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Modelling an interconnected economy - general equilibrium and matching model approaches

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Introduction

This doctoral dissertation consists of three theoretical articles with a common interest in the creation of interconnections in an economy and the role of these interconnections in spreading bad shocks. The spread of bad shocks has been explored in economics literature mainly in the context of contagious bank runs but the following articles focus on different aspects of economy where the effects of deterioration of an economy's characteristics occur.

Interconnections in economies are often necessary for production or they serve as an insurance tool but they at the same time expose the economy to a possible spread of damage. The mechanism is very simple. A negative shock that hits one agent in the economy can spread through interconnections in the economy, such as financial linkages between banks, supply linkages between firms, or personal linkages, like marriage, between agents.

In the following models I study optimal creation of interconnections in two different types of economy and I also study properties of an interconnected economy in general equilibrium.

In the first article I propose a general equilibrium model of an economy where firms are connected through supply linkages crucial for their production. I study the properties of the proposed model and compare it to a benchmark model without linkages. The model with supply linkages exhibits lower aggregate level of production but in case of increase of individual fluctuations of firms the supply linkages help to boost aggregate production, i.e. the production can be increasing at the margin while it is always decreasing in the benchmark model.

The second and the third article are theoretical matching models. In the second article I construct a model in which agents search for partners to establish a pair interaction that brings them profit. The agents differ in their probabilities of exit from the economy. The composition of every pair determines its expected lifetime and profits the agents have from the interaction. The model allows for the study of equilibrium properties of the matching market with entry and exit of agents. Optimal individual decisions of accepting or rejecting each particular type of match are analyzed. It is shown that for a certain range of parameters multiple equilibria exits. Social optimality of agents' decisions is assessed and it is shown that for a range of parameters the social planner is able to impose Pareto improving matchings.

The third article is a matching model between firms and workers. Workers do not differ in their productivity but they differ in their probability of leaving the labor force. Firms choose whether to accept a match with workers depending on their type and they also choose whether and when to fire them. A stationary situation of the economy is considered. It is shown that several types of stable matching can be a stable stationary equilibrium of the economy depending on the parameters of the model. Multiplicity of equilibria occurs for some ranges of parameters. Stable matchings chosen by the agents are shown to be almost always socially suboptimal. The planner's solution is never Pareto improving for the agents.

Contents

Chapter 1

Supply Linkages Creation - a General Equilibrium Approach

1.1 Introduction

This paper studies an equilibrium model of an economy where for an exchange between firms they need to have a specific exchange relationship. In standard macroeconomic literature the exchange between firms happens through a market without any special arrangements. Every period the given firm goes to the market to find a firm to exchange with, without questioning its identity. Microeconomic literature has often a more complicated view on exchange, considering repeated exchange between two specific firms and all the arrangements that can exist between them. In his paper Ben-Porath [2] describes it as follows: "Parties to a transaction can establish rules or norms for their exchange relationship, a common view concerning contingencies, and procedures for settling disputes that can serve them beyond a single transaction. The cost of negotiating and establishing these rules will have to be incurred again if the parties change." Recently also macroeconomists are not completely satisfied with their simplified picture of firms' exchange and have started to consider firms organized in networks rather than anonymous markets.

Relationships between firms can, of course, be of very different nature, going from really tight to really loose financial, commercial or other types of relationships. These different arrangements are often described as Hybrid Organizations [16]. Despite the fact that there is a whole literature dedicated to theory of hybrid forms of firms, due to confidentiality of business data there are only a few empirical studies dealing with the topic.

In France, in 2003, the Ministry of Economy, Finance and Industry organized a Survey of inter-firm relations [18]. This survey provides a strong evidence of inter-firm relations: "In industrial groups 82% of producers are organized in a system of cooperation among firms." The survey shows not only that firms tend to enter industrial relationships, but also that they tend to stay in them. It shows that 52% of relations of independent industrial firms last for more than 5 years. This number is valid for relationships outside of industrial groups. Inside the groups it is up to 79% of relationships that last for more than 5 years.

For the United States there exists an empirical study by Lafontaine and Shaw [14] focused on franchises, one specific form of hybrid organization. They show on their sample of almost twelve thousands firms that the mean for the duration of franchising is 9.4 years. Moreover, this paper also clearly shows that relationships between firms are costly to establish. In this context a franchise fee needs to be payed to establish the relationship. The mean of franchise fees for the given sample of firms is 23 300 US dollars.

The evidence above only confirms a generally accepted idea that inter-firm relations are common and they have often long term character. In this light the market of firms where each period firms find a new exchange partner does not seem to be an entirely appropriate model of a real economy. The following paper proposes a new step from standard economic theory towards a model of an economy with firms interconnected by long term linkages.

A simple dynamic model of economy with heterogeneous firms building supply linkages is presented. The linkages are a necessity for one specific type of exchange in the economy. Building linkages is costly but the linkages can persist over time. Heterogeneity of firms causes that sometimes linkages have to be newly constructed. The aim of the model is to explore the properties of the economy with linkages in a steady state when compared to a standard economy. Because the aim is to capture purely the effect of building linkages, we assume a very simple setting with no storage technology available in the economy. This allows us to study the effect of building linkages isolated because the possibility of storage would have an impact on the properties the economy exhibits.

Models of economies with interconnected firms have two aspects that are different from standard markets. The structure of connections allows for the pooling of the risk as the linkages guarantee a certain stability in supply and demand of goods. Of course, as building linkages is costly, the number of connections each firms has is limited and therefore the risk of losing supply is not fully eliminated. On the other hand, connections allow for transmissions of bad shocks in the economy. If a firm is hit by a bad shock affecting its production then this can provoke a chain reaction where all the firms connected with the given firm can suffer a lower or nonexisting supply. This has a direct negative effect on their production or they can decide to replace the concerned connection by a new one, which is costly.

The positive effect of risk pooling and chain reactions of negative shocks on structures of interconnected firms are effects that we can not observe in standard markets and that is why it is important to see what properties gives the composition of these two effects compared to the standard market economy.

In the past economies with inter-firm relations have been discussed from several angles in the economic literature. A model of buyer-seller networks has been explored by Kranton and Minehart [13]. Their paper shows solid empirical motivation for the existence of network structures in several industries. They ask why these structures arise in real economies. And they answer by showing that networks alow for the pooling of the risk. On the other hand they also show that in a non-cooperative environment the networks that arise are not always socially optimal.

The idea of financial interdependence has been explored by Kiyotaki and Moore [7]. Their study explores the consequences of a small temporary shock to the liquidity of firms that are part of a network interconnected by financial obligations. They conclude that although on an individual level firms are able to deal with this problem by rescheduling the debt, in aggregate it may lead to serious consequences such as a large and persistent fall of production. A similar idea is explored by the same authors in Kiyotaki and Moore [6]. They show that the dynamic interaction between credit limits and asset prices is a strong transmission mechanism by which the shocks can persist, amplify and also transmit to other sectors of the economy. Small temporary shocks to technology or to income distribution can have a large effect, create fluctuations in asset prices and in output of the economy.

The idea of firms interconnected in a network structure has been explored in a paper by Kakade, Kearns, and Ortiz [10]. Their model is a general equilibrium one on a fixed network. The network structure implies existence of local prices that depend on interconnections in the network. The distribution of prices across the network is an equilibrium result. An important fact to note is that the network does not evolve over time, it is fixed and agents can not take any decisions that would change the network's structure.

In empirical literature the idea of interdependence in an industry has been used by Elsinger, Lehar, and Summer [4]. They study vulnerability of the Austrian banking system to a contagion of bankruptcies. Based on Austrian data they identify the network of credit links in between Austrian banks and they use a scenario analysis to test for a possible contagion. Their approach is suitable for testing contagion effects on any type of static network but they do not introduce any dynamics into the model.

This paper is a first step in modelling of firms' connections in a dynamic general equilibrium setting. The novelty of the model is in the endogenous creation and destruction of supply linkages of firms. Linkages are modelled explicitly for each firm but connections between firms are not explicit, they pass through a market, to allow for market clearing as in standard general equilibrium models.

A possible next step in this line of research is to add fixed costs of production to the model in the spirit of the models of Hopenhayn[7] and Hopenhayn[8]. This should allow us to observe endogenous exit of firms in the economy. Considering one of the models of Hopenhayn as a benchmark we can then compare whether the endogenous loss of linkages in the proposed model increases or decreases the fraction of firms that decide to exit the economy. An increase in exits would mean that the linkages cause a spread of bad shocks in the economy. A decrease on the other hand would mean that linkages ensure more stable supply of intermediary good that would prevent from exits.

1.2 The Economy

Time in the economy is discrete and the horizon is infinite. The economy is populated by intermediate production firms and final production firms, as well as by a homogeneous mass of workers. Each of these three groups of agents is of mass 1. Workers are employed by intermediaries who produce an intermediary good. The intermediary good serves as an input in the production of the final producers who produce a consumption good for workers. All the goods are non-storable. Prices are established competitively and the economy is studied in a stationary recursive competitive equilibrium.

The seller-buyer relationship between intermediaries and final producers is different from the standard literature. For every unit of intermediary good sold there must be a special supply linkage created between intermediaries and final producers. Building linkages is costly but they persist over time.

In the model the supply linkages are not modelled explicitly, i.e. we do not have supply linkages between a specific intermediary and a specific final producer, but for every unit of intermediary good sold a given intermediary firm must have supply linkages of the same size created, and similarly for the final producers, for every unit of intermediary good bought a given final producer must have supply linkages of the same size created. Therefore the linkages are not directly connecting firm to firm but they are connecting firm to a market. Building linkages represents communication of special requirements firms have on the intermediary good. To transmit this requirement to the market is costly, so there are set-up costs for linkages, but on the other hand, once the cost is incurred linkages persist over time. Linkages can disappear in two ways, either by a random shock or they disappear when they are not used. For every amount of the intermediary good the supply can be done only through linkages of the same size. When the amount of the supplied intermediary good is smaller than the size of the linkages the redundant ones disappear. Linkages destroyed can be rebuilt at the usual cost.

1.2.1 Workers

There is a unit mass of workers. They supply inelastically labor to intermediary firms. At the same time they are the owners of the firms therefore their income consists of the wages obtained for their work as well as profits of the firms. The proceeds collected in the economy from the costs of link creation are also divided among workers. The income of workers serves them to buy consumption good. Workers have no decision to make, they simply spend all their income on consumption.

1.2.2 Intermediary Firms

Intermediaries are of mass 1. They maximize discounted expected stream of profits. They use labor $n \in N = [0,1]$ as the only input to their production. The intermediaries are heterogeneous. Their production is subject to an idiosyncratic productivity shock $z \in Z = [0,\infty)$ that follows a first-order Markov process, characterized by a transition matrix Q. The production function of intermediaries is given by $f(z,n) = z \cdot n^{\gamma}$, where $\gamma \in (0,1)$ is the production parameter. The result of the production is an intermediate good that serves as input to the production of final producers. The intermediate good is in the model considered to be a numeraire, therefore its price is normalized to 1. The competitive wage paid to the labor force is denoted w.

The intermediate good can be sold to the final producers only through supply linkages, denoted l^i. Creation of linkages is costly. The cost per unit is denoted g. When the linkages are not used in the given period they automatically disappear. This happens at no cost.

The decisions of an intermediary firm in one period is as follows. The firm enters the period with linkages built in previous periods and it observes its productivity shock, so the firm's state is (z, l^i). Based on the state it decides how much to produce knowing that for every unit of the intermediary good the firm produces it needs to have or to build linkages to be able to sell the good. For the linkages built the firm pays per-unit cost and the linkages persist to the next period. The decision on the size of the production is equivalent to the decision on the size of the workforce employed, because the productivity shock is already revealed. While deciding on the size of the production the firm considers not only the size of the present productivity shock but, based on the matrix characterizing the Markov process, also the expected future shocks. Because building linkages is costly, even with a high shock in the present period it is not profitable to invest in an extensive building of linkages if the expected future shocks are low because the linkages will not be fully used in the next period and therefore the unused capacity will disappear.

The decision of an intermediary characterized by the productivity shock z and level of links l^i, and discounting future by a factor $\beta \in (0,1)$, can be formalized in a recursive way as follows

$$\begin{aligned} v^i(z,l^i) = \max_n \Big\{ & z \cdot n^\gamma - n \cdot w - g \cdot \max\{0, z \cdot n^\gamma - l^i\} + \\ & \beta \cdot \sum_{z'} v^i(z', l^{i'}) \cdot Q(z'|z)) \Big\} \\ \text{s.t.} \quad & l^{i'} = z \cdot n^\gamma, \end{aligned}$$

where $Q(z'|z)$ is the conditional probability of the shock z' in the next period, given the present shock is z. This probability is given by the transition matrix Q characterizing the Markov process for the productivity shocks.

As described already above, the linkages with which the firm will start next period, $l^{i'}$, are the same as the linkages at the end of the present period, i.e. the level is equal to the present production that was sold through the linkages.

1.2.3 Final Production Firms

There is a unit mass of final producers. They maximize discounted expected stream of profits. They use intermediate good $i \in I = [0, \infty)$ as input to their production. The production function of intermediaries is non-stochastic and it is given by $f(i) = i^\alpha$, where $\alpha \in (0,1)$ is the production parameter. The result of the production is the consumption good. The price of the consumption good is denoted p.

The intermediate good can be bought from the intermediaries only through supply linkages, denoted l^f. Similarly as for the intermediaries, also for the final producers creation of linkages is costly. The cost per unit stays the same, g, and linkages that are not used disappear at no cost.

Because linkages are the connection between intermediaries and final producers the loss of linkages on one side of the market should be reflected on the other side of the market. The loss of linkages of the final producers is the implication of the individual fluctuations in the production of intermediaries. Every individual final producer perceives such a loss of linkages as a shock. In the model this shock is multiplicative. The fraction $l^f \cdot (1-s)$ represents the linkages that are lost, while $l^f \cdot s$ are the linkages that persist. The evolution of shocks affecting linkages follows a Markov process.

The decisions of a final production firm in one period is as follows. The firm enters the period with linkages built in previous periods and it observes its shock to linkages. Based on these variables it decides how much to produce knowing that for every unit of the intermediary good that should serve as an input to its production it needs to have or to build linkages to be able to buy the good. For the newly built linkages the firm pays per-unit cost. The linkages at the end of the period, i.e. after purchase, will persist to the next period but will again be subject to shocks.

The decision of the final producer entering the period with the linkages l^f and being subject to the shock s can be formalized in a recursive way as follows:

$$v^f(l^f, s) = \max_i \left\{ p \cdot i^\alpha - i - g \cdot \max\{0, i - l^f \cdot s\} + \beta \cdot \sum_{s'} v^f(i, s') \cdot \Pi(s'|s) \right\},$$

where the linkages at the beginning of the next period are equal to the linkages used to buy input in the present period, i.e. $l^{f\prime} = i$.

1.3 Equilibrium

Because there is no aggregate uncertainty in the model we are able to study the economy in a stationary recursive competitive equilibrium. For exposition purposes it is convenient to consider a specific case of the general model, i.e. to assume particular forms of distributions of shocks. We first start with the description of the equilibrium in the general case and afterward we discuss the specific case of the general model.

1.3.1 General Case

In the general case we do not assume any specific form of the transition matrix for the productivity shocks of the intermediaries and we assume that the shocks to linkages of the final producers are idiosyncratic.

Therefore the problem of an intermediary is as follows

$$v^i(z,l^i) = \max_n \left\{ z\cdot n^\gamma - n\cdot w - g\cdot \max\{0, z\cdot n^\gamma - l^i\} + \beta\cdot \sum_{z'} Q(z'|z)\cdot v^i(z', z\cdot n^\gamma) \right\}.$$

The problem of final producers is as follows

$$v^f(l^f, s) = \max_i \left\{ p\cdot i^\alpha - i - g\cdot \max\{0, i - l^f\cdot s\} + \beta\cdot \sum_{s'} \Pi(s'|s)\cdot v^f(i, s') \right\}.$$

In this general case we can not directly solve the optimal closed-form policy functions of intermediaries and final producers and to get results it's necessary to use simulations.

Let us denote the optimal policy function of intermediaries $n(z, l^i)$. This policy function describes the optimal labor force employed in the production depending on the current productivity shock and existing linkages. Let us denote $i(l^f, s)$ the optimal policy function of final producers. This policy describes the optimal level of input used in the production of the final producers depending on the existing linkages and shocks. The described policy functions directly imply the policies for the linkages of both intermediaries and final producers.

$$\begin{aligned} {l^i}'(z, l^i) &= z\cdot (n(z,l^i))^\gamma \\ {l^f}'(l^f, s) &= i(l^f, s) \end{aligned}$$

On the aggregate level the shocks to linkages are determined in an endogenous way. The shocks in the model serve as a transmission mechanism. When intermediaries optimally decide to down-size their linkages this should have an impact on the linkages of the final producers. There is no market mechanism that would inform the final producers about the loss of linkages and therefore they perceive this loss as a shock. Because the linkages are not modelled as a connection between two specific firms but rather they are the connection of firms to a market, the distribution of shocks can not be determined by the model, it is a modelling choice. On the other hand we know exactly what is the individual as well as aggregate loss of linkages of the intermediaries, which is determined endogenously, and this loss should hit the final producers. Therefore the aggregate of the final producers' linkages lost due to shocks is, in equilibrium, equal to the aggregate of the linkages lost due to the down-sizing decisions of intermediaries. The distribution and persistence of the shocks are exogenous.

Because the intermediaries as well as the final producers are heterogeneous we should have a tool to describe this heterogeneity. Let us denote λ^i and λ^f the distributions of intermediaries and final producers. The policy functions determine the laws of motion for these distributions.

$$\begin{aligned} \lambda^{i\prime}(z', l^{i\prime}) &= \sum_{(z,l^i):z\cdot(n(z,l^i))^\gamma = l^{i\prime}} Q(z'|z)\cdot\lambda^i(z,l^i) \\ \lambda^{f\prime}(l^{f\prime}, s') &= \sum_{(l^f,s):i(l^f,s)=l^{f\prime}} \Pi(s'|s)\cdot\lambda^f(l^f,s) \end{aligned}$$

The stationary recursive competitive equilibrium consists of the value functions $v^i(z,l^i)$, $v^f(l^f,s)$, policy functions $n(z,l^i)$, $i(l^f,s)$, prices p, w, and probability measures λ^i, λ^f such that

1. given the prices p and w the policy functions solve the optimization problems of every intermediary and final producer,
2. the probability measures λ^i and λ^f are time invariant,
3. the aggregate of linkages of final producers lost in every period is determined endogenously and it is time invariant, i.e. the shocks s are drawn from such a distribution that the following aggregate equality holds

$$\sum_z\sum_{l^i}\max\left\{0, l^i - l^{i\prime}(z,l^i)\right\}\cdot\lambda^i(z,l^i) = \sum_{l^f}\sum_s(1-s)\cdot l^f\cdot\lambda^f(l^f,s),$$

4. the prices p and w are such that markets clear,
   market for labor:

$$1 = \sum_z\sum_{l^i} n(z,l^i)\cdot\lambda^i(z,l^i),$$

market for intermediary good:

$$\sum_z \sum_{l^i} z \cdot (n(z,l^i))^\gamma \cdot \lambda^i(z,l^i) = \sum_{l^f} \sum_s i(l^f,s) \cdot \lambda^f(l^f,s),$$

market for final good:

$$\sum_{l^f} \sum_s (i(l^f,s))^\alpha \cdot \lambda^f(l^f,s) = C,$$

5. aggregate variables, i.e. intermediate production I, final production F, and consumption C, are constant,

$$\begin{aligned} I &= \sum_z \sum_{l^i} z \cdot (n(z,l^i))^\gamma \cdot \lambda^i(z,l^i) \\ F &= \sum_{l^f} \sum_s (i(l^f,s))^\alpha \cdot \lambda^f(l^f,s) \\ C &= F, \end{aligned}$$

6. aggregate feasibility holds,

$$C \cdot p = w + \Pi_i + \Pi_f + CL_i + CL_f,$$

where Π_i and Π_f are the profits of the intermediaries and the final producers, and CL_i and CL_f are the costs of building linkages paid by the intermediaries and final producers for which the following holds

$$\begin{aligned} \Pi_i &= \Bigg(\sum_z \sum_{l^i} z \cdot (n(z,l^i))^\gamma - n(z,l^i) \cdot w - \\ &\qquad g \cdot \max\left\{0, z \cdot (n(z,l^i))^\gamma - l^i\right\}\Bigg) \cdot \lambda^i(z,l^i) \\ \Pi_f &= \Bigg(\sum_{l^f} \sum_s p \cdot (i(l^f,s))^\alpha - i(l^f,s) - \\ &\qquad g \cdot \max\left\{0, i(l^f,s) - l^f \cdot s\right\}\Bigg) \cdot \lambda^f(l^f,s) \\ CL_i &= \Bigg(\sum_z \sum_{l^i} g \cdot \max\left\{0, z \cdot (n(z,l^i))^\gamma - l^i\right\}\Bigg) \cdot \lambda^i(z,l^i) \\ CL_f &= \Bigg(\sum_{l^f} \sum_s g \cdot \max\left\{0, i(l^f,s) - l^f \cdot s\right\}\Bigg) \cdot \lambda^f(l^f,s). \end{aligned}$$

1.3.2 Specific Example

To make the exposition of the equilibrium easy to follow we take several assumptions that allow us to find an analytical solution of the equilibrium of the model.

