializing function.

Claim 6.11. *$B \restriction M[H]$ is the set of uncountable branches through T.*

Proof. We have already seen that every element of $B \restriction M[H]$ is an uncountable branch through T. For the reverse inclusion, suppose that h is an uncountable branch through T. We identify branches through T with their union, i.e., we think of h as being of the form $h : M[H] \cap \lambda \to 2$. Since T_1 is special and therefore cannot have an uncountable branch, a tail of h must lie inside T_0, i.e., we can find $\alpha_0 < \omega_1$ such that, for all $\alpha \in [\alpha_0, \omega_1)$, we have $h \restriction c(\alpha) \in T_0$. For all such α, there is a unique $b_\alpha \in B \restriction M[H]$ such that $g(b_\alpha) \subsetneq h \restriction c(\alpha) \subsetneq b_\alpha$. By Fodor's Lemma, there is a fixed $t \in T$ and a stationary $R \subseteq \omega_1$ such that $g(b_\alpha) = t$ for all $\alpha \in R$. Since g is injective, there is $b \in B \restriction M[H]$ such that $b_\alpha = b$ for all $\alpha \in R$. Then $h \restriction c(\alpha) \subseteq b$ for all $\alpha \in R$, and since R is cofinal in ω_1, this implies that $h = b$. $\square$

We are now ready to prove that $M[H]$ is an ω_1-guessing model for λ in $V[H]$. Work in $V[H]$, and let $d \subseteq \lambda$ be $M[H]$-approximated. Let $h : \lambda \cap M[H] \to 2$ be the characteristic function of $d \cap M[H]$. Then, for all $\alpha < \omega_1$, we have $d \cap c(\alpha+1) \in M[H]$ and hence $h \restriction c(\alpha+1) \in M[H]$. It follows that $h \restriction c(\alpha+1) \in T$, and hence h is an uncountable branch through T. By Claim 6.11, we have $h \in B \restriction M[H]$, so there is $b \in B \cap M[H]$ such that $b \restriction M[H] = h$. Let $e := \{\eta < \lambda \mid b(\eta) = 1\}$. Then $e \in M[H]$ and $e \cap M[H] = d \cap M[H]$. Therefore, d is guessed by $M[H]$, as desired. $\square$

Theorem 6.12. $\mathsf{PFA(S)[S]}$ *implies* GMP.

Proof. Suppose that V satisfies $\mathsf{PFA(S)}$. We will prove that GMP holds in V^S. Given a cardinal $\lambda \geq \omega_2$, let GMP^λ denote the assertion that, for all sufficiently large regular θ, there are stationarily many $M \in \mathscr{P}_{\omega_2} H(\theta)$ that are ω_1-guessing models for λ. By [31, Proposition 3.2], for every sufficiently large regular θ, if $\mathsf{GMP}^{|H(\theta)|}$ holds, then there are stationarily many ω_1-guessing models $M \in \mathscr{P}_{\omega_2} H(\theta)$. It therefore suffices to prove that GMP^λ holds for all $\lambda \geq 2^\omega$.

Work in V. Fix $\lambda \geq 2^\omega$ and a sufficiently large regular cardinal θ, and let $\mathbb{P} := \mathrm{Add}(\omega, 1) * \dot{\mathrm{Coll}}(\omega_1, 2^\lambda)$. Then $\mathbb{P}$ collapses 2^λ to have cardinality ω_1 and, since $\mathrm{Add}(\omega, 1)$ is ω_1-Knaster and $\dot{\mathrm{Coll}}(\omega_1, 2^\lambda)$ is forced to be countably closed, $\mathbb{P}$ is proper and preserves the fact that S is a Suslin tree. It is proven in [9] that $\mathbb{P}$ has the ω_1-covering and ω_1-approximation properties. Since S is ω_1-distributive, the definition of $\mathbb{P}$ is the same in V and in V^S; in particular, the proof from [9] can be carried out in V^S to show that $\mathbb{P}$ has the ω_1-approximation property in that model as well. We can therefore apply Lemma 6.10 to find a $\mathbb{P} \times S$-name $\dot{T}$ for a tree of height and size ω_1 with no cofinal branches and a $w \in H(\theta)$ such that, for every $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$, if there is $G \subseteq \mathbb{P} * \dot{\mathbb{P}}(\dot{T})$ that is M-generic, then M^S is an ω_1-guessing model for λ in V^S.

