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Two graph games

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Abstract

We isolate two simple infinite determined games on analytic graphs. With their help, we prove that certain large classes of analytic graphs have countable chromatic number.

1 Introduction

Chromatic number of graphs is a notoriously difficult topic. In this paper, we isolate two graph games which can be used to give conceptual and brief proofs of two strengthenings of countable chromatic number for analytic graphs on Polish spaces: the countable coloring number, and left-separation.

Definition 1.1. Let Γ be a graph on a set X.

1. [2] Γ has *countable coloring number* if there is a well-ordering $\leq$ on X such that for every $x \in X$, the set $\{y \in X : y \leq x \text{ and } y \,\Gamma\, x\}$ is finite.
2. If in addition X is a Polish space, Γ is *left-separated* if there is a well-ordering $\leq$ on X such that for every $x \in X$, x is not an accumulation point of the set $\{y \in X : y \leq x \text{ and } y \,\Gamma\, x\}$.

It is easy to see that countable coloring number implies left-separation, which in turn implies countable chromatic number (Corollary 3.3). Our first game provides a brief proof of a previously known dichotomy for countable coloring number.

Definition 1.2. The topological graph Δ_0 is the graph of the space X_0 which is the disjoint union of 2^ω and $2^{<\omega}$ (the former with the usual topology and the latter with discrete topology) connecting points $t \in 2^{<\omega}$ and $x \in 2^\omega$ if $t \subset x$.

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Theorem 1.3. [1] *Let Γ be an analytic graph on a Polish space X. Exactly one of the following occurs:*

1. *there is a continuous injective homomorphism from Δ_0 to Γ;*
2. *Γ has countable coloring number.*

The second game provides a quick way to verify left-separation for many graphs, even though in this case we do not get a dichotomy.

Definition 1.4. The topological graph Δ_1 is the graph of the space X_1 which is the disjoint union of 2^ω and $2^{<\omega}$ connecting points $t \in 2^{<\omega}$ and $x \in 2^\omega$ if $t \subset x$. Unlike for the graph Δ_0 of Definition 1.2, the topology on X_1 is induced by the metric $d(x, y) = 2^{-n}$ where n is the length of the longest common initial segment of x and y, so the initial segments of any point $x \in 2^\omega$ converge to x.

Theorem 1.5. *Let Γ be an analytic graph on a Polish space X. One of the following occurs:*

1. *there is a continuous injective homomorphism from Δ_1 to Γ;*
2. *Γ is left-separated.*

As a fairly immediate corollary, one can conclude that distance graphs on Euclidean spaces are left-separated and therefore countably chromatic, a result of [5]. There are many analytic graphs which satisfy both (1) and (2) of the above theorem–Δ_1 is a trivial example. To partially address this issue, we provide a game-free argument for the following.

Theorem 1.6. *If X is a Polish field, $d > 0$ is a number, and Γ is the union of countably many pairwise disjoint irreflexive-algebraic graphs on X^d, then Γ is left-separated.*

There are many open questions. In particular, we do not know if there is an exact characterization of left-separation of analytic graphs by a determined game and a minimal problematic graph.

For the notation, a graph Γ on a set X is a symmetric irreflexive relation on X. For a set $A \subseteq X$ we write $\Gamma(A) = \{x \in X : \forall y \in A\ y = x \vee y\ \Gamma\ x\}$. If X is a Polish field and $d \geq 1$ is a number, a set $A \subset X^d$ is *algebraic* if there is a finite collection $\{p_i : i \in j\}$ of polynomials with parameters in X and d many free variables such that $A = \{\bar{x} \in X^d : \forall i \in j\ p_i(\bar{x}) = 0\}$. If $Y \subset X$ is a subfield, then the set $A \subset X^d$ is Y-algebraic if the defining polynomials can be selected so that all their coefficients belong to Y. A graph Γ on X^d is *algebraic* if the set $\{\langle \bar{x}, \bar{y} \rangle \in (X^d)^2 : \bar{x}\ \Gamma\ \bar{y} \text{ or } \bar{x} = \bar{y}\}$ is an algebraic subset of X^{2d}. A graph Γ on X^d is *irreflexive-algebraic* if the set $\{\langle \bar{x}, \bar{y} \rangle \in (X^d)^2 : \bar{x}\ \Gamma\ \bar{y}\}$ is an algebraic subset of X^{2d}. Γ is *σ-algebraic* if it is a union of countably many algebraic graphs. The only fact about algebraic sets used in this paper is the Hilbert Basis Theorem: an intersection of an arbitrary collection of algebraic sets is equal to the intersection of a finite subcollection. An amalgamation operation on well-orderings is repeated in several proofs of this paper. If b is a set, $b = \bigcup_{\alpha\in\kappa} c_\alpha$ is an increasing union for some ordinal κ, and for each $\alpha \in \kappa$ a relation $\leq_\alpha$ is a well-ordering on the set c_α, the *amalgamation* of the orderings $\leq_\alpha$ is a relation on b defined as follows. If $x, y \in b$ are elements and $\alpha_x, \alpha_y \in \kappa$ are the smallest ordinals α such that $x \in c_\alpha$ and $y \in c_\alpha$ respectively, we put $x \leq y$ if either $\alpha_x \in \alpha_y$ or ($\alpha_x = \alpha_y$ and $x \leq_\alpha y$) holds. It is immediate that an amalgamation of well-orderings is a well-ordering again.