We assume that the productivity shocks of intermediaries can take only two values, that we will call "high" and "low". We consider a simple transition matrix, we assume that all the conditional probabilities for the productivity shock in the next period, given this period's shock, are equal to 1/2.

With these simplifying assumptions the problem of every intermediary in the steady state is the following

$$v^i(z,l^i) = \max_n \left\{ z\cdot n^\gamma - n\cdot w - g\cdot \max\{0, z\cdot n^\gamma - l^i\} + \beta\cdot\sum_{z'} \frac{1}{2}\, v^i(z', z\cdot n^\gamma) \right\},$$

where z, $z' \in \{z_l, z_h\}$.

For the final producers we assume homogeneity of shocks to linkages. The probability of the shock is assumed to be one. Out of the steady state the firms are uncertain about the size of the shock, which is determined endogenously as a part of the equilibrium.

Since we assume homogeneity of firms with respect to the shock to linkages, s, and because the size of this shock is endogenously determined in the model, in a stationary recursive competitive equilibrium this value is not changing and therefore s loses it's properties of a state variable.

With these simplifying assumptions the problem of every final producer is the following

$$v^f(l^f) = \max_i \left\{ p\cdot i^\alpha - i - g\cdot \max\{0, i - l^f\cdot s\} + \beta\cdot v^f(i) \right\}.$$

Since problems involving link-building as the one described above are not standard in economic theory we will discuss the solution to firms' problems in detail to get some intuition about their optimal behavior.

Intermediary Firms

The intermediaries are subject to the productivity shocks. These shocks are multiplicative constants in the optimization problems of the firms, therefore we can consider a general solution for a fixed z and only then see what is the solution when there are two shocks[1].

To find the solution we have to consider three possible cases, depending on the relation between $z \cdot n^{\gamma}$ and l_0^i.

If $z \cdot n^{\gamma} > l_0^i$, then the optimal size of intermediary production, i.e. also the optimal size of linkages, is $l^{i'} = z^{\frac{1}{1-\gamma}} \left(\frac{\gamma(1-g+\frac{\beta}{2}g))}{w} \right)^{\frac{\gamma}{1-\gamma}}$. If $z \cdot n^{\gamma} < l_0^i$, then the optimal production is $l^{i'} = z^{\frac{1}{1-\gamma}} \left(\frac{\gamma(1+\frac{\beta}{2}g)}{w} \right)^{\frac{\gamma}{1-\gamma}}$. The third option is that $z \cdot n^{\gamma} = l_0^i$. For each of the two possible productivity shocks we get this type of solution. For the sake of simplicity of the exposition in this specific example we suppose that the size of the two shocks is such that the two policy functions do not cross, i.e. $\overline{l_l} < \underline{l_h}$. This assumption guarantees a very simple steady state distribution of firms. Graphically we represent the described policies in the following way.

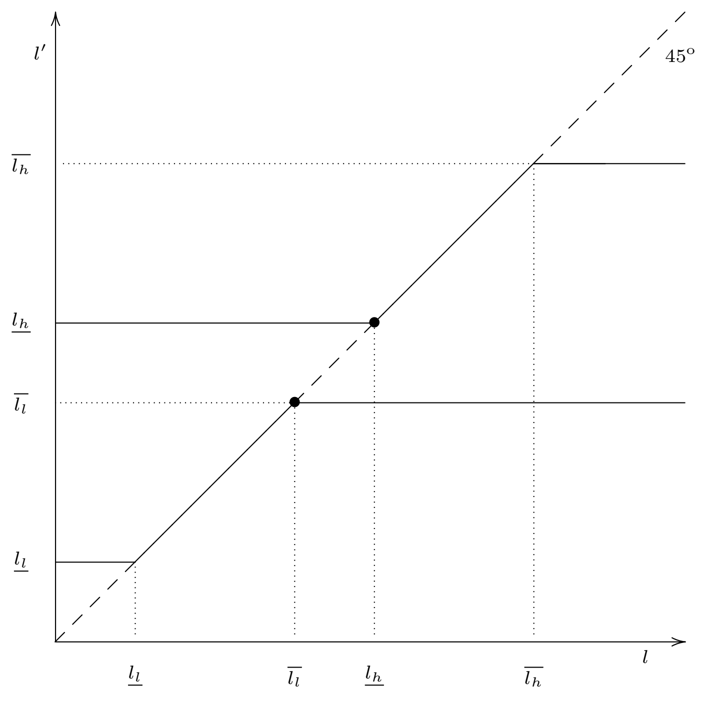


Figure 1

The levels of l' in the graphical representation are as follows

[1] This is possible only because we assume the transition matrix for shocks that guarantees the same expected future for intermediaries independent of their present shock.

$$
\begin{aligned}
\underline{l_l} &= z_l{}^{\frac{1}{1-\gamma}} \cdot \left(\frac{\gamma \cdot (1-g+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}} \\
\overline{l_l} &= z_l{}^{\frac{1}{1-\gamma}} \cdot \left(\frac{\gamma \cdot (1+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}} \\
\underline{l_h} &= z_h{}^{\frac{1}{1-\gamma}} \cdot \left(\frac{\gamma \cdot (1-g+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}} \\
\overline{l_h} &= z_h{}^{\frac{1}{1-\gamma}} \cdot \left(\frac{\gamma \cdot (1+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}}.
\end{aligned}
$$

The intuition for these results is simple. No matter what is the production shock, we have two possible types of firms. Those that need to build up their linkages and those that do not. Because the costs of production in these two cases are different the solutions to their profit-maximizing dynamic problem are different too. We obtain a small firm size $z^{\frac{1}{1-\gamma}}\left(\frac{\gamma(1-g+\frac{\beta}{2}g))}{w}\right)^{\frac{\gamma}{1-\gamma}}$, in the plot denoted $\underline{l}$, as a result of the problem of those firms that need to build up their linkages. We obtain a large firm size $z^{\frac{1}{1-\gamma}}\left(\frac{\gamma(1+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}}$, on the plot denoted $\bar{l}$, for those that are scaling down their linkages to the maximal size that is still profitable. Due to the discontinuity of the costs between the firms that are building up linkages and the firms that are scaling down we obtain a whole region of firm's size where they stay inactive, i.e. they take no action with respect to their initial size of the linkages.

If the policies for the two productivity shocks were crossing it would imply a continuum of types of intermediary firms in the steady state. On the contrary, in our specific example with policies that do not cross we can show that in the steady state of the economy there are only two stationary points $z_l{}^{\frac{1}{1-\gamma}}\left(\frac{\gamma(1+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}}$ and $z_h{}^{\frac{1}{1-\gamma}}\left(\frac{\gamma(1-g+\frac{\beta}{2}g)}{w}\right)^{\frac{\gamma}{1-\gamma}}$.

It is clear that no matter what the initial conditions in the economy are, after the first period the agents with the low productivity shock will have the size of linkages belonging to the interval $[\underline{l_l}, \overline{l_l}]$, and similarly, the agents with the high shock the size belonging to the interval $[\underline{l_h}, \overline{l_h}]$. Because we consider the economy in the steady state and because there is non-zero probability of switch between the high and the low shock we know, that after such a switch the agents will end-up either in $\overline{l_l}$ or in $\underline{l_h}$. After that they continue only to switch between these two points when the shock changes. These two points are therefore the only stationary points for the size of linkages.

The two steady state levels of linkages size correspond to two levels of labor force used in the production. The two levels are as follows

$$\begin{aligned} n_l &= \left(\frac{\gamma \cdot z_l \cdot (1 + \frac{\beta}{2}g)}{w}\right)^{\frac{1}{1-\gamma}} \\ n_h &= \left(\frac{\gamma \cdot z_h \cdot (1 - g + \frac{\beta}{2}g)}{w}\right)^{\frac{1}{1-\gamma}}. \end{aligned}$$

Although we have only two steady state levels of the labor force used by the firms, and therefore only two levels of production, we have three levels of profits. This is a consequence of the fact that the firms that have the high productivity shock in the present period, depending on the shock they had in the previous period, have or do not have to build additional linkages.

The intermediaries that face the low productivity shock never build any linkages because they were either in the same situation in the previous period, or they faced the high shock in the previous period and therefore they have too many linkages built. Therefore these intermediaries make one-period profits that can be expressed in the following way:

$$\Pi_{i_l} = z_l \cdot n_l^{\gamma} - n_l \cdot w.$$

On the other hand the profits of the intermediaries that face the high productivity shock depend on their shock in the previous period.

The intermediaries that face two subsequent high shocks have no linkages to build and their profits can be therefore expressed in the following way:

$$\Pi_{i_{hh}} = z_h \cdot n_h^{\gamma} - n_h \cdot w.$$

The intermediaries that face the high shock in the present period and had the low shock in the previous period have to build some links and their profits can be expressed in the following way:

$$\Pi_{i_{hl}} = z_h \cdot n_h^{\gamma} - n_h \cdot w - (z_h \cdot n_h^{\gamma} - z_l \cdot n_l^{\gamma}) \cdot g.$$

Final Producers

Final producers are subject to loss of linkages caused by fluctuations in individual productions of intermediaries.

To make the exposition easy to follow, we first start by solving the final producers' problem for the limit case when $s = 1$ and then for the general case, which is the case of our interest, $s \in [0, 1)$.

When the shock s is equal to 1 this means that the linkages that were built in the previous period are entirely preserved for the present period. To find the solution we have to consider three possible cases. If $i > l^f$, then the optimal size of input, i.e. also the optimal size of linkages, is $i = (\frac{p \cdot \alpha}{1+g\cdot(1-\beta)})^{\frac{1}{1-\alpha}}$. If $i < l^f$, then the optimal i is $i = (p \cdot \alpha)^{\frac{1}{1-\alpha}}$. The third option is that $i = l^f$. This is a similar type of solution as for the intermediary firms.

Now we consider the case when $s \in [0, 1)$, i.e. in every period a fraction of linkages disappears. Here too there are three cases to be considered. If $i > l^f \cdot s$, i.e. when the initial linkages need to be build up, similarly to the case when $s = 1$, the optimal solution is $i = (\frac{p \cdot \alpha}{1+g\cdot(1-\beta\cdot s)})^{\frac{1}{1-\alpha}}$, denoted i^s_{low}. This solution corresponds to the region $l^f \in [a, b]$ in Figure 2. In the other two cases, because the shocks to linkages play a role now, the intuition differs from the case when $s = 1$. Now in every period a part of linkages disappears so if the firm stays inactive in building its linkages, at some point in time its size will shrink under the lower-bound size level established above and the firm will have to build-up a part of its linkages. This is the case when initially $i = l^f \cdot s$, which corresponds to the region $l^f \in [b, e]$ in Figure 2. Because the linkages are shrinking at some point $l^f \cdot s$ will fall below the lower bound threshold and the firm will have to build up the linkages to get to the size i^s_{low}. The last case to consider is the case $i < l^f \cdot s$, in Figure 2 in the region $l^f \in [e, \infty)$, where the firm initially needs to downsize its linkages. Here, intuitively, firm will not directly downsize to the size i^s_{low}. For several periods it will take advantage of the fact that a fraction of linkages is disappearing in every period. For how many periods the firm will let it's linkages disappear without any up-scaling depends on the parameters of the model and if we denote the number of such periods k then the optimal solution to the firms size is $i = (\frac{p\cdot\alpha\cdot(1+\beta\cdot s^\alpha+(\beta\cdot s^\alpha)^2+)+\ldots+(\beta\cdot s^\alpha)^{k-1}}{1+\beta\cdot s+(\beta\cdot s)^2+\ldots+(\beta\cdot s)^{k-1}-g\cdot(\beta\cdot s)^k})^{\frac{1}{1-\alpha}}$, in the plot denoted i^s_{high}.

In this case, when $s \in [0,1)$, the only stationary point for the size of links is the point i^s_{low}.

The results are plotted in Figure 2.

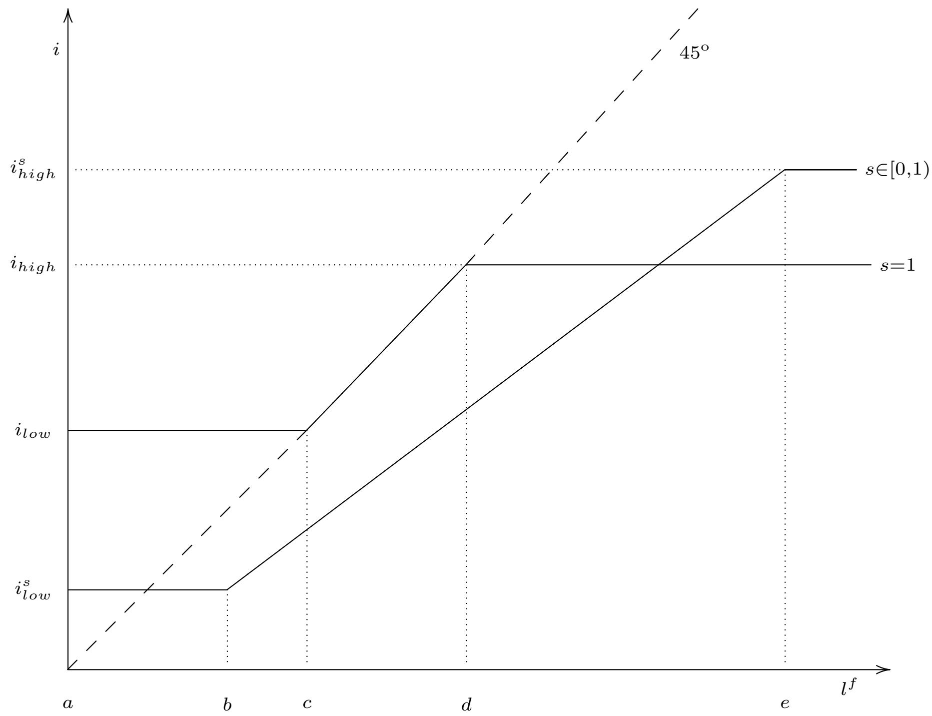


Figure 2

In the plot, as $s \in [0,1)$ increases and approaches to 1 the plot that corresponds to the case $s \in [0,1)$ approaches to the one that corresponds to the case $s = 1$. But only the case $s = 1$ implies a whole interval of stationary points $[i_{low}, i_{high}]$.

In the steady state problem of final producers the loss of part of their linkages in every period is a fixed and known fraction $s \in [0,1)$. The steady state solution to the problem of the final producers is therefore as follows:

$$i = \left(\frac{\alpha \cdot p}{1 + g \cdot (1 - \beta \cdot s)} \right)^{\frac{1}{1-\alpha}}.$$

Now that we know what are the solutions to the firms' problems we can establish equilibrium prices and aggregate levels in the model.

Definition of Equilibrium

Because the structure of the productivity shocks to intermediate production is assumed to be very simple we know exactly what will be the distribution of intermediary firms in the stationary state. We know that 1/2 firms have high productivity shocks and 1/2 have low shocks. Therefore, the overall labor force used by the intermediary firms is the following:

$$\frac{1}{2}\cdot\left(\frac{\gamma\cdot z_h\cdot(1-g+\frac{\beta}{2}g)}{w}\right)^{\frac{1}{1-\gamma}}+\frac{1}{2}\cdot\left(\frac{\gamma\cdot z_l\cdot(1+\frac{\beta}{2}g)}{w}\right)^{\frac{1}{1-\gamma}}.$$

At the same time we know that the workers supply the labor inelastically, and therefore the labor force is equal to 1.

Having both sides of the equation for labor used in the economy we can compute the price of labor

$$w=2^{\gamma-1}\cdot\gamma\cdot\left(\left(z_h\cdot(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}+\left(z_l\cdot(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}.$$

By similar reasoning we can establish the price of the final good p. We know that the final producers will use in their production the following amount of the intermediary good

$$\left(\frac{\alpha\cdot p}{1+g\cdot(1-\beta\cdot s)}\right)^{\frac{1}{1-\alpha}}.$$

At the same time we know that the intermediary goods used by the final producers in their production must be produced by the intermediaries. The production of intermediaries is the following:

$$\frac{1}{2}\cdot z_h\cdot n_h^{\gamma}+\frac{1}{2}\cdot z_h\cdot n_l^{\gamma}.$$

We know that in the steady state the good produced by intermediaries is entirely used in the production of final producers. This equality allows us to compute the price of the final good, p, having in mind that the price of the intermediary good is normalized to 1.

$$p=\left(2^{\gamma-1}\cdot\frac{z_h^{\frac{1}{1-\gamma}}\cdot(1-g+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}+z_l^{\frac{1}{1-\gamma}}\cdot(1+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}}{\left(\left(z_h\cdot(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}+\left(z_l\cdot(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}\right)^{\gamma}}\right)^{1-\alpha}\cdot\frac{1+g\cdot(1-\beta\cdot s)}{\alpha}$$

Now let us turn our attention to the shocks to linkages. The supply linkages in the economy create a direct connection between intermediaries and final producers. If on one side of this connection the intermediary firms optimally decide to down-size their production due to the low productivity shock, this should be reflected on the other side and, in this model, this is done using the shock to linkages.

More specifically, consider an intermediary that had in the previous period high productivity but in the present period its productivity is low. Such intermediary will have to down-size its production. Because this is done at no cost there is no mechanism in the economy that would transmit the information about the down-sizing to the other side of the market, therefore the other side perceives this downsizing as a shock.

Because we have assumed a specific structure of the productivity shocks we know that in our economy, in every period, we have a mass of 1/2 of intermediaries that are hit by the low productivity shock and we also know that 1/2 of these had the high productivity shock in the previous period. Therefore we have 1/4 of intermediaries that are downsizing their links in every period.

The aggregate of the linkages lost due to the down-sizing of the production of some intermediaries is therefore the following:

$$\frac{1}{4} \cdot \left(z_h \cdot n_h^{\gamma} - z_l \cdot n_l^{\gamma} \right).$$

We know that the aggregate of the linkages lost due to the shocks to linkages of the final producers is:

$$(1-s) \cdot \left(\frac{\alpha \cdot p}{1 + g \cdot (1 - \beta \cdot s)} \right)^{\frac{1}{1-\alpha}}.$$

In the steady state the mass of linkages lost on one side of the market should be equal to the mass lost on the other side which allows us to determine the expression for the fraction of linkages persisting to another period

$$s = 1 - \frac{z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} - z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}}}{2 \cdot \left(z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} \right)}.$$

In the stationary recursive equilibrium the policy functions, value functions, prices, and aggregate variables are stable. The policy function of intermediaries determines the law of motion for their distribution[2].