In $V^{\mathbb{P}}$, by Theorem 6.6, $\mathbb{P}(\dot{T}) \times S$ is c.c.c. In particular, it follows that, in V, $\mathbb{P} * \dot{\mathbb{P}}(\dot{T})$ is proper and preserves S. Therefore, by $\mathsf{PFA(S)}$ and Lemma 6.5, the set of $M \in \mathscr{P}_{\omega_2} H(\theta)$ for which $|M| = \omega_1 \subseteq M$ and there exists an M-generic $G \subseteq \mathbb{P} * \dot{\mathbb{P}}(\dot{T})$ is stationary in $\mathscr{P}_{\omega_2} H(\theta)$. Let $\dot{C}$ be an S-name for a club in $(\mathscr{P}_{\omega_2} H(\theta))^{V^S}$, and, using Proposition 2.3, let $\dot{f}$ be an S-name for a function from $[H(\theta)]^2 \to \mathscr{P}_{\omega_2} H(\theta)$ such that $\dot{C}_{\dot{f}} := \{X \in \mathscr{P}_{\omega_2} H(\theta) \mid \forall z \in \mathscr{P}_\omega X \ \dot{f}(z) \subseteq X\}$ is forced to be a subset of $\dot{C}$. We can then find $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w, \dot{f}\} \subseteq M \prec H(\theta)$ and such that there exists an M-generic $G \subseteq \mathbb{P} * \dot{\mathbb{P}}(\dot{T})$.

It follows that M^S is an ω_1-guessing model for λ in V^S. We now show that $M^S \in \dot{C}$; since $\dot{C}$ was an arbitrary name for a club in $\mathscr{P}_{\omega_2} H(\theta)$, this will imply that GMP^λ holds in V^S. To see that $M^S \in \dot{C}$, recall that $\dot{f} \in M$, and therefore, for every S-name $\dot{z} \in M$ for an element of $\mathscr{P}_\omega H(\theta)$, there is an S-name for $\dot{f}(\dot{z})$ in M. It follows that $M^S \in \dot{C}_{\dot{f}} \subseteq \dot{C}$. □

Corollary 6.13. $\mathsf{PFA(S)[S]}$ *implies* IGMP.

We end this subsection by pulling back the previous results from V^S to V, showing that $\mathsf{PFA(S)}$ also implies GMP.

Proposition 6.14. *Suppose that $\mathbb{P}$ is a forcing notion such that $|\mathbb{P}| \leq \omega_1$ and $\mathbb{P}$ has the ω_1-covering property, and suppose that θ is a sufficiently large regular cardinal and $M \in \mathscr{P}_{\omega_2} H(\theta)$ is such that $M \prec H(\theta)$ and $\mathbb{P} \cup \{\mathbb{P}\} \subseteq M$. If $M^{\mathbb{P}}$ is an ω_1-guessing model in $V^{\mathbb{P}}$, then M is an ω_1-guessing model in V.*

Proof. Suppose that $M^{\mathbb{P}}$ is an ω_1-guessing model in $V^{\mathbb{P}}$ but M is not an ω_1-guessing model in V. We can then fix a cardinal $\lambda \in M$ and a set $d \subseteq \lambda$ such that d is (ω_1, M)-approximated but not M-guessed.

Claim 6.15. *In $V^{\mathbb{P}}$, d is $(\omega_1, M^{\mathbb{P}})$-approximated.*

Proof. Fix a condition $p \in \mathbb{P}$ and a $\mathbb{P}$-name $\dot{y} \in M$ for an element of $\mathscr{P}_{\omega_1}\lambda$. Since $\mathbb{P}$ has the ω_1-covering property, and by elementarity, we can find $q \leq_{\mathbb{P}} p$ and $z \in \mathscr{P}_{\omega_1}\lambda \cap M$ such that $q \Vdash_{\mathbb{P}}$ "$\dot{y} \subseteq z$". Then $d \cap z \in M$, so $q \Vdash_{\mathbb{P}}$ "$d \cap \dot{y} = (d \cap z) \cap \dot{y} \in M^{\mathbb{P}}$". The conclusion follows by genericity. □

Therefore, since $M^{\mathbb{P}}$ is an ω_1-guessing model in $V^{\mathbb{P}}$, we can find a $\mathbb{P}$-name $\dot{e} \in M$ for a subset of λ and a condition $p \in \mathbb{P}$ such that $p \Vdash_{\mathbb{P}}$ "$d \cap M^{\mathbb{P}} = \dot{e} \cap M^{\mathbb{P}}$". Note that, since $\mathbb{P} \subseteq M$, we have $M^{\mathbb{P}} \cap \lambda = M \cap \lambda$.