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2 Coloring number

In this section we provide a characterization of countable coloring number of analytic graphs via a simple infinite game. This provides an efficient, forcing-free proof of a dichotomy result of [1]. For the sake of exposition and parallel of the treatment in Section 3, we maintain the proof of the following well-known proposition.

Proposition 2.1. [2] *Let Γ be a graph on a Polish space X. The following are equivalent:*

1. *Γ has countable coloring number;*
2. *Γ has an orientation in which every point has finite outflow.*

Proof. To show that (1) implies (2), choose a well-ordering $\leq$ on X witnessing countable coloring number of Γ. Orient every edge of Γ from the larger to the smaller vertex in the ordering $\leq$ and observe that the outflow of every vertex is finite. To show that (2) implies (1), let $\vec{\Gamma}$ be an orientation of Γ in which every vertex has finite outflow. Say that a set $b \subset X$ is $\vec{\Gamma}$-closed if with every vertex b contains all vertices in its outflow. It is clear that every infinite subset of X is a subset of a $\vec{\Gamma}$-closed set of the same cardinality.

By transfinite induction on the cardinality of an infinite $\vec{\Gamma}$-closed set b, prove that $\Gamma \restriction b$ has countable coloring number. In the end, applying this for $b = X$, (1) will be proved. The case in which b is countable is easy: the well-ordering can be chosen as one of ordertype ω. Now, suppose that the statement has been proved for all infinite $\vec{\Gamma}$-closed sets of cardinality smaller than b. Stratify b as $b = \bigcup_{\alpha\in\kappa} c_\alpha$ as a continuous increasing union of $\vec{\Gamma}$-closed sets of cardinality strictly smaller than b. Use the inductive assumption to find a well-ordering $\leq_\alpha$ on c_α witnessing the fact that $\Gamma \restriction c_\alpha$ has countable coloring number. Let $\leq$ be the well-ordering of b which is the amalgamation of orderings $\leq_\alpha$; we claim that $\leq$ is a witness to b having countable coloring number.

To see this, suppose that $x \in b$ is any vertex. Find the smallest ordinal α such that $x \in c_\alpha$ holds; note that α must be a successor ordinal, $\alpha = \beta + 1$ for some β. Now, consider the set $d = \{y \in b \colon y \leq x \text{ and } y \mathrel{\Gamma} x\}$. The set d is the union of $d_0 = \{y \in c_\beta \colon y \leq x \text{ and } y\ \Gamma\ x\}$ and $d_1 = \{y \in c_\alpha \setminus c_\beta \colon y \leq x$ and $y\ \Gamma\ x\}$. If $y \in d_0$ then $y\ \vec{\Gamma}\ x$ cannot hold by the closure of the set c_β, therefore $x\ \vec{\Gamma}\ y$ must hold. Thus, the set d_0 must be finite, since it is a subset of the finite $\vec{\Gamma}$-outflow of the vertex x. The set d_1 is finite by the choice of the well-ordering $\leq_\alpha$. In sum, the set d is finite as required. $\square$

It is now time to define the first game of the paper.

Definition 2.2. Let Γ be an analytic graph on a Polish space X. Choose a complete compatible metric on the space X. The *coloring game* $G_0(\Gamma)$ is played between Players I and II for infinitely many rounds as follows. In round n Player I plays a point $x_n \in X$ and a basic open ball B_n of radius smaller than 2^{-n}. Player II answers with a point $y_n \in X$. Player I must make sure that his points are all distinct and $y_n \notin \bar{B}_{n+1} \subset B_n$ holds. In the end, let $z \in X$ be the unique point in $\bigcap_n B_n$–the *outcome* of the play. Player I wins if $\forall n \in \omega\ z \Gamma x_n$ holds.

In principle, the game depends on the choice of the complete compatible metric on X. However, the existence of a winning strategy is independent of that choice. This is one of the consequences of the following central theorem.

Theorem 2.3. *Let Γ be an analytic graph on a Polish space X. The game $G_0(\Gamma)$ is determined. In addition,*

1. *Player I has a winning strategy in $G_0(\Gamma)$ if and only if there is an injective continuous homomorphism from Δ_0 to Γ;*
2. *Player II has a winning strategy in $G_0(\Gamma)$ if and only if Γ has countable coloring number.*

Proof. Determinacy of the game $G_0(\Gamma)$ follows from a straightforward unraveling argument. First, use the assumption that the graph Γ is analytic to find a continuous function $k : \omega^\omega \to X^{\omega+1}$ such that its range is the set of all tuples $\langle x_n \colon n \in \omega, z\rangle$ such that for all $n \in \omega$, $x_n\ \Gamma\ z$. Then, consider the game $G_0^u(\Gamma)$ played in the same way as $G_0(\Gamma)$ except Player I in addition plays strings $t_n \in \omega^n$ such that $t_0 \subset t_1 \subset \dots$. In the end, Player I wins if he wins the play of $G_0(\Gamma)$ and in addition $k(\bigcup_n t_n) = \langle x_n \colon n \in \omega, z\rangle$. It is not difficult to see that the unraveled game is closed for Player I and therefore determined [4]. Thus, the following two claims prove the determinacy of the game $G_0(\Gamma)$:

Claim 2.4. *If Player I has a winning strategy in $G_0^u(\Gamma)$, then he has a winning strategy in $G_0(\Gamma)$.*

Proof. Game $G_0^u(\Gamma)$ is clearly more difficult for Player I to win than $G_0(\Gamma)$. $\square$

Claim 2.5. *If Player II has a winning strategy in $G_0^u(\Gamma)$, then he has a winning strategy in $G_0(\Gamma)$.*

Proof. Let σ be a winning strategy for Player II in the unraveled game $G_0^u(\Gamma)$. Let κ be an uncountable cardinal large enough such that X belongs to H_κ, the set of all sets whose transitive closure has cardinality less than κ. To describe the winning strategy for Player II in $G_0(\Gamma)$, as the game progresses, let Player II on the side construct an increasing elementary chain $\langle M_n : n \in \omega \rangle$ of countable elementary submodels of H_κ such that $\sigma \in M_0$ and $x_n \in M_n$ for every $n \in \omega$, and play the points y_n in such a way that $\{y_n : n \in \omega\} = \bigcup_n M_n \cap X$. We claim that this is a winning strategy for Player II.

