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Simple Markovian Equilibria in Dynamic Spatial Legislative Bargaining *

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Abstract

The paper proves, by construction, the existence of Markovian equilibria in a model of dynamic spatial legislative bargaining. Players bargain over policies in an infinite horizon. In each period, a majority vote takes place between the proposal of a randomly selected player and the status-quo, the policy last enacted. This determines the policy outcome that carries over as the status-quo in the following period; the status-quo is endogenous. Proposer recognition probabilities are constant and discount factors are homogeneous. The construction relies on *simple* strategies determined by *strategic bliss points* computed by the *algorithm* we provide. A strategic bliss point is the policy maximizing the dynamic utility of a player with ample bargaining power. Relative to a bliss point, the static utility ideal, a strategic bliss point is a moderate policy. Moderation is strategic and germane to the dynamic environment; players moderate in order to constrain the future proposals of opponents. Moderation is a strategic substitute; when a player's opponents do moderate, she does not, and when they do not moderate, she does. We prove that the simple strategies induced by the strategic bliss points computed by the algorithm deliver a Stationary Markov Perfect equilibrium. Thus we prove its existence in a large class of symmetric games with more than three players and (possibly with slight adjustment) in any three-player game. Because the algorithm constructs *all* equilibria in simple strategies, we provide their general characterization, and we show their generic uniqueness. Finally, we analyse how the degree of moderation changes with changes in the model parameters, and we discuss the dynamics of the equilibrium policies.

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Abstrakt

Tento text dokazuje, konstrukcí, existenci Markov rovnováhy v dynamickém prostorovém modelu legislativního vyjednávání. Hráči v modelu vyjednávají o politikách v nekonečném horizontu. Finální rozhodnutí v každé periodě je výsledkem většinového hlasování mezi návrhem náhodně vybraného hráče a status-quo. Finální rozhodnutí se stává status-quo pro následující kolo vyjednávání, status-quo je tedy endogenní. Konstrukce rovnováhy se opírá o *jednoduché* strategie. Jediný parametr, *strategic bliss point*, plně determinuje každou jednoduchou strategii a nezbytným elementem konstrukce rovnováhy je *algoritmus*, který produkuje profil těchto parametrů. Na rozdíl od politiky maximalizující statický užitek hráčů, strategic bliss point je umírněná politika. Umírněnost hráčů je výsledkem jejich strategické interakce v dynamickém prostředí. Hráči navrhují umírněné politiky, aby omezili své protihráče. Umírněnost je strategickým substitutem, pakliže protihráči daného hráče jsou umírnění, on sám není, a pakliže protihráči daného hráče nejsou umírnění, on sám je. Ukazujeme, že jednoduché strategie a strategic bliss points spolu s algoritmem vedou ke konstrukci, která představuje Stationární Markov Perfect rovnováhu. Jako důsledek, ukazujeme, že tato rovnováha existuje ve velké skupině modelů dynamického prostorového legislativního vyjednávání s více jak třemi hráči a v jakémkoliv modelu s právě třemi hráči. Protože prezentovaný algoritmus je schopen zkonstruovat všechny profily strategic bliss points které podporují rovnováhu v jednoduchých strategiích, poskytujeme její obecnou charakterizaci a ukazujeme, že je obecně unikátní. Dále analyzujeme jak se míra rovnovážné umírněnosti mění s parametry modelu a popisujeme dynamiku rovnovážných politik.

JEL Classification: C73, C78, D74, D78

Keywords: dynamic decision-making; endogenous status-quo; spatial bargaining; legislative bargaining

1 Introduction

Many real world policies and spending programs persist and evolve in time, are determined repeatedly, and their changes are enacted under the shadow of the extant legislation that is revised and becomes the new status-quo. Dynamic legislative bargaining models reflect these features. The models build on static non-cooperative models of legislative bargaining in the spirit of Baron and Ferejohn (1989). In these models negotiations follow a sequential protocol of proposal-making and voting, either in *distributive* bargaining over the allocation of benefits, or in *spatial* bargaining over choices of policies. The static models assume bargaining terminates upon an agreement being reached. The dynamic models instead embed the static decision-making protocol as a stage game in an infinite horizon repeated interaction. In each stage game the status-quo is the policy last enacted, making the current decision future status-quo and inducing a dynamic, not just repeated, strategic situation.

Starting with Baron (1996), the dynamic legislative bargaining literature has been steadily growing (see next section for an overview). Kalandrakis (2004b) was the first to characterize the Markov equilibrium for the dynamic version of the distributive model. In the absence of applicable existence theorems for Markovian equilibria, his characterization constitutes an existence proof as well. In the continuing absence of the existence theorems, and due to the lack of similar characterization for the spatial model, the existence and properties of Markov equilibria in the dynamic spatial model remain unknown.[1]

In this paper we prove, using constructive arguments, the existence of Markov equilibria in a dynamic spatial legislative bargaining model. A group of legislators repeatedly sets policy in a one- or multi-dimensional policy space. The preferences of the legislators are quadratic or Euclidean, characterized by *bliss points*, the most preferred policies. In each period of infinite horizon a randomly selected legislator puts forward a proposal. Majoritarian voting between the proposal and the status-quo determines the winning alternative that yields utility to the legislators and becomes the status-quo for the subsequent period. The status-quo evolves endogenously and depends on the identity of the proposer and the votes of the entire legislature in every period.

We start the equilibrium construction by defining *simple* stationary Markovian proposal strategies. A Markovian proposal strategy maps the state, the status-quo, into a policy proposal. A simple stationary Markovian proposal strategy depends on a single parameter, the policy a player proposes when the status-quo gives her ample bargaining

[1] Baron (1996) and Duggan and Kalandrakis (2012), the two papers closest to providing existence and characterization in the spatial model, are discussed in the next section.

power. In the static setting this parameter would be the player's bliss point, the policy maximizing the static utility of a player. In the dynamic setting this parameter is the *strategic bliss point*, the policy maximizing the dynamic utility of a player. The crux of the construction is an algorithm generating the strategic bliss points.

Two conditions guarantee that the construction, the simple proposal strategies in combination with the algorithm, delivers Markov equilibrium. The first one, sufficient, is stronger than necessary but easy to check. The second, necessary and sufficient, is more involved to verify, but still focuses only on a finite set of points in an otherwise infinite policy space.

Using these tools, we prove, by construction, the existence of Stationary Markov Perfect equilibrium (SMPE) for any *strongly symmetric* dynamic spatial legislative bargaining game with one-dimensional policy space. Existence is assured under a mild condition on the degree of patience of the players, a condition which ceases to bind as the number of the players increases. For games that are *symmetric*, a weaker notion, we prove the same result under a stronger condition on the parameters of the game.[2] Although not generally, the construction can also work for games that are not symmetric. One must, however, specify a meaningful class of asymmetric games for which it does.

One such class are three-player games with one-dimensional policy space. For these, we show that the construction either delivers an SMPE or we can construct it via an easy adjustment to the simple strategies. Therefore, we prove the existence of SMPE for any three-player dynamic spatial legislative bargaining game with one-dimensional policy space. Because the (adjusted) simple strategies are pure, the SMPE is in pure strategies.

For one-dimensional bargaining games with a general number of players, we further demonstrate the multiplicity of SMPE in the simple strategies. This multiplicity is especially severe in symmetric games with many players; adding two players to a symmetric game increases the number of equilibria twofold. With three players, the multiplicity is at its minimum. We prove that for any three-player one-dimensional game, if an SMPE in simple strategies exists, and we provide conditions when it does, it is essentially unique; at most two SMPE in simple strategies exist and if so, they exists under non-generic conditions.[3]

[2] A game is strongly symmetric if the players' bliss points are equidistant from each other and the players have equal recognition probabilities. It is symmetric if pairs of players around the median have bliss points equidistant from the median's bliss point and have equal recognition probabilities. 'Any' game discussed below means for any bliss points, recognition probabilities and discounting. See section 3 for formal definitions.

[3] We stress that any uniqueness statement refers to SMPE in simple strategies and does not imply the uniqueness of an SMPE in general.

In fact, any multiplicity of SMPE in simple strategies is non-generic. We show that *all* profiles of strategic bliss points that support an SMPE in simple strategies are constructed by our algorithm. By analysing the profiles of strategic bliss points produced by the algorithm, we provide general characterization of all SMPE in simple strategies for any one-dimensional dynamic spatial legislative bargaining game. And the analysis shows that the algorithm produces multiple profiles of strategic bliss points under non-generic conditions.

For games with multi-dimensional policy spaces we proceed in a similar manner. We define simple strategies characterized by strategic bliss points, we specify the algorithm producing these strategic bliss points, and we derive conditions guaranteeing that the construction constitutes an SMPE. And we present two classes of games, one in $\mathbb{R}^2$ and one in $\mathbb{R}^n$, that satisfy the conditions.

Moderation and its *strategic substitute* nature are at the core of our equilibrium construction. This is the main insight of the paper. A player moderates when she proposes her strategic bliss point, which is a more moderate policy - closer to the median - than her (static) bliss point. A player moderates in order to constrain opponents; by moving the status-quo closer to median's bliss point, future proposals are constrained to be moderate as well.[4] When the opponents *do* moderate, they are effectively constraining themselves, so that the player has no incentive to moderate. If the opponents *do not* moderate, the player herself has an incentive to do so; that is, moderation is strategic substitute. As a result, all the equilibria that we construct induce asymmetric moderation (in terms of who moderates and to what extent), even if the underlying game is strongly symmetric.

Moderation and its intensity are the result of two opposing forces. The first force is standard; proposals are pushed towards the proposers' stage utility optimum, their bliss points. The second force is strategic; proposals are pushed towards the bliss point of the median player, with proposers aiming to constrain the future policies of all other players. These two forces cancel out at the strategic bliss point. The strategic force gains prominence and the extent of moderation increases with the patience of the players and with the higher probability of recognition of direct opponents - those with bliss points on the other side of the median.

We proceed as follows. The next section surveys the literature on dynamic legislative bargaining. Section 3 introduces our model, notation and solution concept. Sections 4, 5 and 6 are devoted to the analysis of a model with one-dimensional policies. Section 4 describes the simple strategy construction and establishes the conditions which guarantee

[4] The identity of the median and the fact that she is decisive under majority voting rule are results that do not follow immediately.

that it delivers equilibrium. Section 5 examines these conditions for symmetric games. Section 6 investigates three-player games. Section 7 is devoted to the multi-dimensional model. Section 8 concludes. Most of the proofs are in appendix A1. A series of examples introduced throughout the paper are designed to illustrate prominent features of our analysis and of the equilibria we construct.

2 Review of the Literature

The typical dynamic legislative bargaining model with *endogenous status-quo* posits a group of players bargaining in an infinite discrete time horizon with discounting. Each period starts with a status-quo, the policy last enacted. A randomly chosen player makes a proposal after which a vote over a binary agenda, consisting of the status-quo and the proposal, follows. The winning alternative determines players' utility for the period and becomes the status-quo for the next one.[5]

The original formulation of legislative bargaining as a model with endogenous status-quo are Baron (1996) and Epple and Riordan (1987). Baron (1996) analyses a spatial bargaining model in which the players bargain over a one-dimensional policy space. Epple and Riordan (1987) analyse a distributive bargaining model in which the players bargain to distribute a fixed-sized budget among themselves. In the spatial formulation the utility of players varies in all the dimensions of the policy space. In the distributive setting the players only care about their share of the budget.

The model of Baron (1996) is the most closely related to ours. His model is almost identical to our one-dimensional model; he restricts policies to $\mathbb{R}_+$, which we allow for but do not require, and his stage utilities are general, not quadratic. He develops partial equilibrium characterization and provides intuition for the strategic forces at play.[6]

In addition to Baron (1996), several other papers analyse spatial models under special constraints. These include restrictions on the policy space (Dziuda and Loeper, 2012; Fong, 2005), restrictions on number of players (Forand, 2014; Nunnari and Zapal, 2013) or use of numerical computations (Baron and Herron, 2003; Duggan, Kalandrakis, and

[5] The dynamic legislative bargaining models share many features with the static legislative bargaining models we do not survey here. See, for example, Banks and Duggan (2000, 2006a); Cardona and Ponsati (2007, 2011); Cho and Duggan (2003, 2009); Eraslan (2002); Eraslan and McLennan (2013); Eraslan and Merlo (2002); Herings and Predtetchinski (2010); Kalandrakis (2004a, 2006a,b) for theoretical treatment of the static models.

[6] See discussion following Proposition 1 for why the quadratic utilities cannot be dispensed with. Baron (1996) also includes informal discussion of an example of full equilibrium characterization (his Table 1). The discussion following Proposition 2 explains why the profile of strategies in the example cannot constitute an equilibrium.

Manjunath, 2008).[7]

Following Epple and Riordan (1987), the analysis of distributive models has focused on equilibrium characterization and properties (Kalandrakis, 2004b, 2010; Anesi and Seidmann, 2012; Baron and Bowen, 2013) including investigation of models with risk aversion or alternative decision making protocols (Battaglini and Palfrey, 2012; Baron and Bowen, 2013; Bowen and Zahran, 2012; Diermeier, Egorov, and Sonin, 2013; Nunnari, 2012; Richter, 2014). Models combining distributive and spatial aspects with (Baron, Diermeier, and Fong, 2012; Cho, 2004) or without (Bowen, Chen, and Eraslan, 2014) electoral competition usually investigate joint public (spatial) and private (distributive) good determination.[8,9]

General characterization and existence results for Stationary Markov Perfect equilibria, the standard solution concept in the papers surveyed, are scarce. Kalandrakis was the first to provide a characterization of SMPE for the distributive model with three (Kalandrakis, 2004b) or more than five (Kalandrakis, 2010) players. Diermeier and Fong (2011) provide an algorithm leading to SMPE in a model with a persistent agenda setter and a discrete policy space. Duggan and Kalandrakis (2012) provide a very general SMPE existence result assuming noise in the preferences and the status-quo between-period transitions. The noise complicates the equilibrium characterization and is absent in our model.[10,11]

We want to highlight the fact that the endogenous status-quo literature just discussed is related to but distinct from the models with a single decision to be taken and bargaining proceeding through a series of rounds with *evolving default* (Anesi and Seidmann, 2014; Bernheim, Rangel, and Rayo, 2006; Diermeier and Fong, 2009; Vartiainen, 2014). Also related but distinct is the literature with dynamic political economy models (for example

[7] Papers that embed dynamic spatial models in richer economic or political settings include Chen and Eraslan (2013) (agenda formation), Diermeier, Prato, and Vlaicu (2013) (choice of decision-making rules), Levy and Razin (2013) (interest group influence), Piguillem and Riboni (2013a) (capital taxation), Piguillem and Riboni (2013b) (present-biased legislators) or Riboni (2010); Riboni and Ruge-Murcia (2008) (monetary policy).

[8] Electoral competition in combination with legislative bargaining. However, as Forand (2014) and Nunnari and Zapal (2013) illustrate, the difference between electoral competition and legislative bargaining can be merely a difference in labelling.

[9] Two papers, analysing judicial precedents (Anderlini, Felli, and Riboni, 2014) and legislative sunset provisions (Zapal, 2012, chapter 1), are models with endogenous status-quo, where in every period players bargain jointly over policy and, not necessarily equal, status-quo for the next period.

[10] Hortala-Vallve (2011), Penn (2009) and Roberts (2007) characterize equilibria in models with random, not endogenous and strategically chosen, proposals.

[11] Faced with the complex equilibria of the dynamic legislative bargaining models, many authors use, at least partially, numerical computations (Baron and Herron, 2003; Battaglini and Palfrey, 2012; Bowen et al., 2014; Duggan et al., 2008; Piguillem and Riboni, 2013a; Riboni and Ruge-Murcia, 2008, among others) or provide numerical computation techniques tailored to these models (Duggan and Kalandrakis, 2011).

Azzimonti, 2011; Bai and Lagunoff, 2011; Battaglini and Coate, 2007, 2008; Battaglini, Nunnari, and Palfrey, 2012) where the dynamic link stems not from persistent policies but from the accumulation of a durable public good, (public) debt or capital.

3 Model, Notation, Solution Concept

A game $\mathcal{G} = \langle n, \mathbf{x}, \mathbf{r}, \delta, X\rangle$ is fully specified by n, $\mathbf{x}$, $\mathbf{r}$, δ, and X all satisfying the assumptions we introduce next, and which are maintained throughout. $N = \{1, \ldots, n\}$ is the set of players with odd $n \geq 3$. The stage utility of $i \in N$ from policy p is $u_i(p) = -(p - x_i)^2$ where x_i is the bliss point of i. $\mathbf{x} = \{x_1, \ldots, x_n\}$ denotes the profile of bliss points of all the players and we assume all the bliss points are distinct and ordered such that $x_i < x_{i+1}$ for $\forall i \in N \setminus \{n\}$. The median player is denoted by $m = \lceil n/2 \rceil$. The median bliss point is denoted by $x_m = x_{\lceil n/2 \rceil}$.

In each discrete period of infinite horizon, $i \in N$ is recognized to propose policy $p \in X$ where $X \subseteq \mathbb{R}$ is a closed convex interval. If $X \subsetneq \mathbb{R}$ then we require X to be symmetric around x_m and include both $\min\{\mathbf{x}\}$ and $\max\{\mathbf{x}\}$. $\mathbf{r} = \{r_1, \ldots, r_n\}$ with $r_i > 0$ for $\forall i \in N$ is the profile of probabilities of recognition and naturally $\sum_{i=1}^{n} r_i = 1$. Given the status-quo $x \in X$ and policy proposal $p \in X$ by recognized $i \in N$, a majoritarian vote between x and p follows. The winning alternative determines the utility of the players and becomes the new status-quo. The utility of player $i \in N$ from an infinite path of policies $\mathbf{p} = \{p_0, p_1, \ldots\}$ is

$$U_i(\mathbf{p}) = \sum_{t=0}^{\infty} \delta^t u_i(p_t) \tag{1}$$

where $\delta \in [0, 1)$ is the common discount factor.

Define $d(x) = |x - x_m|$ to be the distance of $x \in \mathbb{R}$ from median x_m. $d_a(x) = x_m + d(x)$ is x mapped into the point above the median's bliss point and $d_b(x) = x_m - d(x)$ is x mapped into the point below the median's bliss point. Note that $x \in \{d_b(x), d_a(x)\}$. A similar operation is defined on the space of players' indexes. $d^I(i) = |i - m|$ denotes index 'distance' of $i \in N$ from m. $d_a^I(i) = m + d^I(i)$ and $d_b^I(i) = m - d^I(i)$ is the pair of players index distance $d^I(i)$ from median.

$N_a = \{i \in N | x_i > x_m\}$ is the set of players with bliss points above the median and $N_b = \{i \in N | x_i < x_m\}$ is the set of players with bliss points below the median. Sums of recognition probabilities for the two groups of players are denoted by $r_a = \sum_{i \in N_a} r_i$ and $r_b = \sum_{i \in N_b} r_i$. For $j \in \{1, \ldots, \frac{n-1}{2}\}$, $r_j^e = \sum_{i=1}^{j} r_i$ will denote the sum of recognition probabilities of j most extreme players in N_b. By convention $r_j^e = 0$ when $j = 0$. We will be using this notation in the context of symmetric games and do not need to establish a similar notation for players in N_a. Finally, $f(a^-) = \lim_{x \to a^-} f(x)$ denotes the one-sided limit of a real-valued function from below and $f(a^+) = \lim_{x \to a^+} f(x)$ denotes the one-sided limit of a real-valued function from above.[12]

Definition 1 (Symmetric $\mathcal{G}$). *$\mathcal{G}$ is symmetric if and only if for $\forall i \in N$, $d(x_{d_b^I(i)}) = d(x_{d_a^I(i)})$ and $r_{d_b^I(i)} = r_{d_a^I(i)}$.*

Definition 2 (Strongly symmetric $\mathcal{G}$). *$\mathcal{G}$ is strongly symmetric if and only if $r_i = r_j$ for $\forall i \in N$ and $\forall j \in N$ and $x_i - x_{i-1} = x_{i+1} - x_i$ for $\forall i \in \{2, \ldots, n-1\}$.*

A pure stationary Markov strategy of each $i \in N$ specifying a policy proposal given status-quo x is $\hat{p}_i : X \to X$. We denote by $\hat{\sigma} = (\hat{p}_1, \ldots, \hat{p}_n)$ a profile of pure strategies, reserving notation p_i and $\sigma = (p_1, \ldots, p_n)$ exclusively for the simple strategies defined below (Definition 4).

Any profile of pure stationary Markov strategies $\hat{\sigma} = (\hat{p}_1, \ldots, \hat{p}_n)$ induces a continuation value function of player $i \in N$, $V_i : X \to \mathbb{R}$. $V_i(x|\hat{\sigma})$ denotes the expected utility of i from an infinite future of play according to $\hat{\sigma}$, starting with status-quo x, before the identity of the proposer in the next period has been determined. It can be computed as

$$V_i(x|\hat{\sigma}) = \sum_{j=1}^{n} r_j \left[u_i(\hat{p}_j(x)) + \delta V_i(\hat{p}_j(x)|\hat{\sigma}) \right] \tag{2}$$

and dynamic (expected) utility of i from accepted x, $U_i : X \to \mathbb{R}$, is

$$U_i(x|\hat{\sigma}) = u_i(x) + \delta V_i(x|\hat{\sigma}). \tag{3}$$

We need several assumptions to calculate V_i as in (2). The first one concerns the proposal strategies. We assume that proposals with zero probability of acceptance are never made.[13] The second one concerns the voting strategies. We assume that all players use the stage undominated voting strategies of Baron and Kalai (1993) when voting between the proposed policy $p \in X$ and the status-quo $x \in X$ and vote for p when indifferent between

[12] To avoid any misunderstanding, function f is called increasing (at x) if $f'(x) > 0$ and non-decreasing if $f'(x) \geq 0$. Similarly, $x \in \mathbb{R}$ is positive if $x > 0$ and non-negative if $x \geq 0$.