Claim 6.16. *There are conditions $q_0, q_1 \leq_{\mathbb{P}} p$ and an ordinal $\alpha < \lambda$ such that $q_0 \Vdash_{\mathbb{P}}$ "$\alpha \in \dot{e}$" and $q_1 \Vdash_{\mathbb{P}}$ "$\alpha \notin \dot{e}$".*

Proof. If not, then, letting $e^* := \{\alpha < \lambda \mid \exists q \leq_{\mathbb{P}} p \; [q \Vdash_{\mathbb{P}} \text{“}\alpha \in \dot{e}]\}$, we have that $e^* \in M$ and $p \Vdash_{\mathbb{P}}$ "$\dot{e} = e^*$". But then we would have $e^* \cap M = d \cap M$, contradicting the assumption that d is not M-guessed. □

By elementarity, we can find $q_0, q_1 \leq_{\mathbb{P}} p$ and an ordinal $\alpha \in \lambda \cap M$ such that $q_0 \Vdash_{\mathbb{P}}$ "$\alpha \in \dot{e}$" and $q_1 \Vdash_{\mathbb{P}}$ "$\alpha \notin \dot{e}$". If $\alpha \in d$, then let $q^* = q_1$, and if $\alpha \notin d$, then let $q^* = q_0$. In either case, we have $q^* \leq_{\mathbb{P}} p$ and $q^* \Vdash_{\mathbb{P}}$ "$\dot{e} \cap M \neq d \cap M$", which is a contradiction. □

Corollary 6.17. $\mathsf{PFA(S)}$ *implies* GMP.

Proof. Suppose that $\mathsf{PFA(S)}$ holds in V. Fix a regular $\theta \geq \omega_2$. By the proof of Theorem 6.12, there are stationarily many $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that M^S is an ω_1-guessing model in V^S. By Proposition 6.14, for each such M for which $M \prec H(\theta)$ and $S \cup \{S\} \subseteq M$, we know that M is a guessing model in V. Therefore, GMP holds in V. □

6.2. **Almost Suslin trees.** As mentioned at the end of Section 5, work of Cox and Krueger shows that IGMP follows from the conjunction of GMP and the assertion that all trees of size and height ω_1 with no cofinal branches are special, and that IGMP implies that there are no Suslin trees. Together, these results raise a natural question, asked already in [3], as to whether IGMP implies that all trees of height and size ω_1 with no cofinal branches are special. In this section, we prove that $\mathsf{PFA}(\mathsf{T}^*)$ implies IGMP. Since every special tree contains a stationary antichain, this answers the question negatively. It also positively answers a question of Krueger from [11] as to whether $\mathsf{PFA}(\mathsf{T}^*)$ implies $\neg$wKH, since, as mentioned in Section 3, GMP implies $\neg$wKH.

The following proposition is a slight improvement of [3, Proposition 4.4 and Corollary 4.5]. To simplify the statement, let us say that a tree of height ω_1 is *persistently branchless* if it has no cofinal branches and continues to have no cofinal branches in any forcing extension that preserves ω_1. Note that every special tree T is persistently branchless, and, in fact, it is enough that T_C be special for some cofinal set $C \subseteq \omega_1$.

Proposition 6.18. *Suppose that* GMP *holds and every tree T of size and height ω_1 with no cofinal branches is persistently branchless. Then* IGMP *holds.*

Proof. Fix a regular cardinal $\theta \geq \omega_2$. It suffices to show that every ω_1-guessing model $N \in \mathscr{P}_{\omega_2} H(\theta)$ with $\omega_1 \subseteq N$ is an indestructible ω_1-guessing model. To this end, fix such an N, and fix a cardinal $\lambda \in N$. It suffices to prove that, in any forcing extension of V in which ω_1 is preserved, N continues to be an ω_1-guessing model for λ.

Since N is an ω_1-guessing model with $\omega_1 \subseteq N$, [10, Theorem 1.4] implies that N is internally unbounded. We can therefore fix a $\subseteq$-increasing sequence $\langle X_i \mid i < \omega_1 \rangle$ such that each X_i is an element of $N \cap \mathscr{P}_{\omega_1}\lambda$ and $\bigcup_{i<\omega_1} X_i = N \cap \lambda$. We now define a tree T as in the proof of [3, Proposition 4.4]. The underlying set of T is all pairs $(i, x) \in N$ such that $i < \omega_1$ and $x \subseteq X_i$. We set $(i, x) <_T (j, y)$ if and only if $i < j$ and $y \cap X_i = x$.