Suppose towards a contradiction that Player I has a winning counterplay against this strategy, consisting of points x_n for $n \in \omega$, basic open sets U_n, and an outcome point $z \in X$. Let $u \in \omega^\omega$ be a point such that $k(u) = \langle x_n : n \in \omega, z \rangle$, and $\{y_n : n \in \omega\}$ be the sequence of responses the strategy of Player II produced. Player I will now beat the strategy σ in the game $G_0^u(\Gamma)$ in the following way. His first move is $(x_0, \emptyset, U_0)$, Player II responds with y'_0, following σ. In a subsequent round n Player I chooses $i \in \omega$ large enough so that $(x_n, u \restriction n, U_i)$ is a legal move, Player II follows σ and responds y'_n. Such $i \in \omega$ exists since the model $M = \bigcup_n M_n$ is closed under the strategy σ and it must be the case that $\{y'_k : k < n\} \subset \{y_k : k \in \omega\}$. Since Player II won in the play of the game $G_0^u(\Gamma)$, the two plays produced the same outcome z, and $\{y'_k : k \in \omega\} \subseteq \{y_k : k \in \omega\}$, he must have won the play of the game $G_u(\Gamma)$, contradicting the initial assumption. $\square$

For the right-to-left implication of item (1), let h be the injective continuous homomorphism. Let Player I play the game $G_0(\Gamma)$ in such a way that $x_n = h(t_n)$ for some binary string $t_n \in 2^{<\omega}$ and basic open sets B_n such that $t_0 \subset t_1 \subset \dots$, $y_n \notin B_{n+1}$, and $\{h(z) : z \in 2^\omega \text{ and } t_n \subset z\} \subset B_n$. This is not difficult to do. In the end, the unique point $z \in \bigcap_n B_n$ is equal to $\bigcup_n t_n$, as such is Γ-connected to x_n for all $n \in \omega$, and it is not equal to any point y_n for $n \in \omega$.

For the left-to-right implication of item (1), let σ be a winning strategy for Player I. Two claims will be helpful.

Claim 2.6. *Let p be a finite play of the game $G_0(\Gamma)$ played according to the strategy σ, ended with a move of Player I. Let $a \subset X$ be a finite set. There is an extension played according to the strategy σ such that*

1. *the last point played by I does not belong to a;*
2. *the last open set is disjoint from a.*

Proof. The first item follows from the demand that the points played by I are pairwise distinct. The second follows from the fact that Player II is free to produce the points of the set a among his answers. $\square$

Claim 2.7. *Let p be a finite play of the game $G_0(\Gamma)$ played according to the strategy σ, ended with a move of Player I. There are extensions p_0, p_1 played according to the strategy σ in which the closures of the last open sets played by Player I are disjoint.*

Proof. Produce any infinite extension of the play p using the strategy σ. Let $z_0 \in X$ be the final point of the play. Consider any infinite play of the strategy σ extending p in which Player II plays z_0 as one of his moves. Let z_1 be the final point of this second play. Since $z_0 \neq z_1$ and the diameters of open sets played by the strategy σ must converge to zero, there must be initial segments p_0, p_1 of the two plays in which the last open sets played by the strategy σ are disjoint. □

Now, by tree induction on $t \in 2^{<\omega}$ build finite plays p_t of the game $G_0(\Gamma)$ in which Player I follows his strategy σ, produces final move x_t, B_t, the points x_t for $t \in 2^{<\omega}$ are pairwise disjoint, and for each $t \in 2^{<\omega}$ the sets $B_{t^\frown 0}$ and $B_{t^\frown 1}$ are pairwise disjoint, and they do not contain any of the points x_s for binary strings s shorter than t. This is easily possible using the claims. In the end, let h be the map defined by $h(t) = x_t$ and $h(z) =$ the unique point in $\bigcap_{n\in\omega} B_{z\restriction n}$. It is not difficult to check that this is a continuous injective homomorphism from Δ_0 to Γ.

For the left-to-right direction of item (2), suppose that Player II has a winning strategy σ in the game $G_0(\Gamma)$. The following claim is key.

Claim 2.8. *If $b \subset X$ is a set closed under the strategy σ and $z \in X \setminus b$ is a point, then the set $\{x \in b \colon x \; \Gamma \; z\}$ is finite.*

Proof. Otherwise, Player I can play points x_n in the infinite set $\{x \in b \colon x \; \Gamma \; z\}$ and the basic open neighborhods B_n so that they all contain the point z. In the end of such a play against the strategy σ, he wins, since the set b is closed under the strategy σ, so all points the strategy σ plays are in b and therefore not equal to z. This contradicts the choice of σ. □

Now, by transfinite induction on the infinite cardinality of a set $b \subset X$ closed under the strategy σ argue that $G \restriction b$ has countable coloring number. This is clear for countable cardinality of b since then the witnessing well-ordering of b can be chosen of ordertype ω. Now suppose that $b \subset X$ is an infinite set closed under the strategy σ and for all sets of cardinality smaller than $|b|$ the statement has been proved. Stratify $b = \bigcup_{\alpha\in\kappa} c_\alpha$ as a well-ordered continuous increasing union of sets of smaller cardinality, all of them of closed under the strategy σ. For each $\alpha \in \kappa$ pick a well-ordering $\leq_\alpha$ on c_α witnessing the countable coloring number of $\Gamma \restriction c_\alpha$. Now, let $\leq$ be the amalgamation well-ordering on b and argue that it witnesses the left-separation of the graph $\Gamma \restriction b$.