[13] Given status-quo x, the proposing player whose utility maximizing proposal is x can obtain this utility either by proposing x or by making a proposal she knows would be rejected. We assume she does the former. This assumption does not change the set of equilibria that are observationally (outcome) equivalent and is standard in the dynamic bargaining literature.

p and x.[14] This implies i votes for p rather than x if and only if

$$U_i(p|\hat{\sigma}) \geq U_i(x|\hat{\sigma}). \tag{4}$$

These assumptions imply that any proposed policy is also accepted, making the distinction between proposed and accepted policies superfluous and (2) the valid expression for V_i. Note also that the voting strategies are fully determined by the proposal strategies (along with the assumptions we have made). We abuse notation and terminology somewhat by subsuming the voting strategies into the proposal strategies $\hat{\sigma}$ or σ without changing their notation or name.

The social acceptance set for a given $x \in X$, $\mathcal{A}(x|\hat{\sigma})$, is the set of policies such that

$$\mathcal{A}(x|\hat{\sigma}) = \left\{p \in X | \tfrac{n+1}{2} \leq |\{i \in N | U_i(p|\hat{\sigma}) \geq U_i(x|\hat{\sigma})\}|\right\} \tag{5}$$

and recognized $i \in N$ proposes a policy from $\arg\max_{p \in \mathcal{A}(x|\hat{\sigma})} u_i(p) + \delta V_i(p|\hat{\sigma})$.

Definition 3 (Stationary Markov Perfect Equilibrium)**.** *A stationary Markov perfect equilibrium (SMPE) is a profile of stationary Markov strategies $\hat{\sigma}^* = (\hat{p}_1^*, \ldots, \hat{p}_n^*)$ such that, for $\forall x \in X$ and $\forall i \in N$,*

$$\hat{p}_i^*(x) \in \underset{p \in \mathcal{A}(x|\hat{\sigma}^*)}{\arg\max}\ u_i(p) + \delta V_i\left(p|\hat{\sigma}^*\right)$$

and $i \in N$ votes for proposed $p \in X$ against $x \in X$ if and only if

$$U_i(p|\hat{\sigma}^*) \geq U_i(x|\hat{\sigma}^*).$$

4 Equilibrium Construction with $X \subseteq \mathbb{R}$

The first result we prove greatly simplifies the derivation of decisive coalitions needed to approve any given proposal p. It implies that the acceptance sets $\mathcal{A}$ are determined solely by the shape of the expected utility of the median.

Proposition 1 (Dynamic median voter theorem for $X \subseteq \mathbb{R}$)**.**
For any profile of pure stationary Markov strategies $\hat{\sigma}$, with implied voting such that, for

[14] Stage undominated voting is a standard assumption in voting literature and rules out implausible equilibria that can support arbitrary outcomes that are accepted because no voter is pivotal. Assuming that an indifferent voter casts her vote for the proposed policy avoids any open set complications.

$\forall i \in N$, $i \in N$ votes for proposed $p \in X$ against the status-quo $x \in X$ if and only if $U_i(p|\hat{\sigma}) \geq U_i(x|\hat{\sigma})$, p is accepted if and only if $U_m(p|\hat{\sigma}) \geq U_m(x|\hat{\sigma})$.

Proof. See appendix A1

We stress that Proposition 1, which does not require SMPE, crucially depends on the utility functions being quadratic. The definition of median as the player with x_m comes from the fact that m is decisive in the vote between two deterministic alternatives $x \in X$ and $p \in X$. However, voting between status-quo x and proposed p means voting between two *lotteries*, as each of the alternatives induces a distribution over future policies. That decisiveness of the median in the choice over pure alternatives extends to the choice over lotteries, under quadratic preferences, is a well known result (Banks and Duggan, 2006b). Equally well known is the fact that this result does not extend beyond quadratic utilities (see example following proof of Lemma 2.1 in Banks and Duggan, 2006b).[15]

4.1 Simple Strategies, Strategic Bliss Points

Definition 4 (Simple proposal strategies)**.** *The simple pure stationary Markov proposal strategy of $i \in N$ is*

$$p_i(x|\hat{x}_i) = \begin{cases} \min\{d_a(x), \hat{x}_i\} & \textit{if} \;\; i \in N_a \\ \hat{x}_m & \textit{if} \;\; i = m \\ \max\{d_b(x), \hat{x}_i\} & \textit{if} \;\; i \in N_b \end{cases}$$

where $\hat{x}_i$ is the strategic bliss point of i.

Given a profile of strategic bliss points $\hat{\mathbf{x}} = \{\hat{x}_1, \ldots, \hat{x}_n\}$ a profile of simple proposal (and implied voting) strategies is $\sigma = (p_1, \ldots, p_n)$. With p_i fully determined by $\hat{x}_i$, we abuse terminology somewhat and also call $\hat{x}_i$ the proposal strategy of i and $\hat{\mathbf{x}}$ the profile of strategies.

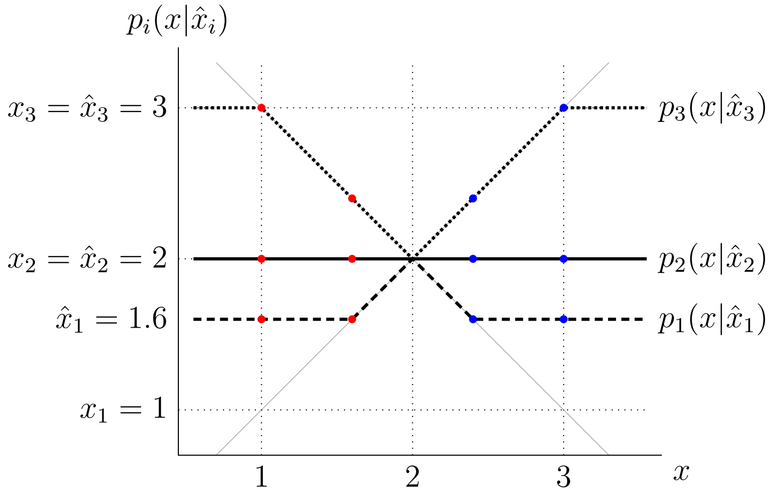
The following example illustrates the shape of the simple strategies in a strongly symmetric $\mathcal{G}$ with three players for a profile of strategic bliss points that constitutes an SMPE, as we prove below.[16]

[15] Alternative voting rules, with a veto player, or decision making protocols, with a representative voter, would not necessitate quadratic stage utilities for the social acceptance set to be driven by preferences of a unique player. Our approach to equilibrium construction would be applicable to these alternative models as well, even with general stage utilities.

[16] Unless specified otherwise, the policy space in all the examples is $X = \mathbb{R}$.

Example 1. *Consider $\mathcal{G}$ with $n = 3$, $x_i = i$ and $r_i = \frac{1}{n}$ for $\forall i \in N$ and $\delta = 0.9$. Figure 1 illustrates the profile of simple strategies induced by these parameters along with a profile of strategic bliss points $\hat{\mathbf{x}} = \{1.6, 2, 3\}$.*

Figure 1: Simple strategies in Example 1



Let us first explain the rationale behind calling $\hat{x}_i$ strategic bliss points. $\hat{x}_i$ is the policy i proposes when the status-quo gives her ample bargaining power, that is, when i is not constrained by the acceptance set of the median. In Lemma 2 below we prove that the acceptance set is $\mathcal{A}(x) = [d_b(x), d_a(x)]$. Hence i can propose $\hat{x}_i$ when $x \notin (1.6, 2.4)$ for $i = 1$ and $x \notin (1, 3)$ for $i = 3$. Not being constrained means i can propose the policy maximizing her dynamic utility U_i, her strategic bliss point. Notice also that meaning of 'ample bargaining power' is relative to the given profile of (equilibrium) strategies inducing the acceptance correspondence $\mathcal{A}$.

The reason $\hat{x}_i$ and x_i differ is because the former policy maximizes dynamic utility $U_i = u_i + \delta V_i$, whereas the latter policy maximizes (static) utility u_i. Take player 1 from Example 1 and suppose the status-quo $x = 1$. We claim $p_1(1) = 1.6$ whereas the policy maximizing u_1 is $x_1 = 1$. With $x = 1$, $\mathcal{A}(1) = [1, 3]$ hence $x_1 = 1$, if proposed, would be accepted. The reason $x_1 = 1 \neq \hat{x}_1 = 1.6$ is that in the dynamic setting player 1 takes into account the impact of her proposal on the distribution of future policies. Two such distributions, induced by proposing $x_1 = 1$ and $p_1(1) = 1.6$, are indicated by the (red) circles to left of $x = 2$ in Figure 1. By proposing $p_1(1) = 1.6$, as opposed to proposing $x_1 = 1$, player 1 foregoes the chance to maximize her static utility but brings future policy of player 3 from $p_3(1) = 3$ to $p_3(1.6) = 2.4$. That is, player 1 *moderates* her proposal, she foregoes (current) static utility, in an attempt to constrain the future policy of player 3, increasing her future utility when she is not in possession of proposal power. The incentive to moderate is purely strategic; absent the intertemporal link created by persistent policies, player 1 would propose $x_1 = 1$.

Furthermore, we claim that player 3 from Example 1 *does not* moderate and her strategic bliss point coincides with her bliss point. Clearly, the strategic force to moderate is present for player 3 as well. Take status-quo $x = 3$. We claim player 3 proposes $p_3(3) = 3$ instead of moderating and proposing, using the same extent of moderation as player 1, $p' = 2.4$. Both $p_3(3) = 3$ and $p' = 2.4$ would be accepted with status-quo $x = 3$ and lead to the distribution over future policies indicated by the (blue) circles to the right of $x = 2$ in Figure 1. The reason player 3 does not moderate is because proposing $p' = 2.4$ or $p_3(3) = 3$ induces the same future policy by player 1, $p_1(2.4) = p_1(3) = 1$. In order to constrain the future policy of player 1, player 3 would have to moderate to some policy in $[2, 2.4)$, which is too costly for her in terms of foregone current utility. In other words, moderation is a *strategic substitute*; when player 1 moderates, the best response for player 3 is not to moderate, and when player 1 does not moderate, player 3 best responds by moderating.[17]

To proceed, given $\hat{\mathbf{x}}$ and induced σ, we define several objects required in the analysis below. By $\mathcal{ND}(\sigma) = \{\hat{x}_m, d_b(\hat{x}_1), d_a(\hat{x}_1), \dots, d_b(\hat{x}_n), d_a(\hat{x}_n)\}$ we denote the set of points such that, for any $x \in \mathcal{ND}(\sigma)$, there exists at least one p_i that is not differentiable with respect to x at x. $\mathcal{D}(\sigma) = X \setminus \mathcal{ND}(\sigma)$ denotes the set such that $x \in \mathcal{D}(\sigma)$ implies that all the strategies are differentiable with respect to x at x.[18]

For $\forall x \in \mathcal{D}(\sigma)$ define $\mathcal{C}(x|\sigma) = \{i \in N | p_i'(x|\hat{x}_i) = 0\}$ to be the set of players who, at x, are on the constant part of p_i (judging by its derivative). Similarly, for $\forall x \in \mathcal{D}(\sigma)$ define $\mathcal{NC}(x|\sigma) = \{i \in N | p_i'(x|\hat{x}_i) \neq 0\}$ to be the set of players who, at x, are on the non-constant part of p_i. It is easy to check that $\mathcal{C}(x|\sigma) \cup \mathcal{NC}(x|\sigma) = N$ for $\forall x \in \mathcal{D}(\sigma)$. We deliberately leave $\mathcal{C}$ and $\mathcal{NC}$ undefined for $x \in \mathcal{ND}(\sigma)$ as the interpretation of constant and non-constant has no meaning at points in $\mathcal{ND}(\sigma)$. Despite $\mathcal{C}$ being a correspondence, define its one-sided limits at any $x \in \mathcal{ND}(\sigma)$, $\mathcal{C}(x^-|\sigma)$ and $\mathcal{C}(x^+|\sigma)$, as $\mathcal{C}(x^-|\sigma) = \{i \in$

[17] The insight that in any SMPE at least one player does not moderate goes beyond the simple strategies considered here. In fact, the following claim can be easily proven. Consider any profile of pure proposal strategies $\hat{\sigma}$ such that, for $\forall i \in N \setminus \{m\}$, $\hat{p}_i(x) = \hat{x}_i$ for $\forall x \notin (d_b(\hat{x}_i), d_a(\hat{x}_i))$ with $d(\hat{x}_i) < d(x_i)$. That is, for $\forall i \in N \setminus \{m\}$, i moderates to $\hat{x}_i$ whenever the status-quo x satisfies $x \leq d_b(\hat{x}_i)$ or $x \geq d_a(\hat{x}_i)$. Then $\hat{\sigma}$ cannot constitute an SMPE. The intuition is, using without loss of generality $d(x_1) \leq d(x_n)$ and $d(\hat{x}_i) \leq d(\hat{x}_n)$ for $\forall i \in N \setminus \{m\}$, that V_n is constant on $X \setminus (d_b(\hat{x}_n), d_a(\hat{x}_n))$, U_n inherits the shape of u_n and thus $U_n(\hat{x}_n) < U_n(x_n)$. That is, n has no incentive to moderate to $\hat{x}_n$.

[18] This is not entirely precise. If $\hat{x}_i = x_m$ for $\forall i \in N$ all p_i are constant and hence differentiable on X. $\mathcal{ND}(\sigma)$ should be understood as the set of points at which some p_i *might not be* differentiable. As we are primarily concerned with taking derivatives when these do not exist, that is with $\mathcal{D}(\sigma)$, this is a mere imprecision in the label for $\mathcal{ND}(\sigma)$.

$N|p'_i(x^-|\hat{x}_i) = 0\}$ and $\mathcal{C}(x^+|\sigma) = \{i \in N|p'_i(x^+|\hat{x}_i) = 0\}$. Similarly, for any $x \in \mathcal{ND}(\sigma)$, $\mathcal{NC}(x^-|\sigma) = \{i \in N|p'_i(x^-|\hat{x}_i) \neq 0\}$ and $\mathcal{NC}(x^+|\sigma) = \{i \in N|p'_i(x^+|\hat{x}_i) \neq 0\}$.[19]

For $\forall x \in \mathcal{D}(\sigma)$ define $r_{nc}(x|\sigma) = \sum_{i \in \mathcal{NC}(x|\sigma)} r_i$ to be the sum of recognition probabilities of players on the non-constant part of their strategy. Splitting r_{nc} into the probabilities of recognition for players in N_a and N_b, we have $r_{nc,a}(x|\sigma) = \sum_{i \in \mathcal{NC}(x|\sigma) \cap N_a} r_i$ and $r_{nc,b}(x|\sigma) = \sum_{i \in \mathcal{NC}(x|\sigma) \cap N_b} r_i$ with $r_{nc}(x|\sigma) = r_{nc,a}(x|\sigma) + r_{nc,b}(x|\sigma)$ for $\forall x \in \mathcal{D}(\sigma)$. These objects are undefined at $x \in \mathcal{ND}(\sigma)$, nevertheless they possess one-sided limits at these points (defined using one-sided limits of $\mathcal{NC}$).[20]

For $\forall i \in N \setminus \{m\}$ define the (possibly empty) sets

$$\begin{aligned}
\mathcal{S}_i(\sigma) &= \begin{cases} \mathcal{ND}(\sigma) \cap (\hat{x}_i, x_i) & \text{if } i \in N_a \\ \mathcal{ND}(\sigma) \cap (x_i, \hat{x}_i) & \text{if } i \in N_b \end{cases} \\
\mathcal{L}_i(\sigma) &= \{x \in \mathcal{D}(\sigma) | U'_i(x|\sigma) = 0\} \\
\mathcal{N}_i(\sigma) &= \begin{cases} ((\mathcal{ND}(\sigma) \cup \mathcal{L}_i(\sigma)) \cap (\hat{x}_i, x_i)) \cup \{x_i, \hat{x}_i\} & \text{if } i \in N_a \\ ((\mathcal{ND}(\sigma) \cup \mathcal{L}_i(\sigma)) \cap (x_i, \hat{x}_i)) \cup \{x_i, \hat{x}_i\} & \text{if } i \in N_b \end{cases}
\end{aligned} \tag{6}$$

with elements of $\mathcal{N}_i(\sigma)$ ordered in increasing (decreasing) order if $i \in N_a$ ($i \in N_b$). $\mathcal{S}_i(\sigma)$ is the set of points in the interval between $\hat{x}_i$ and x_i at which p_j is not differentiable for some $j \in N$. $\mathcal{N}_i(\sigma)$ is a similar set of points adding points of local maxima of $U_i(\sigma)$, $\mathcal{L}_i(\sigma)$, and $\hat{x}_i$ and x_i. We are well aware that all $\mathcal{ND}$, $\mathcal{D}$, $\mathcal{C}$, $\mathcal{NC}$, r_{nc}, $r_{nc,a}$, $r_{nc,b}$, $\mathcal{S}_i$, $\mathcal{L}_i$ and $\mathcal{N}_i$, as well as the previously defined p_i, V_i, U_i and $\mathcal{A}$, are defined relative to $\hat{\mathbf{x}}$ and hence relative to σ. We suppress the dependence of these objects on σ when confusion cannot arise (as we have already done in Example 1).

Lemma 1 (Minimal properties of SMPE $\hat{\mathbf{x}}$). *If a profile of simple stationary Markov strategies σ induced by a profile of strategic bliss points $\hat{\mathbf{x}}$ constitutes an SMPE, then $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$.*

Proof. See appendix A1

[19] One-sided limits of $\mathcal{C}$ and $\mathcal{NC}$ at any $x \in \mathcal{D}(\sigma)$ are defined similarly. It is easy to see that $\mathcal{NC}$ and $\mathcal{C}$ are both piecewise 'constant' on intervals determined by $\mathcal{ND}(\sigma)$ and hence, for $\forall x \in \mathcal{D}(\sigma)$, $\mathcal{C}(x|\sigma) = \mathcal{C}(x^+|\sigma) = \mathcal{C}(x^-|\sigma)$ and $\mathcal{NC}(x|\sigma) = \mathcal{NC}(x^+|\sigma) = \mathcal{NC}(x^-|\sigma)$.

[20] For any profile of strategic bliss points $\hat{\mathbf{x}}$ and σ it induces, $p_i(d_a(x)|\hat{x}_i) = p_i(d_b(x)|\hat{x}_i)$ for $\forall x \in X$ and $\forall i \in N$. Hence, for $\forall x \in \mathcal{D}(\sigma)$, $\mathcal{C}(d_a(x)|\sigma) = \mathcal{C}(d_b(x)|\sigma)$, $\mathcal{NC}(d_a(x)|\sigma) = \mathcal{NC}(d_b(x)|\sigma)$, $r_{nc,a}(d_a(x)|\sigma) = r_{nc,a}(d_b(x)|\sigma)$ and $r_{nc,b}(d_a(x)|\sigma) = r_{nc,b}(d_b(x)|\sigma)$. For $\forall x \in X$, $\mathcal{C}(d_a(x)^-|\sigma) = \mathcal{C}(d_b(x)^+|\sigma)$ and $\mathcal{C}(d_a(x)^+|\sigma) = \mathcal{C}(d_b(x)^-|\sigma)$ and similarly for $\mathcal{NC}$, $r_{nc,a}$ and $r_{nc,b}$ are all easy to check. Furthermore, for any $x \in \mathcal{D}(\sigma)$ and $y \in \mathcal{D}(\sigma)$ such that $d(x) \leq d(y)$, $\mathcal{NC}(y|\sigma) \subseteq \mathcal{NC}(x|\sigma)$ and thus $r_{nc,a}(y|\sigma) \leq r_{nc,a}(x|\sigma)$ and $r_{nc,b}(y|\sigma) \leq r_{nc,b}(x|\sigma)$. Because $\mathcal{NC}$ is piecewise constant on intervals determined by $\mathcal{ND}(\sigma)$, r_{nc}, $r_{nc,a}$ and $_{nc,b}$ are as well.

Lemma 2 (Properties of V_i and U_i induced by $\hat{\mathbf{x}}$)**.** *For any $\hat{\mathbf{x}}$ with $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$ and induced profile of simple strategies σ, for $\forall i \in N$,*

1. *$V_i(d_b(x)|\sigma) = V_i(d_a(x)|\sigma)$ for $\forall x \in X$;*
2. *$U_i(d_b(x)|\sigma) < U_i(d_a(x)|\sigma)$ if $i \in N_a$, $U_i(d_b(x)|\sigma) > U_i(d_a(x)|\sigma)$ if $i \in N_b$ and $U_m(d_b(x)|\sigma) = U_m(d_a(x)|\sigma)$, for $\forall x \in X \setminus \{x_m\}$;*
3. *U_i is continuous on X;*
4. *$U_i''(x|\sigma) < 0$ for $\forall x \in \mathcal{D}(\sigma)$;*
5. *$U_m(x|\sigma) > U_m(y|\sigma)$ for $\forall x \in X$, $\forall y \in X$ such that $d(x) < d(y)$;*
6. *$\mathcal{A}(x|\sigma) = [d_b(x), d_a(x)]$ for $\forall x \in X$.*

Proof. See appendix A1

In addition to several technical properties of V_i and U_i induced by $\hat{\mathbf{x}}$, Lemma 2 demonstrates the shape of the social acceptance set $\mathcal{A}$. Because $p_i(x|\hat{x}_i) \in [d_b(x), d_a(x)]$ for $\forall i \in N$ and $\forall x \in X$ whenever $\hat{\mathbf{x}}$ satisfies the requirements of the lemma, any proposal generated by a simple strategy based on such $\hat{\mathbf{x}}$ belongs to the social acceptance set induced by $\hat{\mathbf{x}}$.