[2]Note that we have assumed that the transition probabilities are equal to $\frac{1}{2}$ for all the possible transitions between the shocks.

$$\lambda(z', l^{i\prime}) = \sum_{(z,l^i):l^{i\prime}(z,l^i)=l^{i\prime}} \frac{1}{2} \cdot \lambda(z, l^i)$$

The stationary recursive competitive equilibrium of the described economy consist of the value functions $v^i(z, l^i)$, $v^f(l^f)$, policy functions $n(z, l^i)$, $i(l^f)$, prices p, w, and probability measure λ such that

1. given the prices p and w the policy functions solve the optimization problems of every intermediary and final producer,

2. the probability measure λ is time invariant,

3. the fraction of linkages of every final producer that persist in every period is determined endogenously and it is time invariant,

$$s = 1 - \frac{z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} - z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}}}{2 \cdot \left(z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} \right)},$$

4. the prices p and w are such that markets clear,

   market for labor:

$$1 = \frac{1}{2} \cdot n_h + \frac{1}{2} \cdot n_l,$$

   market for intermediary good:

$$\frac{1}{2} \cdot z_h \cdot n_h^\gamma + \frac{1}{2} \cdot z_l \cdot n_l^\gamma = \left(\frac{\alpha \cdot p}{1 + g \cdot (1 - \beta \cdot s)} \right)^{\frac{1}{1-\alpha}},$$

   market for final good:

$$\left(\frac{\alpha \cdot p}{1 + g \cdot (1 - \beta \cdot s)} \right)^{\frac{\alpha}{1-\alpha}} = C,$$

5. aggregate variables, i.e. intermediate production I, final production F, and consumption C, are constant,

$$\begin{aligned}
I &= 2^{\gamma-1} \cdot \frac{z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}}}{\left(\left(z_h \cdot (1 - g + \frac{\beta}{2} g) \right)^{\frac{1}{1-\gamma}} + \left(z_l \cdot (1 + \frac{\beta}{2} g) \right)^{\frac{1}{1-\gamma}} \right)^\gamma} \\
F &= \left(2^{\gamma-1} \cdot \frac{z_h^{\frac{1}{1-\gamma}} \cdot (1 - g + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}} \cdot (1 + \frac{\beta}{2} g)^{\frac{\gamma}{1-\gamma}}}{\left(\left(z_h \cdot (1 - g + \frac{\beta}{2} g) \right)^{\frac{1}{1-\gamma}} + \left(z_l \cdot (1 + \frac{\beta}{2} g) \right)^{\frac{1}{1-\gamma}} \right)^\gamma} \right)^\alpha \\
C &= F,
\end{aligned}$$

6. aggregate feasibility holds

$$C \cdot p = w + \frac{1}{4} \cdot \Pi_{i_{hh}} + \frac{1}{4} \cdot \Pi_{i_{hl}} + \frac{1}{2} \cdot \Pi_{i_l} + \Pi_f + CL_i + CL_f,$$

where CL_i and CL_f are the costs of link building paid by the intermediaries and final producers.

1.4 Results

This chapter discusses in details the results for the specific model introduced in the previous chapter. Because this link building model is a completely new type of theoretical model of production, despite the fact that this paper has no aspirations to calibrate the proposed model, it is still good to make sure that the relative levels of aggregates and relative prices we obtain in the equilibrium are of plausible magnitudes. The parameters of the model are fixed to standard values. The discount factor $\beta = 0.95$, the parameter of the intermediary production, in which the input is labor, $\gamma = 2/3$, the parameter of the final production $\alpha = 1/3$. The parameters specific for the proposed model we fix as follows: productivity shocks are $z_h = 1$ and $z_l = 0.5$, the cost of link building $g = 0.5$, the same as numeraire, which is the price of the intermediary good.

With these parameters fixed we obtain the following equilibrium results. The fraction of linkages preserved to the next period $s = 0.76$, the wage $w = 0.46$ and the final good price $p = 2.70$, keeping in mind that the intermediary good is numeraire in the model. For the aggregates, the labor is by assumption equal to 1, the intermediary production $I = 0.79$ and the final production is 0.92. This means that the model is sustainable in the long term, it does not collapse and does not explode. Moreover, the results are not particularly sensitive to changes in parameters, as can be seen in the table provided in the appendix.

1.4.1 Comparison with Benchmark Model

In this section we discuss what are the consequences of building linkages in the proposed model by comparing it to a benchmark model. We do the comparison with the model without linkages. Important to note is the fact that in the benchmark model, by construction, we do not have any intertemporal variable. In the proposed model linkages are the only intertemporal element of the model.

For the benchmark model we assume the same characteristics as for the link-building model. We assume homogeneous workers and homogeneous final producers. We assume heterogeneous intermediaries characterized in the same way as in the proposed model. The benchmark model, in mathematical terms, corresponds to the limit case of the proposed model when the parameter g equals to zero.

The results of the benchmark model are summarized in the following table.

	benchmark model
n_h	$(\frac{z_h \cdot \gamma}{w})^{\frac{1}{1-\gamma}}$
n_l	$(\frac{z_l \cdot \gamma}{w})^{\frac{1}{1-\gamma}}$
w	$2^{\gamma-1} \cdot \gamma \cdot (z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}})^{1-\gamma}$
I	$(\alpha \cdot p)^{\frac{1}{1-\alpha}}$
p	$2^{(\gamma-1)(1-\alpha)} \cdot (z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}})^{(1-\gamma)(1-\alpha)} \cdot \frac{1}{\alpha}$

When we plug-in general equilibrium prices the optimal results, in terms of numeraire, which is the intermediary good, are the following.

	benchmark model
n_h	$\frac{2 \cdot z_h^{\frac{1}{1-\gamma}}}{(z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}})^{1-\gamma}}$
n_l	$\frac{2 \cdot z_l^{\frac{1}{1-\gamma}}}{(z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}})^{1-\gamma}}$
I	$2^{\gamma-1} \cdot (z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}})^{1-\gamma}$

Because the final production is always the same monotone function of the input, i^{α}, it is enough to compare the proposed model with the benchmark in terms of the aggregate intermediary production I, for the final production the same qualitative properties hold, it is only the magnitude that is different.

To compare the models in terms of the intermediary good produced we consider the ratio of the productions in the two models $\frac{I_B}{I}$, where I_B denotes the intermediary production in the benchmark and I denotes the production in the proposed model. After a straightforward simplification the ratio is the following:

$$\frac{I_B}{I} = \frac{z_h{}^{\frac{1}{1-\gamma}} + z_l{}^{\frac{1}{1-\gamma}}}{z_h{}^{\frac{1}{1-\gamma}}(1-g+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}} + z_l{}^{\frac{1}{1-\gamma}}(1+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}} \cdot \left(\frac{\left(z_h(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}} + \left(z_l(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}}{z_h{}^{\frac{1}{1-\gamma}} + z_l{}^{\frac{1}{1-\gamma}}}\right)^{\gamma}.$$

This ratio helps us to determine in which of the two models the intermediary production is higher. If the ratio is greater than one, then the benchmark production is higher, if on the other hand it is lower than one, then building linkages helps to increase the production over the level of the benchmark production.

To determine whether the ratio is greater than one it is enough to realize that the benchmark model is a limit of the link-building model, when the cost of link-building is equal to zero. Therefore, if the function $I(g)$ is monotone we will be able to compare the two intermediate productions. The derivative of the intermediate production I with respect to the cost g is the following:

$$\frac{\partial I}{\partial g} = \frac{2^{1+\gamma} \cdot \gamma \cdot \left(\left(z_h(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}} + \left(z_l(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}\right)^{-1-\gamma} \cdot (z_h z_l)^{\frac{1}{1-\gamma}}}{\left(4+4g(-1+\beta)+g^2\beta(-2+\beta)\right) \cdot (-1+\gamma)} \cdot \left((1-g+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}(1+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}} - (1+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}(1-g+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}}\right)$$

This derivative is negative for every possible combination of parameters, this is due to the negative denominator in the fraction above. The negativity can be seen once we realize that in order to have policy functions that do not cross parameter g must satisfy the following inequality:

$$g < \frac{1-(z_l/z_h)^{1/\gamma}}{1-\frac{\beta}{2}\left(1-(z_l/z_h)^{1/\gamma}\right)}$$

From the fact that $\partial I/\partial g < 0$ we conclude that the intermediate production is a decreasing function of the cost g for all the possible combinations of the parameters of the model. Therefore we conclude that since the intermediate production in the benchmark model is equal to $I(g=0)$ and the intermediate production in the link-building model is equal to $I(g>0)$, the ratio $\frac{I_B}{I}$ is greater than one.

We have established that the intermediate production is lower in the link-building model than in the benchmark model. That is an intuitive result becasue in the link-building model marginal costs are higher due to the building and re-building of links after a bad shock.

Now let us focus on the composition effect between the firms with a low production shock and the firms with a high production shock. In both benchmark as well as in the link-building model we have the same equilibrium fractions of firms with high and low shock, i.e. one half of the firms have a high shock and one half have a low shock. Therefore it is not the composition but the size of the two types of production that makes the difference between the models. We want to know whether the benchmark production is higher because both the firms with high productivity as well as the firms with low productivity produce more or is it just one of the firms' type that drives the result?

For the benchmark model the equilibrium intermediate productions of the firms with high productivity shock and the firms with the low productivity shock are the following:

$$(i_l)_B = \frac{2^\gamma \cdot {z_l}^{\frac{1}{1-\gamma}}}{\left({z_h}^{\frac{1}{1-\gamma}} + {z_l}^{\frac{1}{1-\gamma}}\right)^\gamma}$$

$$(i_h)_B = \frac{2^\gamma \cdot {z_h}^{\frac{1}{1-\gamma}}}{\left({z_h}^{\frac{1}{1-\gamma}} + {z_l}^{\frac{1}{1-\gamma}}\right)^\gamma}.$$

For the model with linkages the equilibrium intermediate production of the firms with high productivity shock and the firms with the low productivity shock are the following:

$$i_l = 2^\gamma \cdot {z_l}^{\frac{1}{1-\gamma}} \cdot \left(\frac{(1+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}}}{\left(z_h(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}} + \left(z_l(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}}\right)^\gamma$$

$$i_h = 2^\gamma \cdot {z_h}^{\frac{1}{1-\gamma}} \cdot \left(\frac{(1-g+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}}}{\left(z_h(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}} + \left(z_l(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}}\right)^\gamma.$$

Using simple algebra we can establish that $(i_l)_B < i_l$ and $(i_h)_B > i_h$. The intuition for these two results is quite simple. The firms that are hit by the low shock in the model with linkages do not want to destroy too many linkages because destruction today implies costly reconstruction in the future, once hit by a high productivity shock. Therefore the firms will try to save as many linkages as possible because they bring them future value.

On the other hand, once hit by the high productivity shock the firms in the model with linkages may want to build up linkages, which is at additional per unit cost g. This cost decreases their production with respect to the level of the production in the benchmark model.

We can conclude that the second effect is predominant because in aggregate the link building model exhibits lower level of intermediary production than the benchmark model.

The analysis above illustrates negative effect of network structures. In the economy where linkages are necessary for exchange the fall of production of part of the intermediaries causes lost of linkages that translates into additional costs for building new linkages and lower production. So the negative shock to intermediaries' production transmits as a lower and more costly production of the final producers.

From the results established above we can see that the benchmark model and the model with linkages exhibit different equilibrium levels of production. It is important to see how much of this difference comes from the differences in the technology assumed in the two models, i.e. building linkages, and what is the effect of the general equilibrium, which implies different prices in the two models.

We therefore consider the benchmark and the model with linkages out of equilibrium, with the same level of wages, and we obtain the following result:

$$\left(\frac{I_B}{I}\right)_{fixed\ wage} = \frac{{z_h}^{\frac{1}{1-\gamma}} + {z_l}^{\frac{1}{1-\gamma}}}{{z_h}^{\frac{1}{1-\gamma}}(1-g+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}} + {z_l}^{\frac{1}{1-\gamma}}(1+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}}.$$

Also with the fixed wages the level of intermediary production in the benchmark model is higher than in the link building model. This relative difference is purely technological. On the other hand if we go back to the result already established for the relative difference of intermediary productions taking into account respective general equilibrium prices we will see that the following holds:

$$\left(\frac{I_B}{I}\right)_{GE\ wages} = \left(\frac{I_B}{I}\right)_{fixed\ wage} \cdot \left(\frac{\big(z_h(1-g+\frac{\beta}{2}g)\big)^{\frac{1}{1-\gamma}} + \big(z_l(1+\frac{\beta}{2}g)\big)^{\frac{1}{1-\gamma}}}{{z_h}^{\frac{1}{1-\gamma}} + {z_l}^{\frac{1}{1-\gamma}}}\right)^{\gamma}.$$

Therefore we can separate the effect of technology from the effect of the general equilibrium. As a matter of fact, the effect of general equilibrium is, naturally, proportional to the difference in wages:

$$\frac{w_B}{w} = \left(\frac{z_h^{\frac{1}{1-\gamma}} + z_l^{\frac{1}{1-\gamma}}}{\left(z_h(1-g+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}} + \left(z_l(1+\frac{\beta}{2}g)\right)^{\frac{1}{1-\gamma}}} \right)^{1-\gamma}.$$

For the effect of the general equilibrium we conclude that the term belongs to the interval $(0,1)$ and depends on all the parameters of the model. Intuitively, the effect approaches one when the cost of link building g approaches zero.

Because the general equilibrium effect is smaller than 1 it is clear that the general equilibrium prices help the model with linkages to approach the levels of production of the benchmark model by diminishing the gap between the productions under the fixed wages.

1.4.2 Comparative Statics of the Model with Linkages

Now let us discuss the comparative statics of the proposed model with linkages. In the model there are two parameters that can have a major influence on the behavior of the model: the cost of creation of linkages g and the relative difference in productivity shocks, i.e. the ratio z_l/z_h.

We have already seen in the previous section that the intermediary production is a monotone function of the cost g. As long as the ratio z_l/z_h is considered, this represents the relative changes in productivity shocks, which cause fluctuations in output of the intermediaries. It is interesting to see the sensitivity of the output to a change in the ratio of shocks. We present the results both for the benchmark model as well as for the model with linkages.

For the benchmark model we obtain the following derivative

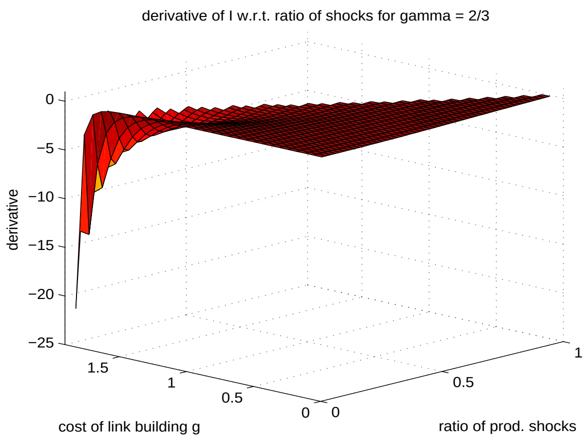
$$\frac{\partial I_B}{\partial(z_l/z_h)} = 2^{\gamma-1} \cdot z_h \cdot (z_l/z_h)^{\frac{\gamma}{1-\gamma}} \left(1 + (z_l/z_h)^{\frac{1}{1-\gamma}}\right)^{-\gamma}.$$

Clearly, the derivative $\frac{\partial I_B}{\partial(z_l/z_h)}$ is always positive.

For the model with linkages the derivative is the following

$$\frac{\partial I}{\partial(z_l/z_h)} = \frac{2^{\gamma-1}\cdot z_h\cdot(z_l/z_h)^{\frac{\gamma}{1-\gamma}}}{-1+\gamma}\Bigg((z_l/z_h)^{\frac{1}{1-\gamma}}(-1+\gamma)(1+\frac{\beta}{2}g)^{\frac{1+\gamma}{1-\gamma}} - (1+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}(1-g+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}} + \gamma(1-g+\frac{\beta}{2}g)^{\frac{\gamma}{1-\gamma}}(1+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}}\Bigg) \Bigg((1-g+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}} + (z_l/z_h)^{\frac{1}{1-\gamma}}(1+\frac{\beta}{2}g)^{\frac{1}{1-\gamma}}\Bigg)^{-1-\gamma}.$$

Due to the second part of the expression above, the derivative $\frac{\partial I}{\partial(z_l/z_h)}$ can be both negative as well as positive. That is an important result. It implies that for a range of parameters an increase in the fluctuations, which is the decrease of the ratio z_l/z_h, leads to an increasing additional intermediary production. This can never be observed in the benchmark model where the derivative of the intermediary production with respect to the ratio z_l/z_h is always positive. We can see from the following plot that the derivative in the link building model gets negative for a range of parameters where the cost of link building is high and the ratio of production shocks is small[3].



This property is observed in the model with linkages and not in the benchmark as it is caused by the interaction between the technology parameter γ and the cost of building linkages g, which is not present in the benchmark model. More specifically, in the benchmark model, when fluctuations increase the production of firms with low shock goes down, the production of firms with high

[3]The plot area is restricted to such g and z_l/z_h that the policy functions for high and low productivity shock do not cross and the prices are computable.

shock goes up and while the overall effect is positive, the marginal effect is always negative, i.e. the additional production is decreasing with increase in fluctuations. On the other hand, in the model with linkages, the marginal effect can be both positive as well as negative, i.e. the additional production can be both increasing and decreasing with increase in fluctuations. It is because of the composition of the production of firms with high and low shocks?

In the model with linkages the role of firms with high shock is diminished, as the shock is always multiplied by the factor $1-g+\frac{\beta}{2}g<1$, while the role of firms with low shock is amplified because of multiplication by the factor $1+\frac{\beta}{2}g>1$. These multipliers cause the difference in the composition of the overall effect in the benchmark model and model with linkages.

From this result we can conclude that in an environment where fluctuations in production of big magnitude are present and costly building of linkages is necessary for supply of goods, these linkages are an arrangement that helps to boost intermediate production when an increase in fluctuations occurs. This is a clear positive effect that the economy with linkages has compared to the anonymous market economy.

1.5 Conclusions and Possible Extensions

This paper presents a new type of general equilibrium model in which we model supply linkages between firms that are often present in real economies. It is shown that this type of model has properties different from a standard production model. The model exhibits both positive as well as negative properties we expect network structures to have. The aggregate production in the proposed model is lower than in the benchmark economy. That is a direct consequence of transmission of bad shocks in the economy through linkages that imply rebuilding of linkages, which increases the marginal cost of production. On the other hand the proposed model has better behavior than the benchmark with respect to fluctuations in production. In case of increase in fluctuations the links help to boost aggregate production, which is always marginally decreasing in the benchmark model.

The model should be further explored in a more general setting without any restricting assumptions on shocks, with continuum of types of firms and a convex cost function for building linkages. Such a model has a more complicated distribution of firms that can have a significant influence on the aggregates of the economy. The generalized model is not tractable analytically and therefore requires simulation of all the results.

It would be also interesting to explore the model with more than two shocks. In the presence of several, or continuum of, shocks the economy with linkages should exhibit a certain degree of stability. While in the benchmark model the output is expanded or shrunken as an immediate response to a shock, in the model with linkages this structure of the economy creates disincentives to diminish output once the links are created and also, if the links are not created, their cost is a disincentive for expansion[4].

Another possible extension of the suggested model is to model the supply structure explicitly and therefore assume a certain network structure of the economy and possibly allow this structure to dynamically change.

The model can be also enriched in terms of entry and exit of firms. It can help to explore whether the persistent supply relationships in the economy help to avoid exits or they, due to a possible chain reaction, help to propagate bad shocks causing increasing number of exits from the economy.

[4]These effects are present also in this paper and are visible on the comparison of outputs of firms with low and high shock in the benchmark model and the presented model, but the stability should become more obvious in the model with continuum of shock where firms when hit only by a slightly different shock than in the previous period should be reluctant to change the level of production.

1.A Appendix

1.A.1 Sensitivity of the Model w.r.t. Parameters

The following Table summarizes several possible parameterizations of the proposed model with linkages. In the upper part of the table parameters of the model are fixed. The lower part then present prices and aggregates computed for the given fixed parameters. The presented parameterizations of the model differ only in one parameter at a time, the parameter is highlighted in bold font.