Suppose that $x \in b$ is an arbitrary point and argue for the finiteness of the set $a = \{y \in b \colon y \leq x \text{ and } y \; \Gamma \; x\}$. Let $\alpha \in \kappa$ be the first ordinal such that $x \in c_\alpha$ holds. By the continuity of the stratification, the ordinal α is a successor of some β. The set $a \cap c_\beta$ is finite by Claim 2.8 applied to c_β and x. The set $a \setminus c_\beta$ is a subset of $\{y \in c_\alpha \colon y \leq_\alpha x \text{ and } y \; \Gamma \; x\}$ which is finite by the choice of the well-ordering $\leq_\alpha$. It follows that the set $a = (a \cap c_\beta) \cup (a \setminus c_\beta)$ is finite as required.

The right-to-left direction of item (2) is proved in the contrapositive. If Player II has no winning strategy, then Player I must have it, and by (1) there must be a continuous homomorphic injection of Δ_0 to Γ. To conclude that Γ does not have countable coloring number, it is then enough to show that Δ_0 does not have it. Indeed, suppose towards a contradiction that $\vec{\Delta}_0$ is an orientation of Δ_0 in which every vertex has finite outflow. By a counting argument, there must be a point $x \in 2^\omega$ which does not belong to the outflow of any point in $2^{<\omega}$. By another counting argument, there must be an initial segment $t \subset x$ which does not belong to the outflow of x. Since $\{t, x\}$ is an edge in Δ_0, this is a contradiction. $\square$

Theorem 1.3 is now an immediate corollary.

Example 2.9. [3] If a is a countable set of positive reals and Γ is the graph on $\mathbb{R}^2$ connecting points whose distance belongs to a, then Γ has countable chromatic number. To construct a winning strategy for Player II, just make sure as Player I produces points x_n for $n \in \omega$, if n_0, n_1 are distinct numbers and r_0, r_1 are elements of a, the (at most two) points in $\mathbb{R}^2$ which have distance r_0 from x_{n_0} and distance r_1 from x_{n_1} are on the sequence Player II plays. This is easy to do, and it constitutes a winning strategy for Player II.

3 Left-separation

Left-separation of graphs on Polish spaces is characterized by a tool parallel to orientation with finite outflows. This tool has the advantage of abstracting away from well-orderings.

Definition 3.1. Let Γ be a graph on a Polish space X. A Γ-*neighborhood assignment* is a function π which to every point $x \in X$ assigns an open neighborhood $\pi(x) \subset X$ of x so that if x, y are Γ-related points then either $x \notin \pi(y)$ or $y \notin \pi(x)$.

Proposition 3.2. *Let Γ be a graph on a Polish space X. The following are equivalent:*

1. *Γ is left-separated;*
2. *there is a Γ-neighborhood assignment.*

Proof. To show that (1) implies (2), choose a well-ordering $\leq$ on X witnessing left-separation of Γ. For every point $x \in X$, let $\pi(x) \subset X$ be any open neighborhood of x containing no points of the set $\{y \in X : y \leq x \text{ and } y \mathrel{\Gamma} x\}$. Clearly, this is a Γ-neighborhood assignment. To show that (2) implies (1), let π be a Γ-neighborhood assignment. Let d be a compatible metric on the space X. For every number $n \in \omega$, let $A_n = \{x \in X : \pi(x)$ contains an open ball of radius 2^{-n} around $x\}$. Clearly, The sets A_n increase with respect to inclusion as n grows, and $X = \bigcup_n A_n$. Let $\leq$ be any well-ordering on X in which A_n is an initial segment for every $n \in \omega$. We claim that $\leq$ witnesses left-separation of the graph Γ.

Indeed, suppose that $y \leq x$ are Γ-connected points in the space X and n is the smallest number such that $x \in A_n$ holds. Since $y \in A_n$ and π is a Γ-neighborhood assignment, $d(x, y) \geq 2^{-n}$ must hold. Thus, the open 2^{-n}-ball around x contains no points of the set $\{y \in X : y \leq x \text{ and } y \, \Gamma \, x\}$ as desired. $\square$

Corollary 3.3. *Let Γ be a graph on a Polish space X. If Γ is left-separated, then it is countably chromatic.*

Proof. Let π be a Γ-neighborhood assignment. Shrinking the values of π if necessary, we may assume that they are all basic open sets. Then π is in fact a Γ-coloring with a countable range. $\square$

It is important to understand that, unlike the coloring number, the definition of left-separation and Γ-neighborhood assignments depends on the topology of the underlying space.

Example 3.4. Let Δ_2 be a graph on 2^ω connecting points x, y if x contains only finitely many 1's, y contains infinitely many 1's, and all 1's of x appear in $x \cap y$, the longest common initial segment of x and y. Then Δ_2 is not left-separated.