Precisely as in [3], T is a tree of size and height ω_1 with at most ω_1-many cofinal branches. Let B be the set of all cofinal branches through T, and, using Lemma 5.4, find a Baumgartner function $g : B \to T$.

Now, as in the proof of Lemma 6.10, let $T_0 := \{t \in T \mid \exists b \in B \ g(b) <_T t \in b\}$, and let $T_1 := T \setminus T_0$. Since T_0 contains a tail of every $b \in B$, it follows that T_1 has no cofinal branches and is therefore, by assumption, persistently branchless.

Now let W be any forcing extension of V in which ω_1 is preserved, and let $d \in \mathscr{P}^W\lambda$ be (ω_1, N)-approximated in W. Then, for each $i < \omega_1$, we have $(i, d \cap X_i) \in T$, and the set $c = \{(i, d \cap X_i) \mid i < \omega_1\}$ is a cofinal branch through T. Since T_1 is persistently branchless in V, it must be the case that a tail of c lies inside T_0. Then, precisely by the argument in the proof of Claim 6.11, we must in fact have $c \in B$, i.e., c is already in V. Then $d \cap N = \bigcup_{i<\omega_1} d \cap X_i$ is also in V and is clearly (ω_1, N)-approximated there, since d is (ω_1, N)-approximated in W. Therefore, since N is a guessing model in V, $d \cap N$ is N-guessed, i.e., there is $e \in N$ such that $e \cap N = d \cap N$. Clearly, this set e witnesses that d is N-guessed in W. Therefore, N is an ω_1-guessing model in W; since W was arbitrary, N is an indestructible ω_1-guessing model in V. $\square$

For the rest of this subsection, let T^* denote an almost Suslin Aronszajn tree. In [11, Definition 1.7], Krueger introduces the notion of a forcing poset $\mathbb{P}$ being *T^*-proper*. Although being T^*-proper is a weakening of the conjunction of being proper and T^*-preserving, Krueger proves that $\mathsf{PFA}(\mathsf{T}^*)$ is equivalent to $\mathsf{FA}(\mathcal{C})$, where $\mathcal{C}$ is the class of T^*-proper forcing posets [11, Proposition 2.5]. We will not need the definition of T^*-properness here, but just note the following facts:

Fact 6.19. *(1)* [11, Theorem 2.2] *T^*-properness is preserved under countable support iteration.*

(2) [11, Proposition 3.1] *Every strongly proper forcing poset is T^*-proper. In particular, Cohen forcing is T^*-proper.*

(3) [11, Proposition 1.11] *Every ω_1-closed forcing poset is T^*-proper.*

(4) [23, §IX, Lemma 4.6] *For every tree T of size and height ω_1 with no cofinal branches, there is a T^*-proper forcing $\mathbb{Q}(T)$ that adds an unbounded subset $A \subseteq \omega_1$ and a specializing function $f : T_A \to \omega$.*

As noted at the end of [11], Fact 6.19(4) shows that $\mathsf{PFA}(\mathsf{T}^*)$ implies that for every tree T of size and height ω_1 with no cofinal branches, there is an unbounded $A \subseteq \omega_1$ such that T_A is special; in particular, T is persistently branchless. Therefore, by Proposition 6.18, to show that $\mathsf{PFA}(\mathsf{T}^*)$ implies IGMP, it suffices to show that $\mathsf{PFA}(\mathsf{T}^*)$ implies GMP.

Theorem 6.20. $\mathsf{PFA}(\mathsf{T}^*)$ *implies* IGMP.

Proof. Suppose that $\mathsf{PFA}(\mathsf{T}^*)$ holds. As noted immediately before the statement of the theorem, it suffices to prove that GMP holds. Since all of the ideas of this proof are transparently present in the arguments of Subsection 6.1 of this paper and Section 4 of [31], we provide only a sketch.