We now specify the algorithm that derives a profile of strategic bliss points $\hat{\mathbf{x}}$. The simple strategies in combination with $\hat{\mathbf{x}}$ from the algorithm need not constitute an SMPE. At this stage we view $\hat{\mathbf{x}}$ and the profile of strategies σ it induces as a candidate for SMPE.

Algorithm 1 (Strategic bliss points with $X \subseteq \mathbb{R}$)**.** *For the set of players $\mathbb{P}_t$ in step t of the algorithm, denote $r_{t,a} = \sum_{i \in \mathbb{P}_t \cap N_a} r_i$ and $r_{t,b} = \sum_{i \in \mathbb{P}_t \cap N_b} r_i$.*

step 0 *Set $\hat{x}_m = x_m$ and $\mathbb{P}_1 = N \setminus \{m\}$*

step t *For $i \in \mathbb{P}_t$ compute*

$$\hat{x}_{i,t} = \begin{cases} x_i + 2\delta r_{t,b}(x_m - x_i) & \text{if } i \in N_a \\ x_i + 2\delta r_{t,a}(x_m - x_i) & \text{if } i \in N_b \end{cases}$$

Define $\mathbb{R}_t = \{i \in \mathbb{P}_t | (x_i - x_m)(\hat{x}_{i,t} - x_m) \leq 0\}$
If $\mathbb{R}_t = \emptyset$, select one $j \in \arg\min_{i \in \mathbb{P}_t} d(\hat{x}_{i,t})$, set $\hat{x}_j = \hat{x}_{j,t}$
If $\mathbb{R}_t \neq \emptyset$, select one $j \in \mathbb{R}_t$, set $\hat{x}_j = x_m$
Set $\mathbb{P}_{t+1} = \mathbb{P}_t \setminus \{j\}$ and if $\mathbb{P}_{t+1} \neq \emptyset$, proceed to step $t+1$

It is immediately clear that the algorithm finishes in $n-1$ steps and produces a full profile of strategic bliss points $\hat{\mathbf{x}}$ with $\hat{x}_i \in [x_m, x_i]$ if $i \in N_a$ and $\hat{x}_i \in [x_i, x_m]$ if $i \in N_b$. Further, it is easy to check that $\hat{x}_i \leq \hat{x}_{i+1}$ for $i \in N \setminus \{n\}$ and that $\hat{x}_i = x_i$ for $i = 1$ or $i = n$ but not both (unless $\delta = 0$).

The intuition behind the algorithm is as follows. It starts with a full set of players apart from the median. It conjectures that strategy of all these players will be characterized by strategic bliss points equal to $+\infty$ for $i \in N_a$ and $-\infty$ for $i \in N_b$, that is players in N_a proposing $d_a(x)$ and players in N_b proposing $d_b(x)$. Calculating U_i for this conjectured strategy, the algorithm computes $\hat{x}_{i,1}$ which is a policy at which U_i attains its maximum. At $\hat{x}_{i,1}$ it ceases to be optimal for i to propose $d_a(x)$ or $d_b(x)$ and the best response, for any status-quo further from x_m relative to $\hat{x}_{i,1}$, is to propose $\hat{x}_{i,1}$. The algorithm then drops the player with $\hat{x}_{i,1}$ closest to x_m as the first player for whom, moving status-quo away from x_m, the conjectured strategy ceases to be a best response. Proceeding to step 2, the algorithm conjectures that the strategy of all players not previously dropped will be characterized by bliss points equal to $+\infty$ and $-\infty$ and continues similarly.

There are two possible complications. The first one arises when the algorithm arrives at $\hat{x}_{i,t}$ and $\hat{x}_{j,t}$ with $d(\hat{x}_{i,t}) = d(\hat{x}_{j,t})$ and both i and j belong to $\arg\min_{i \in \mathbb{P}_t} d(\hat{x}_{i,t})$. This implies $i \in N_a$ and $j \in N_b$ or vice versa, the algorithm requires exactly one of the players to be dropped, but we have not specified which one. This reflects the strategic substitute nature of moderation and is the sole reason for SMPE multiplicity. If i is dropped then j does not want to moderate and the algorithm retains j. If j is dropped then i does not want to moderate and is retained. That, say, i is retained means that the algorithm might eventually produce $\hat{\mathbf{x}}$ with i moderating as well. But this moderation is driven by other players still in the algorithm. In Example 1 dropping j meant i was retained as the sole player, in which case the algorithm produces $\hat{x}_i = x_i$. Example 1 (continued) below illustrates this complication and highlights the fact that the profile of strategic bliss points the algorithm produces need not be unique.

The second complication arises when $2\delta r_a \geq 1$ or $2\delta r_b \geq 1$ (both cannot hold simultaneously as $r_a + r_b = 1 - r_m < 1$). Suppose $2\delta r_a \geq 1$ holds. Then $\mathbb{R}_t \neq \emptyset$, $\mathbb{R}_t \subseteq N_b$ and $\mathbb{R}_t \cap N_a = \emptyset$ for $\forall t \in \{1, \ldots, \frac{n-1}{2}\}$ which means that the algorithm sequentially drops all the N_b players in steps $t \in \{1, \ldots, \frac{n-1}{2}\}$ and $\hat{x}_i = x_m$ for $\forall i \in N_b$. That is, the proposal strategies of all the N_b players are identical to the proposal strategy of the median player. Intuitively, when the N_a players are very likely to propose, the strategic force pushing the N_b players towards moderation is very strong, dominates any concerns for current utility and the greatest extent of constraint the N_b players can impose on the N_a players is by proposing x_m. When this happens, the algorithm also produces $\hat{x}_i = x_i$ for $\forall i \in N_a$, that is, the N_a players do not moderate. Example 2 below illustrates this complication.

The strategic bliss point of player i from Algorithm 1, via the simple strategy $p_i(x|\hat{x}_i)$, determines the extent of moderation of player i. From the algorithm, unless $\hat{x}_i = x_m$, $\hat{x}_i = x_i + 2\delta r(x_m - x_i)$ where r is the probability of recognition of i's opponents. Player i thus moderates to a larger extent with increasing δ and r. Both variables reinforce the strategic incentive to moderate and $\hat{x}_i$ increases when $i \in N_b$ and decreases when $i \in N_a$.

Example 1 (continued)**.** *In step* 0 *the algorithm drops the median player and sets* $\hat{x}_2 = x_2 = 2$*. In step* 1 *the algorithm computes* $\hat{x}_{1,1} = 1.6$ *and* $\hat{x}_{3,1} = 2.4$ *and, by dropping player* 1*, produces* $\hat{x}_3 = \hat{x}_{3,2} = 3$ *as already anticipated in Figure 1, which used* $\hat{\mathbf{x}} = \{1.6, 2, 3\}$*. Notice that dropping player* 3 *in step* 1 *would produce a profile of strategic bliss points* $\hat{\mathbf{x}} = \{1, 2, 2.4\}$*, which is distinct but symmetric around* x_m*.*

Example 2 (Players proposing identically as median)**.** *Consider* $\mathcal{G}$ *with* $n = 5$*,* $x_i = i$ *for* $\forall i \in N$*,* $\mathbf{r} = \{0.4, 0.4, 0.1, 0.05, 0.05\}$ *and* $\delta = 0.9$*. It is easy to confirm that* $\mathbb{R}_1 = \{4, 5\}$ *with the algorithm dropping player* 4 *and* $\mathbb{R}_2 = \{5\}$ *with the algorithm dropping player* 5*. After two more steps, the algorithm produces* $\hat{\mathbf{x}} = \{1, 2, 3, 3, 3\}$*.*

The following parametrization is taken from Duggan and Kalandrakis (2007). They numerically compute an SMPE in a model with preference and status-quo transition noise our setup lacks, but our methodology is fully applicable to the noise-less version of their model.

Example 3 (Duggan and Kalandrakis (2007) parametrization)**.** *Consider* $\mathcal{G}$ *with* $n = 5$*,* $\mathbf{x} = \{1, 1.5, 2, 2.8, 3\}$*,* $r_i = \frac{1}{n}$ *for* $\forall i \in N$ *and* $\delta = 0.9$*. The algorithm eliminates players* 2*,* 1*,* 4*, and* 5 *in steps* 1 *through* 4 *respectively and produces a unique profile of strategic bliss points* $\hat{\mathbf{x}} = \{1.72, 1.86, 2, 2.8, 3\}$*.*

The following lemma summarizes the key properties of any profile of strategic bliss points produced by Algorithm 1. The real significance of the lemma arises from Proposition 2 that follows.

Lemma 3 (Characterization of $\hat{\mathbf{x}}$ from Algorithm 1)**.** *Let* $\hat{\mathbf{x}}$ *be a profile of strategic bliss points produced by Algorithm 1. Then*

1. *if* $\delta = 0$*, then* $\hat{\mathbf{x}} = \mathbf{x}$*;*

2. *if* $\delta \in (0, 1)$ *and* $1 \leq 2\delta r_g$ *for some* $g \in \{a, b\}$*, then* $\hat{x}_i = x_m$ *for* $\forall i \in N \setminus N_g$ *and* $\hat{x}_i = x_i$ *for* $\forall i \in N_g$*;*

3. *if* $\delta \in (0,1)$, $1 > 2\delta r_a$ *and* $1 > 2\delta r_b$, *then* $\hat{x}_i < \hat{x}_{i+1}$ *for* $\forall i \in N \backslash \{n\}$ *and* $d(\hat{x}_i) \neq d(\hat{x}_j)$ *for* $\forall i \in N$, $\forall j \in N$, $i \neq j$.

Proof. See appendix A1

Proposition 2. *Let* $\hat{\mathbf{X}}$ *be the set of profiles of strategic bliss points produced by Algorithm 1. If* σ *induced by* $\hat{\mathbf{x}}$ *constitutes an SMPE, then* $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$.

Proof. See appendix A1

Proposition 2 states that if a profile of strategic bliss points $\hat{\mathbf{x}}$ that induces SMPE σ exists, then $\hat{\mathbf{x}}$ is produced by Algorithm 1. Lemma 3 thus not only characterizes any $\hat{\mathbf{x}}$ produced by Algorithm 1, it also constitutes a characterization of SMPE in simple proposal strategies.[21] In addition, Proposition 2 implies that $\#\hat{\mathbf{X}}$, the number of different profiles of strategic bliss points produced by the algorithm, puts an upper bound on the number of SMPE in simple proposal strategies. If Algorithm 1 produces a unique $\hat{\mathbf{x}}$, then an SMPE in simple strategies is either unique or fails to exist.[22]

From the way the algorithm constructs $\hat{\mathbf{x}}$, $\#\hat{\mathbf{X}} \geq 2$ is possible only if it in step t arrives at $\hat{x}_{i,t}$ and $\hat{x}_{j,t}$ with $d(\hat{x}_{i,t}) = d(\hat{x}_{j,t})$. The equality rewrites as $d(x_i)(1 - 2\delta r_{t,b}) = d(x_j)(1 - 2\delta r_{t,a})$ and is non-generic. That is, a perturbation of $\mathbf{x}$ by $\epsilon > 0$, $\mathbf{x}(\epsilon)$, exists, such that Algorithm 1 applied to $\mathcal{G}(\epsilon) = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X\rangle$ produces unique $\hat{\mathbf{x}}(\epsilon)$. In fact, any $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ can be approached by unique $\hat{\mathbf{x}}(\epsilon)$. The following lemma states this result formally and its proof constructs the claimed perturbation.

Lemma 4. *Fix any* $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ *from Algorithm 1 applied to* $\mathcal{G}$. *Then a perturbation of* $\mathbf{x}$ *by* $\epsilon > 0$, $\mathbf{x}(\epsilon)$, *and* $\bar{\epsilon} > 0$ *exist, such that* $\lim_{\epsilon \to 0} \mathbf{x}(\epsilon) = \mathbf{x}$ *and Algorithm 1 applied to* $\mathcal{G}(\epsilon) = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X\rangle$, *for* $\forall \epsilon \leq \bar{\epsilon}$, *produces a unique profile of strategic bliss points* $\hat{\mathbf{x}}(\epsilon)$ *such that* $\lim_{\epsilon \to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.

Proof. See appendix A1

[21] The lemma states that even when $\mathcal{G}$ is strongly symmetric and $\delta \in (0,1)$, no two strategic bliss points can be the same distance from the median bliss point. The reason is the strategic substitute nature of moderation. If $n = 5$, player 2 starts moderating when the status-quo is distance $d(\hat{x}_2)$ from $x_3 = x_m$. It cannot be SMPE for player 4 to start moderating at $d(\hat{x}_4) = d(\hat{x}_2)$; if player 2 starts at $d(\hat{x}_2)$ it is optimal for player 4 to start at $d(\hat{x}'_4) > d(\hat{x}_2)$, if player 4 starts at $d(\hat{x}_4)$ it is optimal for player 2 to start at $d(\hat{x}'_2) > d(\hat{x}_4)$. Lemma 3 with Proposition 2 imply that the example of full equilibrium characterization in Baron (1996, based on strategic bliss points in his equation (18) and summarized in his Table 1) cannot constitute an SMPE.

[22] We stress that all the uniqueness statements pertain to SMPE in simple strategies and should be read as referring to the uniqueness of SMPE in the class of SMPE in simple strategies.

4.2 Necessary and Sufficient Conditions

We are ready to state two conditions that guarantee that a profile of strategic bliss points $\hat{\mathbf{x}}$ from Algorithm 1 induces SMPE σ.[23]

Definition 5 (Condition $\mathbb{S}$, sufficient)**.** *A profile of strategic bliss points $\hat{\mathbf{x}}$ from Algorithm 1 that induces σ satisfies condition $\mathbb{S}$ if and only if, for $\forall i \in N \setminus \{m\}$ and $\forall x \in \mathcal{S}_i(\sigma)$,*

$$\begin{aligned} x - x_i - 2\delta r_{nc,b}(x^+|\sigma)(x_m - x_i) &\geq 0 \quad \text{if } \; i \in N_a \\ x - x_i - 2\delta r_{nc,a}(x^-|\sigma)(x_m - x_i) &\leq 0 \quad \text{if } \; i \in N_b. \end{aligned} \tag{$\mathbb{S}$}$$

Definition 6 (Condition $\mathbb{N}$, necessary and sufficient)**.** *A profile of strategic bliss points $\hat{\mathbf{x}}$ from Algorithm 1 that induces σ satisfies condition $\mathbb{N}$ if and only if, for $\forall i \in N \setminus \{m\}$ and denoting elements of $\mathcal{N}_i(\sigma)$ by $\{z_0, z_1, \ldots\}$,*

$$\begin{aligned} &\sum_{j=1}^{J} \Big[T_i(x|\sigma)\Big]_{z_j^-}^{z_{j-1}^+} \geq 0 \quad \text{for } \; \forall J \in \{1, \ldots, |\mathcal{N}_i(\sigma)| - 1\} \text{ if } \; i \in N_a \\ &\sum_{j=1}^{J} \Big[T_i(x|\sigma)\Big]_{z_j^+}^{z_{j-1}^-} \geq 0 \quad \text{for } \; \forall J \in \{1, \ldots, |\mathcal{N}_i(\sigma)| - 1\} \text{ if } \; i \in N_b \end{aligned} \tag{$\mathbb{N}$}$$

where

$$T_i(x|\sigma) = -\frac{2}{1 - \delta r_{nc}(x|\sigma)} \left[\frac{x^2}{2} - c_i(x|\sigma)x\right]$$

$$c_i(x|\sigma) = \begin{cases} x_i + 2\delta r_{nc,b}(x|\sigma)(x_m - x_i) & \text{if } \; i \in N_a \\ x_i + 2\delta r_{nc,a}(x|\sigma)(x_m - x_i) & \text{if } \; i \in N_b. \end{cases}$$

Proposition 3 (SMPE under $\mathbb{S}$ and $\mathbb{N}$ conditions)**.** *A profile of strategic bliss points $\hat{\mathbf{x}}$ from Algorithm 1 induces SMPE σ*

1. *if $\hat{\mathbf{x}}$ satisfies condition $\mathbb{S}$;*
2. *if and only if $\hat{\mathbf{x}}$ satisfies condition $\mathbb{N}$.*

Proof. See appendix A1

The reason both $\mathbb{S}$ and $\mathbb{N}$ guarantee that the simple strategies induced by $\hat{\mathbf{x}}$ constitute an SMPE is the following. First note that player $i \in N_a$ would never propose policy $p < x_m$

[23] Both conditions apply to profiles of strategic bliss points from Algorithm 1. By Proposition 2 this is without loss of generality as the algorithm constructs all $\hat{\mathbf{x}}$ that support an SMPE for given $\mathcal{G}$. An alternative method would be to state both conditions for general $\hat{\mathbf{x}}$ such that $\hat{x}_i \in [\min\{x_m, x_i\}, \max\{x_m, x_i\}]$ for $\forall i \in N$, which is a property of any $\hat{x}_i$ from Algorithm 1 and hence, by Proposition 2, of any $\hat{\mathbf{x}}$ that induces SMPE σ.

due to symmetry, around x_m, of the acceptance sets $\mathcal{A}$ and of the continuation value functions V_i. Furthermore, in the proof of the proposition we show that U_i is increasing on $[x_m, \hat{x}_i]$ and decreasing on $[x_i, +\infty)$. However, for the simple strategy with $\hat{x}_i$ to be the best response to the strategies of the other players, U_i must be decreasing on $[\hat{x}_i, x_i]$ as well. From Lemma 2 we know U_i is piecewise strictly concave, which means ensuring that the right derivative of U_i is non-positive, at any point in $\mathcal{ND}$ that falls into $(\hat{x}_i, x_i)$, suffices for U_i to be decreasing on $[\hat{x}_i, x_i]$. This is what condition $\mathbb{S}$ does. When it holds, U_i is increasing on $[x_m, \hat{x}_i]$ and decreasing on $[\hat{x}_i, +\infty)$, implying that proposing $d_a(x)$ when the status-quo x is such that $\hat{x}_i \notin \mathcal{A}(x)$ and proposing $\hat{x}_i$ otherwise is optimal for i.

Note that condition $\mathbb{S}$ is stronger than necessary. It ensures that U_i is decreasing on $[\hat{x}_i, x_i]$ while for $\hat{x}_i$ to be optimal for $i \in N_a$, only $U_i(\hat{x}_i) \geq U_i(x)$ for $\forall x \geq \hat{x}_i$ is required. This is what condition $\mathbb{N}$ does. It only looks at a finite set of points using the fact that U_i is piecewise quadratic and $U_i(x) - U_i(y) = \left[\int \frac{\partial}{\partial z} U_i(z) dz\right]_y^x$.

Despite the fact that both conditions guaranteeing the existence of SMPE only need to be checked at a finite set of points, their disadvantage is that they apply to the strategic bliss points from Algorithm 1. Relating these conditions directly to the parameters defining $\mathcal{G}$ is non-trivial due to the complicated mapping from n, $\mathbf{x}$, $\mathbf{r}$ and δ to $\hat{\mathbf{x}}$. This is why in the next section we look at symmetric environments. Putting enough structure on the parameters defining $\mathcal{G}$ will allow us to relate (mainly) condition $\mathbb{S}$ to these parameters.

We have explained that the incentive of the players to moderate is driven by their concern about the future policy outcomes. It is natural to conjecture that when the players are almost myopic, the strategic bliss points $\hat{\mathbf{x}}$ differ little from $\mathbf{x}$ and hence induce SMPE σ. The following proposition derives conditions such that the conjecture is indeed true.

Proposition 4 (Condition $\mathbb{N}$ holds for small δ). *If $r_i \in [\frac{r_j}{2}, 2r_j]$ for every pair of players $\{i, j\}$ with $d(x_i) = d(x_j)$, then $\bar{\delta} \in (0, 1)$ exists, such that for $\forall \delta \leq \bar{\delta}$ any $\hat{\mathbf{x}}$ from Algorithm 1 satisfies condition $\mathbb{N}$.*

Proof. See appendix A1

Before we proceed we provide two examples. The first shows that despite the apparent complexity of conditions $\mathbb{S}$ and $\mathbb{N}$ these can be simple to verify. The second example shows that whether these conditions are satisfied or not can depend non-monotonically on δ. It is also easy to see that both of the conditions hold in Examples 2 and 3.

Example 1 (continued). *With $\mathbf{x} = \{1, 2, 3\}$ and $\hat{\mathbf{x}} = \{1.6, 2, 3\}$, the set of points at which differentiability of (at least some of) the proposal strategies might fail is $\mathcal{ND} =$*

$\{1, 1.6, 2, 2.4, 3\}$. *The subset of players on the non-constant part of their strategy is*

$$\mathcal{NC}(x) = \begin{cases} \{1,3\} & \text{for } x \in (1.6, 2) \cup (2, 2.4) \\ \{3\} & \text{for } x \in (1, 1.6) \cup (2.4, 3) \\ \emptyset & \text{for } x \in (-\infty, 1) \cup (3, +\infty) \end{cases}$$

which induces $r_{nc,a}(x) = \frac{1}{3}$ *for* $x \in (1,2) \cup (2,3)$ *and* $r_{nc,b}(x) = \frac{1}{3}$ *for* $x \in (1.6,2) \cup (2,2.4)$ *with both* $r_{nc,a}$ *and* $r_{nc,b}$ *equal to* 0 *for any other* $x \in X \setminus \mathcal{ND}$.