It is obvious that if 2^ω is equipped with a larger topology in which the set of its rational points is clopen, the graph Δ_2 becomes left-separated.

Proof. Suppose that π is a neighborhood assignment on 2^ω; shrinking π if necessary, we assume that its values are in fact clopen sets. It will be enough to find a point y and points x_n for $n \in \omega$ such that $\lim_n x_n = y$ and for all $n \in \omega$ $x_n \Gamma y$ and $y \in \pi(x)$ both hold. Then $\pi(y)$ must contain all but finitely many points x_n, violating the definitory property of a Γ-neighborhood assignment.

To find the points x_n, just construct them so that each of them has exactly n many unit entries, all unit entries of x_n appear in $x_n \cap x_{n+1}$, and $x_n \in \bigcap_{m \in n} \pi(x_m)$. This is easy to accomplish. In the end, let $y = \lim_n x_n$. It is clear from the construction that y is Γ-connected with every point x_n, and belongs to $\pi(x_n)$ as well. This concludes the proof. $\square$

There is a natural graph game associated with existence of a Γ-neighborhood assignment. Even though this time the game does not provide a full characterization, it still provides interesting information difficult to obtain otherwise.

Definition 3.5. Let Γ be a graph on a Polish space X. The game $G_1(\Gamma)$ proceeds as follows. Players I and II alternate for ω many rounds, playing points $x_n \in X$ and $y_n \in X$ for $n \in \omega$. Player I wins if the points x_n converge to some point $z \in X$ (this point is referred to as the *outcome* of the play), and for each $n \in \omega$, $z \, \Gamma \, x_n$ and $z \neq y_n$ both hold.

Theorem 3.6. *Let Γ be an analytic graph on a Polish space X. The game $G_1(\Gamma)$ is determined. Moreover,*

1. *Player I has a winning strategy if and only if there is a continuous homomorphic injection from Δ_1 to Γ;*

2. *if Player II has a winning strategy then Γ is left-separated.*

Proof. The determinacy proof is very close to that of Theorem 2.3. We only indicate the unraveled version of the game and leave the rest to the reader. Fix a continuous function $k\colon \omega^\omega \to X^{\omega+1}$ such that its range consists exactly of points $\langle x_n \colon n \in \omega, z\rangle$ such that $z = \lim_n x_n$ and for every $n \in \omega$, $x_n\ \Gamma\ z$. The unraveled game $G_1^u(\Gamma)$ is played just as $G_1(\Gamma)$, except Player I in addition plays finite sequences t_n of natural numbers of length at least n such that y_n does not belong to the closure of the set $\{k(u)(\omega) \colon t_{n+1} \subset u\} \subset X$; in addition, the sequences t_n are required to form an inclusion increasing sequence. Player I wins if $\langle x_n \colon n \in \omega\rangle = k(\bigcup_n t_n) \restriction \omega$. This is a closed game for Player I, therefore determined. It is more difficult for Player I than the original game $G_1(\Gamma)$, and Player II has a winning strategy in the original game if and only if he has a winning strategy in the unraveled game.

The right-to-left implication of item (1) is easy. Let $\pi\colon X_1 \to X$ be the injective continuous homomorphism of Δ_1 to Γ. Let Player I play points $x_n = \pi(t_n)$ where $t_n \in 2^n$ form an inclusion increasing sequence such that $y_n \notin \pi''\{z \in 2^\omega \colon t_{n+1} \subset z\}$. This is clearly a winning strategy for Player I.

The left-to-right implication of item (1) is notationally more difficult. Let σ be a winning strategy for Player I. Let d be a compatible complete metric on X. For a finite play p of the game let $z(p)$ be the set of all points that Player I can produce as outcomes in plays extending p and observing the strategy σ. The following claims will streamline the construction.

Claim 3.7. *Let $a \subset X$ be a finite set and p a finite play observing the strategy σ. There is an extension q of p observing σ such that $z(q)$ is disjoint from a.*

Proof. If this failed for some set a and a play p, Player II could win against the strategy σ from p by listing all points in a among his points and finding an infinite play in which the strategy σ produces infinitely many points arbitrarily close to points in a. Note that the outcome of the play would have to belong to the set a. $\square$

Claim 3.8. *Let $\varepsilon > 0$ be a positive real and p a finite play of the game observing the strategy σ. There is an extension q of p observing σ such that $z(q)$ has metric diameter smaller than ε.*

Proof. If this failed for some $\varepsilon > 0$ and p, then Player II could find a counterplay starting with p in which the strategy σ produces points x_n for $n \in \omega$ in such a way that for every n there is $m \geq n$ such that $d(x_m, x_n) > \varepsilon/2$. Such a sequence of points cannot converge, and Player II would win the play, contradicting the choice of the strategy σ. $\square$

Claim 3.9. *Let p be a finite play of the game observing the strategy σ. There are extensions q_0, q_1 of p observing the strategy σ such that the closures of the sets $z(q_0)$ and $z(q_1)$ are disjoint.*

Proof. Find two infinite extensions r_0, r_1 of p such that for every $\varepsilon > 0$ there is an initial segment $q_{\varepsilon 0} \subset r_0$ such that $z(q_{\varepsilon 0}) < \varepsilon$, similarly for r_1, and in addition, the outcome z_0 of r_0 is one of the points Player II played in the play r_1. This is possible by the previous claim. The outcome z_1 of r_1 is distinct from z_0, and they are at a distance $\delta > 0$. Find finite intermediate stages q_0, q_1 or r_0, r_1 such that the sets $z(q_0), z(q_1)$ both have diameters smaller than $\delta/4$. It is not difficult to see that q_0, q_1 work as desired. $\square$