As in the model, also in the table the aggregate labor is assumed to be equal to 1 and the price of the intermediary good is also equal to 1.

β	**0.95**	0.95	0.95	0.95	0.95	0.95	**0.99**
α	**1/3**	**2/3**	1/3	1/3	1/3	1/3	1/3
γ	**2/3**	2/3	2/3	2/3	2/3	2/3	2/3
$\mathbf{z_h}$	**1**	1	1	1	1	1	1
$\mathbf{z_l}$	**0.5**	0.5	**0.9**	**0.7**	0.5	0.5	0.5
g	**0.5**	0.5	**0.1**	**0.1**	**0.3**	**0.1**	0.5
s	0.7603	0.7603	0.9711	0.7953	0.6869	0.6325	0.7582
w	0.4555	0.4555	0.6300	0.5692	0.4879	0.5281	0.4607
p	2.7046	1.3487	2.5479	2.5188	2.6521	2.5078	2.6720
I	0.7875	0.7875	0.9502	0.8738	0.8146	0.8245	0.7884
F	0.9234	0.8527	0.9831	0.9560	0.9339	0.9377	0.9238

As we can concluded from the table, the model is not particularly sensitive to changes in parameters, and relative prices as well as relative size of aggregates are of reasonable magnitude.

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Chapter 2

Stable Matchings in an Economy with Strong and Weak Agents

2.1 Introduction

This paper describes the properties of the stationary equilibrium of a particular matching market. Agents in the economy meet at random and they have possibility to create a match. It is assumed that agents are heterogenous in their probabilities of leaving the economy, which effectively means destruction of their match. Probabilities of exit influence lifetime of matches and therefore profit the matches bring to the agents.

The stationary situation of the economy with exit and entry of agents is studied. Optimal individual decisions of accepting or rejecting each particular type of match are analyzed and the social optimality of these decisions is assessed.

In economic literature the models where markets are modelled as a place for social contacts between individuals are referred to as search models. Search models have been used to study the properties of labor markets, money markets and also so called marriage markets. The basic model of reference in the labor market literature is Mortensen and Pissarides [10]. They developed a model of two-sided matching between vacant jobs and unemployed workers that was able to explain reasonably well the job creation and job destruction observed in the United States. Mortensen-Pissarides aggregate matching function was afterwards widely used in macroeconomic models of job search. Since then the research moved towards empirically more appealing models that allow on-the-job search. From this literature the model of matching between employers and workers by Kiyotaki and Lagos [6] is close in its spirit to the model presented in this paper. The matching model of Kiyotaki and Lagos helps to explain several features of labor market like the size and persistence of changes in income of workers due to job-to-job transitions, the length of job tenures and unemployment duration.

The matching models have been used also in the money market literature in order to explain why fiat currency can function as medium of exchange. For reference see Kiyotaki and Wright [7], [8].

The marriage market literature has its origins in the paper of Gale and Shapley [5]. They study the equilibrium properties of a particular two-sided matching market, the marriage market. They assume that every man has preferences over women and every woman has preferences over men and they study properties of the set of stable matchings in the economy. The "marriage" model was then extended in many ways, especially by assuming different degrees of transferability of the utility within pairs (e.g. Burdett and Wright [3]).

An interesting two-sided matching model is proposed also by Burdett and Coles [1]. They assume that the agents are ex-ante heterogenous, each is characterized by a real number which is in fact the utility of the spouse after they agree to marry. In this setting the authors are able to observe an equilibrium sorting of agents into clusters based on the numbers by which they are characterized. Burdett and Coles focus their attention only on the process of match creation, i.e. once a match is created the agents leave the market and are replaced by new agents.

The model presented in this paper also focuses on the process of match creation. Agents are assumed to be of two ex-ante types - strong agents and weak agents. The types differ by their exogenous probabilities of leaving the market. Since, by assumption, the matched agents do not have an opportunity to meet other agents, the matches split only due to exogenous reasons, i.e. when one of the partners exits the economy. The strength of the agents has therefore a direct impact on the expected lifetime of a match.

Matching enables interaction of agents, which is modelled like a production process. Proceeds from the production are split between the members of the pair. Single agents can not produce but they have prospects of being matched in the future. Optimal behavior of agents imply that a match is created only when both partners find it profitable, taking into account the outside option of staying single. Once the match is created, the matched agents do not have any incentives to walk away because their outside options do not change over time. Agents' optimal decisions of creating or rejecting a match are studied in an environment where agents differ only in their probabilities of exit from the economy but not in their productivity.

Possible extension of the proposed model are to assume that agents can search while matched or to allow for interaction of more than two agents. These extensions may allow the model to be suitable as the model of search on the labor market.

2.2 The Model

Time in the economy is discrete and the horizon is infinite. The economy is populated by agents of unit mass. There are two observable types of agents - strong ones and weak ones. We will refer to these characteristics as ex-ante types of agents. The mass of the strong agents is A and the mass of the weak agents is $1 - A$. The strength of agents is measured by their probability of exit from the economy. Let w denotes the probability of exit of a weak agent in a given period and s the probability of exit of a strong agent in a given period ($w > s$).

The interaction between agents, modelled like a production process, is happening in pairs. By assumption a pair of agents produces $2 \cdot \pi$ units of goods which they split. The pair bargains over splitting the proceeds from production.

Agents in the economy can be not only matched but also single. Those that are matched produce every period until one member of the pair exits. The other member is left single and the exiting one is replaced by a single agent of the same type, i.e. of the same strength as the exited one. Note that since no new information is revealed over time, we are focusing on the stationary situation of the economy, and the agents can not search for a new partner while matched, the agents do not have any incentives to walk away from the match once it was formed, i.e. the exits in the economy are only exogenous.

Single agents enter a market of singles. On this market in every period a fraction m of the singles is randomly proposed matching into pairs. This fraction is fixed, it is not dependent in any way on the searching behavior of individual agents in the economy.

The matching offer can come only once per period. Agents individually choose whether to accept the proposed match or stay single for another period. If both agents accept the match is created. Agents' individual decisions determine the types of possible equilibria. The conditions under which agents accept the proposed matches are discussed. The analysis is performed with respect to 4 parameters: the fraction of strong agents in the population A, the probability of being matched when single m, the probability of exit of the weak agents from the economy w, and the probability of exit of the strong agents from the economy s.

There can be up to 6 types of agents in each population. Three types of strong agents: a strong one matched with another strong, denoted ss type; a strong one matched with a weak one, sw type; and a non-matched strong, so type; and similarly three types of weak agents: ws type, ww type, and wo type. We will refer to these characteristics as ex-post types of agents.

2.2.1 The Bargaining Procedure

When two agents meet and have a possibility to create a match, they enter a bargaining procedure. The bargaining takes into account that the outside option of agents is to stay single for another period. When the two agents are of a different strength their values of being single differ. Therefore if they create a match the split of the proceeds from their production will be uneven.

As was already stated, the one-period production of a pair is independent of the composition of the pair. Pairs differ only in their expected lifetime, which depends on the composition of the particular pair. Therefore the pairs differ in their expected future profits.

Agents do not have any decisive power over the production, the only decision agents face is whether to match with a proposed partner when the matching situation occurs. Intuitively it is in the interest of both sides to match because only the pair interaction brings profits to agents. But due to bargaining it can happen that not every proposed match is accepted. The basic trade-off of the model is between the expected profit extracted from the particular match and the expected lifetime of the match[1]. Based on this trade-off agents may be willing to reject a certain type of the proposed match. As an example, consider the case when the probability of matching in each period is relatively high (m is high), the fraction of weak agents in the economy is high (A is small), and the difference between the strength of weak and strong agents is big ($w - s$ is big). Then when two strong agents meet they may consider rejecting the proposed match because they know that in the next period they have a high probability to be matched with weak agents and because of the big difference between the strengths they will be able to extract a lot of profit from the weak agents through the bargaining procedure.

The bargaining is assumed to be a take-it-or-leave-it offer. When two single agents meet one of them is randomly chosen to suggest the split of the expected proceeds from the future production of the pair (each of the agents is chosen with probability $\frac{1}{2}$). This agent will offer to his counterpart the smallest share possible so that the counterpart still accepts the offer, i.e. the profit the counterpart would have today when taking an outside option of staying single. The value of the outside option is the discounted value of being single (of the corresponding type) in the next period (it is discounted by the time factor β but also by the probability that the agent survives till the next period). But this value sums up all the future profits of the particular type of agent. The

[1] Either the agents can gain high one-period profits but the pair has a short life expectancy, or they have lower one-period profits with high life expectancy.

proposing agent will offer only the part of this value that corresponds to the present period[2].

This means that, for example, in the case of the match of a strong and a weak agent when the strong agent is proposing the split, the outside value of his weak counterpart is $\beta(1-w)v_{wo}$, where v_{wo} denotes the value of being wo type at the beginning of every period. The value is the sum of all expected future profits of a weak single agent that come from the possible matches in the future. The exact formula for the value will be stated later on.

The outside option differs from the value v_{wo} because the agent was in the present period already proposed a match and if he rejects it his profit in the present period is 0 and the value comes only from the future prospects given the agent will survive till the next period. The present's part of the outside value is $(1-\beta(1-w))\beta(1-w)v_{wo}$[3]. The strong agent therefore has to offer $(1-\beta(1-w))\beta(1-w)v_{wo}$ to the weak one and he will take $2\pi-(1-\beta(1-w))\beta(1-w)v_{wo}$. This happens with probability 1/2. With the same probability he will get $(1-\beta(1-s))\beta(1-s)v_{so}$ when the weak agent is proposing the split, and the weak one will take $2\pi-(1-\beta(1-s))\beta(1-s)v_{so}$.

Note that when the agents of the same type meet they both have the same bargaining power and therefore they must split the proceeds of the production equally, i.e. both agents get exactly π. Also note that the split of profits, as described above, is in fact a Nash bargaining result.

The bargaining procedure makes agents indifferent between accepting a proposed match and taking the outside option. When computing agents values of being a certain type the possibilities that the agent is proposing the split and the agent is accepting the split are taken into account. Since this expected profit from the bargaining can be either greater or smaller than the outside option for some parameters, agents will reject the proposed match and for other parameters they will accept it.

[2] One-period profit as a flow variable while the value of agents is the corresponding stock variable.

[3] $(1-\beta(1-w))\beta(1-w)v_{wo}(1+\beta(1-w)+(\beta(1-w))^2+(\beta(1-w))^3+...)=\beta(1-w)v_{wo}$

2.2.2 Values and Distributions

Because no searching while matched is allowed the agents have no incentives to walk away from a match once they have accepted it. Moreover, every period a particular type of agent faces the same prospects. Consequently, we can express the value of being a certain type recursively. The values for the six types, under the assumption that each agent would accept the proposed match, are:

$$
\begin{aligned}
v_{ss} &= \pi + \beta\Big((1-s)^2 \cdot v_{ss} + (1-s)s \cdot v_{so}\Big) \\
v_{sw} &= 1/2\Big(2\pi - (1-\beta(1-w))\beta(1-w) \cdot v_{wo} + (1-\beta(1-s))\beta(1-s) \cdot v_{so}\Big) + \\
&\quad \beta\Big((1-s)(1-w) \cdot v_{sw} + (1-s)w \cdot v_{so}\Big) \\
v_{so} &= m \cdot \Big(S \cdot v_{ss} + W \cdot v_{sw}\Big) + (1-m)\beta(1-s) \cdot v_{so} \\
v_{ww} &= \pi + \beta\Big((1-w)^2 \cdot v_{ww} + (1-w)w \cdot v_{wo}\Big) \\
v_{ws} &= 1/2\Big(2\pi - (1-\beta(1-s))\beta(1-s) \cdot v_{so} + (1-\beta(1-w))\beta(1-w) \cdot v_{wo}\Big) + \\
&\quad \beta\Big((1-w)(1-s) \cdot v_{ws} + (1-w)s \cdot v_{wo}\Big) \\
v_{wo} &= m \cdot \Big(S \cdot v_{ws} + W \cdot v_{ww}\Big) + (1-m)\beta(1-w) \cdot v_{wo}
\end{aligned}
$$

where we assume that the value of exit is 0, β is a factor by which agents discount the future.

Under the assumption that the Law of Large Numbers holds, in a matching situation agents will face a weak or a strong counterpart with probabilities that are proportional to the fractions of weak and strong agents that are single. The probabilities, and also the fractions of weak and strong agents in the pool of single agents, are denoted W and S respectively.
The probability S is equal to

$$S = d_{so}/(d_{so} + d_{wo})$$

and the probability W is equal to

$$W = d_{wo}/(d_{so} + d_{wo}) = 1 - S$$

where $d_{..}$ are distribution fractions of agents of indicated types.

The system of value functions can be rewritten in a matrix form as

$$\mathbf{V} \cdot \mathbf{v} = \boldsymbol{\pi}$$

where $\mathbf{v'} = (v_{ss}, v_{sw}, v_{so}, v_{ws}, v_{ww}, v_{wo})$, $\boldsymbol{\pi'} = (-\pi, -\pi, 0, -\pi, -\pi, 0)$ and $\mathbf{V}$ is the matrix implied by the system of equations. The matrix equation can be analytically solved and we get the value functions dependent only on parameters of the model.

$$\mathbf{v} = \mathbf{V}^{-1} \cdot \boldsymbol{\pi}$$

The distribution of agents across types $\mathbf{distr} = (d_{ss}, d_{sw}, d_{so}, d_{ws}, d_{ww}, d_{wo})$ evolves in time according to the vector equation

$$\mathbf{distr_{t+1}} = \mathbf{distr_t} \cdot \mathbf{Q}.$$

$\mathbf{Q}$ is a transition matrix that describes movement of agents across the states.

We are looking for a stationary distribution $\mathbf{distr}^*$, i.e. distribution that is stable in time

$$\mathbf{distr}^* = \mathbf{distr}^* \cdot \mathbf{Q}$$

Since agents are ex-ante of two strengths we will look for two stationary distributions, one for each type. Note that the fraction of ws type in the stationary distribution of weak agents must be the same as the fraction of sw type in the stationary distribution of strong agents.

Under the assumptions that the Law of Large Numbers holds and every agent accepts the proposed matching the transition matrices for strong and weak agents are QS and QW. The interpretation is that an element q_{ij} is the probability that the next period the agent will be of type j given that today he is of type i.

$$QS = \begin{pmatrix} (1-s)^2 & 0 & (1-s)s+s \\ 0 & (1-s)(1-w) & (1-s)w+s \\ mS(1-s)^2 & mW(1-s)(1-w) & \substack{(1-m)+mS((1-s)s+s) \\ +mW((1-s)w+s)} \end{pmatrix}$$

$$QW = \begin{pmatrix} (1-w)^2 & 0 & (1-w)w+w \\ 0 & (1-s)(1-w) & (1-w)s+w \\ mW(1-w)^2 & mS(1-w)(1-s) & \substack{(1-m)+mS((1-w)s+w) \\ +mW((1-w)w+w)} \end{pmatrix}$$

The types are ordered ss, sw, so in QS matrix and ww, ws, wo in QW matrix.

When computing the stationary distributions of weak and strong agents we have to take into account that $d_{ss} + d_{sw} + d_{so} = A$, and $d_{ww} + d_{ws} + d_{wo} = 1 - A$ must hold.

It is important to note that S and W are functions of the stationary distribution, therefore the system of equations describing the stationary distribution is not linear.

2.2.3 Stable Equilibria

This section in detail describes what is understood, in the context of this model, to be a stable equilibrium, and how different types of equilibria occur.

A set of agents' matching strategies together with corresponding values form the stable equilibrium if they constitute a Nash equilibrium and the strategies are evolutionary stable[4]. This means that none of the agents has incentives to change his strategy, given the strategy of the other agents. At the same time the strategies are resistent to small invasions, i.e. if there exists a fraction ε of agents that have decided to deviate in their strategy the agents playing the equilibrium strategy do not find it profitable to join the group of deviants.

In the context of the model there are up to six types of agents in the stationary situation of the economy. Fewer types can occur in the stable equilibrium depending on the strategies of agents. For all the possible strategies we have to check whether they constitute a Nash equilibrium and then check whether they are evolutionary stable. It is easy to see that, for example, some trivial Nash equilibria will not be stable from the evolutionary point of view.

The agents of both weak and strong type have 4 possible strategies[5]:

1. reject every match proposed

[4]Only pure strategies are discussed in this paper.
[5]Agents of the same ex-ante type use the same strategy.

2. accept only the match with an agent of the same ex-ante type

3. accept only the match with an agent of the opposite ex-ante type

4. accept all the proposed matches

These strategies can lead to several types of matchings. Each of the matchings can be characterized by the types of agents that exist in the matching. When respecting the assumptions of the model, the candidates for equilibria are[6]:

1. the matching consisting of types *so*, and *wo*

2. the matching consisting of types *ww*, *so*, and *wo*

3. the matching consisting of types *sw*, *ws*, *so*, and *wo*

4. the matching consisting of types *sw*, *ws*, *ww*, *so*, and *wo*

5. the matching consisting of types *ss*, *so*, and *wo*

6. the matching consisting of types *ss*, *ww*, *so*, and *wo*

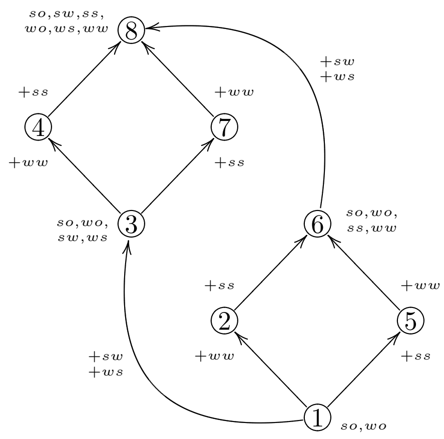
7. the matching consisting of types *ss*, *sw*, *ws*, *so*, and *wo*

8. the matching consisting of types *ss*, *sw*, *ws*, *ww*, *so*, and *wo*

There are no other possibilities that can occur. If the assumptions of the model are respected the single agents must always be present, and if the type *sw* exists so must the type *ws*.

Graphically we can represent the matchings as follows:

[6]Further on we will refer to the matchings based on the number assigned to them here.



The system of equations for the values of the types that has been described in the previous section applies to the matching 8. The rest of the matchings are less complex. They can be described by systems of equations similar to the one that applies for matching 8. The systems of equations are simpler because some of the nonexisting types are not present and the values of these types of agents can be considered to be zero. Similarly, the transition matrices have to be adjusted to the fact that some of the types of agents do not exist in matchings $1-7$.

We consider the stationary situations of the above described matchings, i.e. the situation when the values of the types of agents as well as the distribution are stationary. A stationary matching is considered to be a stable equilibrium if it is a Nash equilibrium with strategies that are evolutionary stable. It is easy to see that from the eight candidates for stable equilibrium matchings matchings 1, 2, and 5, i.e. the matchings that leave the whole populations of weak and/or strong agents single, are not evolutionary stable. Take as an example matching 2, where all the strong agents are single. Assume there exists a small group of strong agents that decide to deviate in their strategy and match with a strong agent if they meet one that is willing to match with them. Then the strategy of not accepting any match proposed is evolutionary unstable, because it assigns agents the value of 0, while the strategy of accepting a match with another strong agent has a positive value implied by the fact that the match, unlike single agents, is productive[7].

2.2.4 Multiple Stable Equilibria, Pareto Dominance

The theoretical matching model described in the previous sections has 4 parameters: m, A, w and s. The goal is to describe how the existence of different stable equilibria depend on the parameters of the model. Naturally, for a given combination of the parameters, several stable equilibria can occur, i.e. parameters are such, that several combinations of weak and strong agents' strategies lead to Nash equilibria that are also evolutionary stable. In this situation the question of Pareto dominance of one equilibrium over another one occurs. It is important to notice that the stationary equilibria are never directly comparable because they never consist of the same types of agents. Therefore it is not enough to compare the value of a certain type of agents in one equilibrium with the value of the same type of agents in the other equilibrium. A Pareto dominating equilibrium, in the context of this model, is such that all the values of a fixed ex-ante type of agent in this equilibrium are higher that the values of the same ex-ante type in another equilibrium. The condition, though it seems rather strict, is necessary because the ex-post types of agents as well as their distribution in the two equilibria are different and therefore one type of agent in the first equilibrium can become a different type in the other equilibrium[8].