Because $\mathcal{S}_1 = \mathcal{ND} \cap (1, 1.6) = \emptyset$ *and* $\mathcal{S}_3 = \mathcal{ND} \cap (3,3) = \emptyset$ *and because* $\mathcal{L}_1 = \mathcal{L}_3 = \emptyset$, *we have* $\mathcal{N}_1 = \{1, 1.6\}$ *and* $\mathcal{N}_3 = \{3\}$. *Conditions* $\mathbb{S}$ *and* $\mathbb{N}$ *hold, which, by Proposition 3, implies* σ *induced by* $\hat{\mathbf{x}} = \{1.6, 2, 3\}$ *constitutes an SMPE.*

Example 4 (Non-monotonic failure of $\mathbb{S}$ and $\mathbb{N}$ conditions)**.** *Consider* $\mathcal{G}$ *with* $n = 7$, $x_i = i$ *and* $r_i = \frac{1}{n}$ *for* $\forall i \in N$ *and* $\delta = 0.5$. *Then Algorithm 1 produces eight different profiles of strategic bliss points* $\hat{\mathbf{x}}$ *(depending on the selection of players to drop). For every* $\hat{\mathbf{x}}$, *condition* $\mathbb{S}$, *and by implication condition* $\mathbb{N}$, *holds. For the same* $\mathcal{G}$ *with* $\delta = 0.9$ *the number of* $\hat{\mathbf{x}}$ *from Algorithm 1 reduces to two but both fail both* $\mathbb{S}$ *and* $\mathbb{N}$ *conditions. For the same* $\mathcal{G}$ *with* $\delta = 0.95$ *there are again two possible* $\hat{\mathbf{x}}$ *and for both condition* $\mathbb{S}$ *fails while condition* $\mathbb{N}$ *holds.*

5 Equilibrium Existence in Symmetric Games

Recall that $\mathcal{G}$ is symmetric if any pair of players $\{d_b^I(i), d_a^I(i)\}$ has equal recognition probabilities and bliss points at the same distance from x_m. This implies $r_a = r_b < \frac{1}{2}$ and that r_j^e, the sum of the recognition probabilities of the $j < m$ most extreme players $\{1, \ldots, j\}$, is equal to the sum of the recognition probabilities of players $\{d_a^I(j), \ldots, n\}$.

The definition that follows guarantees that Algorithm 1 drops players $\{m-1, m+1\}$ in steps $t \in \{1, 2\}$. In step $t = 1$, the algorithm offers an option to drop either one of these two players, and in step $t = 2$ drops the player not eliminated in step $t = 1$. In steps $t \in \{3, 4\}$ the algorithm drops players $\{m-2, m+2\}$ in a similar manner and the same happens in any steps $\{t, t+1\}$ with t odd. This is what condition $\mathbb{G}_1$ ensures. The resulting structure of $\hat{\mathbf{x}}$ along with symmetry of $\mathcal{G}$ allows us to write condition $\mathbb{G}_2$ which, as we prove in Proposition 5, guarantees that $\hat{\mathbf{x}}$ satisfies condition $\mathbb{S}$ and hence induces SMPE σ. Notice that both conditions are written in terms of parameters of $\mathcal{G}$.

Definition 7 (Pairwise moderation inducing $\mathcal{G}$)**.** $\mathcal{G}$ *induces pairwise moderation if and* *only if $\mathcal{G}$ is symmetric, for* $\forall i \in \{1, \dots, \frac{n-3}{2}\}$

$$\frac{1-2\delta r_i^e}{1-2\delta r_{i+1}^e} \leq \frac{x_m - x_i}{x_m - x_{i+1}} \tag{$\mathbb{G}_1$}$$

and for $\forall i \in \{1, \dots, \frac{n-3}{2}\}$ *and* $\forall j \in \{1, \dots, i\}$

$$\frac{1-2\delta r_{j-1}^e}{1-2\delta r_j^e} \leq \frac{x_m - x_j}{x_m - x_{i+1}}. \tag{$\mathbb{G}_2$}$$

The complexity of the conditions defining pairwise moderation inducing $\mathcal{G}$ is driven by our attempt to write them for a general class of symmetric games as much as possible.[24] In fact, any symmetric $\mathcal{G}$ induces pairwise moderation if the players are sufficiently impatient.

Lemma 5. *For any symmetric $\mathcal{G}$, $\bar{\delta} \in (0,1)$ exists such that $\mathcal{G}$ induces pairwise moderation for $\forall \delta \leq \bar{\delta}$.*

Proof. Conditions $\mathbb{G}_1$ and $\mathbb{G}_2$ clearly hold for $\delta = 0$. In both conditions, the right hand side is strictly greater than unity, the left hand side is equal to unity for $\delta = 0$ and is increasing in δ. □

There are two conditions defining pairwise moderation inducing $\mathcal{G}$ and we explained their rationale above. However, condition $\mathbb{G}_2$ proves to be redundant in certain 'well behaved' games satisfying 'monotonicity' of the recognition probabilities or of the distances between the bliss points of adjacent players.

Lemma 6. *If condition $\mathbb{G}_1$ co-defining pairwise moderation inducing $\mathcal{G}$ holds, then $\mathbb{G}_2$ in the same definition holds if at least one of the following conditions are satisfied.*

1. $r_i \leq r_{i+1}$ *for* $\forall i \in \{1, \dots, \frac{n-3}{2}\}$
2. $x_i - x_{i-1} \leq x_{i+1} - x_i$ *for* $\forall i \in \{2, \dots, \frac{n-3}{2}\}$ *and* $\frac{1}{1-2\delta r_1} \leq \frac{x_m - x_1}{x_m - x_2}$

Proof. See appendix A1

For strongly symmetric games with equidistant bliss points and equal recognition probabilities, the conditions defining pairwise moderation inducing $\mathcal{G}$ become trivial to verify.

[24] To understand $\mathbb{G}_1$ and $\mathbb{G}_2$, after dropping player $m-1$ in step 1, Algorithm 1 in step 2 calculates $\hat{x}_{m+1,2} = x_{m+1} + 2\delta r_{m-2}^e(x_m - x_{m+1})$ and $\hat{x}_{m-2,2} = x_{m-2} + 2\delta r_{m-1}^e(x_m - x_{m-2})$. $\mathbb{G}_1$ is then the general version of the condition ensuring $m+1$ is dropped, $d(\hat{x}_{m+1,2}) \leq d(\hat{x}_{m-2,2})$. When the algorithm drops player $d_a^I(j)$ at a further step, $d_b(\hat{x}_{d_a^I(j)}) \in \mathcal{S}_{m-1}$, among other values, needs to satisfy condition $\mathbb{S}$. With $d_b(\hat{x}_{d_a^I(j)}) = x_j + 2\delta r_j^e(x_m - x_j)$, the condition requires $d_b(\hat{x}_{d_a^I(j)}) - x_{m-1} - 2\delta r_{j-1}^e(x_m - x_{m-1}) \leq 0$, which rewrites as $\mathbb{G}_2$. Because, say, $\mathbb{G}_1$ rewrites as $\frac{2\delta r_{i+1}}{1-2\delta r_{i+1}^e} \leq \frac{x_{i+1}-x_i}{x_m - x_{i+1}}$, both conditions put an upper bound on the incentive to moderate driven by δ and r_{i+1}.

Lemma 7. *Symmetric $\mathcal{G}$ with $n = 3$ induces pairwise moderation. Strongly symmetric $\mathcal{G}$ with $n \geq 5$ and $\delta \leq \frac{n}{n+1}$ induces pairwise moderation.*

Proof. Symmetric $\mathcal{G}$ with $n = 3$ obviously induces pairwise moderation as it is symmetric and the parametric conditions in Definition 7 apply only for $n \geq 5$.

For strongly symmetric $\mathcal{G}$, $r_i^e = \frac{i}{n}$ and $x_m - x_i = (\frac{n+1}{2} - i)(x_m - x_{m-1})$ for any $i \in \{1, \ldots, \frac{n-1}{2}\}$. Substituting into $\mathbb{G}_1$ in Definition 7, which by Lemma 6 suffices, gives $\delta \leq \frac{n}{n+1}$. □

To state the main result of this section we need the following definition. As we explained above, condition $\mathbb{G}_1$ ensures that Algorithm 1 drops pairs of players $\{d_b^I(i), d_a^I(i)\}$ in pairs of steps $\{t, t+1\}$. For knife edge cases when condition $\mathbb{G}_1$ holds with equality, the algorithm offers the option, in step $t = 1$, to drop players $\{m-1, m+1\}$ and dropping $m+1$, in step $t = 2$, offers the option to drop players $\{m-1, m+2\}$. At this point, for $\hat{\mathbf{x}}$ to have the structure underlying Proposition 5, we have to ensure that player $m-1$ is dropped in step $t = 2$. That is, we need to ensure that if $i \in N_a$ is dropped in $t = 1$ then $i \in N_b$ is dropped in $t = 2$ and vice versa, whenever the algorithm faces multiple players to be dropped. A similar selection is necessary at any step $t \geq 3$.

Definition 8 (Pairwise path through Algorithm 1)**.** *A selection of which players to drop, whenever a non-unique option arises, in Algorithm 1 is called a pairwise path if and only if, in step $t \geq 2$, $i \in N_a$ is dropped when $j \in N_b$ has been dropped in step $t-1$ and $i \in N_b$ is dropped when $j \in N_a$ has been dropped in step $t-1$.*

Proposition 5 (SMPE with pairwise moderation)**.** *Assume $\mathcal{G}$ induces pairwise moderation. Then*

1. *if $\delta \in (0, 1)$, $2^{(n-1)/2}$ distinct profiles of strategic bliss points $\hat{\mathbf{x}}$ produced by pairwise paths through Algorithm 1 exist, if $\delta = 0$, $\hat{\mathbf{x}} = \mathbf{x}$;*

2. *σ induced by any of these profiles of strategic bliss points constitutes an SMPE;*

3. *σ induced by any of these profiles of strategic bliss points satisfies condition $\mathbb{S}$ and, for $\forall i \in N$, U_i is single peaked on X.*

Proof. See appendix A1

Proposition 5 is the main result of this section. It proves the existence of an SMPE in the large class of games that induce pairwise moderation. To construct an SMPE all that is needed is a profile of strategic bliss points from Algorithm 1 and simple strategies.

As already anticipated, the result relies on the fact that pairwise moderation inducing $\mathcal{G}$ delivers $\hat{\mathbf{x}}$ which satisfies condition $\mathbb{S}$. Using Lemma 7, Proposition 5 implies the existence of SMPE in any symmetric $\mathcal{G}$ with $n = 3$ and any strongly symmetric $\mathcal{G}$ with $n \geq 5$ and $\delta \leq \frac{n}{n+1}$, a condition which virtually ceases to bind as n increases.

The following examples substantiate our claim that Proposition 5 in fact applies to a large class of games that are not strongly symmetric. The first two examples assume monotonicity in the recognition probabilities (Example 5) or in the distance between bliss points of adjacent players (Example 6). Example 7 takes a strongly symmetric $\mathcal{G}$ and increases the median player's recognition probability. Example 8 also takes a strongly symmetric $\mathcal{G}$ but increases the distance of bliss points between players $\{d_b^I(j) - 1, d_b^I(j)\}$ and between players $\{d_a^I(j), d_a^I(j) + 1\}$. This produces a $\mathcal{G}$ with three 'clusters' of players, one around m and two 'extreme' clusters. Note also that all the examples state conditions guaranteeing that the underlying $\mathcal{G}$ induces pairwise moderation. All the conditions put an upper bound on the patience of the players, collapse to $\delta \leq \frac{n}{n+1}$ when $\mathcal{G}$ becomes strongly symmetric, which is allowed by all the examples, and effectively cease to bind when n increases.[25]

Example 5 (More extreme players less/more likely to propose)**.**

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} = x_{i+1} - x_i$ for $\forall i \in \{2, \ldots, n-1\}$ and $r_i \leq r_{i+1}$ for $\forall i \in \{1, \ldots, \frac{n-3}{2}\}$. Then condition $\mathbb{G}_1$ co-defining pairwise moderation inducing $\mathcal{G}$ holds if and only if it holds for $i = \frac{n-3}{2}$;[26] when $\mathbb{G}_1$ holds then $\mathbb{G}_2$ holds as well; and $\mathcal{G}$ induces pairwise moderation if and only if $\delta \leq \frac{1}{2(r_a + r_{m-1})}$, which does not bind if $r_{m-1} \leq \frac{1}{2} - r_a = \frac{r_m}{2}$.

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} = x_{i+1} - x_i$ for $\forall i \in \{2, \ldots, n-1\}$ and $r_i \geq r_{i+1}$ for $\forall i \in \{1, \ldots, \frac{n-3}{2}\}$. Then condition $\mathbb{G}_1$ co-defining pairwise moderation inducing $\mathcal{G}$ holds if and only if it holds for $i = 1$; when $\mathbb{G}_1$ holds and $\delta \leq \frac{1}{r_1(n-1)}$ then $\mathbb{G}_2$ holds as well; and $\mathcal{G}$ induces pairwise moderation if and only if $\delta \leq \min\{\frac{1}{2r_1 + (n-1)r_2}, \frac{1}{r_1(n-1)}\}$.

Example 6 (Increasing/decreasing extremism)**.**

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} \geq x_{i+1} - x_i$ for $\forall i \in \{2, \ldots, \frac{n-1}{2}\}$ and $r_i = r_{i+1}$ for $\forall i \in \{1, \ldots, \frac{n-1}{2}\}$. Then condition $\mathbb{G}_1$ co-defining pairwise moderation inducing $\mathcal{G}$ holds if and only if it holds for $i = \frac{n-3}{2}$; when $\mathbb{G}_1$ holds then $\mathbb{G}_2$ holds as well;

[25] Examples 5, 6 and 7 also show that the conditions on δ need not bind at all.

[26] This claim, as well as the similar claim for $i = 1$ below, does not follow immediately. We feel that a formal proof is unnecessary, but are ready to provide it. To outline the idea, the proof uses the monotonicity of the recognition probabilities and the equidistance of players' bliss points. For the following example, a similar proof uses the monotonicity of the distances between players' strategic bliss points and equal recognition probabilities.

and $\mathcal{G}$ induces pairwise moderation if and only if $\delta \leq \frac{n(x_{m-1}-x_{m-2})}{(n-1)(x_{m-1}-x_{m-2})+2(x_m-x_{m-1})}$, *which does not bind if* $x_{m-1} - x_{m-2} \geq 2(x_m - x_{m-1})$.

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} \leq x_{i+1} - x_i$ *for* $\forall i \in \{2, \ldots, \frac{n-1}{2}\}$ *and* $r_i = r_{i+1}$ *for* $\forall i \in \{1, \ldots, \frac{n-1}{2}\}$. *Then condition* $\mathbb{G}_1$ *co-defining pairwise moderation inducing $\mathcal{G}$ holds if and only if it holds for* $i = 1$*; when* $\mathbb{G}_1$ *holds then* $\mathbb{G}_2$ *holds as well; and $\mathcal{G}$ induces pairwise moderation if and only if* $\delta \leq \frac{n(x_2-x_1)}{2(x_m-x_1+x_2-x_1)}$.

Example 7 (Arbitrary median's recognition probability)**.**

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} = x_{i+1} - x_i$ *for* $\forall i \in \{2, \ldots, n-1\}$ *and* $r_i = \frac{1-r_m}{n-1}$ *for* $\forall i \in N \setminus \{m\}$. *Then condition* $\mathbb{G}_1$ *co-defining pairwise moderation inducing $\mathcal{G}$ either holds or fails for* $\forall i \in \{1, \ldots, \frac{n-3}{2}\}$*; when* $\mathbb{G}_1$ *holds then* $\mathbb{G}_2$ *holds as well; and $\mathcal{G}$ induces pairwise moderation if and only if* $\delta \leq \frac{n-1}{n+1}\frac{1}{1-r_m}$, *which does not bind if* $r_m \geq \frac{2}{n+1}$.

Example 8 (Clusters of players)**.**

Assume $\mathcal{G}$ is symmetric with $n \geq 5$, $x_i - x_{i-1} = d$ *for* $\forall i \in \{2, \ldots, m\} \setminus \{j\}$, $x_j - x_{j-1} = d + e$ *with* $e \geq 0$ *where* $2 \leq j \leq m$ *and* $r_i = r_{i+1}$ *for* $\forall i \in \{1, \ldots, \frac{n-1}{2}\}$. *Then condition* $\mathbb{G}_1$ *co-defining pairwise moderation inducing $\mathcal{G}$ holds if and only if it holds for* $\forall i = \{1, \ldots, j-2\} \cup \{\frac{n-3}{2}\}$*; when* $\mathbb{G}_1$ *holds then* $\mathbb{G}_2$ *holds as well; and $\mathcal{G}$ induces pairwise moderation if and only if* $\delta \leq \frac{n}{(n+1)+2\frac{e_j}{d}}$ *where* $e_j = 0$ *if* $j = 2$ *and* $e_j = e$ *if* $j \in \{3, \ldots, m\}$.

Proposition 5 shows that $2^{(n-1)/2}$ SMPE exist for any $\mathcal{G}$ that induces pairwise moderation, all based on profiles of strategic bliss points delivered by Algorithm 1. In Lemma 4, we have shown that the multiplicity of $\hat{\mathbf{x}}$ from Algorithm 1 is non-generic and can be perturbed away. The lemma, however, is silent about the ability of the perturbed $\hat{\mathbf{x}}(\epsilon)$ to support SMPE $\sigma(\epsilon)$. Our next proposition shows that it is indeed possible to perturb $\mathbf{x}$ without upsetting the ability of the profile of strategic bliss points from Algorithm 1 to support an SMPE.

Proposition 6. *Assume $\mathcal{G}$ induces pairwise moderation. Fix any $\hat{\mathbf{x}}$ produced by pairwise path through Algorithm 1. Then a perturbation of $\mathbf{x}$ by $\epsilon > 0$, $\mathbf{x}(\epsilon)$, and $\bar{\epsilon} > 0$ exist, such that $\lim_{\epsilon \to 0} \mathbf{x}(\epsilon) = \mathbf{x}$ and Algorithm 1 applied to $\mathcal{G}(\epsilon) = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X \rangle$, for $\forall \epsilon \leq \bar{\epsilon}$, produces a unique profile of strategic bliss points $\hat{\mathbf{x}}(\epsilon)$ that satisfies condition $\mathbb{S}$ and $\lim_{\epsilon \to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.*

Proof. See appendix A1

In addition to showing the non-generic nature of the multiplicity of SMPE in pairwise moderation inducing $\mathcal{G}$, Proposition 6 shows that the equilibrium correspondence mapping $\mathcal{G}$ into the set of SMPE in simple strategies is upper hemicontinuous in $\mathbf{x}$, SMPE exists as $\mathbf{x}(\epsilon) \to \mathbf{x}$ and continues to exist at the limit of the sequence, at $\mathbf{x}$. However, it fails lower hemicontinuity. Only one of the equilibria that exists at $\mathbf{x}$ can be approached by the unique SMPE that exists along $\mathbf{x}(\epsilon) \to \mathbf{x}$.

Throughout this section, our focus has been on games that induce pairwise moderation. For strongly symmetric $\mathcal{G}$, this implied restricted focus on $\delta \leq \frac{n}{n+1}$. Despite the fact that this condition becomes rather weak as n increases, a natural question arises about the existence and properties of SMPE as $\delta \to 1$. The following proposition shows that for high enough δ, at least two SMPE supported by $\hat{\mathbf{x}}$ from Algorithm 1 exist.

Proposition 7 (Patient players in strongly symmetric $\mathcal{G}$).
Assume $\mathcal{G}$ is strongly symmetric, $n \geq 5$, and $\delta \geq \bar{\delta}(n)$ where

$$\bar{\delta}(n) = \max\left\{\tfrac{n}{n+1}, \tfrac{n}{n-3}\left[2\tfrac{n-2}{n-1} - \sqrt{\tfrac{n^3-n^2-n-7}{(n-1)^3}}\right]\right\} < 1.$$

Then

1. *two profiles of strategic bliss points $\hat{\mathbf{x}}$ produced by Algorithm 1 exist with, for $g \in \{a, b\}$, $\hat{x}_m = x_m$, $\hat{x}_i = x_i$ for $\forall i \in N_g$ and $d(\hat{x}_i) \in (0, d(x_{m-1}))$ for $\forall i \in N \backslash (N_g \cup \{m\})$;*
2. *σ induced by any of these profiles of strategic bliss points constitutes an SMPE.*

Proof. See appendix A1

Suppose $g = a$ in part 1 of the proposition. Then for any $\delta \geq \bar{\delta}(n)$, $\hat{\mathbf{x}}$ that induces SMPE σ exists. This $\hat{\mathbf{x}}$ is characterized by $\hat{x}_i = x_i$ for $\forall i \in N_a \cup \{m\}$ and $\hat{x}_i \in (x_{m-1}, x_m)$ for $\forall i \in N_b$. In words, all the players in N_b moderate while none of the players in N_a do. In the proof of the proposition we show that $\hat{x}_i = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for the moderating players. As a result $\lim_{n\to\infty}\lim_{\delta\to 1}\hat{x}_i = x_m$ so that proposal behaviour of any moderating player in an SMPE of large $\mathcal{G}$ with patient players resembles the proposal behaviour of the median player.