Now, by a tree induction on $t \in 2^{<\omega}$ produce points x_t and plays p_t observing the strategy σ so that $s \subset t$ implies $p_s \subset p_t$, x_t is among the points produced by the strategy σ in p_t, the points x_t for $t \in 2^{<\omega}$ are pairwise distinct, and for every $n \in \omega$ the sets $z(p_t)$ for $t \in 2^n$ are pairwise disjoint and contain none of the points x_s for $s \in 2^n$. This is easily possible by the three above claims. In the end, let $h\colon 2^{<\omega} \cup 2^\omega \to X$ be the map defined as follows. $h(t) = x_t$ for $t \in 2^{<\omega}$, and $h(y)$ =the outcome of the play $\bigcup_n p_{y \restriction n}$ for $y \in 2^\omega$. It is not difficult to see that the map h is a continuous injective homomorphism of Δ_1 to Γ.

For the second item, suppose that σ is a winning strategy for Player II. The following is immediate.

Claim 3.10. *Let $b \subset X$ be a set closed under the strategy σ and $z \in X \setminus b$ be a point. Then z is not an accumulation point of the set $\{y \in b\colon z \; \Gamma \; y\}$.*

Proof. Otherwise, Player I can construct a winning counterplay using a sequence of points in b whose limit is z. $\square$

Now, by induction on cardinality of an infinite set $b \subseteq X$ closed under the strategy σ prove that $\Gamma \restriction b$ is left-separated. Applied to the set $b = X$, this provides item (2). Now, the induction statement is immediate for a countable set b, since the well-ordering $\leq$ can have ordertype ω in this case. Now suppose that $b \subseteq X$ is a set closed under σ such that left-separation has been proved for all infinite sets closed under the strategy σ of smaller cardinality.

Stratify $b = \bigcup_{\alpha \in \kappa} c_\beta$ as a continuous increasing union of sets closed under σ of strictly smaller cardinality than b. For each ordinal $\alpha \in \kappa$ use the induction assumption to select a well-ordering $\leq_\alpha$ on c_α witnessing the fact that $\Gamma \restriction c_\alpha$ is left-separated. Amalgamate the well-orderings $\leq_\alpha$ for $\alpha \in \kappa$ to obtain a well-ordering $\leq$ on b. Now, argue that the resulting well-ordering $\leq$ on the set b witnesses left-separation of $\Gamma \restriction b$.

To see this, let $x \in b$ be an element and α_x be the smallest ordinal $\alpha \in \kappa$ such that $x \in c_{\alpha+1}$. Then, the set $\{y \in b\colon y \leq x \text{ and } y \; \Gamma \; x\}$ is a subset of the union of the sets $\{y \in c_\alpha\colon x \; \Gamma \; y\}$ and $\{y \in c_{\alpha+1}\colon y \leq_{\alpha+1} x \text{ and } x \; \Gamma \; y\}$. The former set does not have x as an accumulation point by Claim 3.10 applied to the set c_α; the latter set does not have x as an accumulation point by the choice of $\leq_{\alpha+1}$. This concludes the induction step. $\square$

Theorem 1.5 is now an immediate corollary. There are other colorful corollaries.

Corollary 3.11. *Let X be a Polish field. Let Γ be a closed σ-algebraic graph on X^d for some $d \geq 1$. Exactly one of the following occurs:*

1. Γ *contains a perfect clique;*

2. *there is a* Γ*-neighborhood assignment.*

Proof. It is immediate that (1) implies the negation of (2); we need to show that the negation of (1) implies (2). For every closed set $A \subset X^d$, let $h(A) = \{x \in A \colon \forall y \in A\ x = y \lor x \Gamma y\}$. Since Γ is a closed graph and A is closed, $h(A)$ is closed. The set $h(A)$ is a Γ-clique by its definition. If there is a closed set A for which $h(A)$ is uncountable, then $h(A)$ contains a perfect clique and item (1) prevails. Thus, assume that for every closed set A, $h(A)$ is countable and work to find a winning strategy for Player II in the game $G_1(\Gamma)$, verifying item (2).

Let $\Gamma = \bigcup_n \Gamma_n$ be a presentation of Γ as a union of countably many algebraic graphs on X^d. Consider the strategy for Player II in which, if Player I produces points x_i for $i \in \omega$, Player II enumerates all points in sets $h(A)$, where A runs through all possible sets of the form $\bigcap_{i \in a} \Gamma_{n_i}(x_i)$ where $a \subset \omega$ is a finite set and $\langle n_i \colon i \in a \rangle$ is a tuple of natural numbers. Since the sets $h(A)$ are all countable, this is easy for Player II to do. We will argue that this is a winning strategy for Player II.