2.2.5 Social Optimality

The proposed model allows for the study of social optimality of the equilibria implied by the agents' optimal decisions. Assume that in the model there exists a social planner who's goal is to maximize the aggregate welfare of the economy. Assume the social planner has no means how to change the meeting technology in the economy but he can influence agents' decisions by imposing rules on which

[7]The value of match with another strong agent reflects the fact that the match produces every period until the exit of one of the matched agents, as well as the fact that the probability of meeting an agent who is deviating is $m \cdot S \cdot \varepsilon$, where ε is the fraction of deviants in the population of single strong agents S.

[8]Note that it is enough to do the comparison of the values for the strong and the weak agents separately as the ex-ante types of agents can not change over time.

match they have to accept and which match they have to reject. The only way of dissolution of a match is exit of one of the matched agents from the economy. The stability of matchings is therefore imposed on the agents. This implies that the matchings that would be unstable from the point of view of agents can be chosen as optimal by the social planner.

The task of the social planner is simple. For each combination of parameters of the model and for each of the matchings 1 - 8 described in section 2.2.3 he computes the stationary distribution and the values of the types of agents. Using these the planner determines aggregate welfare Ω, which is defined as a weighted average of the values of types of agents with the weights that are the corresponding fractions of the distribution of agents.

$$\Omega = d_{ss} \cdot v_{ss} + d_{sw} \cdot v_{sw} + d_{so} \cdot v_{so} + d_{ww} \cdot v_{ww} + d_{ws} \cdot v_{ws} + d_{wo} \cdot v_{wo}$$

The agents' decisions are then, for the given combination of parameters, considered to be socially optimal if the set of equilibria stable under the given combination of parameters contains the stationary matching preferred by the planner.

In cases where the social optimum differs from the optima chosen by agents the natural question one can ask is whether the planner's solution is Pareto improving for the agents.

By the same argument as in the previous section, the matchings are directly incomparable between each other because in every comparison at least one type of agents is missing or is redundant. Moreover, the distribution of agents in each of these matchings is completely different so it is not clear whether the agent of a certain type in one matching will be of the same type in the other matching. The only possibility how to make sure that one matching is Pareto improving when compared with another matching is to make sure that all the values of types of agents in one matching are higher than all the values in the other matching.

2.3 Results

The stationary distributions as well as the values of types of agents in each of the 8 matchings mentioned in section 2.2.3 can be analytically expressed as the functions of parameters of the model. The values are homogenous of degree 1 in π.

To solve for the stationary state, first the stationary distribution has to be obtained. By a simple fixed point argument the stationary distribution exists and is unique[9] for every well defined transition matrix, which is the case of the transition matrices described in the theoretical section of the paper. Details of computation of the distribution can be found in Appendix 2.A.1.

After the stationary distribution is obtained the results enter the computation of the values of types of agents. The analytical solution to the system of value functions can be obtained but due to their complexity the analysis that follows is based on the results of a computation of the values for each particular combination of the parameters of the model. In the computation the profit π is fixed and is equal to 1. Since the values are homogenous of degree 1 in π this choice of the numeric value of π is not restrictive. The discount factor is $\beta = 0.95$, which is a standard value. The computation is performed for different fixed values of A, m, s, and w between zero and one, more precisely for eleven cases: for every 0.1 point in the interval $(0, 1)$ and for the extreme cases 0.001 and 0.999[10]

2.3.1 Results of Agents' Optimal Decisions:

Agents strategies may lead to 8 possible types of matching. For each of the 8 cases we have to check whether the strategies lead to a Nash equilibrium and if yes then whether they are evolutionary stable.

When agents follow any of the strategies except for the strategy "accept all the proposed matches" the resulting matching will consist of single agents of

[9] The theoretical background discussed in detail can be found in Stokey, Lucas, and Prescott [12].

[10] Values 0 and 1 for probabilities pose problems in computation of distribution of agents. Since the model assumes that $w > s$ all the computations take that into account.

both ex-ante types and at most one type of match for each of the ex-ante types, i.e. the resulting matching can be 1, 2, 3, 5, or 6, as described in the section 2.2.3. In these matchings the value of being matched is necessarily higher than the value of being single. The reasoning behind this fact is simple. Matches are productive, single agents are not. Because for each ex-ante type there does not exist more than one type of match, the strategic waiting for a better match can not be profitable. Therefore the value of being matched consists of today's profit from production plus the prospects of being matched (in the same type of match) or being single. The single agents have zero profits in all the subsequent periods until they become matched, therefore their value is smaller than the value of the matched agents.

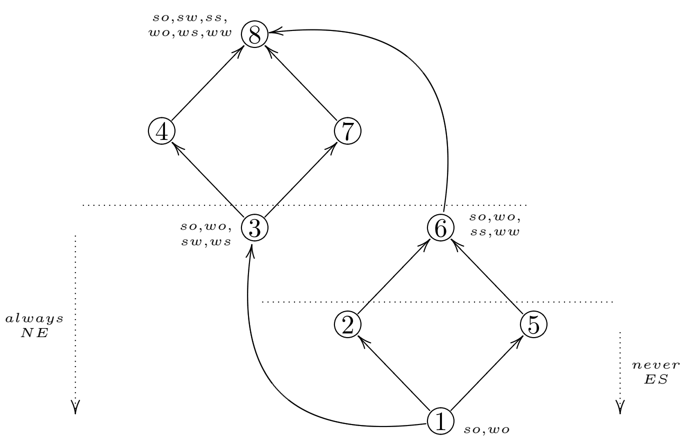
In the cases of matchings 1, 2, and 5 where the whole populations of strong and/or weak agents are single their values are necessarily equal to 0 because they are not producing and they have no prospects of being able to produce in the future.

From the considerations above it follows that for each of the matchings 1, 2, 3, 5, or 6, the values of the matched agents are higher than the values of the single agents of the same ex-ante type. Therefore the strategies leading to each of these matchings, the invariant distributions and the values of the agents in these matching, constitute Nash equilibria. Another straightforward conclusion is that matchings 1, 2, and 5 are not evolutionary stable. For each of these if there exists a small group of agents that decide to accept the match with their own type, they will make profit by playing this deviating strategy[11]. Therefore the candidates for stable matchings for each combination of parameters are matchings 3, 4, 6, 7, and 8. For each of these the evolutionary stability of the corresponding strategies has to be checked.

If either the population of weak and/or the population of strong agents plays the strategy of accepting all the matches proposed the strategic rejection of certain type of match becomes an issue. Therefore, for some combinations of parameters the matchings resulting from this strategy may not be a Nash equilibrium. For example, strong agents may find it profitable to reject the

[11]Value of deviating is positive. The reasoning behind this fact has been discussed at the end of section 2.2.3

match with a weak agent in a situation where the population of strong agents is big and meetings are common, because they have good prospects of quickly meeting a strong agent with whom they will produce for many more periods, in the expected terms. Another strategic consideration, when the population of weak agents is big, may be to wait for a weak agent from whom the strong agents can extract a lot of profits through the bargaining procedure.



The results of computations of stable equilibria with all the combinations of values of the 4 parameters of the model are summarized in Appendix 3.A.1 in Table 2.3.

To get the intuition behind the results we will focus our attention to one limit case of the model. We consider that strong agents do not die, i.e. the probability s is equal to 0. Even with this limit assumption we are able to obtain all the interesting combinations of stable equilibria that the general model implies.

Moreover, when the probability of meeting a new partner m is low the matches in the economy are very valuable, therefore the only equilibrium strategy is to accept all the matches proposed, leading to the equilibrium matching 8, which is therefore for a majority of combinations of the parameters the only matching that forms stable equilibrium. For the higher probability of being matched multiple stable equilibria occur. It is due to the fact that both matching 3 and matching 6 are Nash equilibria. If the agents do not have incentives to deviate from these two towards the same equilibrium, then necessarily we obtain multiple equilibria. We will therefore focus our attention on the cases when meeting are common, i.e. m is relatively high.

As we have already concluded, matchings 3 and 6 are always two coexisting Nash equilibria and they may be excluded from the results only in the cases when the strategies leading to this equilibria are not stable. The computations show, that each of the matchings 3, 4, 6, 7 and 8 can be a stable equilibrium for certain range of parameters. Matching 3 therefore creates multiple equilibrium with matching 6 and in some cases with matching 8. Matching 4 always coexists with matching 6. This happens for the range of parameters where matching 4 is Nash equilibrium. Agents will always deviate from matching 3 towards matching 4. Matching 6 can coexist either with matching 7 or matchings 7 and 4 at the same time.

When the probability of meeting m is very high, matchings 3 and 6 form stable equilibria basically for the whole range of probability w. In this range matchings 4, 7 and often also matching 8 do not form Nash equilibria and therefore matchings 3 and 6 are necessarily stable. When the probability of meeting decreases, matching 4 becomes Nash equilibrium, especially for higher values of probability w. Weak agents will deviate in their strategy from matching only with strong agents towards the strategy of accepting every match proposed and therefore matching 3 will not be stable anymore and will be replaced by matching 4. The reasoning behind this is as follows. When the probability w increases, keeping other parameters fixed, value v_{ws} goes down due to decreased life expectancy of the couple. This has an impact on the value of being single v_{wo}, which also goes down and therefore value v_{ww} can easily become greater than v_{wo}, which makes matching 4 a Nash equilibrium. Once 4 is Nash the agents will always deviate from 3 towards 4.

The threshold for the probability w at which agents deviate from matching 3 towards matching 4 moves closer to one as the fraction of strong agents in the economy A gets larger. It is because the probability of meeting a weak agent goes down, which decreases the value of ww pairs and therefore matching 4 is Nash equilibrium only for higher values of w.

When the probability of meeting a new partner m decreases towards values $m = 0.6 - 0.7$, then for probability w close to one not only matchings 4 and 6 are stable equilibria. Matching 7 joins them in the multiple equilibrium. The high probability w decreases the value of strong agents in mixed pairs v_{sw}, that as a consequence decreases the value of singles v_{so} and the value of pairs of strong agents can therefore easily become greater than the value of being single. At the same time pairs of weak agents are not particularly attractive, because the high probability w and relatively low probability of meeting m imply very low value of pairs ww, in fact lower than the value of staying single, which makes matching 7 form a stable equilibrium. At the same time the probability of meeting is still high enough so that weak agents can reject certain matches. With a further decrease of probability of matching agents have to accept all the matches proposed and this implies matching 8.

When 7 is stable equilibrium agents always have incentives to deviate from 3 towards 7. Other types of deviations are not very common. That is why we can observe coexistence of matching 7 either with matching 6, or with both matching 4 and matching 6.

One more type of coexisting stable equilibria occurs. It is matching 3 with matching 8. This happens for high fractions of strong agents in the economy A and low probabilities w. The usual starting point is matchings 3 and 6 that are Nash. Matching 3 is stable because for the low values of w and low probability to meet a weak agent, there are no incentives to create weak pairs, therefore matching 4 is not Nash. On the other hand, given this parametrization, matching 8 also forms Nash equilibrium and agents will always deviate from matching 6 towards matching 8. This is an implication of the fact that strong agents may improve their value by exploiting the bargaining procedure is the sw pair and weak agents improve lifetime of their pair in the ws match. On the other hand matching 3 is stable with respect to matching 8 as pairs ww are not present in matching 3 and they are not an improvement for weak agents in matching 3.

Despite the fact that this analysis of equilibria is provided for a fixed value of probability of exit of strong agents, similar results can be obtained without this restrictive assumption. The analysis without any restrictions has been performed and summary of the equilibria together with their characteristic range of parameters can be found in Appendix 2.A.2 in Table 2.3. Illustrative plots of the equilibria showing the change of equilibria with the change of parameters is provided in Appendix 2.A.3.

Among the coexisting equilibria two types of Pareto dominance occur but they are present only for small ranges of parameters. Matching 4 dominates matching 6, typically for small values of the parameter A, and matching 8 dominates matching 3, typically for the values of A in the middle range.

2.3.2 Results of Planner's Optimal Decisions:

In the planner's problem, when studying which are socially optimal matchings, matchings 3, 4, 7, and 8 can be socially optimal depending on the combination of parameters. It is important to realize that for a matching to be socially optimal several factors play a role. The welfare of the economy depends on both the stationary distribution and the values of types. So, if in the stationary distribution fraction of one type of agents is high and at the same time the value of this type is relatively high this can be crucial for the matching to be socially optimal.

For the planner the split of profits within a pair does not play any role because the social value of every pair is the same since every pair produces 2π per period. The planner is therefore concerned only about the expected lifetime of every type of producing pair. This is precisely the reason why matching 6 can not be optimal for the planner's problem. In matching 6 the mixed pairs are not present. The pairs are only of the ss-type or of the ww-type. This is inefficient from the planner's point of view because the planner would prefer to take advantage from the longer expected lifetime of the strong agents and by matching them with weak agents improve the lifetime of the weak agents' pairs. So the planner will first use the strong agents for the mixed pairs and only in the situation when there are many strong agents he will allow for the "luxury" of ss-type of pairs. Therefore, if ss-type is present in the matchings imposed by the planner so must be the mixed types. That is why matching 6 is never socially optimal.

The ranges of parameters under which the 4 above listed types of matching are socially optimal are summarized in Table 2.4 in the Appendix. Briefly the ranges of the parameters can be characterized as follows.

The matching 3 is socially optimal typically when meetings are common ($m \geq 0.7$), strong agents are in a minority in the economy ($A \leq 0.5$) and strong agents are very strong (s very low). Under this parametrization the planner uses strong agents for increasing the lifetime of matches of the weak agents, i.e. strong agents are matched with weak agents in mixed pairs. Because the strong agents are very strong and the probability of meeting is high the planner will forbid ww-type of matches, he will prefer to let the weak agents wait for a strong partner that will dramatically prolong the expected lifetime of the couple.

For the similar range of parameters as in the previous case, with the difference that there is even less strong agents in the economy ($A \leq 0.3$), the planner will allow also for ww-type of matches, implying matching 4. Because there are too few strong agents in the economy to meet a strong counterpart is difficult. Therefore more profitable than to wait for a strong agent is to allow weak agents to create short lasting ww-type couples.

Matching 7 is socially optimal when there are many strong agents in the economy, i.e. when A is large ($A \geq 0.5$). Because there are many strong agents in the economy the planner wants to use them to improve the expected lifetime of pairs where weak agents are present, i.e. the planner will allow for the mixed pairs. And as the strong agents are in the majority the planner can also allow for creation of ss-type of pairs. On the other hand he will forbid ww-type of pairs in the cases when the meetings are common (m is high). Under such parametrization the planner prefers to let the weak agents wait for a strong counterpart rather then to let them create short-lasting ww pairs. In expected terms, the waiting time for a strong counterpart should be short as meetings are common and strong agents are in majority in the economy.

Matching 8 is typically socially optimal in the cases when the strong agents are not too special, i.e. they are either not too strong (s is relatively high) or they are almost the same as the weak agents ($w - s$ is small). In these cases the planner has no reason to forbid any type of match. Matching 8 is also characteristic for the parameterizations with m small, i.e. when meetings of single agents are rare. Under these circumstances the planner can not afford to forbid any type of match as the expected waiting time for another match is too long.

The equilibria implied by agents' decisions are now to be compared with the socially optimal matchings. For each combination of the parameters it should be assessed whether the stable equilibria chosen by the agents contain the socially optimal matching. If that is not the case then the agents' decisions lead to a socially suboptimal matching. The reason why agents' decisions may lead to the socially suboptimal result is simple. Agents by taking optimal individual decisions end up in the stable equilibrium. The planner, on the other hand, can implement a matching that is not evolutionary stable or not even a Nash equilibrium. That means he can, for example, force agents to create matching where the value of being single is higher than one of the values of being matched. Of course, the agents would choose to reject such type of match and they would prefer to stay single. That would lead to a lower fraction of matched agents which could have in the end a negative impact on the overall welfare of the economy.

Computations suggest that about 18% of agents' choices under all possible parameterizations are not socially optimal. The types of socially suboptimal choices the agents make are briefly summarized in Table 2.1. The table shows all possible types of suboptimal choices that have been obtained in the numeric computations. The most common socially suboptimal choices of the agents are the multiple equilibrium of matchings 3 and 6, and the multiple equilibrium of matchings 4 and 6. In both cases the planner's choice is typically matching 8, though for small ranges of parameters it can be also other matchings, as summarized in Table 2.1.

In some of the cases of agents' socially suboptimal decisions the planner can achieve a Pareto improvement by implementing the matching optimal from his point of view. The Pareto improvement, as already described in the previous section, is achieved when all the values of types in the improving matching are higher than the values in the matching chosen by the agents. In this way improvement is guaranteed for all the agents as the distribution of agents changes with changing the type of matching. Note that there are two types of Pareto improvement that can be considered. First type is the one where the Pareto improving matching is improvement of all the stable matchings for the given set of parameters. The other, a weaker version, is such that the socially optimal

Socially suboptimal equilibria	
Agents' choice	Planner's choice
8	**3**
	4
	7
7	**8**
4,6	**3**
	8
3,8	**7**
3,6	**4**
	7
	8
6,7	**8**
4,6,7	**8**

Table 2.1: Socially suboptimal choices of agents

matching is Pareto improving at least one of the stable matchings chosen by the agents.

The Pareto improvement by the social planner can be achieved rarely, in approximately 14% of the cases when agents do not behave socially optimally. In most of the cases the improving matching is matching 8, in minority of the cases it is matching 7. The Pareto improvement is typically achievable for high values of the parameter m, and it is briefly summarized in Table 2.2.

Pareto improvement by planner	
Agents' choice	Planner's choice
6	**7**
6	**8**
3	**8**

Table 2.2: Pareto improving matchings

The planner can also achieve the improvement of all the matchings that are stable from the agents' point of view. For a small range of parameters under which the coexistence of matchings 6 and 3 occurs the planner can achieve a Pareto improving situation by imposing matching 8.

2.4 Conclusions

The paper studies the stationary situation of the matching market in the economy where agents differ in their probabilities of exit. In this economy the agents' optimal profit maximizing decisions can lead to several types of stable equilibrium matching. The model produces 5 different types of stable equilibria. Multiple equilibria are possible for certain ranges of parameters. Overall we observe 7 types of multiple or simple stable equilibria.

The optimal behavior of the agents can be summarized as follows. For the lower probabilities of meeting a potential partner agents tend to accept every proposed match. For the higher probabilities of meeting the profit extractions from the bargaining procedure play a significant role and therefore some of the proposed matches are rejected. The agents' optimal decision leads, particularly for higher probabilities of being matched, to socially suboptimal matchings. Socially optimal can be 4 of the 5 matchings resulting from the agents' decisions. This is an implication of the fact that the social planner does not care about the division of the profits within pairs.

In approximately 18% of the cases the social planner is able to improve welfare of the economy by imposing matchings, usually with more types of matched agents, that are unstable from the point of view of agents. Some of the planner's decisions lead to the Pareto improvement for all the agents in the economy.

The results of this paper suggest that the presence of the social planner in the organization of matching markets may be beneficial for the overall welfare of the economy and may have also a Pareto improving effect for all the agents in the economy.

2.A Appendix

2.A.1 Results for the Stationary Distribution

The following results for the stationary distribution hold for the case when all the matches are accepted, i.e. for matching 8.