From Lemma 7 we know that symmetric $\mathcal{G}$ with $n = 3$ induces pairwise moderation which, by Proposition 5, implies SMPE existence for any δ. For strongly symmetric $\mathcal{G}$ with $n \geq 5$, the same lemma requires $\delta \leq \frac{n}{n+1}$. Because $\bar{\delta}(n) = \frac{n}{n+1}$ for $n = 5$, Propositions 7, 5 and Lemma 7 jointly imply SMPE existence in any strongly symmetric $\mathcal{G}$ with $n = 5$. When $\delta \leq \frac{n}{n+1}$, any $\hat{\mathbf{x}}$ produced by a pairwise path through Algorithm 1 induces SMPE σ. When $\delta \geq \frac{n}{n+1}$, any $\hat{\mathbf{x}}$ from Proposition 7 induces SMPE σ.

Corollary 1. *An SMPE exists in strongly symmetric $\mathcal{G}$ with $n = 5$.*

For strongly symmetric $\mathcal{G}$ with more than five players, $\bar{\delta}(n) > \frac{n}{n+1}$ for any $n \geq 7$, leaving a gap in the range of discount factors covered by Propositions 5 and 7.[27] On the other hand, from the arguments presented in the proof of Proposition 7, when $\delta > \frac{n}{n+1}$ then Algorithm 1 produces exactly two profiles of strategic bliss points, the two profiles supporting SMPE in Proposition 7. From Proposition 2 we then know that exactly two SMPE in simple strategies exist.

Corollary 2. *Exactly two SMPE in simple proposal strategies exist in strongly symmetric $\mathcal{G}$ if $n = 5$ and $\delta > \frac{n}{n+1}$ or if $n \geq 7$ and $\delta \geq \bar{\delta}(n)$.*

Recalling Lemma 4, the duplicity of SMPE the corollary shows is non-generic. Each SMPE is supported by $\hat{\mathbf{x}}$ that can be approached by the sequence of unique profiles of strategic bliss points Algorithm 1 produces for perturbed $\mathcal{G}$. The following proposition shows that it is indeed possible to approach $\hat{\mathbf{x}}$ in such a way that, along the sequence, all profiles of strategic bliss points support an SMPE.

Proposition 8. *Assume $\mathcal{G}$ is strongly symmetric with $n = 5$ and $\delta > \frac{n}{n+1}$ or with $n \geq 7$ and $\delta \geq \bar{\delta}(n)$. Fix $\hat{\mathbf{x}}$ to be one of the two profiles of strategic bliss points from Algorithm 1. Then a perturbation of $\mathbf{x}$ by $\epsilon > 0$, $\mathbf{x}(\epsilon)$, and $\bar{\epsilon} > 0$ exist, such that $\lim_{\epsilon \to 0} \mathbf{x}(\epsilon) = \mathbf{x}$ and Algorithm 1 applied to $\mathcal{G}(\epsilon) = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X \rangle$, for $\forall \epsilon \leq \bar{\epsilon}$, produces a unique profile of strategic bliss points $\hat{\mathbf{x}}(\epsilon)$ that satisfies condition $\mathbb{N}$ and $\lim_{\epsilon \to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.*

Proof. See appendix A1

The proposition shows that for any strongly symmetric $\mathcal{G}$ with patient players, an essentially unique SMPE in simple strategies exists. Because $\lim_{\epsilon \to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$, we know that in the SMPE, all players with bliss points on one side of the median moderate while their opponents do not. The perturbation required for SMPE uniqueness is constructed in the proof of the proposition, but its structure is very simple. $\mathbf{x}(\epsilon)$ used to approach $\hat{\mathbf{x}}$ with $\hat{x}_i = x_i$ for $\forall i \in N_a \cup \{m\}$ is $x_i(\epsilon) = x_i$ for $\forall i \in N \setminus \{m-1\}$ and $x_{m-1}(\epsilon) = x_{m-1} - \epsilon$. If the most moderate player in N_b, $m-1$, has a stronger incentive to moderate than the most moderate player in N_a, $m+1$, in the unique SMPE in simple strategies all players in N_b moderate while none of the players in N_a do.

[27] Example 4, strongly symmetric $\mathcal{G}$ with $n = 7$, illustrates this gap. For $n = 7$, $\frac{n}{n+1} = 0.875$ and $\bar{\delta}(n) \approx 0.924$. When $\delta = 0.5$, eight profiles of strategic bliss points exist, each supporting SMPE by Proposition 5. When $\delta = 0.95$, two profiles of strategic bliss points exist, each supporting SMPE by Proposition 7.

5.1 Comparative Statics and Policy Dynamics

Given the SMPE characterization from Proposition 5 the comparative statics of change in the model parameters are almost immediate. To state the next proposition denote by $p(x|\sigma)$ the policy adopted in the period starting with the status-quo x when the profile of proposal strategies is σ. $p(x|\sigma)$ is random variable with realizations fully determined by the identity of the proposing player.

Proposition 9 (Comparative statics with pairwise moderation)**.** *Assume $\mathcal{G}$ induces pairwise moderation. Then, for any pair of profiles of strategic bliss points $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ produced by pairwise path through Algorithm 1 and the induced (SMPE) σ and σ' and $\forall x \in X$, $\mathbb{E}[d(p(x|\sigma))] = \mathbb{E}[d(p(x|\sigma'))]$. Moreover, if conditions $\mathbb{G}_1$ and $\mathbb{G}_2$ hold strictly, marginal impact of (symmetry of $\mathcal{G}$ preserving)*

1. *increase in δ,*
2. *increase in r_i compensated by decrease in r_m,*
3. *decrease in $d(x_i)$,*

on $\mathbb{E}[d(p(x|\sigma))]$ is non-positive.

Proof. See appendix A1

Proposition 9 implies that the average distance of $p(x|\sigma)$ from the bliss point of the median player is independent of the specific equilibrium from Proposition 5 considered. In addition, the proposition shows that a marginal increase in δ or r_i and a marginal decrease in $d(x_i)$ brings the policy proposed in any such equilibrium closer to the bliss point of the median player. The key driving force behind the result is the stronger incentive of all the players to moderate and propose policies closer to x_m. This manifests in the strategic bliss points moving (weakly) closer to x_m and is easily seen from the fact that $d(\hat{x}_i) = d(x_i)(1-2\delta r)$ where $r \in [0, \frac{1}{2})$ is the probability that Algorithm 1 used to compute $\hat{x}_i$.[28]

To describe the dynamics of the policies, denote by $\mathbf{p}(x|\sigma) = \{p_0, p_1, \ldots\}$ the path of policies generated by play according to SMPE σ starting with status-quo x, which we denote by p_{-1}. Depending on whether we view $\mathbf{p}(x|\sigma)$ as generated by deterministic sequence of proposers or not, it is a sequence of numbers or of random variables.

[28] Proposition 9 requires conditions $\mathbb{G}_1$ and $\mathbb{G}_2$ to hold strictly in order to ensure that marginal change of the model parameters preserves pairwise moderation inducing $\mathcal{G}$.

Proposition 10 (Policy dynamics with pairwise moderation)**.** *Assume $\mathcal{G}$ induces pairwise moderation. Then, for any profile of strategic bliss points $\hat{\mathbf{x}}$ produced by pairwise path through Algorithm 1 and induced (SMPE) σ, for $\forall x \in X$ and $\forall t \in \{0, 1, \ldots,\}$, viewing $\mathbf{p}(x|\sigma) = \{p_0, p_1, \ldots\}$ as deterministic*

1. $d(p_t) \leq d(p_{t-1})$,

2. *either* $d(p_t) = d(\hat{x}_i)$ *for some* $i \in N$ *or* $d(p_t) = d(p_{t-1})$,

and viewing $\mathbf{p}(x|\sigma) = \{p_0, p_1, \ldots\}$ as sequence of random variables

3. $\mathbb{P}[d(p_t) > 0] = (1 - r_m)^{t+1}$ *if* $x \neq x_m$,

4. $\mathbb{P}[d(p_t) = d(p_{t-1})]$ *is non-decreasing in* t,

5. $\mathbb{P}[p_t > x_m | p_{t-1} \neq x_m] = \mathbb{P}[p_t < x_m | p_{t-1} \neq x_m] = r_a$.

Proof. See appendix A1

Thus, Proposition 10 says that over time adopted policies move closer to the bliss point of the median player x_m. In every period, p_t is either equal to the strategic bliss point of some player, or its distance from x_m equals the distance of the status-quo policy from x_m. For p_t to stay away from x_m only non-median players have to be proposing in all periods up to t, which happens with probability $(1 - r_m)^{t+1}$. The expected number of periods it takes for p_t to reach x_m starting from $x \neq x_m$ is $\sum_{t=1}^{\infty} t \cdot (1 - r_m)^{t-1} r_m = \frac{1}{r_m}$. That is, the policies in the model converge to x_m, but the expected length of the convergence phase can be arbitrarily long.

Part 4 of the proposition says that convergence of p_t slows down over time. With the status-quo policy approaching x_m, an increasing number of players is constrained by the acceptance of the median player, they cannot propose their strategic bliss point and propose, in period t, $d_b(p_{t-1})$ or $d_a(p_{t-1})$ instead. Slower convergence, however, does not mean p_t does not vary in time. In fact, as long as the status-quo policy differs from x_m, p_t is as likely to be above x_m as it is likely to be below. These fluctuations around x_m are result of players in N_a replacing players in N_b, or vice versa, in the proposer role.

6 Equilibrium Existence in Three-player Games

The goal of this section is to study in more detail equilibria in games with three players. We construct an SMPE for any $\mathcal{G}$ with $n = 3$ and arbitrary $\mathbf{r}$ and $\mathbf{x}$. The construction relies heavily on the simple proposal strategies with strategic bliss points from Algorithm 1, possibly with a slight adjustment. Throughout the section, let us, if $d(x_1) \neq d(x_3)$, define $e \in \{1,3\}$ to be the more extreme player and $-e = \{1,3\} \setminus \{e\}$ to be the less extreme player, such that $d(x_e) > d(x_{-e})$.

Definition 9 (Adjusted simple proposal strategies)**.** *The adjusted simple pure stationary Markov proposal strategy of $i \in N$ is*

$$p_i^a(x|\hat{x}_i, \vec{x}) = \begin{cases} p_i(x|\hat{x}_i) & \text{if } x \in [d_b(\vec{x}), d_a(\vec{x})] \\ p_i(x|x_i) & \text{if } x \notin [d_b(\vec{x}), d_a(\vec{x})] \end{cases}$$

where $\hat{x}_i$ is the strategic bliss point of i and $\vec{x}$ is called the point of adjustment. Adjusted simple strategy of $i \in N$ is denoted by $\vec{\sigma}_i = (\hat{x}_i, \vec{x})$.

Figure 2: Adjusted simple strategies

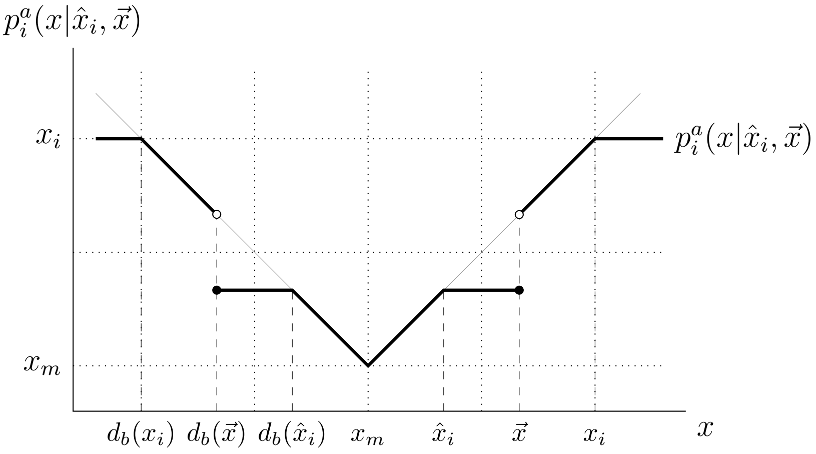


Figure 2 illustrates the adjusted simple proposal strategies from Definition 9. These strategies resemble the unadjusted ones except at $\vec{x}$, i switches from proposing policy $\hat{x}_i$ to proposing policy $d_a(x)$.[29] The adjustment is necessary for the SMPE construction in the case when the strategic bliss point from Algorithm 1 of e satisfies $d(\hat{x}_e) < d(x_e)$. This implies that $\hat{x}_{-e} = x_{-e}$ and is due to the fact that even though e is the more extreme player in terms of the distance of her bliss point from x_m, the recognition probability of $-e$ is large enough for e to have an incentive to moderate to a larger extent.

This in turn implies $\mathcal{S}_e \neq \emptyset$ as $d(\hat{x}_e) < d(x_{-e}) < d(x_e)$. Thus, moving x away from x_m, the first player to switch to the constant part of her strategy is e, $d(\hat{x}_e)$ far from x_m, and

[29] The figure is drawn for $i \in N_a$. If $i \in N_b$ the switch is to proposing policy $d_b(x)$.

the second player to switch is $-e$, $d(x_{-e})$ far from x_m. At this point the continuation value functions of all the players become constant and the dynamic utilities inherit the shape of the stage utilities. Moving x further away from x_m toward x_e, U_e increases, implying failure of condition $\mathbb{S}$, and might reach x_a such that $U_e(\hat{x}_e|\sigma') = U_e(x_a|\sigma')$ where σ' is induced by $\hat{\mathbf{x}} = \{\hat{x}_e, \hat{x}_2, \hat{x}_{-e}\}$. Any further increase in $U_e(x|\sigma')$ then implies that σ' cannot constitute an SMPE due to the failure of condition $\mathbb{N}$.

However, if we adjust the simple strategy of e, $\hat{x}_e$, and allow her to switch, at x_a, from proposing $\hat{x}_e$ to proposing $d_a(x_a)$ if $e \in N_a$ or to $d_b(x_a)$ if $e \in N_b$, the resulting $\vec{\sigma}_e = (\hat{x}_e, x_a)$ will be the best response to the proposal strategies of the other players. That the profile of strategies $\sigma'' = (\vec{\sigma}_e, \hat{x}_2, \hat{x}_{-e})$ generated by replacing strategy of e in σ' is an SMPE is related to a longer argument left for proofs of the propositions below. Heuristically, the jump in the policy e proposes further away from x_m induces downward jumps in the dynamic utilities of m and $-e$. For m, this has no impact on either her optimal proposal strategy or on the $\mathcal{A}$ generated by her voting strategy. For $-e$, for status-quo x_a she is on the constant part of her strategy proposing $\hat{x}_{-e} = x_{-e}$ as $d(x_{-e}) < d(x_a) < d(x_e)$. The downward jump in U_{-e} then only reinforces the optimality of $\hat{x}_{-e}$. Notice also that because x_a is defined by $U_e(\hat{x}_e|\sigma') = U_e(x_a|\sigma')$, it is intuitive that σ'' will give rise to a continuous U_e, despite the discontinuity in the proposal strategy of e. What remains is to specify the exact location of the point of adjustment x_a.

Definition 10 (Point of adjustment)**.** *For $\mathcal{G}$ with $n = 3$ and $d(x_1) \neq d(x_3)$ define point of adjustment x_a as*

$$x_a = \begin{cases} x_e + (m-e)\sqrt{4\delta r_{-e} d(x_e)^2 - \frac{\delta r_{-e}}{1-\delta r_{-e}}(d(x_e) + d(x_{-e}))^2} & \text{if } \delta r_{-e} < \frac{1}{2} \\ x_e + (m-e)\sqrt{\frac{1}{1-\delta r_{-e}} d(x_e)^2 - \frac{\delta r_{-e}}{1-\delta r_{-e}}(d(x_e) + d(x_{-e}))^2} & \text{if } \delta r_{-e} \geq \frac{1}{2} \end{cases}$$

and note $x_a \in \mathbb{C}$, $x_a = x_e$ and $d(x_a) < d(x_e)$ as $d(x_e) < d(x_{-e})T_e$, $d(x_e) = d(x_{-e})T_e$ and $d(x_e) > d(x_{-e})T_e$ respectively, where

$$T_e = \begin{cases} \frac{1}{2\sqrt{1-\delta r_{-e}} - 1} & \text{if } \delta r_{-e} < \frac{1}{2} \\ \frac{\sqrt{\delta r_{-e}}}{1-\sqrt{\delta r_{-e}}} & \text{if } \delta r_{-e} \geq \frac{1}{2}. \end{cases}$$

We explained above that the need for the adjusted simple proposal strategies arises in cases when $-e$ is very likely to propose, which creates strong incentives for e to moderate. When e is the player who is more likely to propose, then Algorithm 1 produces $\hat{x}_e = x_e$

as $-e$ has stronger incentive to moderate relative to e, due to both $d(x_e) > d(x_{-e})$ and $r_e > r_{-e}$. In this case $\hat{\mathbf{x}}$ from Algorithm 1 induces SMPE σ without need for further adjustments. A similar lack of complications arises when $d(x_1) = d(x_3)$ as the incentives to moderate are determined purely by r_1 and r_3. The following definition formalizes when the need for adjustment arises and allows us to state the two propositions below.

Definition 11 (Condition $\mathbb{E}$)**.** *$\mathcal{G}$ with $n = 3$ satisfies condition $\mathbb{E}$ if and only if, whenever $\mathcal{A}_e$ holds, then $\mathcal{B}_e$ holds, where*

$$\begin{aligned} &\mathcal{A}_e : d(x_1) \neq d(x_3) \wedge d(x_e)(1 - 2\delta r_{-e}) \leq d(x_{-e})(1 - 2\delta r_e) \\ &\mathcal{B}_e : d(x_e) \leq d(x_{-e})T_e. \end{aligned} \tag{$\mathbb{E}$}$$

Proposition 11. *Assume condition $\mathbb{E}$ holds in $\mathcal{G}$ with $n = 3$. Then*

1. *an SMPE in simple proposal strategies exists with $\hat{\mathbf{x}}$ produced by Algorithm 1;*
2. *an SMPE in adjusted simple proposal strategies exists if and only if, in condition $\mathbb{E}$, $\mathcal{A}_e$ holds and $\mathcal{B}_e$ holds with equality; it is characterized by $\hat{\mathbf{x}}$ from Algorithm 1 (dropping e in step 1, if given an option) and $\vec{\sigma}_e = (\hat{x}_e, x_e)$;*
3. *if and only if $d(x_1) = d(x_3)$ or $d(x_e)(1 - 2\delta r_{-e}) \geq d(x_{-e})(1 - 2\delta r_e)$, $\hat{\mathbf{x}}$ produced by Algorithm 1 (dropping $-e$ in step 1, if given an option) induces U_1 that is single peaked on $\{x \in X | x \leq x_m\}$ (on X if $\delta r_1 \leq \frac{1}{2}$) and U_3 that is single peaked on $\{x \in X | x \geq x_m\}$ (on X if $\delta r_3 \leq \frac{1}{2}$).*

Proof. See appendix A1

Proposition 12. *Assume condition $\mathbb{E}$ fails in $\mathcal{G}$ with $n = 3$. Then*

1. *an SMPE in adjusted simple proposal strategies exists with $\hat{\mathbf{x}}$ from Algorithm 1 (dropping e in step 1, if given an option) and $\vec{\sigma}_e = (\hat{x}_e, x_a)$;*
2. *an SMPE in simple proposal strategies exists if and only if $d(x_1)(1-2\delta r_3) = d(x_3)(1-2\delta r_1)$; it is characterized by $\hat{\mathbf{x}}$ from Algorithm 1 (dropping $-e$ in step 1).*

Proof. See appendix A1

Parts 1 of the two propositions jointly imply the existence of an SMPE for any three-player $\mathcal{G}$. It is constructed using either the simple strategies or their adjusted version if necessary.

Corollary 3. *An SMPE exists in $\mathcal{G}$ with $n = 3$.*

We know from Proposition 2 that whenever an SMPE in simple strategies exists, Algorithm 1 produces the profile of strategic bliss points that supports it. For $\mathcal{G}$ with $n = 3$, Algorithm 1 produces two distinct profiles of strategic bliss points if and only if $\delta \in (0, 1)$ and $d(x_1)(1-2\delta r_3) = d(x_3)(1-2\delta r_1)$. If in addition condition $\mathbb{E}$ holds, two SMPE exist in simple strategies. Failure of any of these three conditions implies that an SMPE in simple strategies is either unique or fails to exist. Because $d(x_1)(1 - 2\delta r_3) = d(x_3)(1 - 2\delta r_1)$ fails upon perturbation of $\mathbf{x}$ or $\mathbf{r}$, the multiplicity of SMPE in simple strategies is non-generic.