To this end, suppose that points x_i for $i \in \omega$ converge to z, and $\forall i\ x_i\ \Gamma\ z$ holds. For each $i \in \omega$ let $n_i \in \omega$ be a number such that $x_i\ \Gamma_{n_i}\ z$ holds. Use the Hilbert Basis Theorem to find a finite set $a \subset \omega$ such that $A = \bigcap_{i \in \omega} \Gamma_{n_i}(x_i)$ is equal to the set $\bigcap_{i \in a} \Gamma_{n_i}(x_i)$. The set $h(A)$ is countable, and all of its elements are among the points the strategy for Player II produced when confronted with the points x_i for $i \in \omega$. Thus, it will be enough to show that $z \in h(A)$ holds. And indeed, if $y \in A$ is any point distinct from z, then y is Γ-connected to every point $x_i \in \omega$, and by the closure of the graph Γ, it is also connected to z as well. $\square$

Corollary 3.12. *Let X be a Polish field. Let Γ be a closed σ-algebraic graph on X^d for some $d \geq 1$. Suppose that $\Gamma = \bigcup_m \Gamma_m$ is a union of algebraic graphs such that*

(*) *for any distinct points $x, y \in X$ there is an open neighborhood O of x such that for every $z \in O$ and every $m \in \omega$, at least one of $x\ \Gamma_m\ z$, $y\ \Gamma_m\ z$ fails.*

Then there is a Γ*-neighborhood assignment.*

Proof. By Theorem 2.3, it is enough to produce a winning strategy for Player II in the game $G_1(\Gamma)$. To do this, let Player I play a sequence of points $x_n \in X$ for $n \in \omega$. Player II plays in such a way that if $n_0 \in \omega$ is a number and m_n for $n \in n_0$ are numbers and the set $\bigcap_{n \in n_0} \Gamma_{m_n}(x_n)$ is a singleton, then the unique point in the intersection appears on the sequence played by Player II. This is easily possible as there are only countably many finite strings of natural numbers.

In the end, Player II must have won. To see this, suppose that $z = \lim_n x_n$ and the point z is Γ-connected to every point x_n for $n \in \omega$. It follows that there must be numbers m_n for $n \in \omega$ such that $z \, \Gamma_{m_n} \, x_n$ holds. By the Hilbert Basis Theorem, there must be a number n_0 such that $C = \bigcap_{n \in \omega} \Gamma_{m_n} x_n = \bigcap_{n \in n_0} \Gamma_{m_n} x_n$. The set C must be the singleton $\{z\}$: if it contained another point w different from z, there would be an open neighborhood $O \subset X$ of z witnessing (*) of the assumptions for Γ, there would have to be a number $n \in \omega$ such that $x_n \in O$, and then the point x_n and the number m_n would violate the choice of the set O. It follows that the point z has appeared on the sequence of points that Player II produced, resulting in a victory for Player II. □

Example 3.13. Let $d \geq 1$ be a number, let a be a countable set of positive reals, and let Γ be the graph on $\mathbb{R}^d$ connecting points x, y if their Euclidean distance belongs to the set a. Clearly $\Gamma = \bigcup_m \Gamma_m$ where $\{r_m : m \in \omega\}$ is any listing of the set a and Γ_m connects points of Euclidean distance r_m. Then (*) is satisfied: for distinct points $x, y \in \mathbb{R}^n$ choose O to be any open ball around x whose radius is smaller than half of the distance between x and y. By Corollary 3.12, the graph Γ is left-separated. This reproves a result of Schmerl [6, Theorem 4].

In the last theorem of this paper, we provide a proof of left-separation for σ-algebraic graphs which works even in some cases where Player I has a winning strategy in the game G_1.

Theorem 3.14. *Let $d > 0$ be a natural number. Let X be a Polish field and Γ be a σ-algebraic graph on X^d. Suppose that Γ is a countable disjoint union of irreflexive-algebraic graphs. Then Γ is left-separated.*

Proof. Fix a partition $\Gamma = \bigcup_n \Gamma_n$ into pairwise disjoint irreflexive-algebraic graphs. Let $Y \subset X$ be a subfield. Say that π is a *suitable* map on Y if

- $\mathrm{dom}(\pi)$ consists of all Y-algebraic subsets of X^d;
- for every Y-algebraic set A, $\pi(A) \subset X^d$ is an open neighborhood of A;
- $A \subseteq B$ implies $\pi(A) \subseteq \pi(B)$;
- for every $A \in \mathrm{dom}(\pi)$, for all but finitely many $n \in \omega$, $\pi(\Gamma_n(A)) \cap A = 0$.

By induction on cardinality of a subfield $Y \subset X$ we now show that there is a suitable map on Y. To conclude the proof of the theorem from this, let π be a suitable map on X. Let χ be any neighborhood assignment on X^d such that $\chi(x) \subset \pi(\{x\})$ and $\chi(x)$ is disjoint from the finitely many sets $\Gamma_n(\{x\})$ for which $\pi(\Gamma_n(\{x\}))$ does not contain the point x. It is not difficult to see, that this will be the desired Γ-neighborhood assignment.

Suppose first that Y is countable, and enumerate nonempty Y-algebraic sets as $\{A_i : i \in \omega\}$. By recursion on $i \in \omega$ build the values $\pi(A_i)$ so that (a) for $i, j \in \omega$, $A_j \subseteq A_i$ implies $\pi(A_j) \subseteq \pi(A_i)$; and (b) for every $j < i$, either $A_j \cap A_i \neq 0$, or $\pi(A_i) \cap A_j = 0$, or there is $k < j$ such that $A_k \subseteq A_i$ and $\pi(A_k) \cap A_j \neq 0$. To see how this is done, suppose the values of $\pi(A_j)$ for $j < i$ have been constructed. Let $O \subset X^d$ be an open neighborhood of the set A_i which is disjoint from A_j for all $j < i$ such that $A_j \cap A_i = 0$, and let $\pi(A_i) = (O \cup \bigcup\{\pi(A_j) : j < i \text{ and } A_j \subseteq A_i\}) \cap \bigcap\{\pi(A_j) : j < i \text{ and } A_i \subseteq A_j\}$.