When solving for the distribution, the system of six equations together with two constraints, $d_{ss} + d_{sw} + d_{so} = A$ and $d_{ww} + d_{ws} + d_{wo} = 1 - A$, can be narrowed down to the system of two quadratic equations with two unknowns that has two sets of solution. It can be shown that only one of these will give positive results for all combinations of parameters s, w, m, A. The system of quadratic equations is:

$$d_{so}^2(1-m+\frac{m}{2s-s^2})+d_{so}d_{wo}(1-m+\frac{m}{s+w-sw})-Ad_{so}-Ad_{wo}=0$$

$$d_{wo}^2(1-m+\frac{m}{2w-w^2})+d_{so}d_{wo}(1-m+\frac{m}{s+w-sw})-(1-A)d_{so}-(1-A)d_{wo}=0.$$

Then the solution of the system that is plausible, i.e. that gives positive fractions of d_{so} and d_{wo}, is:

$$d_{so}=\frac{-xy+y^2-2Ay^2+2Axz+(x-y)\sqrt{(1-2A)^2y^2-4(-1+A)Axz}}{2x(-y^2+xz)}$$

$$d_{wo}=-\frac{y^2-2Ay^2-2xz+2Axz+yz+(y-z)\sqrt{(1-2A)^2y^2-4(-1+A)Axz}}{2z(-y^2+xz)}$$

where x, y, z stand for

$$x=1-m+\frac{m}{2s-s^2}$$

$$y=1-m+\frac{m}{s+w-sw}$$

$$z=1-m+\frac{m}{2w-w^2}.$$

The other four fractions of the stationary distribution can be expressed, using d_{so} and d_{wo}, like this:

$$d_{ss}=m\cdot\frac{d_{so}^2}{d_{so}+d_{wo}}\cdot\frac{(1-s)^2}{2s-s^2}$$

$$d_{sw} = d_{ws} = m \cdot \frac{d_{so}d_{wo}}{d_{so} + d_{wo}} \cdot \frac{(1-s)(1-w)}{s + w - sw}$$
$$d_{ww} = m \cdot \frac{d_{wo}^2}{d_{so} + d_{wo}} \cdot \frac{(1-w)^2}{2w - w^2}$$

In a similar manner the results for all other types of matching, i.e. matchings 3, 4, 6, and 7 can be obtained. As a matter of fact, the same system of quadratic equations can be used for the computation, only the expressions for x, y, and z will change. One or two of them will be equal to 1, depending on which matching we consider.

2.A.2 Summary of the Equilibria

Table 2.3 summarizes the coexisting equilibria implied by agents' decision and provides rough intervals for the parameters under which the equilibria occur. The intervals of parameters are only the estimates done based on the plots done for 121 combinations of values of the parameters A and m and plotted for the approximation of a continuous range of the parameters w and s, where $w \geq s$.

As stated and explained in section 3.3.1, for each combination of the parameters one of the matchings 8 and 6 is always stable equilibrium. For the combinations of parameters where $m \leq 0.7$ it is the matching 8, for the extreme value of m being close to 1 it is matching 6. In between these values the division of the space of parameters s and w is approximately described by the line $s = \max(0.05 + 2.5(m - 0.6) - 0.5Aw, 0)$. For the values of s higher than this threshold matching 8 is the stable equilibrium, for the lower values it is matching 6. Matchings 3, 4 and 7 then can be ordered on the plots from the left to the right,i.e. matching 3 occurs for the lower values of w, matching 4 for the middle range and matching 7 for high values of w.

It is important to note that the table summarizing the coexistence of the equilibria has only an informative character and the inequalities described by the table do not hold for some exceptional cases of the combination of parameters. Also, because equilibrium matching 8 forms the stable matching for the majority of the combinations of parameters it is not described in detail. On the contrary, all the other cases are described in a great detail and it is implied that the rest of the combinations of parameters describes matching 8 in case when it is the only stable equilibrium matching.

Coexisting equilibria	Values of the parameters		
	A	m	w, s ($s \leq w$)
7	0.999	0.999	$w \geq 0.6$, $s \leq 0.3w-0.03$, $s \geq 0.125w+0.075$
6, **7**	0 - 0.1	0.7 - 0.8	$w \geq m+0.1$ $s \in [0.4\cdot(m-0.7)+0.1-A, 0.4\cdot(m-0.7)+0.3-2\cdot A]$
4, **6**, **7**	0.1 - 0.9	0.7 - 0.9	$w \geq m+0.1-0.2\cdot A$ $s \in [0.1+(m-0.7)\cdot 3-0.4\cdot A, 0.2+(m-0.7)\cdot 3-0.4\cdot A]$
4, **6**	0.001 - 0.5	0.6 - 0.9	$w \in [0.1,0,3]$, $s \to 0$ for $m=0.6$
			$w \leq 0.8+A$, $s \leq 0.25\cdot w\cdot(1-A)$ for $m=0.7$
			$\forall w$, $s \in [(0.5-A)\cdot w, (0.6-A)\cdot w]$ for $m=0.8$
			$\forall w$, $s \in [0.8\cdot w-A\cdot 0.5\cdot w, 0.85\cdot w-A\cdot 0.5\cdot w]$ for $m=0.9$
	0.5 - 0.9	0.7 - 0.9	$w \geq A-0.4$, $s \to 0$ for $m=0.7$
			$w \geq A-0.3$, $s \leq 0.35\cdot w-(A-0.5)\cdot 0.5$ $s \leq 0.4-0.35\cdot m-((A-0.5)\cdot 0.5)\cdot(1-w)$ for $m=0.8$
			$w \geq A-0.2$, $s \leq (1.15-A)\cdot w$, $s \leq 1.05-A$ for $m=0.9$
3, **6**	0.001 - 0.5	0.8 - 0.999	$\forall w$, $s \leq (0.6-A)\cdot w$ for $m=0.8$
			$\forall w$, $s \leq (0.9-A)\cdot w$ for $m=0.9$
			$\forall w$, $\forall s$ for $m=0.999$
	0.5 - 0.999	0.7 - 0.999	$w \in [0.05, A-0.4]$, $s \to 0$ for $m=0.7$
			$w \in [0, A-0.3]$, $s \leq (0.9-A)\cdot w$ for $m=0.8$
			$w \in [0, 0.55+(A-0.6)\cdot 0.5]$, $s \leq (1.15-A)\cdot w$ for $m=0.9$
			$\forall w$, $\forall s$ for $m=0.999$
3, **8**	0.3 - 0.4	0.6 - 0.9	$w \in [0,(m-0.5)\cdot 0.5]$, $s \doteq w$
	0.6 - 0.999	0.3 - 0.9	$w \in [0, 0.5\cdot(A-0.5)+0.4\cdot(m-0.3)]$, $s \to 0$ for $m \leq 0.7$
			$w \in [0, 0.5\cdot(A-0.5)+0.4\cdot(m-0.3)]$ $s \geq ((0.9-A)+(m-0.8)\cdot 2.5)\cdot w$ for $m \geq 0.8$
8	$\forall A$	0.001 - 0.9	all the ranges not covered by the previous cases

Table 2.3: Agents' optimal choice

Table 2.4 summarizes socially optimal matchings and typical parameters for which they occur. Similarly to the case of agents' optimal decisions, the intervals of parameters are the estimates done based on the plots done for 121 combinations of values of the parameters A and m and plotted for the approximation of a continuous range of the parameters w and s, where $w \geq s$. Socially optimal can be matchings 3, 4, 7, and 8.

Social optima	Values of the parameters		
	A	m	w, s (s≤w)
3	0.2	0.8-0.999	w∈[0.35-0.5(m-0.8), 0.45+(m-0.8)], s→0
	0.3	0.7-0.999	w∈[0.1+6s, 0.3+1.5(m-0.7)-9s], s≤0.1(m-0.6)
	0.4	0.5-0.999	w∈(4s, 0.08+0.8(m-0.5)-3s] s≤0.1(m-0.6) for m≥0.7, s→0 otherwise
	0.5	0.999	w∈[0.02, 0.1], s→0
4	0.001	0.7-0.999	∀w, s≤w[0.5(m-0.7)+0.05]
	0.1	0.7	w∈[0.05, 0.45], s→0
		0.8-0.999	∀w, s≤0.5w(m-0.7)
	0.2	0.8	w∈[0.1, 0.35], s→0
		0.9-0.999	∀w≥0.05, s≤0.05+w(m-0.9) except w∈[0.25, 0.55+(m-0.9)], s→0
	0.3	0.999	w∈[0.8, 0.95], s→0
7	0.5	0.999	w∈[0.1, 0.25], s≤0.02
	0.6	0.8-0.999	w∈[0.05,0.2+0.7(m-0.8)-s], s≤0.01+0.2(m-0.8)
	0.7	0.7	w∈[0.05,0.2], s=0.02
		0.8-0.999	w∈[0,0.06+0.2(m-0.8)] & s→0 w∈[5s-0.1,0.5m-2.5s+0.05], s∈(0,0.04+0.1(m-0.8)]
	0.8	0.6-0.999	w∈[0,0.04+0.4(m-0.6)] & s→0 w∈[2.5s,0.5m-2.5(s-0.02)], s∈(0,0.15m-0.07]
	0.9	0.4-0.999	w∈[0,0.4m-0.1] & s→0
		0.6-0.999	w∈[0.05+2.5(s-0.02),0.5m+0.1-2.5(s-0.02)] s∈(0,0.15m-0.07]
	0.999	0.4-0.999	w∈[0,m-0.2] & s→0
		0.5-0.999	w∈[0.05+2.5(s-0.02),m-5s],s∈(0,0.16m-0.06]
8	0.001 - 0.999	∀m	ranges not covered by the cases **3**, **4**, and **7**

Table 2.4: Planner's optimal choice

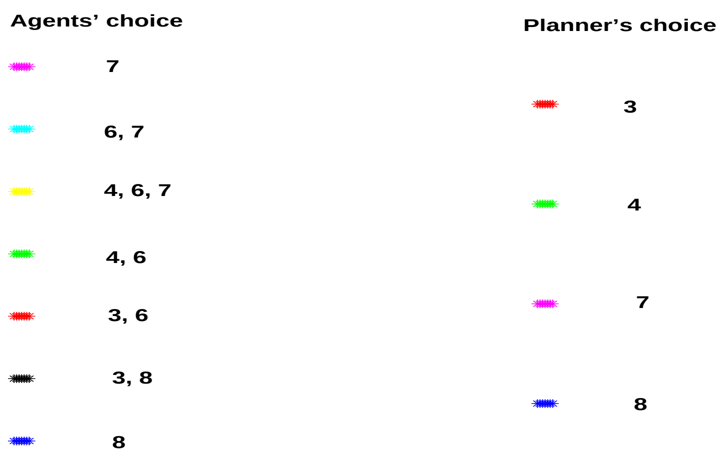
2.A.3 Plots of Equilibria

This section contains illustrative plots of equilibria implied by individual decisions as well as the social planner's equilibria. The plots are done for the case discussed in the paper, i.e. $s = 0$. The value of parameter A is fixed and the particular value is stated on each of the plots. The whole range of parameter w is covered together with higher range of parameter m. The plots show on purpose higher values of parameter m where we can observe many different types of equilibria with changing values of the parameters w and A.

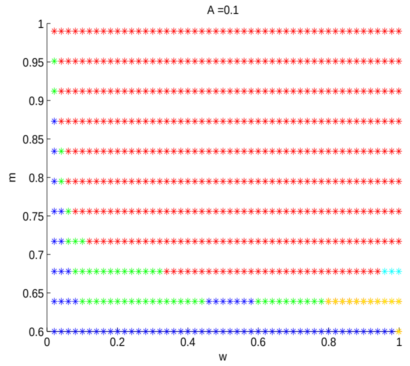
For each possible combination of parameters the type of equilibrium is computed and then it is plotted in the color reserved for that particular type of equilibria. The table preceding the plots should help orientation among different types of equilibria in both agents' and planner's problems.

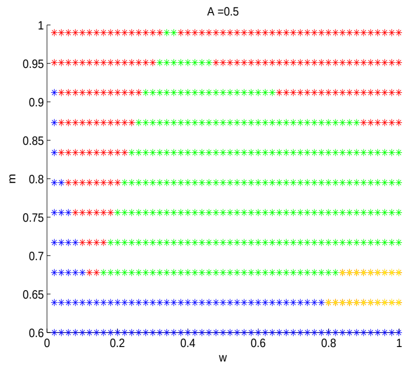
On the horizontal axis we have parameter w, the vertical axis represents parameter m.

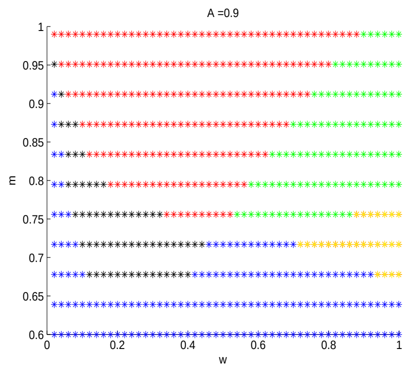
Different types of equilibria are distinguished by different colors as follows:



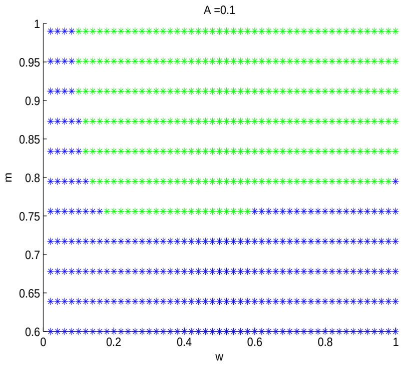
Equilibria of agents' problem:

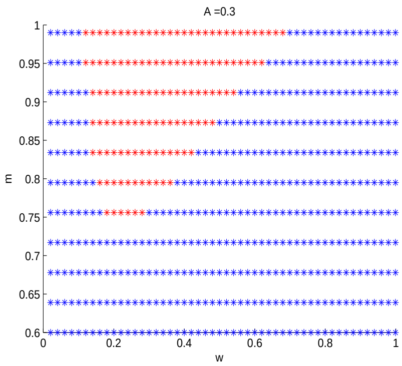


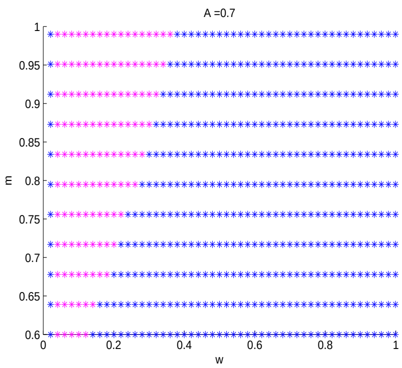




Equilibria of planner's problem:







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Chapter 3

Stable Firm-Worker Matchings in an Economy with Ageing Workers

3.1 Introduction

This paper is inspired by the research that has been done on equilibrium properties of two-sided matching markets. Traditionally, the labor market or marriage market are studied. These markets are considered to be two-sided because matches are created between agents from two distinct disjoint populations. Good examples of such populations are firms and workers, men and women. The match is defined as a long-term relationship of two agents, each coming from a different population. It is assumed that being in a match is a profitable activity, and agents maximize their expected future profits. The economy is studied in a stationary situation when the matching between the two populations is stable, i.e. no matched individual prefers to be single, and no single individual, when having an opportunity to match, would choose the match over the option of staying single.

This paper focuses on a labor market matching. The firms are assumed to be identical, workers on the other hand are of two types. The types do not differ in their productivity but they differ in their probabilities of leaving the economy. Workers' probabilities of exit influence the lifetime of matches and therefore profit from the matches. The proposed model allows for the study of equilibrium properties of a simple labor market with entry and exit of agents. Optimal individual decisions of accepting or rejecting each particular type of match as well as firing decisions of firms are analyzed and social optimality of these decisions is assessed.

In the past two-sided matching models have been used to address questions related to labor markets and marriage markets. The two sided matching models have been widely used to study both macro and microeconomic problems. Mortensen and Pissarides [10] developed a model of two-sided matching between vacant jobs and unemployed workers. The model was able to explain reasonably well job creation and job destruction observed in the United States. Mortensen-Pissarides aggregate matching function has since then been widely used in macroeconomic models of job search.

A model of matching between employers and workers by Kiyotaki and Lagos [6] helped to explain several features of the labor market like the size and persistence of changes in income of workers due to job-to-job transitions, the length of job tenures and unemployment duration.

In micro-oriented literature, the discussion on two-sided matchings started with the "marriage" model of Gale and Shapley [5]. They assumed that every man has preferences over women and every woman has preferences over men and they studied properties of the set of stable matchings in the economy. The marriage model was then extended in many ways, especially by assuming different degrees of transferability of the utility within pairs (e.g. Burdett and Wright [3]).

An interesting two-sided matching model was proposed by Burdett and Coles [1]. They assumed that the agents are ex-ante heterogenous, each is characterized by a real number which is in fact the utility of the spouse after they agree to marry. In this setting the authors were able to observe an equilibrium sorting of agents into clusters based on the numbers by which they are characterized. Burdett and Coles focused their attention only on the process of match creation, i.e. once a match is created the agents leave the market and are replaced by new agents.

The following model focuses on the process of match creation but allows also endogenous dissolution of matches. Firms are homogenous in the model. The workers are heterogenous. They do not differ in their productivity but they do differ in their probabilities of leaving the workforce. Workers can be young or old, with the assumption that young with a certain probability become old. The type of the worker has therefore impact on the expected lifetime of the worker-firm match.

Matching enables production. Proceeds from the production are split between the members of the match. Non-matched agents can not produce but they have prospects of being matched in the future. Optimal behavior of agents imply that a match is created only when both partners find it profitable, taking into account the outside option of staying single. Firms, depending on the parameters of the model, may have incentives to fire their worker when he becomes old. Optimal decisions of creating, rejecting and dissolution of a match are studied and they are compared to the optimal decision of social planner.

3.2 The Model

Time in this economy is discrete and the horizon is infinite. The economy consists of a mass 1 of firms and a mass 1 of workers. Every firm needs to employ one worker in order to be able to produce. The proceeds from the production are split between the firm and the worker.

The characteristics of the agents are as follows. The firms are ex-ante identical. They are characterized by their probability of bankruptcy, denoted b. Bankrupted firms are replaced by new firms, characterized by the same probability of bankruptcy, so that the mass of firms in the economy is kept constant.

The workers are ex-ante of two types - workers with lower probability of exit from the economy, we will call them young workers, and workers with higher probability of exit, we will call them old workers. The types do not differ in their productivity. The young workers are also characterized by their probability of becoming old. Naturally, the old workers can not become young, they can only exit. The workers that exit the economy are replaced by young workers so that the mass of workers in the economy stays constant.

In mathematical terms, we will assume a simple structure "without memory", i.e. probability that a young worker exits labor force is y and that is true in every period. Probability that he stays in the labor force is $1 - y$. This case is further divided into two possibilities. The worker either ages, becomes old, which happens with probability $(1 - y)a$, or the agent stays young for another period, which happens with probability $(1 - y)(1 - a)$. Probability that an old worker exits the labor force is denoted o, $(o > y)$. Consequently, probability that an old worker stays in the economy for another period is $1 - o$.

Workers and firms are matched to pairs at random. Probability of meeting a partner for match is m and it is the same for the workers and the firms. When firm meets a worker it chooses whether to accept the worker for the match, in which case the pair starts to produce, or reject the match and search for a worker for another period[1]. Only non-matched firms and workers can be matched, i.e. there is no "on-the-job" search. The firms can decide to fire the workers that aged, i.e. became old in the given period[2]. The dissolutions of pairs happen also due to exits of firms and workers from the economy. The surviving part of a dissolved pair becomes again a searching worker or firm and waits for a new match.

The timing in one period of the economy is as follows. First, the state of the world is revealed. Bankrupted firms and workers that are out of the labor force leave the economy and they are replaced by new firms and young workers. Firms, if they decided to do so, fire the old workers and start to look for young ones. The firms whose workers exited join the pool of searching firms, the workers whose firms went bankrupt join the pool of searching workers. A fraction m of the pool of searching firms at random meets a fraction m of searching workers. In every

[1]The agents will always accept the match with a firm because being in the match, unlike being single, brings profit, and because all the firms are the same, i.e. there are no strategic reasons why not to accept the proposed match. The young agents have no incentives to wait because their contract will automatically change once they age, as ageing is observable by firms

[2]Firms can have incentives for dissolution of their match only when the characteristics of the worker they employ changes. In other cases they would not create the match at all rather than split afterwards.

pair the firm decides whether to match and produce with the proposed worker or whether to search for another period. Every firm-worker pair produces 2π of good in every period and these proceeds from production are divided between the firm and the worker based on the Nash bargaining solution, taking into account that the outside option for both the firm and the worker is to stay single and search for another production partner.