Corollary 4. *If it exists, an SMPE in simple proposal strategies is essentially unique in $\mathcal{G}$ with $n = 3$.*

7 Equilibrium Construction with $X \subseteq \mathbb{R}^{n'}$

This section extends the model to policy spaces of multiple dimensions. The policy space is $X \subseteq \mathbb{R}^{n'}$. Any element of X, a policy $\vec{p}$, a status-quo $\vec{x}$ or an i's bliss point $\vec{x}_i$, is a vector in $\mathbb{R}^{n'}$ with components denoted by superscripts, such that $\vec{x} = (x^1, \ldots, x^{n'}) \in X$. When $X \subsetneq \mathbb{R}^{n'}$, then we require X to be the Cartesian product $X = \bigtimes_{j=1}^{n'} X_j$ where each $X_j \subseteq \mathbb{R}$ is a closed convex interval that is symmetric around x_m^j (m defined below) and includes both $\min_{i\in N} \{x_i^j\}$ and $\max_{i\in N} \{x_i^j\}$. The stage utility of $i \in N$ from policy $\vec{p}$ is $u_i(\vec{p}) = -\sum_{j=1}^{n'}(p^j - x_i^j)^2$ where x_i^j is the most preferred policy of i on dimension j. Using $||\cdot||$ to denote Euclidean norm (distance), $u_i(\vec{p}) = -||\vec{p} - \vec{x}_i||^2$.[30]

Denote by m the player with bliss point $\vec{x}_m$ in the majority core. In order for the majority core to exist we assume that the Plott (1967) condition holds. As is well known, for an odd number of players this condition is both sufficient and necessary (Austen-Smith and Banks, 2000) for the core existence and implies that it consists of a single alternative, $\vec{x}_m$. The Plott (1967) condition states that for any $i \in N \setminus \{m\}$, $i^r \in N \setminus \{m, i\}$ exists such that $\alpha\vec{x}_i + (1 - \alpha)\vec{x}_{i^r} = \vec{x}_m$ for some $\alpha \in (0, 1)$. That is, for any player, another player exists such that the line connecting their bliss points passes through $\vec{x}_m$. This special arrangement of bliss points is also called *radial symmetry* and that is why, for any $i \in N \setminus \{m\}$, we denote by $i^r \in N \setminus \{m, i\}$ the player with a bliss point on the line connecting $\vec{x}_i$ and $\vec{x}_m$. For simplicity, we assume that exactly three players, i, m and i^r, lie on each such line and, without loss of generality, we set $\vec{x}_m$ as an origin of X

[30] The rest of the model extends naturally and we do not (re)define the proposal strategies, value functions, dynamic utilities, social acceptance correspondence and SMPE due to space considerations. We continue to use $\mathbf{x} = \{\vec{x}_1, \ldots, \vec{x}_n\}$ for the profile of bliss points and $\hat{\mathbf{x}}$ for the profile of strategic bliss points as well as $\mathcal{G} = \langle n, \mathbf{x}, \mathbf{r}, \delta, X\rangle$.

such that $\vec{x}_m = (0, \ldots, 0) = \mathbf{0}$.[31] For any $i \in N \setminus \{m\}$ and $j \in N \setminus \{m\}$, we denote by $\cos(i,j) = \frac{\vec{x}_i' \vec{x}_j}{||\vec{x}_i|| \cdot ||\vec{x}_j||}$ angle between $\vec{x}_i$ and $\vec{x}_j$ (on the plane determined by $\vec{x}_i$, $\vec{x}_j$ and $\vec{x}_m$).

Definition 12 (Orthogonal strongly symmetric $\mathcal{G}$)**.** *$\mathcal{G}$ is orthogonal strongly symmetric if and only if $r_i = \frac{1}{n}$ for $\forall i \in N$, $||\vec{x}_i|| = b > 0$ for $\forall i \in N \setminus \{m\}$ and $cos(i,j) = 0$ for $\forall i \in N \setminus \{m\}$ and $\forall j \in N \setminus \{i, i^r, m\}$.*

Definition 13 (Equiangular $\mathcal{G}$ on circle)**.** *$\mathcal{G}$ is equiangular on a circle if and only if $r_i = \frac{1}{n}$ for $\forall i \in N$, $||\vec{x}_i|| = b > 0$ for $\forall i \in N \setminus \{m\}$, $\vec{x}_1 = (b, 0)$ and $\cos(i, 1) = \cos((i-1)\alpha)$ for $\forall i \in N \setminus \{n\}$ where $\alpha = \frac{2\pi}{n-1}$.*

7.1 Simple Strategies, Strategic Bliss Points

The dynamic median voter theorem from Proposition 1 extends to multi-dimensional policy space and again implies that the social acceptance sets $\mathcal{A}$ are determined by the median's expected utility.

Proposition 13 (Dynamic median voter theorem for $X \subseteq \mathbb{R}^{n'}$)**.**
For any profile of pure stationary Markov strategies $\hat{\sigma}$, with implied voting such that, for $\forall i \in N$, $i \in N$ votes for proposed $\vec{p} \in X$ against the status-quo $\vec{x} \in X$ if and only if $U_i(\vec{p}\,|\hat{\sigma}) \geq U_i(\vec{x}\,|\hat{\sigma})$, $\vec{p}$ is accepted if and only if $U_m(\vec{p}\,|\hat{\sigma}) \geq U_m(\vec{x}\,|\hat{\sigma})$.

Proof. See appendix A1

Definition 14 (Simple proposal strategies)**.** *The simple pure stationary Markov proposal strategy of $i \in N$ is*

$$\vec{p}_i(\vec{x}\,|\hat{k}_i) = \vec{x}_i \cdot \min\left\{\hat{k}_i, \frac{||\vec{x}||}{||\vec{x}_i||}\right\}$$

where $\hat{k}_i\vec{x}_i$ is the strategic bliss point of i with $\hat{k}_i \geq 0$.

With a strategic bliss point of i, $\hat{k}_i\vec{x}_i$, fully determined by $\hat{k}_i$, we also call $\hat{k}_i$ a strategic bliss point since no confusion arises. The profile of strategic bliss points then refers to $\hat{\mathbf{x}} = \{\hat{k}_1\vec{x}_1, \ldots, \hat{k}_n\vec{x}_n\}$ or $\hat{\mathbf{k}} = \{\hat{k}_1, \ldots, \hat{k}_n\}$. Given $\hat{\mathbf{x}}$ or $\hat{\mathbf{k}}$ the profile of simple proposal (and implied voting) strategies is $\sigma = (\vec{p}_1, \ldots, \vec{p}_n)$. Since $\vec{p}_i$ is fully determined by $\hat{k}_i\vec{x}_i$ or $\hat{k}_i$, we also call $\hat{k}_i\vec{x}_i$ or $\hat{k}_i$ the proposal strategy of i and $\hat{\mathbf{x}}$ or $\hat{\mathbf{k}}$ the profile of strategies.

The simple strategies in $\mathbb{R}^{n'}$ are analogous to the simple strategies in $\mathbb{R}$. For any status-quo $\vec{x}$ close to the bliss point of the median player, $\vec{x}_m = \mathbf{0}$, player i proposes a

[31] The model is shift and rotation invariant, hence the normalization $\vec{x}_m = \mathbf{0}$. By the same argument, setting $\vec{x}_1$ to lie on the coordinate axis of $\mathbb{R}^2$ in the examples below entails no loss of generality.

policy on the ray starting at $\vec{x}_m$ and passing through $\vec{x}_i$, i-ray for short. The distance between the proposed policy and $\vec{x}_m$ is equal to the distance between the status-quo $\vec{x}$ and $\vec{x}_m$. For any status-quo $\vec{x}$ far away from $\vec{x}_m$, player i still proposes a policy on the i-ray, but at distance $\hat{k}_i||\vec{x}_i||$ from $\vec{x}_m$. From Definition 14, in this case $\hat{k}_i||\vec{x}_i|| \leq ||\vec{x}||$. That is, player i moderates and proposes $\hat{k}_i\vec{x}_i$ instead of proposing $\vec{x}_i\frac{||\vec{x}||}{||\vec{x}_i||}$, which would be a policy at distance $||\vec{x}||$ from $\vec{x}_m$. The strategic bliss point $\hat{k}_i$ is then relative to $||\vec{x}_i||$ distance at which i switches from proposing $\vec{x}_i\frac{||\vec{x}||}{||\vec{x}_i||}$ to proposing $\hat{k}_i\vec{x}_i$, the distance of status-quo at which i starts moderating.

Given $\hat{\mathbf{k}}$ and induced σ we define several objects required in the analysis below. By $\mathcal{ND}(\sigma) = \{0, \hat{k}_1||\vec{x}_1||, \ldots, \hat{k}_n||\vec{x}_n||\}$ we denote the set of distances such that, for any $x \in \mathcal{ND}(\sigma)$, there exists at least one $\vec{p}_i$ that is not differentiable, along the i-ray, with respect to x at x.[32] $\mathcal{D}(\sigma) = \mathbb{R}_{\geq 0} \setminus \mathcal{ND}(\sigma)$ denotes the complement of $\mathcal{ND}(\sigma)$, the set of distances such that all the strategies are differentiable. For $i \in N \setminus \{m\}$ the set of elements in $\mathcal{ND}(\sigma)$ rescaled by $||\vec{x}_i||$ is $\mathcal{ND}_i(\sigma) = \{x/||\vec{x}_i||\, | x \in \mathcal{ND}(\sigma)\}$.

Denote by $\vec{p}_i'(x|\hat{k}_i) = \frac{\partial}{\partial x}\left[\vec{p}_i(x\frac{\vec{x}_i}{||\vec{x}_i||}|\hat{k}_i)\right]$ the derivative of $\vec{p}_i$ along the i-ray and note that $\vec{p}_i'(x|\hat{k}_i) \neq 0$ for $x \in (0, \hat{k}_i||\vec{x}_i||)$ and $\vec{p}_i'(x|\hat{k}_i) = 0$ for $x > \hat{k}_i||\vec{x}_i||$. When $i = m$, there is no i-ray and, as a convention, we choose an arbitrary i-ray with $i \in N \setminus \{m\}$, which implies $\vec{p}_m'(x|\hat{k}_m) = 0$.[33] For $\forall x \in \mathcal{D}(\sigma)$, define $\mathcal{C}(x|\sigma) = \{i \in N|\vec{p}_i'(x|\hat{k}_i) = 0\}$ and $\mathcal{NC}(x|\sigma) = \{i \in N|\vec{p}_i'(x|\hat{k}_i) \neq 0\}$. $\mathcal{C}(x|\sigma)$ and $\mathcal{NC}(x|\sigma)$ are sets of players who, at distance x from the origin, are on the constant and the non-constant part of $\vec{p}_i$ (judging by its derivative) respectively. Naturally, $\mathcal{C}(x|\sigma) \cup \mathcal{NC}(x|\sigma) = N$ for $\forall x \in \mathcal{D}(\sigma)$. Despite $\mathcal{C}$ being a correspondence, define its one-sided limits $\mathcal{C}(x^+|\sigma) = \{i \in N|\vec{p}_i'(x^+|\hat{k}_i) = 0\}$, for $\forall x \in \mathcal{ND}(\sigma)$, and $\mathcal{C}(x^-|\sigma) = \{i \in N|\vec{p}_i'(x^-|\hat{k}_i) = 0\}$, for $\forall x \in \mathcal{ND}(\sigma) \setminus \{0\}$. One-sided limits of $\mathcal{NC}(x|\sigma)$, $\mathcal{NC}(x^-|\sigma)$ and $\mathcal{NC}(x^+|\sigma)$ are defined similarly.[34] For $i \in N \setminus \{m\}$, define $\mathcal{NC}_i(x|\sigma) = \mathcal{NC}(x||\vec{x}_i||\,|\sigma)$ for any $x \geq 0$ such that $x||\vec{x}_i|| \in \mathcal{D}(\sigma)$. One-sided limits of $\mathcal{NC}_i(x|\sigma)$, $\mathcal{NC}_i(x^-|\sigma)$ at any $x > 0$ and $\mathcal{NC}_i(x^+|\sigma)$ at any $x \geq 0$, are defined using one-sided limits of $\mathcal{NC}$.[35]

[32] This is not entirely precise. If $\hat{\mathbf{k}} = \mathbf{0}$ all $\vec{p}_i$ are constant and hence differentiable on X. $\mathcal{ND}(\sigma)$ should be understood as the set of distances at which some $\vec{p}_i$ *might not* be differentiable along the i-ray. We are concerned with taking derivatives when these do not exist, so this is a mere imprecision in the label for $\mathcal{ND}(\sigma)$.

[33] To avoid unnecessary repetition and because there is only a minimal chance of confusion, we use a similar convention for any expression involving expansion or derivative of U_i or V_i along the i-ray when $i = m$. It is taken to mean expansion or derivatation along an arbitrary i-ray with $i \in N \setminus \{m\}$, i.e., $U_m(k\vec{x}_i)$ or $V_m(k\vec{x}_i)$ as k varies or derivatation with respect to it.

[34] $\mathcal{NC}$ and $\mathcal{C}$ are both piecewise 'constant' on intervals determined by $\mathcal{ND}(\sigma)$ and hence, for $\forall x \in \mathcal{D}(\sigma)$, $\mathcal{C}(x|\sigma) = \mathcal{C}(x^+|\sigma) = \mathcal{C}(x^-|\sigma)$ and $\mathcal{NC}(x|\sigma) = \mathcal{NC}(x^+|\sigma) = \mathcal{NC}(x^-|\sigma)$.

[35] The difference between $\mathcal{NC}$ and $\mathcal{NC}_i$ is their domain. The former has distance as its domain, the latter has distance relative to $||\vec{x}_i||$ as its domain.

For $\forall x \in \mathcal{D}(\sigma)$ define $r_{nc}(x|\sigma) = \sum_{i \in \mathcal{NC}(x|\sigma)} r_i$ to be the sum of the recognition probabilities of players on the non-constant part of their strategy, at distance x from the origin. $r_{nc}(x|\sigma)$ is undefined at $x \in \mathcal{ND}(\sigma)$ but possesses one-sided limits at these points (defined using one-sided limits of $\mathcal{NC}$).[36]

Finally, for $\forall i \in N \setminus \{m\}$ define the (possibly empty) sets

$$\begin{aligned}
&\mathcal{S}_i(\sigma) = \mathcal{ND}_i(\sigma) \cap (\hat{k}_i, 1) \\
&\mathcal{L}_i(\sigma) = \{k \geq 0 | \tfrac{\partial}{\partial k} [U_i(k\vec{x}_i|\sigma)] = 0 \wedge k||\vec{x}_i|| \in \mathcal{D}(\sigma)\} \\
&\mathcal{N}_i(\sigma) = ((\mathcal{ND}_i(\sigma) \cup \mathcal{L}_i(\sigma)) \cap (\hat{k}_i, 1)) \cup \{\hat{k}_i, 1\}
\end{aligned} \tag{7}$$

with elements of $\mathcal{N}_i(\sigma)$ ordered in increasing order. $\mathcal{S}_i(\sigma)$ is the set of points in the $(\hat{k}_i, 1)$ interval at which $\vec{p}_j$ is not differentiable, along j-ray, for some $j \in N$. $\mathcal{N}_i(\sigma)$ is a similar set of points adding points of local maxima of $U_i(\sigma)$ along the i-ray, $\mathcal{L}_i(\sigma)$, and $\{\hat{k}_i, 1\}$. We are well aware that all $\mathcal{ND}$, $\mathcal{ND}_i$, $\mathcal{D}$, $\mathcal{C}$, $\mathcal{NC}$, $\mathcal{NC}_i$, r_{nc}, $\mathcal{S}_i$, $\mathcal{L}_i$ and $\mathcal{N}_i$ are defined relative to $\hat{\mathbf{k}}$ and hence relative to σ. We suppress the dependence of these objects on σ when confusion cannot arise.

Lemma 8 (Properties of V_i and U_i induced by $\hat{\mathbf{k}}$)**.** *For any $\hat{\mathbf{k}}$ with $\hat{k}_i \geq 0$ for $\forall i \in N \setminus \{m\}$ and $\hat{k}_m = 0$ and induced profile of simple strategies σ, for $\forall i \in N$,*

1. *$V_i(\vec{x}\,|\sigma) = V_i(\vec{y}\,|\sigma)$ for $\forall \vec{x} \in X$ and $\forall \vec{y} \in X$ with $||\vec{x}|| = ||\vec{y}||$;*
2. *$U_i(k\vec{x}_i\,|\sigma) > U_i(\vec{y}\,|\sigma)$, if $i \in N \setminus \{m\}$, for any $k \geq 0$ and $\vec{y} \in X$ such that $k||\vec{x}_i|| = ||\vec{y}||$ but $k\vec{x}_i \neq \vec{y}$;*
3. *U_i is continuous on X;*
4. *$\frac{\partial^2}{\partial^2 k} [U_i(k\vec{x}_i\,|\sigma)] < 0$ for $\forall k \geq 0$ such that $k||\vec{x}_i|| \in \mathcal{D}(\sigma)$;*
5. *$U_m(\vec{x}\,|\sigma) > U_m(\vec{y}\,|\sigma)$ for $\forall \vec{x} \in X$, $\forall \vec{y} \in X$ such that $||\vec{x}|| < ||\vec{y}||$;*
6. *$\mathcal{A}(\vec{x}\,|\sigma) = \{\vec{p} \in X|\, ||\vec{p}|| \leq ||\vec{x}||\}$ for $\forall \vec{x} \in X$.*

Proof. See appendix A1

Lemma 8 is the close analog of Lemma 2. Its most important implication is the shape of the social acceptance correspondence. For any status-quo $\vec{x} \in X$, the set of accepted policies, when proposed, is the set of policies weakly closer to $\vec{x}_m$ relative to $\vec{x}$. As a result,

[36] For any profile of strategic bliss points $\hat{\mathbf{k}}$ and σ it induces, because players are switching from the non-constant to the constant part of their strategy with increasing $x \in \mathbb{R}_{\geq 0}$, for any $x \in \mathcal{D}(\sigma)$ and $y \in \mathcal{D}(\sigma)$ such that $x \leq y$, $\mathcal{NC}(y|\sigma) \subseteq \mathcal{NC}(x|\sigma)$ and hence $r_{nc}(y|\sigma) \leq r_{nc}(x|\sigma)$.

any proposal generated by a simple strategy based on $\hat{\mathbf{k}}$ that satisfies the requirement of the lemma belongs to the social acceptance set induced by $\hat{\mathbf{k}}$. Furthermore, part 2 of the lemma implies that any dynamic utility maximizing policy, for player i, must lie on the i-ray. This is a consequence of the value functions being constant on the hypersphere of the given radius and the stage utility, on the same hypersphere, having the maximum on the i-ray. The final element needed in the construction is to determine the strategic bliss points. This is what Algorithm 2 does.

Algorithm 2 (Strategic bliss points with $X \subseteq \mathbb{R}^{n'}$)**.**

step 0 *Set* $\hat{k}_m = 0$ *and* $\mathbb{P}_1 = N \setminus \{m\}$

step t *For* $i \in \mathbb{P}_t$ *compute*

$$\hat{k}_{i,t} = 1 - \delta \sum_{j \in \mathbb{P}_t} r_j [1 - \cos(i, j)]$$

Define $\mathbb{R}_t = \{i \in \mathbb{P}_t | \hat{k}_{i,t} \leq 0\}$
If $\mathbb{R}_t = \emptyset$*, select one* $j \in \arg\min_{i \in \mathbb{P}_t} \hat{k}_{i,t} ||\vec{x}_i||$*, set* $\hat{k}_j = \hat{k}_{j,t}$
If $\mathbb{R}_t \neq \emptyset$*, select one* $j \in \mathbb{R}_t$*, set* $\hat{k}_j = 0$
Set $\mathbb{P}_{t+1} = \mathbb{P}_t \setminus \{j\}$ *and if* $\mathbb{P}_{t+1} \neq \emptyset$*, proceed to step* $t + 1$

The way in which Algorithm 2 derives the strategic bliss points is closely related to Algorithm 1. With a one-dimensional policy space the opponents of player i are players with bliss points on the opposite side of the median's bliss point. From Algorithm 1, the strength of player i's incentive to moderate, driven by the presence of her opponents, is $2\delta r$ where r is the probability of recognition of the opponents. With a multi-dimensional policy space, players other than player i are her opponents to a certain degree, which is captured by the term $[1 - \cos(i, j)]$. For i^r, the strength of i's incentive to moderate is $2\delta r_{i^r}$ as $[1 - cos(i, j)] = [1 - \cos \pi] = 2$. Players with bliss points orthogonally located relative to $\vec{x}_i$ add half as much to the incentive to moderate as $[1 - \cos(i, j)] = [1 - \cos \frac{\pi}{2}] = 1$. Finally, players on the same i-ray, namely i herself, add nothing to the incentive to moderate as $[1 - \cos(i, j)] = [1 - \cos 0] = 0$.

Example 9 (Simplest example in $\mathbb{R}^2$)**.** *Consider* $\mathcal{G}$ *with* $n = 5$*,* $r_i = \frac{1}{n}$ *for* $\forall i \in N$*,* $\delta = 0.9$ *and the following bliss points*

player	*1*	*2*	*3*	*4*	*5*
x_i^1	*2*	*-2*	*0*	*0*	*0*
x_i^2	*0*	*0*	*2*	*-2*	*0*

In step 1 the algorithm computes $\hat{k}_{i,1} = 0.28$ for $i \in \{1, \ldots, 4\}$. Dropping player 1, in step 2 the algorithm computes $\hat{k}_{i,2} = 0.46$ for $i \in \{3, 4\}$. Dropping player 3, in step 3 the algorithm computes $\hat{k}_{i,3} = 0.82$ for $i \in \{2, 4\}$. Finally, dropping player 2, in step 4 the algorithm computes $\hat{k}_{i,4} = 1$ for $i \in \{4\}$. The selection regarding which players to drop produces

player	*1*	*2*	*3*	*4*	*5*
$\hat{k}_i$	*0.28*	*0.82*	*0.46*	*1*	*0*

The algorithm allowed for four players to be dropped in step 1 and two in steps 2 and 3. Since the number of alternatives in steps 2 and 3 does not depend on the selection in the earlier steps, there are $4 \cdot 2 \cdot 2 = 16$ different profiles of strategic bliss points the algorithm can produce.