First note that this choice of $\pi(A_i)$ preserves the recursion hypothesis. Item (a) is satisfied at i by the definition of $\pi(A_i)$ and item (a) below i. To verify item (b), suppose that $j < i$ is a number such that $A_i \cap A_j = 0$ and $\pi(A_i) \cap A_j \neq 0$. Then, by the definition of $\pi(A_i)$, there must be some $l < i$ such that $A_l \subset A_i$ and $\pi(A_l) \cap A_j \neq 0$. There are two cases. If $l < j$ then let $k = l$ and observe that k witnesses (b) for j and i. If $l > j$, find $k < j$ witnessing (b) for j and l and observe that k witnesses (b) for j and i.

Second, observe that the map π is then suitable on Y. To see this, let $A \subset X^d$ be a nonempty Y-algebraic set. Find $j \in \omega$ such that $A = A_j$. The sets $\Gamma_n(A)$ are all Y-algebraic and disjoint from A_j by the irreflexivity assumptions. Since the graphs Γ_n are pairwise disjoint, for all but finitely many $n \in \omega$ there is no $k < j$ such that $A_k \subseteq \Gamma_n(A)$. Item (b) then shows that for all such n, $\pi(\Gamma_n(A))$ must be disjoint from A as desired.

Now suppose that $Y \subset X$ is a subfield of uncountable cardinality κ and for all subfields of smaller cardinality suitable maps have been produced. Express $Y = \bigcup_{\beta \in \alpha} Y_\beta$ as a continuous increasing union of subfields of smaller cardinality, for some ordinal α. By recursion on $\beta \in \alpha$ build suitable maps π_β on Y_β in such a way that $\gamma \in \beta$ implies $\pi_\gamma \subset \pi_\beta$. At limit ordinals β, take a union and observe that all items above are preserved. Now, suppose that π_β has been constructed, and work to find $\pi_{\beta+1}$.

First, use the induction hypothesis to find a suitable map π on $Y_{\beta+1}$. The difficulty now is that π may not extend π_β. Thus, define $\pi_{\beta+1}$ in the following way. Let $\pi_{\beta+1}(A) = (\pi(A) \cap \pi_\beta(B)) \cup \bigcup\{\pi_\beta(C) : C \subseteq A \text{ is a } Y_\beta\text{-algebraic set}\}$, where B is the smallest Y_β-algebraic set containing A as a subset.

It is immediate that $\pi_{\beta+1}$ is a map which preserves inclusion, and it extends π_β. Now, let $A \subset X^d$ be an $Y_{\beta+1}$-algebraic set, and work to show that for all but finitely many $n \in \omega$, the set $\pi_{\beta+1}(\Gamma_n(A))$ is disjoint from A. Let $B \subset X$ be the smallest Y_β-algebraic set containing A as a subset.

Claim 3.15. *For every number $n \in \omega$, $\Gamma_n(B)$ is the largest Y_β-algebraic subset of $\Gamma_n(A)$.*

Proof. Suppose that $C \subseteq \Gamma_n(A)$ is an Y_β-algebraic set. Consider the set $D = \Gamma_n(C)$. D is Y_β-algebraic and it is a superset of A. Therefore, $B \subseteq D$, which implies that $C \subseteq \Gamma_n(D) \subseteq \Gamma_n(B)$ holds as desired. □

Thus, for every number $n \in \omega$, $\pi_{\beta+1}(\Gamma_n(A)) \subseteq \pi(\Gamma_n(A)) \cup \pi_\beta(\Gamma_n(B))$. For all but finitely many $n \in \omega$, the first entry of this union is disjoint from A and the other from B by the choice of π and π_β. Since $A \subseteq B$ holds, for all but finitely many n the union is disjoint from A as desired. This concludes the transfinite recursion step and the proof of the theorem. □

4 Questions

Many questions regarding Γ-neighborhood assignments remain unresolved. In the following, Γ stands for an analytic graph on a Polish space. However, the questions are already interesting for smaller classes of graphs such as σ-algebraic graphs on Euclidean spaces.

Question 4.1. Are the following equivalent:

1. Γ has countable chromatic number;
2. Γ is left-separated in a suitable Polish topology on X.

Question 4.2. Are the following equivalent:

1. Γ is left-separated;
2. there is no continuous homomorphic injection from Δ_2 to Γ.

Question 4.3. Is the statement "Γ is left-separated" absolute between models of set theory?

Question 4.4. For an analytic set $A \subset X$, is the statement "$\Gamma \cap (A \times A)$ is left-separated" $\mathbf{\Pi}^1_1$?

References

[1] Francis Adams and Jindřich Zapletal. Cardinal invariants of closed graphs. *Israel Journal of Mathematics*, 227:861–888, 2018.

[2] Paul Erdős and András Hajnal. On chromatic number of graphs and set systems. *Acta Math. Acad. Sci. Hung.*, 17:61–99, 1966.

[3] Paul Erdős and Péter Komjáth. Countable decompositions of $\mathbb{R}^2$ and $\mathbb{R}^3$. *Discrete and Computational Geometry*, 5:325–331, 1990.

[4] David Gale and F. M. Stewart. *Infinite games with perfect information*, pages 245–266. Annals of Mathematics Studies 28. Princeton University Press, 1953.

[5] Péter Komjáth. A decomposition theorem for $\mathbb{R}^n$. *Proc. Amer. Math. Soc.*, 120:921927, 1994.

[6] Péter Komjáth. The list-chromatic number of infinite graphs defined on euclidean spaces. *Discrete Comput. Geom.*, 45:497502, 2011.