In a stationary situation of the economy there are up to three types of firms and 4 types of workers. The firm can be either matched with a young worker, or with an old worker or it can be searching. Both young and old workers can be either matched with a firm of they can be searching. Consequently, we can have 7 possible types of agents in a stationary equilibrium. Denote fy the firm that is matched with a young worker, fo the firm that is matched with an old worker, and fs the firm that is searching for a worker. Similarly, denote yf the young worker that is matched with a firm, and ys the searching young worker. The notation of will be further on used for the old worker matched with a firm, and os will stand for the old searching worker.

3.2.1 The Bargaining Procedure

When a worker and a firm meet they enter a bargaining procedure. The bargaining takes into account that the outside option of both is to stay single for another period. Since the values of non-matched firms and non-matched workers differ when a firm-worker match is created they split the proceeds of the production unevenly.

As was already stated, the one-period production of a pair is independent of the type of worker. Pairs differ only in their expected lifetime, which depends on the composition of the particular pair. Therefore the pairs differ in their expected future profits.

Firms do not have any decisive power over the production, the only decision they face is whether to match with a proposed worker when the matching situation occurs. Intuitively it is in the interest of both sides to match because only the pair interaction brings agents profits. But due to bargaining it can happen that not every proposed match is accepted. The basic trade-off of the model is between the expected profit from bargaining and the expected lifetime of the match. In pairs where firms have much lower probability of exit than workers the firms gain a lot through bargaining but the expected lifetime of matches is small, on the other hand if workers have low probability of exit, firms do not gain much in bargaining but lifetime of matches is long. Based on this trade-off firms may decide to reject a certain type of worker they meet. For example, in the situation when the meetings are common (the probability m is high) and the old workers leave the workforce often (the probability o is high), if the firms are characterized by a low probability of bankruptcy (b is low) then they have a lot of bargaining power over the old workers and they extract a lot of profits from the production. In such a situation the firms may decide to reject matches with the young workers that would take part of the profits due to their stronger bargaining position. It is simply more profitable to be matched with old workers whose turnover is higher but because of the high probability of meeting a new worker when non-matched the firms can be sure to be producing in almost every period and extracting profits from the bargaining over the production. Therefore this is the case when the extracting of profits from the old agents more than offsets the effect of the longer expected lifetime of matches with the young workers.

The bargaining is assumed to be a take-it-or-leave-it offer. When a firm meets a worker one of them is randomly chosen to suggest the split of the expected proceeds from the future production of the pair (each of the agents is chosen with probability $\frac{1}{2}$). This agent will offer to his counterpart the smallest share possible so that the counterpart still accepts the offer, i.e. the profit the counterpart would have today when taking an outside option of staying single. The values of the outside options are the discounted values of being non-matched (of the corresponding agent) in the next period (discounted by the time factor β but also by the probability that the agent survives till the next period). But these values sum all the future profits of the particular type of agent. That is why only present parts of these values are taken into account. The outside options differ from the values of being non-matched because the agents are proposed in each period at most one match. If they reject the profit in the present period is 0 and the value of a particular agent comes only from the future prospects given the agent will survive till the next period. Note that the split of profits, as described above, is in fact the Nash bargaining result.

3.2.2 Values and Distributions

As already discussed, in the stationary situation of the economy there are up to 7 types of agents. In every period every type of agent faces the same prospects that is why it is convenient to express their values in a recursive way. The values of the agents, denoted $v_{..}$, are expressed as discounted future proceeds from production. The discount factor β is the same for the workers and the firms.

In the case when all 7 possible types are present in the stationary state of the economy the values are:

$$
\begin{aligned}
v_{fy} = 1/2\Big(&2\pi - (1-\beta(1-y)(1-a))\beta(1-y)(1-a)\cdot v_{ys} - \\
&(1-\beta(1-y)a)\beta(1-y)a\cdot v_{os} + (1-\beta(1-b))\beta(1-b)\cdot v_{fs}\Big) + \\
&\beta\Big((1-y)(1-a)(1-b)\cdot v_{fy} + (1-y)a(1-b)\cdot v_{fo} + y(1-b)\cdot v_{fs}\Big) \\
v_{fo} = 1/2\Big(&2\pi - (1-\beta(1-o))\beta(1-o)\cdot v_{os} + (1-\beta(1-b))\beta(1-b)\cdot v_{fs}\Big) + \\
&\beta\Big((1-o)(1-b)\cdot v_{fo} + o(1-b)\cdot v_{fs}\Big) \\
v_{fs} = m\cdot &\Big(Y\cdot v_{fy} + O\cdot v_{fo}\Big) + (1-m)\beta(1-b)\cdot v_{fs} \\
v_{yf} = 1/2\Big(&2\pi - (1-\beta(1-b))\beta(1-b)\cdot v_{fs} + (1-\beta(1-y)(1-a))\cdot \\
&\beta(1-y)(1-a)\cdot v_{ys} + (1-\beta(1-y)a)\beta(1-y)a\cdot v_{os}\Big) + \\
&\beta\Big((1-y)(1-a)(1-b)\cdot v_{yf} + (1-y)(1-a)b\cdot v_{ys} + \\
&(1-y)a(1-b)\cdot v_{of} + (1-y)ab\cdot v_{os}\Big) \\
v_{ys} = m\cdot &v_{yf} + (1-m)\beta\Big((1-y)(1-a)\cdot v_{ys} + (1-y)a\cdot v_{os}\Big) \\
v_{of} = 1/2\Big(&2\pi - (1-\beta(1-b))\beta(1-b)\cdot v_{fs} + (1-\beta(1-o))\beta(1-o)\cdot v_{os}\Big) + \\
&\beta\Big((1-o)(1-b)\cdot v_{of} + (1-o)b\cdot v_{os}\Big)
\end{aligned}
$$

$v_{os} = m \cdot v_{of} + (1-m)\beta(1-o) \cdot v_{os}$

The value of exit is naturally considered to be 0.

Under the assumption that the Law of Large Numbers holds, in a matching situation firms will face a young or an old worker with probabilities that are proportional to the fractions of young and old unemployed workers. The probabilities, and also the fractions of young and old workers in the pool of unemployed workers, are denoted Y and O respectively.

The probabilities Y and O are endogenously determined in the model and they can be expressed as follows

$$Y = d_{ys}/(d_{ys} + d_{os})$$

$$O = d_{os}/(d_{ys} + d_{os}) = 1 - Y$$

where $d_{..}$ are distribution fractions of workers of indicated types.

The system of value functions can be rewritten in a matrix form as

$$\mathbf{V} \cdot \mathbf{v} = \boldsymbol{\pi}$$

where $\mathbf{v'} = (v_{fy}, v_{fo}, v_{fs}, v_{yf}, v_{ys}, v_{of}, v_{os})$, $\boldsymbol{\pi'} = (-\pi, -\pi, 0, -\pi, 0, -\pi, 0)$ and $\mathbf{V}$ is the matrix implied by the system of equations. The matrix equation can be analytically solved and we get the value functions dependent only on parameters of the model.

$$\mathbf{v} = \mathbf{V}^{-1} \cdot \boldsymbol{\pi}$$

The distribution of firms and workers across types $\mathbf{distr} = (d_{fy}, d_{fo}, d_{fs}, d_{yf}, d_{ys}, d_{of}, d_{os})$ evolves in time according to the vector equation

$$\mathbf{distr_{t+1}} = \mathbf{distr_t} \cdot \mathbf{Q}$$

where $\mathbf{Q}$ is a transition matrix that describes movement of agents across the states.

We are looking for a stationary distribution $\mathbf{distr}^*$, i.e. distribution that is stable in time

$$\mathbf{distr}^* = \mathbf{distr}^* \cdot \mathbf{Q}.$$

Since firms can never become workers and the workers can never become firms the evolution of these two populations are independent, which means that we look for two stationary distributions, one for firms and the other one for workers. On the contrary we can not separate young and old workers since their fractions are determined endogenously in the model.

Under the assumptions that the Law of Large Numbers holds the transition matrices for firms and workers are QF and QW. The interpretation is that an element $q_{ij} \in \mathbf{Q}$ is the probability that the next period the agent will be of type j given that today he is of type i. The presented matrices are for the situation where all the proposed matches are accepted.

$$QF = \begin{pmatrix} (1-b)(1-y)(1-a) & (1-b)(1-y)a & (1-b)y+b \\ 0 & (1-b)(1-o) & (1-b)o+b \\ mY(1-b)(1-y)(1-a) & \substack{mY(1-b)(1-y)a+ \\ mO(1-b)(1-o)} & \substack{mY(1-b)y+mO(1-b)o+ \\ (1-m)(1-b)+b} \end{pmatrix}$$

$$QW = \begin{pmatrix} (1-y)(1-a)(1-b) & (1-y)(1-a)b+y & (1-y)a(1-b) & (1-y)ab \\ m(1-y)(1-a)(1-b) & \substack{m(1-y)(1-a)b+ \\ (1-m)(1-y)(1-a)+y} & m(1-y)a(1-b) & \substack{m(1-y)ab+ \\ (1-m)(1-y)a} \\ 0 & o & (1-o)(1-b) & (1-o)b \\ 0 & o & m(1-o)(1-b) & \substack{m(1-o)b+ \\ (1-m)(1-o)} \end{pmatrix}$$

The types are ordered fy, fo, fs in QF matrix and yf, ys, of, os in QW matrix.

Because the probabilities Y and O are endogenous and they enter the computation of the stationary distribution of the firms, the stationary distribution of the workers has to be computed first.

3.2.3 Stable Equilibria

This section in detail describes what is understood, in the context of this model, to be a stable equilibrium, and how different types of equilibria occur.

A set of agents' matching strategies together with corresponding values form a stable equilibrium if they constitute a Nash equilibrium and strategies are evolutionary stable. This means that none of the agents has incentives to change their strategy, given the strategy of the other agents, and at the same time the strategies are resistent to small invasions, i.e. if there exists a fraction ε of agents that have decided to deviate in their strategy the agents playing the equilibrium strategy do not find it profitable to join the group of deviants.

In the context of the model there are up to 7 types of agents in the stationary situation of the economy. But also fewer types can occur in the stable equilibrium depending on the strategies of agents. For all the possible strategies of agents we have to check whether they constitute a Nash equilibrium and then check whether strategies leading to such equilibrium are evolutionary stable. It is easy to see that, for example, some trivial Nash equilibria are a result of strategies not stable from the evolutionary point of view.
The firms have 4 possible strategies[3]:

1. reject every worker they meet

2. accept only the match with young workers

3. accept only the match with old workers

4. accept match with both types of workers

The workers have 2 possible strategies:

1. reject every firm they meet

2. accept every firm they meet

These strategies can lead to several types of matchings. Each of the matchings can be characterized by the types of agents that exist in the matching. When respecting the assumptions of the model, the candidates for equilibria are[4]:

1. the matching consisting of types fs, ys and os

[3] Only pure strategies of agents are considered in this paper.
[4] Further on we will refer to the matchings based on the numbers assigned to them here.

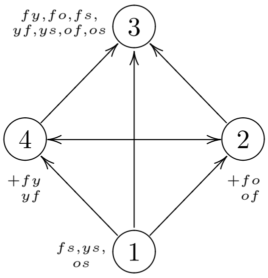
2. the matching consisting of types fs, ys, os, fo, and of

3. the matching consisting of types fs, ys, os, fo, of, fy, and yf

4. the matching consisting of types fs, ys, os, fy, and yf

Despite there are 16 possible combinations of the agents' strategies, there are no other than the 4 mentioned matchings that can occur. If the assumptions of the model are respected, due to the exogenous exits, the non-matched firms and workers must be always present.

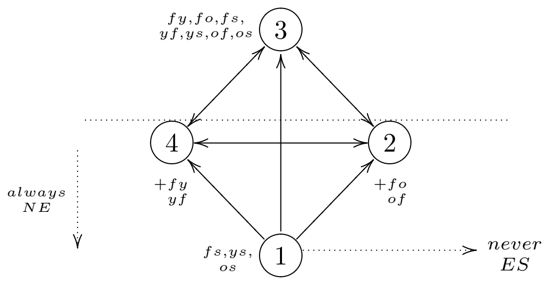
Graphically we can represent the matchings as follows:



The system of equations for the values of the types that has been described in the previous section applies to matching 3. The rest of the matchings are less complex. They can be described by similar systems of equations as is the one that applies for matching 3. The systems of equations are simpler because some types of agents that are not present in these matchings and the values of these types of agents are therefore zero. Similarly the transition matrices have to be adjusted to the fact that some of the types of agents do not exist in matchings 1, 2 and 4.

We consider the stationary situations of the above described matchings, i.e. the situation when the values of the types of agents as well as the distribution are stationary. A stationary matching is considered to be a stable equilibrium if it is a Nash equilibrium with strategies that are evolutionary stable. It is easy to see that, from the 4 candidates for stable equilibrium matchings, matching 1, even though it is Nash, results from the strategies that are not evolutionary stable. It is because all the agents are non-matched and therefore their values are necessarily 0. By deviation to any of the strategies where the agents match they will achieve positive value. Therefore matching 1 is never a stable equilibrium.

Matchings 2 and 4 are always Nash. The reasoning behind this is simple. The firms in matchings 2 and 4 create only one type of match, i.e. they either match with young workers or with old workers. Having a worker brings profit so the value of the matched firm is higher than the value of the unmatched firm, which only has prospects of being matched in the future. Strategic waiting for a better match does not make any sense because the match that would come would be the same as the existing one. Workers, no matter what their strategy is, can end up only employed or unemployed. Since all the types of employment are the same, due to the fact that firms are homogenous, the strategic waiting for a better match can not be profitable and the value of being matched with a firm is higher than the value of being unemployed. From these considerations we can conclude that matchings 2 and 4 are always Nash. This is not always true for matching 3 where the firms' strategic waiting for a better match can play a role. Therefore, for matching 3 we should verify for which sets of parameters the model constitutes Nash equilibrium. After that the evolutionary stability of the matching strategies has to be verified.



When we focus on the evolutionary stability of the strategies we can see that the strategy "reject every firm/worker" is not evolutionary stable neither for firms nor for workers. These strategies imply the value of firms and workers equal to zero. The firms and workers can always do better, i.e. have a positive value, by deviating to any other strategy.

Since the agents have only two strategies one of which is not stable it leaves them with the strategy of accepting every firm they meet. On the other hand the firms have three strategies left. As a matter of fact each of them leads to different type of matching. The strategy "accept only the match with young workers" implies matching 4, the strategy "accept only the match with old workers" implies matching 2, and the strategy "accept match with both types of workers" leads to matching 3. For these firms strategies the evolutionary stability has to be verified because they can have incentives to deviate from one to the remaining two, which would mean a deviation from one type of matching to another one.

3.2.4 Multiple Stable Equilibria and Pareto Dominance

The theoretical matching model described in the previous sections has 5 parameters: m, y, a, o, and b. The goal is to describe how the existence of different stable equilibria depend on the parameters of the model. Naturally, for a given combination of the parameters, several stable equilibria can occur, i.e. parameters are such, that several sets of agents' strategies lead to Nash equilibria that are also evolutionary stable. In this situation the question of Pareto dominance of one equilibrium over another one occurs. It is important to notice that the stationary equilibria are never directly comparable because they never consist of the same types of agents. Therefore it is not enough to compare the value of a certain type of agents in one equilibrium with the value of the same type of agents in the other equilibrium. A Pareto dominating equilibrium, in the context of this model, is such that all the values of firms and workers in this equilibrium are higher that the values in another equilibrium. The condition, though it seems rather strict, is a necessary one because the types of agents as well as the distribution of agents in the two equilibria are different and therefore one type of agent in the first equilibrium can become a different type in the other equilibrium[5].

3.2.5 Social Optimality

The proposed model allows us to study social optimality of the equilibria implied by the agents' optimal decisions. Assume that in the model there exists a social planner who's goal is to maximize the aggregate welfare of the economy. Assume

[5]The agents can change their type of match, not their ex-ante type, i.e. it is enough to compare the values for firms, young worker and old workers separately.

the social planner has no means how to change the meeting technology in the economy but he can influence agents' decisions by imposing rules on what type of match they have to accept and what type they have to reject. Moreover he can force the agents to stay in the created matches until their exit. The stability of matchings is therefore imposed on the agents. This implies that the matchings that would be unstable from the point of view of agents can be chosen as optimal by the social planner.

The task of the social planner is simple. For each combination of parameters of the model and for each of the matchings 1 - 4 described in section 3.2.3 he computes the stationary distributions and the values of the types of agents. Using these he determines aggregate welfare Ω, which is defined as a weighted average of the values of types of agents with the weights that are the corresponding fractions of the distribution.

$$\Omega = d_{fy} \cdot v_{fy} + d_{fo} \cdot v_{fo} + d_{fs} \cdot v_{fs} + d_{yf} \cdot v_{yf} + d_{ys} \cdot v_{ys} + d_{of} \cdot v_{of} + d_{os} \cdot v_{os}$$

The agents' decisions are then, for the given combination of parameters, considered to be socially optimal if the set of equilibria stable under the given combination of parameters contains the stationary matching preferred by the planner.

In cases where the social optimum differs from the optima chosen by agents the natural question one can ask is whether the planner's solution is Pareto improving for the agents.

By the same argument as in the previous section, the matchings are directly incomparable among each other because in every comparison at least one type of agents is missing or is redundant. Moreover, the distribution of agents in each of these matchings is completely different so it is not clear whether the agent of a certain type under one matching will be of the same type under the other matching. The only possibility how to make sure that one matching is Pareto improving when compared with another matching is to make sure that all the values of types of agents in one matching are lower than all the values in the other matching.

A simple conclusion that can be made without any computation is that matching 4, though it theoretically can be socially optimal, can never be Pareto improving because in this matching the value of the old workers, because they are never matched with firms, is zero. On the other hand the values of the old workers in matchings 2 and 3 are positive, which follows from the fact that their probability of being employed is non-zero.

3.3 Results

The stationary distributions as well as the values of types of agents in each of the 4 matchings mentioned in section 3.2.3 can be analytically expressed as the functions of parameters of the model. The values are homogenous of degree 1 in π.

Although analytical expressions for value functions and also fractions of distributions can be obtained the expression are very complex and therefore analytical comparative statics would be tedious. That is why some parts of the following sections rely on numeric computations for particular combinations of parameters.

First, the stationary distributions has to be obtained. By a simple fixed point argument the stationary distributions exist and are unique[6] for all well defined transition matrices, which is the case of the transition matrices described in the theoretical section of the paper. The stationary distribution of workers has to be computed first because the fractions of non-matched young and old workers enter the computation of the stationary distribution of the firms.

Once the stationary distributions are obtained the results enter the computation of the values of types of agents. The analytical solutions can be obtained but due to their complexity the analysis that follows is based on the results of a computation of the values for each particular combination of the parameters of the model. In the computation the profit π is fixed and is equal to 1. Since the values are homogenous of degree 1 in π this choice of the numeric value of π is not restrictive. The discount factor is $\beta = 0.95$, which is a standard value. The computation is performed for different fixed values of m, y, o, a, and b. Since the model assumes that $o > y$ all the computations take that into account. For the parameters a, and b only three values are considered: 0.1, 0.5, and 0.9, i.e.

[6]The theoretical background of stationary distributions discussed in detail can be found in Stokey, Lucas, and Prescott [12].

these probabilities are either low, medium or high. The probability of matching m can take 11 values between zero and one. And the probabilities of exit y and o take up to 51 values, i.e. the aim is to describe the dependence of the results on these parameters as if they were continuous variables.

3.3.1 Results of Agents' Optimal Decisions:

Agents strategies may lead to 4 possible types of equilibria. For each of the 4 cases we have to check whether the strategies lead to a Nash equilibrium and if yes then whether they are evolutionary stable.

As already discussed in section 3.2.3, matchings 1, 2, and 4 are always Nash equilibria and the matching 1 is never evolutionary stable.