Claim 6.21. *Suppose that λ is an uncountable cardinal and θ is a sufficiently large regular cardinal. Assume that $\mathbb{P}$ satisfies the ω_1-covering and ω_1-approximation properties and collapses 2^λ to have size ω_1. Then there is a $\mathbb{P}$-name $\dot{T}_1$ for a tree of height and size ω_1 with no cofinal branches and a $w \in H(\theta)$ such that, letting $\dot{\mathbb{Q}}(\dot{T}_1)$ be a name for the poset in Fact 6.19(4), we have the following: for every $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$, if there is $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}(\dot{T}_1)$ that is M-generic, then M is a guessing model for λ.*

Proof. The proof is exactly as in the proof of [31, Lemma 4.6], all of the ideas of which are also present in the proof of 6.10 above, so we omit it. The only difference between the proof of this claim and that of [31, Lemma 4.6] is that in that paper the forcing $\dot{\mathbb{Q}}(\dot{T}_1)$ is replaced by the ccc forcing to fully specialize $\dot{T}_1$. However, it is evident from that proof that adding this complete specialization function is not necessary; it suffices to specialize $\dot{T}_1$ restricted to some cofinal set of levels, which is precisely what $\dot{\mathbb{Q}}(\dot{T}_1)$ does. □

As in the proof of 6.12, it suffices to show that GMP^λ holds for all uncountable cardinals λ. Therefore, fix such a λ. Let $\mathbb{P} := \mathrm{Add}(\omega, 1) * \dot{\mathrm{Coll}}(\omega_1, 2^\lambda)$. Again as in the proof of 6.12, $\mathbb{P}$ has the ω_1-covering and ω_1-approximation properties and collapses 2^λ to have size ω_1. We can therefore apply Claim 6.21 to find a $\mathbb{P}$-name $\dot{T}$ for a tree of height and size ω_1 with no cofinal branches and a $w \in H(\theta)$ such that, for every $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$, if there is $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}(\dot{T})$ that is M-generic, then M is a guessing model for λ.

Since $\mathbb{P}$ is a two-step iteration in which the first iterand is strongly proper and the second is forced to be ω_1-closed, Fact 6.19 implies that $\mathbb{P}$ is T^*-proper. Also by Fact 6.19, $\dot{\mathbb{Q}}(\dot{T})$ is forced by $\mathbb{P}$ to be T^*-proper, so $\mathbb{P} * \dot{\mathbb{Q}}(\dot{T})$ is T^*-proper. Therefore, by $\mathsf{PFA(T^*)}$ and Lemma 6.5, the set of $M \in \mathscr{P}_{\omega_2} H(\theta)$ such that $\omega_1 \cup \{w\} \subseteq M \prec H(\theta)$ and there exists an M-generic $G \subseteq \mathbb{P} * \dot{\mathbb{Q}}(\dot{T})$ is stationary in $\mathscr{P}_{\omega_2} H(\theta)$. But then, by the previous paragraph, every element of this stationary set is a guessing model for λ, so GMP^λ holds, and, since λ was arbitrary, GMP holds, as desired. □

Our result shows that IGMP does not imply that all trees of size and height ω_1 with no cofinal branches are special. However, IGMP implies that every such tree is special on a dense subset.

Proposition 6.22. *Assume* IGMP. *Let T be a tree of size and height ω_1. Then there is a dense subset S of T such that S is special.*

Proof. Assume IGMP. Let T be a tree of size and height ω_1. Note that the converse of Proposition 6.18 holds as well: IGMP implies that every tree of size and height ω_1 with no cofinal branches is persistently branchless by [3, Theorem 3.6]. Hence T is persistently branchless and therefore T is not ω_1-distributive.

Let $\dot{g}$ be a T-name for a new function from ω into the ordinals. Using the name $\dot{g}$ we can define a dense subset of T containing all nodes of T which decide the value of $\dot{g}(n)$ for some $n \in \omega$ and are minimal such; formally $S = \{t \in T \mid \exists n \in \omega((t \parallel \dot{g}(n)) \text{ and } \forall s < t(s \nparallel \dot{g}(n)))\}$. Note that S is dense since $\dot{g}$ is a name for a new function. It is straightforward to define a specialization function f from S to ω: for $s \in S$, set $f(s) = n$, where n is the minimal $n \in \omega$ such that s decides the value of $\dot{g}(n)$ and no $t < s$ decides the value of $\dot{g}(n)$. It is ease to verify that f is a specialization function for S. □

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(Lambie-Hanson) Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague 1, 115 67, Czech Republic

Email address: `lambiehanson@math.cas.cz`

URL: `https://users.math.cas.cz/~lambiehanson/`

(Stejskalová) Charles University, Department of Logic, Celetná 20, Prague 1, 116 42, Czech Republic

Email address: `sarka.stejskalova@ff.cuni.cz`

URL: `logika.ff.cuni.cz/sarka`

Institute of Mathematics, Czech Academy of Sciences, Žitná 25, Prague 1, 115 67, Czech Republic