7.2 Necessary and Sufficient Conditions

Any profile of strategic bliss points $\hat{\mathbf{k}}$ from Algorithm 2 induces profile of strategies σ. To check that σ constitutes an SMPE we define the following two conditions analogous to conditions $\mathbb{S}$ and $\mathbb{N}$ from the one-dimensional model.

Definition 15 (Condition $\mathbb{S}'$, sufficient)**.** *A profile of strategic bliss points $\hat{\mathbf{k}}$ from Algorithm 2 that induces σ satisfies condition $\mathbb{S}'$ if and only if, for $\forall i \in N \setminus \{m\}$ and $\forall x \in \mathcal{S}_i(\sigma)$,*

$$1 - x - \delta \sum_{j \in \mathcal{NC}_i(x^+|\sigma)} r_j[1 - \cos(i, j)] \leq 0. \tag{$\mathbb{S}'$}$$

Definition 16 (Condition $\mathbb{N}'$, necessary and sufficient)**.** *A profile of strategic bliss points $\hat{\mathbf{k}}$ from Algorithm 2 that induces σ satisfies condition $\mathbb{N}'$ if and only if, for $\forall i \in N \setminus \{m\}$ and denoting elements of $\mathcal{N}_i(\sigma)$ by $\{z_0, z_1, \ldots\}$,*

$$\sum_{j=1}^{J} \Big[T_i(x|\sigma)\Big]_{z_j^-}^{z_{j-1}^+} \geq 0 \text{ for } \forall J \in \{1, \ldots, |\mathcal{N}_i(\sigma)| - 1\} \tag{$\mathbb{N}'$}$$

where

$$T_i(x|\sigma) = -\frac{2||\vec{x}_i||^2}{1 - \delta \sum_{j \in \mathcal{NC}_i(x|\sigma)} r_j} \left[\frac{x^2}{2} - c_i(x|\sigma)x\right]$$
$$c_i(x|\sigma) = 1 - \delta \sum_{j \in \mathcal{NC}_i(x|\sigma)} r_j[1 - \cos(i, j)].$$

Proposition 14 (SMPE under $\mathbb{S}'$ and $\mathbb{N}'$ conditions)**.** *A profile of strategic bliss points $\hat{\mathbf{k}}$ from Algorithm 2 induces SMPE σ*

1. if $\hat{\mathbf{k}}$ *satisfies condition* $\mathbb{S}'$*;*

2. if and only if $\hat{\mathbf{k}}$ *satisfies condition* $\mathbb{N}'$*.*

Proof. See appendix A1

The reason both conditions $\mathbb{S}'$ and $\mathbb{N}'$ guarantee that the profile of strategies σ induced by $\hat{\mathbf{k}}$ constitutes an SMPE is analogous to the one-dimensional model. By Lemma 8 it is sufficient to focus on the shape of the dynamic utility of player i along the i-ray, that is on $U_i(k\vec{x}_i\,|\sigma)$ as $k \geq 0$ varies. Condition $\mathbb{S}'$ then checks that at any point in $(\hat{k}_i, 1)$ where U_i is not differentiable, the right derivative of U_i is non-positive. By piecewise strict concavity of U_i this implies that U_i is decreasing as a function of k on $(\hat{k}_i, 1)$. The best response of player i is then to propose $\hat{k}_i\vec{x}_i$. Condition $\mathbb{S}'$ focuses only on the $(\hat{k}_i, 1)$ interval due to U_i increasing on $[0, \hat{k}_i]$ and decreasing on $[1, +\infty)$. The former is by construction and follows from the way Algorithm 2 determines $\hat{k}_i$ while the latter holds for any $\hat{\mathbf{k}}$.

Condition $\mathbb{S}'$ is stronger than necessary. When it fails, σ possibly still constitutes an SMPE when condition $\mathbb{N}'$ holds. The latter condition verifies that $U_i(\hat{k}_i\vec{x}_i) \geq U_i(k\vec{x}_i)$ for $\forall k \geq \hat{k}_i$. It only looks at a finite set of points using the fact that U_i is piecewise quadratic and $U_i(k\vec{x}_i) - U_i(l\vec{x}_i) = \left[\int \frac{\partial}{\partial z} U_i(z\vec{x}_i) dz\right]_l^k$.

Both conditions guarantee existence of an SMPE and only need to be checked at a finite set of points. Their disadvantage is that they apply to the strategic bliss points from Algorithm 2. Relating $\mathbb{S}'$ and $\mathbb{N}'$ to the parameters defining $\mathcal{G}$ is non-trivial due to complicated mapping from n, $\mathbf{x}$, $\mathbf{r}$ and δ to $\hat{\mathbf{x}}$. That is why in the next two subsections we look at orthogonal strongly symmetric and equiangular games. Putting enough structure on the parameters defining $\mathcal{G}$ will allow us to relate (mainly) condition $\mathbb{S}$ to these parameters.

Before proceeding, we provide several examples. Example 9 (continued) below illustrates that the verification of the conditions can be straightforward. Verification of the conditions in the subsequent Example 10 is more involved, but still possible due to their focus on a finite set of points. Finally, Examples 11 and 12 show that verification of the two conditions is possible even in partially parameterized $\mathcal{G}$.

Example 9 (continued)**.** *With* $\mathbf{x} = \{(2,0), (-2,0), (0,2), (0,-2), (0,0)\}$ *and* $\hat{\mathbf{k}} = \{0.28, 0.82, 0.46, 1, 0\}$, $\mathcal{ND} = \{0, 0.56, 0.92, 1.64, 2\}$ *and for* $i \in N\backslash\{m\}$ $\mathcal{ND}_i = \{0, 0.28, 0.46, 0.82, 1\}$*. The subset* *of players on the non-constant part of their strategy is*

$$\mathcal{NC}(x) = \begin{cases} \{1,2,3,4\} & \textit{for} \quad x \in (0, 0.56) \\ \{2,3,4\} & \textit{for} \quad x \in (0.56, 0.92) \\ \{2,4\} & \textit{for} \quad x \in (0.92, 1.64) \\ \{4\} & \textit{for} \quad x \in (1.64, 2) \\ \emptyset & \textit{for} \quad x \in (2, \infty) \end{cases}$$

which can be used to derive $\mathcal{NC}_i$ for $i \in N \setminus \{m\}$ from $\mathcal{NC}_i(x) = \mathcal{NC}(2x)$. To verify condition $\mathbb{S}'$, $\mathcal{S}_i = \emptyset$ for $i \in \{2,4\}$, $\mathcal{S}_1 = \{0.46, 0.82\}$ and $\mathcal{S}_3 = \{0.82\}$. Using $\mathcal{NC}_i$ for $i \in \{1,3\}$, $\mathcal{NC}_i(x^+) = \{2,4\}$ for $x = 0.46$ and $\mathcal{NC}_i(x^+) = \{4\}$ for $x = 0.82$. From here it is matter of simple algebra to verify that condition $\mathbb{S}'$ holds. The results we prove in the following subsection also imply that any of the 16 different profiles of strategic bliss points Algorithm 2 can produce for this example satisfy condition $\mathbb{S}'$ and also that we could have used any $\delta = (0,1)$ in this example without changing its results. This follows from the fact that the current $\mathcal{G}$ is orthogonal strongly symmetric.

Example 10 (Duggan and Kalandrakis (2011) parametrization)**.** *Consider $\mathcal{G}$ with $n = 9$, $r_i = \frac{1}{n}$ for $\forall i \in N$, $\delta = 0.7$ and bliss points*

player	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
x_i^1	*-0.8*	*0.3*	*-0.2*	*0.9*	*0.1*	*-0.15*	*0.3*	*-0.9*	*0*
x_i^2	*0*	*0*	*0.2*	*-0.9*	*0.6*	*-0.9*	*0.2*	*-0.6*	*0*

Algorithm 2 produces a unique set of bliss points (numbers rounded)

player	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
$\hat{k}_i$	*0.79*	*0.51*	*0.38*	*1*	*0.50*	*0.94*	*0.48*	*0.91*	*0*

for which conditions $\mathbb{S}'$ and $\mathbb{N}'$ hold.

Example 11 (Non-orthogonal players in $\mathbb{R}^2$)**.** *Consider $\mathcal{G}$ with $n = 5$, $r_i = \frac{1}{n}$ for $\forall i \in N$, $\delta \in (0,1)$ and, for $\alpha \in (0, \frac{\pi}{2})$, the following bliss points*

player	*1*	*2*	*3*	*4*	*5*
x_i^1	*1*	*-1*	$\cos\alpha$	$-\cos\alpha$	*0*
x_i^2	*0*	*0*	$\sin\alpha$	$-\sin\alpha$	*0*

Algorithm 2 in step 1 computes $\hat{k}_{i,1} = 1 - \delta\frac{4}{5}$ *for* $i \in N \setminus \{m\}$*. Dropping player 1 gives* $\hat{k}_1 = 1 - \delta\frac{4}{5}$*. In step 2 the algorithm drops player 3 with* $\hat{k}_3 = 1 - \frac{\delta}{5}(3 - \cos(\pi - \alpha))$*. In step 3 the algorithm computes* $\hat{k}_{2,3} = \hat{k}_{4,3} = 1 - \frac{\delta}{5}(1 - \cos\alpha)$ *and dropping player 4 produces* $\hat{k}_4 = 1 - \frac{\delta}{5}(1 - \cos\alpha)$ *and* $\hat{k}_2 = 1$*.*

With these strategic bliss points $\mathcal{S}_i = \emptyset$ *for* $i \in \{2, 4\}$*,* $\mathcal{S}_1 = \{\hat{k}_3, \hat{k}_4\}$ *and* $\mathcal{S}_3 = \{\hat{k}_4\}$*. Computing* $\mathcal{NC}$ *is straightforward using the fact that the algorithm dropped players in the order 1, 3, 4 and 2. Hence, for* $i \in N \setminus \{m\}$*,* $\mathcal{NC}_i(x^+) = \{2, 4\}$ *for* $x = \hat{k}_3$ *and* $\mathcal{NC}_i(x^+) = \{2\}$ *for* $x = \hat{k}_4$*. From here, it is matter of simple algebra to confirm that condition* $\mathbb{S}'$ *holds for any* $\delta \in (0, 1)$ *and* $\alpha \in (0, \frac{\pi}{2})$*.*

Had we dropped player 2 in step 3 of the algorithm, we would have $\mathcal{S}_1 = \{\hat{k}_3, \hat{k}_2\}$ *and* $\mathcal{S}_3 = \{\hat{k}_2\}$ *with* $\hat{k}_2 = 1 - \frac{\delta}{5}(1 - \cos\alpha)$*, that is with the same value as before, and* $\mathcal{NC}_i(x^+) = \{4\}$ *for* $x = \hat{k}_2$*. Condition* $\mathbb{S}'$ *would still hold. Had we dropped any other player than player 1 in step 1 of the algorithm, we would face the same duplicity but condition* $\mathbb{S}'$ *would still hold.*

Example 12 (Players at varying distances in $\mathbb{R}^2$)**.** *Consider* $\mathcal{G}$ *with* $n = 5$*,* $r_i = \frac{1}{n}$ *for* $\forall i \in N$*, bliss points*

player	*1*	*2*	*3*	*4*	*5*
x_i^1	d_x	$-d_x$	*0*	*0*	*0*
x_i^2	*0*	*0*	d_y	$-d_y$	*0*

where $\frac{d_x}{d_y} = d_r > 1$ *and* $\delta \le \frac{5(d_r - 1)}{3d_r - 2}$*. Note that the assumption on* δ *is not binding if* $d_r \ge \frac{3}{2}$*. Algorithm 2 in step 1 computes,* $\hat{k}_{i,1} = 1 - \delta\frac{4}{5}$ *for* $i \in N \setminus \{m\}$ *and offers the option to drop players 3 and 4 due to* $\hat{k}_{i,1} d_y < \hat{k}_{j,1} d_x$ *for any* $i \in \{3, 4\}$ *and* $j \in \{1, 2\}$*. Dropping player 4 produces* $\hat{k}_4 = 1 - \delta\frac{4}{5}$*. In step 2 the algorithm computes* $\hat{k}_{3,2} = 1 - \delta\frac{2}{5}$ *and* $\hat{k}_{i,2} = 1 - \delta\frac{3}{5}$ *for* $i \in \{1, 2\}$*, drops player 3 due to* $\hat{k}_{3,2} d_y \le \hat{k}_{i,2} d_x$ *for* $i \in \{1, 2\}$ *by assumption on* δ*, and produces* $\hat{k}_3 = 1 - \delta\frac{2}{5}$*. Steps 3 and 4 then produce, dropping player 1 in the former,* $\hat{k}_1 = 1 - \delta\frac{2}{5}$ *and* $\hat{k}_2 = 1$*.*

With these strategic bliss points $\mathcal{S}_i = \emptyset$ *for* $i \in \{1, 2\}$*,* $\mathcal{S}_4 = \{\hat{k}_3, \hat{k}_1 d_r\}$ *and* $\mathcal{S}_3 = \{\hat{k}_1 d_r\}$ *(if* $\hat{k}_1 d_r \ge 1$*, then* $\hat{k}_1 d_r$ *does not belong to* $\mathcal{S}_3$ *and* $\mathcal{S}_4$*). Computing* $\mathcal{NC}_i$ *for* $i \in \{3, 4\}$ *gives* $\mathcal{NC}_i(x^+) = \{1, 2\}$ *for* $x = \hat{k}_3$ *and* $\mathcal{NC}_i(x^+) = \{2\}$ *for* $x = \hat{k}_1 d_r$*. From here, it is matter of simple algebra to confirm that condition* $\mathbb{S}'$ *holds. A similar argument shows that it holds for any* $\hat{\mathbf{k}}$ *produced by alternative selection of players to drop in steps 1 and 3 of the algorithm.*

7.3 Orthogonal Strongly Symmetric Games

Recall that $\mathcal{G}$ is orthogonal strongly symmetric if the recognition probabilities of all the players are equal, for every player $i \in N \setminus \{m\}$ exactly one player i^r exists with bliss point on the opposite side of $\vec{x}_m = \mathbf{0}$ relative to $\vec{x}_i$ and for every other player $j \in N \setminus \{i, i^r, m\}$, $cos(i,j) = 0$. This implies that policy space X in $\mathcal{G}$ with n players is $X \subseteq \mathbb{R}^{\frac{n-1}{2}}$. In addition, $r_i = \frac{1}{n}$ for $\forall i \in N$ and $||\vec{x}_i|| = b > 0$ for $\forall i \in N \setminus \{m\}$. $\mathcal{G}$ in Example 9 satisfies this definition while $\mathcal{G}$ in Examples 10, 11 and 12 do not.

Proposition 15 (SMPE in orthogonal strongly symmetric $\mathcal{G}$)**.** *Assume $\mathcal{G}$ is orthogonal strongly symmetric. Then*

1. *if $\delta \in (0,1)$, $2^{(n-1)/2}\left(\frac{n-1}{2}!\right)^2$ distinct profiles of strategic bliss points $\hat{\mathbf{k}}$ produced by Algorithm 2 exist, if $\delta = 0$, $\hat{\mathbf{k}} = \mathbf{1}$;*

2. *σ induced by any of these profiles of strategic bliss points constitutes an SMPE;*

3. *σ induced by any of these profiles of strategic bliss points satisfies condition $\mathbb{S}'$ and, for $i \in N$, $U_i(k\vec{x}_i|\sigma)$ is single peaked (in k) on $\mathbb{R}_{\geq 0}$.*

Proof. See appendix A1

7.4 Equiangular Games on a Circle

We have defined equiangular $\mathcal{G}$ to be in $\mathbb{R}^2$ with the bliss points of all the players at the same distance from $\vec{x}_m$ and arranged such that the angle between the bliss points of any adjacent players is $\alpha = \frac{2\pi}{n-1}$. The players are indexed such that $\vec{x}_1 = (b, 0)$ and $\vec{x}_i$ are arranged, with increasing i, counter-clockwise on a circle of radius b, which implies $m = n$.

Proposition 16 (SMPE in equiangular $\mathcal{G}$)**.** *Assume $\mathcal{G}$ is equiangular on a circle with radius $b > 0$. Then*

1. *if $\delta \in (0,1)$, $2^{(n-3)}(n-1)$ distinct profiles of strategic bliss points $\hat{\mathbf{k}}$ produced by Algorithm 2 exist, if $\delta = 0$, $\hat{\mathbf{k}} = \mathbf{1}$;*

2. *σ induced by any of these profiles of strategic bliss points constitutes an SMPE;*

3. *σ induced by any of these profiles of strategic bliss points satisfies condition $\mathbb{S}'$ and, for $i \in N$, $U_i(k\vec{x}_i|\sigma)$ is single peaked (in k) on $\mathbb{R}_{\geq 0}$;*

4. *$\lim_{n\to\infty} \hat{k}_i = 1 - \delta + \delta\left[\frac{\gamma - \sin\gamma}{2\pi}\right]$ for i Algorithm 2 drops after $\frac{\gamma}{2\pi}$ fraction of players has been already dropped.*

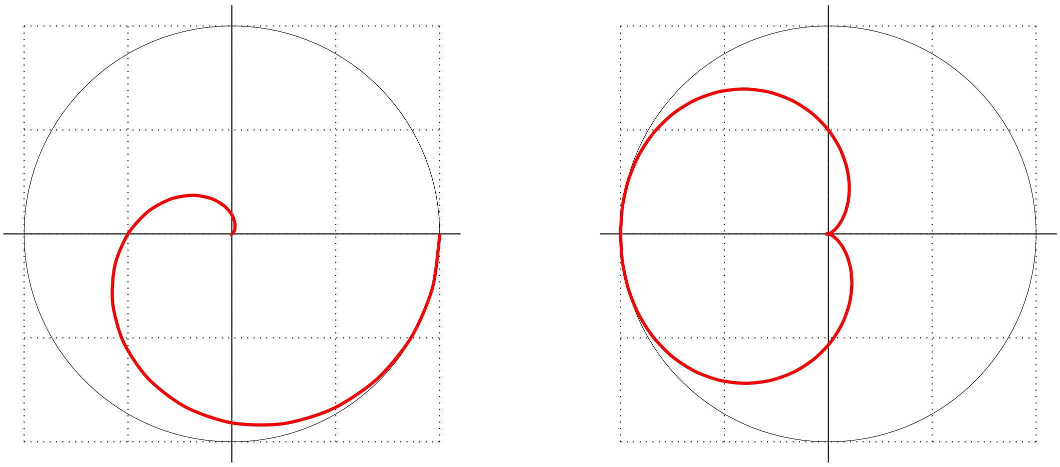
Proof. See appendix A1

The key insight that allows us to prove Proposition 16 is the special structure of the strategic bliss points Algorithm 2 produces for any equiangular $\mathcal{G}$. In step 1 the algorithm offers an option to drop players $\{1, \dots, n-1\}$. Dropping player 1, the algorithm in the next step offers an option to drop players $\{2, n-1\}$. Intuitively, dropping player 1 in step 1 means that the opponent of 1^r moderates, weakening considerably the incentive of 1^r to do so. On the other hand, dropped player 1 is closely allied with players 2 and $n-1$ for whom the incentive to moderate changes only slightly as a result of player 1 being dropped. Dropping player $n-1$, the players to drop in the following step are $\{2, n-2\}$ and so on. In other words, if $\mathbb{P}_t$ is the set of players still in the algorithm in step $t \geq 2$, the set of players that can be dropped is $\{\min \mathbb{P}_t, \max \mathbb{P}_t\}$. This puts just enough structure on the resulting profile of strategic bliss points for us to prove that the induced σ constitutes an SMPE.

Despite the fact that the algorithm can produce a large number of distinct profiles of strategic bliss points, systematic selection of players to drop might generate easy to describe $\hat{\mathbf{k}}$. Figure 3 shows two such $\hat{\mathbf{k}}$ in the limit as $\delta \to 1$ and $n \to \infty$. Panel 3a shows $\hat{\mathbf{k}}$ generated by systematically selecting $\min \mathbb{P}_t$ as the player to be dropped. Panel 3b then shows $\hat{\mathbf{k}}$ generated by alternatively selecting $\min \mathbb{P}_t$ and $\max \mathbb{P}_t$ as the players to be dropped.[37]

Figure 3: Strategic bliss points in equiangular $\mathcal{G}$ limit as $n \to \infty$ and $\delta \to 1$

(a) Counter-clockwise dropping (b) Alternating dropping



[37] In polar coordinates, panel 3a can be expressed as $\frac{\theta - \sin\theta}{2\pi}$ for $\theta \in [0, 2\pi]$ and the upper branch of panel 3b as $\frac{\theta - \sin\theta\cos\theta}{\pi}$ for $\theta \in [0, \pi]$. See proof of Proposition 16 in appendix A1 for details.

8 Conclusion

This paper provided insights and techniques for studying equilibria in dynamic spatial legislative bargaining. Our results show that the structure of equilibria in these models is simple and intuitive, once we address the formal difficulties for which we have provided a resolution. We hope that our results will foster further research into dynamic spatial legislative bargaining, which we feel has unjustifiably been lagging behind the study of the distributive dynamic models.