The computations done for each particular combination of the parameters of the model show the following. Matching 3 is not a Nash equilibrium in the cases when the probability of being matched is very high, i.e. the parameter m is close to one, and the probabilities of exit of workers y and o are low. The intuition behind this result is simple. Meetings are common, therefore the firms can afford to strategically reject some matches because it is probable that they will not wait too long for a new match and therefore will not lose profits from many periods. Also, since the workers of both types stay in the economy for a relatively long time the profit lost due to strategic waiting will be compensated after creation of a match as the match should, in expected terms, produce for quite a long time.

Each of the matching 2, 3, and 4 is stable equilibrium under a range of parameters. Multiple stable equilibria occur. Matching 2 coexists for a range of parameters with matching 3, and for a different range of parameters with matching 4, we denote these multiple equilibria as $(2, 3)$ and $(2, 4)$. There is a range of parameters under which none of the matchings is stable. This happens for the range of parameters where matching 3 is not Nash and agents have always incentives to deviate from matching 2 to matching 4 and back from matching 4 to matching 2. We denote this unstable situation as situation 0.

The stable equilibria together with the parameters that are characteristic for them are briefly summarized Table 3.1. A detailed summary is provided in the Appendix in Table 3.4.

Stable equilibria	Characteristic parameters
2,3	very high m, high a, low b, high y, o
2,4	∀ a,b,m except very low m, low y, o
2	high m,∀ a,b, very low y, o
3	∀ a,b,m, medium and high y, o
4	very high m, low a, ∀ b, low y, o
0	very high m, low a, ∀ b, very low y, o

Table 3.1: Stable equilibria

In general we can conclude that for the agents' decisions the parameter m plays the most important role. For very low values of m it is stable equilibrium 3 that prevails independently of the values of other parameters of the model. It is a consequence of the fact that meetings are rare and therefore firms prefer to take any worker rather than strategically wait. As the probability of being matched m grows the strategic waiting for a better match may become a profitable strategy because the loss of profits due to the waiting is not substantial. For the exposition purposes we fix several parameters of the model in order to discuss the intuition behind results but the full analysis is provided in the Appendix.

Let us focus on the stationary state of the economy where meetings are common, i.e. m is high, and young workers rarely die and firms almost never go bankrupt, i.e. probabilities y and b are very low. In this situation matchings 2 and 4 are stable for lower range of probability o and matching 3 is stable for high values of probability o. When probability of exit of old workers o is high, the difference between old and young workers is significant. Both types can be profitable for firms. Young bring longevity to production pairs, old bring high one-period profits for firms through the bargaining procedure. Moreover, replacing workers is relatively easy and therefore firms have no reason not to take every worker that they meet, which implies matching 3.

For the low probabilities of exit of old workers o the situation is different. Young and old workers are not very different. Young workers don't bring significant improvement of lifetime of a pair compared to old workers, old workers on the other hand don't bring significantly higher profits to firms through bargaining. Depending on which of these two sources of profit weighs more in the value of firms, the firms will prefer only one type of workers. As it turns out, when the probability o is close to zero, it is the old workers that firms will choose to employ, and therefore matching 2 is a stable equilibrium. When probability o increases, the firms will prefer young workers and they will fire workers once they age; therefore matching 4 is stable equilibrium. There is a significant range of parameters where both matchings 2 and 4 are stable. For very high values of probability of meeting m matchings 2 and 4 become unstable in a sense that agents have incentives to deviate from matching 2 to matching 4 and then back to matching 2. It is caused by the fact that values of firms with and without workers in these two matchings are almost the same, which is a consequence of the fact that firms almost never go bankrupt and meetings are common. For firms it is profitable to deviate in their strategy and start to employ the opposite type of worker than any other firm. The deviation in their strategy brings them higher value than the values of other firms, but this increased value is only a consequence of the fact that they have decided to employ the type of workers no other firm employed, and therefore such a worker was easy to meet. Once all the firms repeat the deviating strategy their value is driven down by higher scarcity of unemployed. This reasoning makes firms unstable between two strategies. Because we have assumed that identical firms use identical strategies and we considered only pure strategies, under these assumptions we obtain a region of parameters for which matching 3 is not Nash and matchings 2 and 4 are Nash but not stable.

Despite the fact that in the explanation above we have focused on a particular range of parameters the full range has been explored and table summarizing all the results can be found in the Appendix. Illustrative plots of the equilibria discussed above can also be found in the Appendix.

In situations of multiple equilibria Pareto dominance of one of the equilibria never occurs.

3.3.2 Results of Planner's Optimal Decisions:

In the planner's problem, when studying what is the socially optimal matching we obtain that three of the 4 possible matchings can be socially optimal. Naturally, matching 1 is excluded from the debate as this matching implies values zero for all types of agents and therefore is never optimal from the point of view neither of agents nor the planner.

Interestingly enough we observe that only rarely the stable equilibrium matching chosen by the agents coincides with the matching chosen by the planner. While in the case of agents' choice stable equilibrium 3 is the most common, in the case of the planner's choice equilibrium 3 is the rarest choice.

Socially optimal equilibria are never Pareto improving[7] for the agents. The intuition behind this, concerning the equilibrium 4, is obvious. Since in equilibrium 4 only young agents are working the value of the old agents, which are non-matched and have no prospects to be matched, is zero. In every other type of equilibria their value is positive as the implication of the fact that with a certain probability they can be matched. That's why the planner's choice of equilibrium 4 must be always Pareto inferior to the agents' choice as at least the old agents are necessarily better off in any other type of matching.

Another reason why the planner's choice is Pareto inferior to the agent's choice is the following. Agent's often choose equilibrium 3 where every type of worker can work. At the same time the planner chooses usually matching 4 or 2, where one type of workers does not work. The case of matching 4 has been already discussed. For the case of matching 2 a similar argument can be used. In this matching only the old workers are working. Therefore the value of the young workers comes from the fact that with some probability they will age and they will be employed. Of course, the young workers in matching 3 do not wait and can be employed directly in the given period, moreover their bargaining power over the splitting of the proceeds from production is higher than the bargaining power of the old workers, which implies higher one-period profit and therefore all this sums up to higher value of a young worker in matching 3 than in matching 2.

[7]They are not a Pareto improvement of any alternative

Table 3.2 summarizes the types of socially non-optimal choices the agents make.

Socially suboptimal equilibria	
Agents' choice	Planner's choice
2,4	**3**
2	**3** **4**
3	**2** **4**
4	**2** **3**
0	**2** **4**

Table 3.2: Socially suboptimal choices of agents

The planner's choice of equilibria depends on several factors. The equilibrium fractions of agents are the first factor and the values of agents are the second factor. The planner does not care about division of profits between members of pairs, because both members of the pair contribute to the economy's wealth. The planner on the other hand cares about the lifetime of pairs. It is also important to realize that the fraction of old agents plays an important role in establishing social optimum in the economy. This fraction is endogenous and changes with types of matching and changes in parameters of the model.

The planner chooses equilibrium 2, i.e. the equilibrium with only old workers employed, as the socially optimal in the cases when the probability of meeting m is high and typically for the lower range of values of the probability of exit of the old workers o. Because the probability of exit of young workers is lower than the probability of exit of old workers, the differences between the workers are not substantial. Because the workers have low probability of exit their value whether matched or not is relatively high. Moreover, the young workers even though they are not working have value implied by the fact that at some point they will age and therefore will be able to find a firm to work for. All these factors together imply that matching 2 with only old workers employed is socially optimal.

The range of the parameters y and o for which equilibrium 2 is socially optimal gets larger when parameter a increases. The intuition is simple. Parameter a is in the model responsible for ageing of the young workers. If this parameter increases it means that the endogenous fraction of the old agents in the economy increases, which means that the firms meet the old agents even more often than before.

The planner chooses equilibrium 4 as the social optimum typically when meetings are rare, i.e. the probability m is low, and the probability of exit y is in a lower range. The firms find it more profitable to employ a young worker with whom they will produce for many periods because with an old worker they would be forced to replace him soon, but the search takes a long time and means big loss of profits.

Matching 4 is also socially optimal in the cases when the probability of exit of old workers o is high, independent of the probability of matching m. Naturally in these cases the matching with young workers brings higher welfare of the economy because they guarantee long expected lifetime of producing pairs.

Equilibrium 3 is rarely socially optimal. The parameters typical for this equilibrium are a, b and y low. The range in which this equilibrium is socially lies between the areas of equilibrium 2 and equilibrium 4.

A brief summary of the socially optimal equilibria is provided in Table 3.3.

Planner's equilibria	Characteristic parameters
2	m high; o, y low
3	m high; a, b, y low
4	m low; y low y high o high

Table 3.3: Socially optimal equilibria

3.4 Final Remarks

In the model of worker-firm matching it is desirable not only to allow for hiring and firing decisions of the firms but also to allow for job-to-job transitions, i.e. workers and firms should continue to search even when they are already matched. In the context of the model presented in this paper, because the firms are homogenous, the workers never have incentives to leave the existing match. Therefore if we want to study the job-to job transitions of the workers we have to assume heterogeneity of the firms. It can be done by extending the proposed model by assuming 2 or more types of firms.

Another interesting extension of the model is to allow firms to hire more than one worker but then we are departing from the classical setting of the two-sided matching models and the analysis may get much more complicated than in the presented model.

The last extension, bringing more realism into the model, would be to assume a more realistic structure of ageing, i.e. to assume a Markov process that would guarantee that young workers age for sure after staying young for a certain number of periods. This extension brings serious computational complications to the model. We would have to take into account that there are several types of young workers depending on their “distance” from the period when they age. On the other hand Markov probabilities of ageing and of exit would bring a lot of realism into the model and it would be interesting to see hiring and firing decisions of firms especially those concerned workers of relatively old age but not close to exit.

3.5 Conclusions

The paper studies the stationary situation of the matching market in an economy with firms and workers that are ageing. In this economy the agents’ optimal profit maximizing decisions can lead to several types of stable equilibrium matching. All theoretically possible stable matchings turn out to be the stable equilibrium of the model for a certain range of parameters. Multiple stable equilibria occur but within the coexisting equilibria there is no Pareto dominance of one over the other. There exists a small range of parameters for which none of the matching is stable.

The optimal behavior of the agents can be summarized as follows. For the lower probabilities of meeting a potential partner agents tend to accept every proposed match. For the higher probabilities of meeting the profit extractions from the bargaining procedure play a significant role and therefore some of the proposed matches are rejected. There exits a range of parameters where firms employ only young workers and they fire them once they age, but also a range of parameters where firms employ only old workers.

The proposed model suggests that the individual profit-maximizing behavior often leads to the matchings that are suboptimal from the point of view of the social planner maximizing the general welfare of the economy. This is a direct implication of the fact that the social planner does not care about the division of the profits within pairs. Socially optimal matchings are never Pareto improving for the agents.

The results of this paper suggest that the presence of the social planner in the organization of matching markets may be beneficial for the overall welfare of the economy if the planner is able to impose matchings that are not stable equilibria from the point of view of individual agents.

3.A Appendix

3.A.1 Summary of the Equilibria

The following table summarizes the multiple as well as unique stable equilibria implied by agents' decision and provides approximate intervals for the parameters under which the equilibria occur. The intervals of parameters are only the estimates done based on the plots done for 99 combinations of values of the parameters a, b, and m and plotted for the approximation of a continuous range of the parameters o and y, where $o \geq y$. When an area of parameters is described by an inequality it is always understood that the parameters must satisfy also the conditions that they are from the interval $[0, 1]$. In the table below the following notation is used. The arrow $\searrow$ means that the parameter is going down, while the arrow $\nearrow$ means that the parameter is going up. When talking about parameters y and o, the threshold values are considered.

Coexisting	Values of the parameters			
equilibria	m	a	b	o, y (y≤o)
2,3	0.999	0.5	0.1	∀ y,o: o≥0.8, y≥0.6, y≤0.75o
		0.9	0.1	∀ y,o: y,o≥0.5, y≤0.7o+0.15
2,4	0.3	0.1, 0.5	∀	∀ y,o: y,o≤0.1
	0.4	0.1, 0.5	∀	∀ y,o: y,o≤0.15
	0.5	∀	∀	∀ y,o: o≤0.25, y≤-o+0.25
	0.6-0.9	∀	∀	∀ y,o: y≤-o+0.1+0.5(m-0.5) o∈[0.05,0.3], y≥-o+0.05 y,o↗ with a↗, y,o↘ with b↗
	0.999	0.1	0.1	∀ y,o: o≥0.65, y≤0.5
			0.5, 0.9	∀ y,o: o≥0.65, y≤3o-2, y≤0.8
		0.5, 0.9	0.1	∀ y,o: y≥-o+0.2,y≤0.6
			0.5, 0.9	∀ y,o: y≥-o+0.2, y≤0.9

Coexisting equilibria	Values of the parameters			
	m	a	b	o, y (y≤o)
2	0.6	0.1	0.1	∀ y,o: y,o≤0.05
	0.7, 0.8	0.1	∀	∀ y,o: o≤0.15+(m-0.7)0.5, o↘ with b↗ y≤-o+0.15+(m-0.7)0.5
	0.9	0.1, 0.5	∀	∀ y,o: o≤0.1, y≤-o+0.1
	0.999	0.1, 0.9	∀	∀ y,o: y, o ≤0.05
		0.5	∀	∀ y,o: o≤0.25, y≤-o+0.25
3	0.001-0.2	∀	∀	∀o , ∀y≤o
	0.3	0.1, 0.5	∀	o≥0.1, ∀y
		0.9	∀	∀o , ∀y≤o
	0.4	0.1, 0.5	∀	o≥0.15, ∀y
		0.9	∀	∀o , ∀y≤o
	0.5-0.9	∀	∀	∀ y,o: y≥-o+0.1+0.5(m-0.5) y,o↗ with a↗, y,o↘ with b↗
	0.999	∀	∀	∀ y,o: y, o ≥0.5
4	0.9	0.1	0.1	∀ y,o: o≤0.4, y≥-o+0.3, y≤-o+0.5
			0.5	∀ y,o: y≥-o+0.2, y≤-o+0.7, y≥4o-1.8
			0.9	∀ y,o: y≥-o+0.2, y≤-0.5o+1, y≥o-0.35
	0.999	0.1	0.1	∀ y,o: y≥-o+0.3, y≤0.5, o≤0.65
			0.5, 0.9	∀ y,o: y≥-o+0.3, y≥3o-2, y≤0.8
0	0.9	0.1	0.1	∀ y,o: y≤-o+0.3, y≥-o+0.1
			0.5	∀ y,o: y≤-o+0.2, y≥-o+0.1
	0.999	0.1	∀	∀ y,o: y≤-o+0.3, o,y≥ 0.05

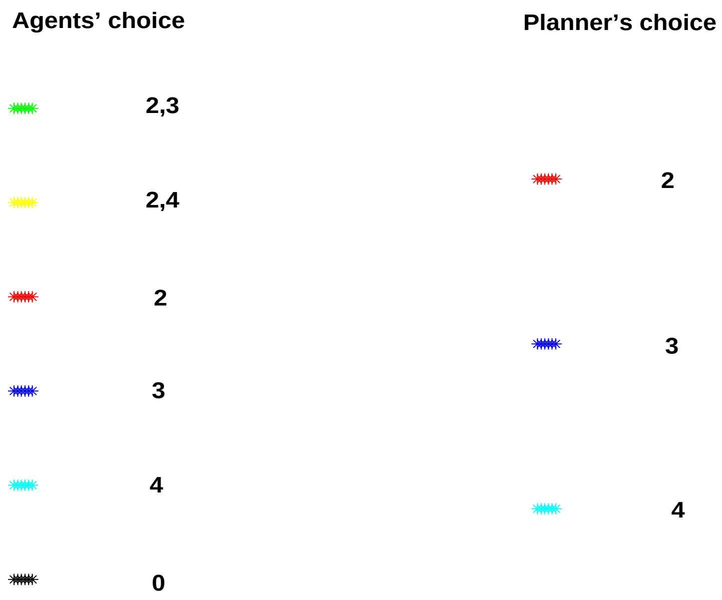
Table 3.4: Agents' optimal choice

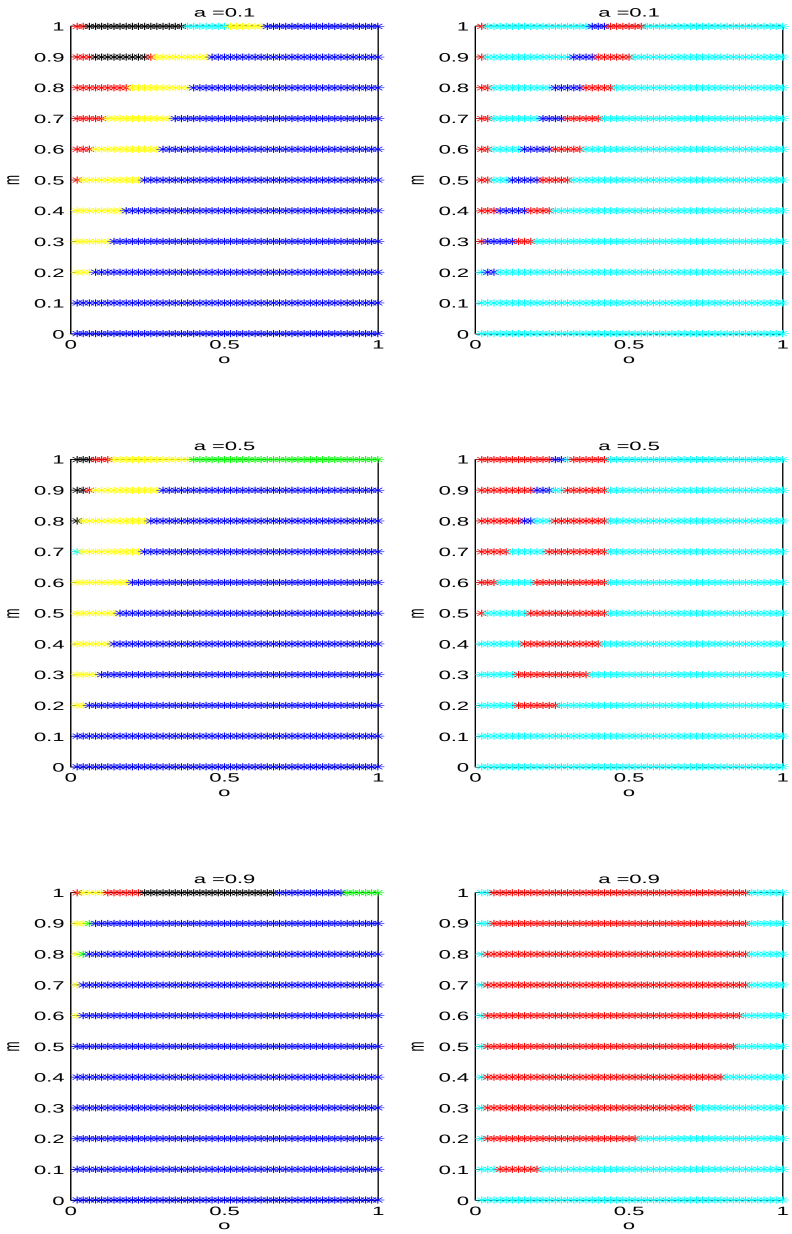
3.A.2 Plots of Equilibria

This section contains illustrative plots of stable equilibria implied by individual decisions as well as the social planner's equilibria. On each of the plots on the left hand side we have the plot of agents' equilibria and on the right hand side the plot of the planner's equilibria. The plots are done for fixed values of the parameters $b = 0.001$, and $y = 0.001$, i.e. firms almost never go bankrupt, young agents almost never die, and for the whole ranges of the parameters o and m. The values of probability of ageing a are stated on the plots. For each possible combination of parameters the type of equilibrium is computed and then it is plotted in the color reserved for that particular type of equilibria. The table preceding the plots should help orientation among different types of equilibria in both agents' and the planner's problem.

On the horizontal axis we have the parameter o, the vertical axis represents the parameter m.

Different types of equilibria are distinguished by different colors as follows:




a =0.1
a =0.1
a =0.5
a =0.5
a =0.9
a =0.9
m
o

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