In order for the dynamic, spatial or distributive, legislative bargaining models to find a stable place in political economics, they need to provide novel insights and further our understanding of policy determination relative to their static precursors. In this paper, for the most part we have failed to stress and comment on the behaviour of policies generated by equilibrium play, focusing instead on the existence of equilibria and relying on reader's ingenuity. Common themes emerging from our analysis are the convergence to the policy preferred by the median player, the convergence path alternation of policies around this policy and asymmetric tendency for moderation towards this policy.

The first theme implies, seemingly in our opinion, that the study of the dynamic models does not warrant the increased complexity of the analysis; in static models the median's optimal policy is typically the strong point of attraction. To dispute this claim, we have shown that the convergence phase can be arbitrarily long. Alternation and moderation along the convergence path, predictions about the evolution of policies, are then distinctive to the dynamic models.

Moderation and asymmetric incentive to do so are likewise specific to the dynamic models. These observations can, for example, explain why in the US the Democratic party is sometimes referred to as 'the party of the people' while the Republican party bears 'the grand old party' moniker. Taking the symmetric three-player dynamic bargaining model studied in section 4, as the probability of recognition of the median player vanishes, we approach a model with two parties proposing policies subject to approval by the median, who is devoid of any proposal power. Re-interpreting the model as one with an electorate and two parties, equilibrium in this model will have exactly one of the parties moderating. If, in addition, the parties become arbitrarily patient, the moderating party will propose policies that almost coincide with the most preferred policy of the electorate, despite the parties being completely symmetric.

A wider use of dynamic bargaining models requires deeper formal understanding of their properties and a large(r) set of existing results. In this respect our analysis raises more questions than it answers. Our approach to equilibrium construction fails when the conditions $\mathbb{N}$ and $\mathbb{N}'$ fail. The existence and properties of equilibria when the conditions fail thus remain open questions. The fact that the adjusted simple strategies can be used to establish equilibrium existence for three-player games strongly suggests that a similar approach could prove fruitful even when the number of players is larger. We have extensively investigated this possibility, but so far failed to prove the desired result. Another open question we leave for further work is the closer link between the necessary conditions for the existence of equilibrium in simple strategies and the parameters of a game. We have provided this link for symmetric one-dimensional games and two highly restricted classes of multi-dimensional games, clearly leaving scope for future work.

The equilibrium construction we provide is in pure proposal strategies, something we view in a positive light. Nevertheless, more general models might require, in order for the equilibria to exist, the use of mixed strategies. From Kalandrakis (2012) we know that mixed strategy equilibria exist in three-player, using our terminology, strongly symmetric games (the first adjective is most likely not needed for his result) and possess interesting properties. Whether mixing can be used to establish the general existence result in a dynamic spatial legislative bargaining model remains an open question.

Finally, our contribution relies heavily on the existence of a unique player who is decisive for the acceptance of any policy, on the existence of median player. Quadratic utilities in the one-dimensional setting and Euclidean utilities along with radial symmetry assumption in the multi-dimensional setting ensure the median exists, raising the natural question of the effect of its nonexistence, when, as an example, alternative utility functions are used or the radial symmetry fails and the existence of the median is either not guaranteed or is known to fail.

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A1 Proofs

A1.1 Proof of Proposition 1

The proposition is an implication of Banks and Duggan (2006b). We present full proof in order to demonstrate dependence of the result on the quadratic utilities used. The key fact we will use is that for any random variable z with mean μ_z and variance σ_z^2 and for quadratic utility with bliss point x_i, we have $\mathbb{E}[-(z-x_i)^2] = -[\sigma_z^2 + (\mu_z - x_i)^2]$. Note also that $\frac{\partial}{\partial x_i}[-[\sigma_z^2 + (\mu_z - x_i)^2]] = 2(\mu_z - x_i)$, which is linear in x_i.

Now fix any profile of pure stationary Markov strategies $\hat{\sigma}$. Consider two policies p_0 and p_0' generating stochastic sequence, via $\hat{\sigma}$, of policies $\mathbf{p} = \{p_0, p_1, \ldots\}$ and $\mathbf{p}' = \{p_0', p_1', \ldots\}$ respectively. The utility of player i from voting either for p_0 or p_0' is

$$U_i(p_0|\hat{\sigma}) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (p_t - x_i)^2\right] \quad U_i(p_0'|\hat{\sigma}) = \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (p_t' - x_i)^2\right]. \tag{A1}$$

Differentiating the difference in utility from the two policies with respect to x_i gives

$$\frac{\partial[U_i(p_0|\hat{\sigma}) - U_i(p_0'|\hat{\sigma})]}{\partial x_i} = \mathbb{E}\left[2\sum_{t=0}^{\infty} -\delta^t (p_t' - p_t)\right] \tag{A2}$$

which is independent of x_i and hence $U_i(p_0|\hat{\sigma}) - U_i(p_0'|\hat{\sigma})$ is linear in x_i.

Now assume $U_m(p_0|\hat{\sigma}) \geq U_m(p_0'|\hat{\sigma})$. Then $U_i(p_0|\hat{\sigma}) \geq U_i(p_0'|\hat{\sigma})$ either for $\forall i \in N_a$ or $\forall i \in N_b$ and p_0 is accepted. Conversely, if $U_m(p_0|\hat{\sigma}) < U_m(p_0'|\hat{\sigma})$, then $U_i(p_0|\hat{\sigma}) < U_i(p_0'|\hat{\sigma})$ either for $\forall i \in N_a$ or $\forall i \in N_b$ and p_0 is rejected. This implies that p_0 is accepted if and only if $U_m(p_0|\hat{\sigma}) \geq U_m(p_0'|\hat{\sigma})$, that is, when the median player (weakly) prefers p_0 to p_0'. □

A1.2 Proof of Lemma 1

By Proposition 1, for any profile of strategies $\hat{\sigma}$, proposal $p \in X$ is accepted under status-quo $x \in X$ if and only if m votes for p. Because m can enforce x_m as an outcome in any future period by rejecting any proposal $p \neq x_m$ when status-quo is x_m, for any SMPE $\hat{\sigma}$ we have $V_m(x_m|\hat{\sigma}) = 0$. This implies $U_m(x_m|\hat{\sigma}) > U_m(x|\hat{\sigma})$ for $\forall x \in X \setminus \{x_m\}$ and, by Proposition 1, $\mathcal{A}(x_m|\hat{\sigma}) = \{x_m\}$. Any SMPE $\hat{\sigma}$ thus has to satisfy $\hat{p}_i(x_m) = x_m$ for $\forall i \in N$, or, in terms of the simple strategies, $p_i(x_m|\hat{x}_i) = x_m$ for $\forall i \in N$, which rewrites as $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$. $\square$

A1.3 Proof of Lemma 2

To see part 1, any simple strategy p_i with any bliss point $\hat{x}_i \in \overline{\mathbb{R}}$ satisfies $p_i(d_b(x)|\hat{x}_i) = p_i(d_a(x)|\hat{x}_i)$ for $\forall x \in X$. The claim then follows from (2).[38] Part 2 follows easily from the symmetry of V_i for $\forall i \in N$ about x_m and asymmetry of the stage utilities for $\forall i \in N \setminus \{m\}$ and symmetry of u_m.

To prove part 3, continuity of the dynamic utilities U_i on X, fix $\hat{\mathbf{x}}$ with $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$ and induced profile of strategies σ. As $U_i(x|\sigma) = u_i(x) + \delta V_i(x|\sigma)$, we need to prove the continuation value functions V_i are continuous. From $p_i(x|\hat{x}_i) \in \{d_b(x), d_a(x)\}$ for any $i \in \mathcal{NC}(x|\sigma)$ and $x \in \mathcal{D}(\sigma)$ and from symmetry of V_i about x_m, we can write (2) for $\forall x \in \mathcal{D}(\sigma)$

$$V_i(x|\sigma) = \frac{\sum_{j \in N} r_j u_i(p_j(x|\hat{x}_j)) + \delta \sum_{j \in \mathcal{C}(x|\sigma)} r_j V_i(p_j(x|\hat{x}_j)|\sigma)}{1 - \delta r_{nc}(x|\sigma)} \tag{A3}$$

which is continuous, for $\forall i \in N$, by continuity of $p_j(x|\hat{x}_j)$ for $\forall j \in N$, constancy of $p_j(x|\hat{x}_j)$ for $\forall j \in \mathcal{C}(x|\sigma)$ and by local, that is on any interval induced by $\mathcal{ND}(\sigma)$, constancy of $\mathcal{C}(x|\sigma)$ and $r_{nc}(x|\sigma)$.

What remains is, for $\forall i \in N$, $V_i(x^-|\sigma) = V_i(x|\sigma) = V_i(x^+|\sigma)$ for any $x \in \mathcal{ND}(\sigma)$. For $x = x_m$ the claim follows from $p_j(x_m^-|\hat{x}_j) = p_j(x_m|\hat{x}_j) = p_j(x_m^+|\hat{x}_j) = x_m$ for $\forall j \in N$, $\mathcal{C}(x_m^-|\sigma) = \mathcal{C}(x_m^+|\sigma)$, $r_{nc}(x_m^-|\sigma) = r_{nc}(x_m^+|\sigma)$, $V_i(x_m^-|\sigma) = V_i(x_m^+|\sigma)$ (by part 1) and $V_i(x_m^-|\sigma) = V_i(x_m|\sigma) = \frac{u_i(x_m)}{1-\delta}$.

For $x \in \mathcal{ND}(\sigma) \setminus \{x_m\}$ let us focus on cases when $x > x_m$. When $x < x_m$ the argument is symmetric and hence omitted. First notice that $p_j(x^-|\hat{x}_j) = p_j(x|\hat{x}_j) =$

[38] We do not rule out $\hat{x}_i = \pm\infty$. The meaning of, say, $\hat{x}_i = \infty$ in p_i is player $i \in N_a$ proposing $d_a(x)$ for any status-quo x. We can use (2) since, when $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$, any proposal generated by the simple proposal strategy p_i of any $i \in N$ is always accepted, which in turn follows from the properties of the social acceptance correspondence $\mathcal{A}$ proved in part 6. As is standard, for now we conjecture that part 6 holds and then confirm that it is the case.

$p_j(x^+|\hat{x}_j)$ for $\forall j \in N$ and $\forall x \in X$ so that the first sum in the numerator of (A3) is continuous. Now use i) $V_i(p_j(x^-|\hat{x}_j)|\sigma) = V_i(p_j(x^+|\hat{x}_j)|\sigma)$ equal to $V_i(x^-|\sigma)$ for $\forall j \in N_a$ and to $V_i(d_b(x)^+|\sigma)$ for $\forall j \in N_b$ when $j \in \mathcal{C}(x^+|\sigma) \setminus \mathcal{C}(x^-|\sigma)$ (players who switch from non-constant to constant part of their strategy at x), ii) $V_i(x^-|\sigma) = V_i(d_b(x)^+|\sigma)$ (by part 1), iii) $\mathcal{C}(x^-|\sigma) \cap \mathcal{C}(x^+|\sigma) = \mathcal{C}(x^-|\sigma)$ (players switch to proposing constant policy at x), iv) $r_{nc}(x^-|\sigma) = r_{nc}(x^+|\sigma) + \sum_{j\in\mathcal{C}(x^+|\sigma)\setminus\mathcal{C}(x^-|\sigma)} r_j$ and v) $V_i(p_j(x^-|\hat{x}_j)|\sigma) = V_i(p_j(x|\hat{x}_j)|\sigma) = V_i(p_j(x^+|\hat{x}_j)|\sigma)$ for $\forall j \in \mathcal{C}(x^-|\sigma) \cap \mathcal{C}(x^+|\sigma)$ (players that propose constant policy in the neighbourhood, below and above, of x) to rewrite (A3), for any $i \in N$,

$$\begin{aligned}
&V_i(x^+|\sigma) = \\
&= \frac{\sum_{j\in N} r_j u_i(p_j(x^+|\hat{x}_j)) + \delta \sum_{j\in\mathcal{C}(x^+|\sigma)} r_j V_i(p_j(x^+|\hat{x}_j)|\sigma)}{1-\delta r_{nc}(x^+|\sigma)} \\
&= \frac{\sum_{j\in N} r_j u_i(p_j(x^-|\hat{x}_j)) + \delta \begin{bmatrix} \sum_{j\in\mathcal{C}(x^-|\sigma)} r_j V_i(p_j(x^-|\hat{x}_j)|\sigma) \\ \sum_{j\in\mathcal{C}(x^+|\sigma)\setminus\mathcal{C}(x^-|\sigma)} r_j V_i(x^-|\sigma) \end{bmatrix}}{1-\delta r_{nc}(x^-|\sigma) + \delta \sum_{j\in\mathcal{C}(x^+|\sigma)\setminus\mathcal{C}(x^-|\sigma)} r_j} \\
&= \frac{V_i(x^-|\sigma)(1-\delta r_{nc}(x^-|\sigma)) + V_i(x^-|\sigma)\delta \sum_{j\in\mathcal{C}(x^+|\sigma)\setminus\mathcal{C}(x^-|\sigma)} r_j}{1-\delta r_{nc}(x^-|\sigma) + \delta \sum_{j\in\mathcal{C}(x^+|\sigma)\setminus\mathcal{C}(x^-|\sigma)} r_j} \\
&= V_i(x^-|\sigma).
\end{aligned} \tag{A4}$$

To prove $V_i(x|\sigma) = V_i(x^-|\sigma)$, we have, from $V_i(p_j(x|\hat{x}_i)|\sigma) = V_i(p_j(x^-|\hat{x}_i)|\sigma)$ for $\forall j \in \mathcal{C}(x^-|\sigma)$ and $V_i(p_j(x|\hat{x}_j)|\sigma) = V_i(x|\sigma)$ for $\forall j \in \mathcal{NC}(x^-|\sigma)$,

$$\begin{aligned}
V_i(x|\sigma) &= \sum_{j\in N} r_j \left[u_i(p_j(x|\hat{x}_j)) + \delta V_i(p_j(x|\hat{x}_j)|\sigma)\right] \\
&= \sum_{j\in N} r_j u_i(p_j(x^-|\hat{x}_j)) + \delta \sum_{j\in\mathcal{C}(x^-|\sigma)} V_i(p_j(x^-|\hat{x}_j)|\sigma) \\
&+ \delta r_{nc}(x^-|\sigma) V_i(x|\sigma) \\
&= V_i(x^-|\sigma)(1-\delta r_{nc}(x^-|\sigma)) + \delta r_{nc}(x^-|\sigma) V_i(x|\sigma)
\end{aligned} \tag{A5}$$

and the claim, for any $i \in N$, follows.

Part 4, $U_i''(x|\sigma) < 0$ for $\forall x \in \mathcal{D}(\sigma)$, follows from $u_i(x)'' = -2$, the only non-constant term in (A3) being $\frac{\sum_{j\in\mathcal{NC}(x|\sigma)} r_j u_i(p_j(x|\hat{x}_j))}{1-\delta r_{nc}(x|\sigma)}$, $u_i''(p_j(x|\hat{x}_j)) = -2[p_j'(x|\hat{x}_j)]^2$ and $p_j'(x|\hat{x}_j) = \pm 1$ for $j \in \mathcal{NC}(x|\sigma)$. Thus we have $U_i''(x|\sigma) = -2 + \delta \frac{-2r_{nc}(x|\sigma)}{1-\delta r_{nc}(x|\sigma)} = \frac{-2}{1-\delta r_{nc}(x|\sigma)} < 0$ for any $x \in \mathcal{D}(\sigma)$ and $i \in N$.

To prove part 5, we only need to show that $U_m(x|\sigma)$ is increasing for $x < x_m$ and decreasing for $x > x_m$. For any $i \in N$ and $x \in \mathcal{D}(\sigma)$ we have, using (A3) and $p_j'(x|\hat{x}_j) = \pm 1$

for $\forall j \in \mathcal{NC}(x|\sigma)$ depending on $x \gtrless x_m$ and $j \in N_a$ or $j \in N_b$ in obvious manner,

$$U_i'(x|\sigma) = \begin{cases} \dfrac{-2[x - x_i - 2\delta r_{nc,a}(x|\sigma)(x_m - x_i)]}{1 - \delta r_{nc}(x|\sigma)} & \text{if } \ x < x_m \\ \dfrac{-2[x - x_i - 2\delta r_{nc,b}(x|\sigma)(x_m - x_i)]}{1 - \delta r_{nc}(x|\sigma)} & \text{if } \ x > x_m. \end{cases} \tag{A6}$$

Evaluating the derivative for m shows that U_m is, for $\forall x \in \mathcal{D}(\sigma)$, increasing for $x < x_m$ and decreasing for $x > x_m$. By continuity of U_m the claim follows.

Finally part 6, $\mathcal{A}(x|\sigma) = [d_b(x), d_a(x)]$ for $\forall x \in X$, is a consequence of part 5 and of Proposition 1. □

A1.4 Proof of Lemma 3

Let $\hat{\mathbf{x}}$ be a profile of strategic bliss points from Algorithm 1. To see part 1, if $\delta = 0$, the algorithm in step t computes $\hat{x}_{i,t} = x_i$ for $\forall t \in \{1, \dots, n-1\}$ and $\forall i \in N$. Hence $\mathbb{R}_t = \emptyset$ for $\forall t \in \{1, \dots, n-1\}$ since the condition defining $\mathbb{R}_t$, $(x_i - x_m)(\hat{x}_{i,t} - x_m) \leq 0$, rewrites as $(x_i - x_m)^2 \leq 0$ and is violated. The algorithm thus sets $\hat{x}_i = x_i$ in every step $t \in \{1, \dots, n-1\}$ and because $\hat{x}_m = x_m$, $\hat{\mathbf{x}} = \mathbf{x}$ follows.

To prove part 2, assume $1 \leq 2\delta r_a$. When $1 \leq 2\delta r_b$ the argument is symmetric and omitted. $1 \leq 2\delta r_a$ implies $1 > 2\delta r_b$; $1 \leq 2\delta r_b$ and $1 \leq 2\delta r_a$ sum to $1 \leq \delta(r_a + r_b)$, which contradicts $\delta < 1$ and $r_a + r_b = 1 - r_m < 1$. In step 0, the algorithm produces $\hat{x}_m = x_m$. In step 1, the algorithm computes $\hat{x}_{i,1}$ for $\forall i \in N \setminus \{m\}$ using $r_{1,a} = r_a$ and $r_{1,b} = r_b$. Now notice that, in general step t of the algorithm, $(x_i - x_m)(\hat{x}_{i,t} - x_m)$ used to construct $\mathbb{R}_t$ rewrites as $(x_i - x_m)^2(1 - 2\delta r_{t,a})$ if $i \in N_b$ and as $(x_i - x_m)^2(1 - 2\delta r_{t,b})$ if $i \in N_a$. In step 1 this means $\mathbb{R}_1 = N_b$ when $1 \leq 2\delta r_a$ and $1 > 2\delta r_b$. At this point the algorithm drops one of the players in $\mathbb{R}_1 = N_b$, say j', and sets $\hat{x}_{j'} = x_m$, which implies that $\mathbb{P}_2 = N_a \cup N_b \setminus \{j'\}$ and hence $r_{2,a} = r_a$ and $r_{2,b} = r_b - r_{j'}$. Clearly $\mathbb{R}_2 = N_b \setminus \{j'\}$, the algorithm in step 2 drops $j'' \in \mathbb{R}_2 \subsetneq N_b$ and sets $\hat{x}_{j''} = x_m$, which implies $\mathbb{P}_3 = N_a \cup N_b \setminus \{j', j''\}$ and hence $r_{3,a} = r_a$ and $r_{3,b} = r_b - r_{j'} - r_{j''}$. The algorithm continues in similar manner, dropping $j \in N_b$ and setting $\hat{x}_j = x_m$, until step $\frac{n-1}{2}$, in which it drops last player from N_b. This implies $\mathbb{P}_{\frac{n-1}{2}+1} = N_a$ and hence $r_{\frac{n-1}{2}+1,a} = r_a$ and $r_{\frac{n-1}{2}+1,b} = 0$. For the remaining steps the algorithm thus sets $\hat{x}_i = x_i$ for all $i \in N_a$.

To prove part 3, because the algorithm is dropping players and $r_{t,a}$ and $r_{t,b}$ are sums of recognition probabilities of the players that remain in the algorithm, $r_{t,a} \geq r_{t+1,a}$ and $r_{t,b} \geq r_{t+1,b}$ for $\forall t \in \{1, \dots, n-2\}$. $1 > 2\delta r_a$ and $1 > 2\delta r_b$ with $r_{1,a} = r_a$ and $r_{1,b} = r_b$ thus imply $1 > 2\delta r_{t,a}$ and $1 > 2\delta r_{t,b}$ for $\forall t \in \{1, \dots, n-1\}$. For any step $t \in \{1, \dots, n-1\}$

of the algorithm, this implies $\mathbb{R}_t = \emptyset$, $\hat{x}_{i,t} > x_m$ if $i \in N_a$ and $\hat{x}_{i,t} < x_m$ if $i \in N_b$ and hence $\hat{x}_{m-1} < \hat{x}_m = x_m < \hat{x}_{m+1}$. To prove $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N \setminus \{n\}$, we thus need to show $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N_a \setminus \{n\}$ and $\forall i \in N_b \setminus \{m-1\}$. We do so for $i \in N_a \setminus \{n\}$. For $i \in N_b \setminus \{m-1\}$ the argument is similar and omitted. Note that, if $i \in N_a$ and $t \in \{1, \ldots, n-1\}$, $\frac{\partial \hat{x}_{i,t}}{\partial x_i} = 1 - 2\delta r_{t,b} > 0$ and $\frac{\partial \hat{x}_{i,t}}{\partial r_{t,b}} = 2\delta(x_m - x_i) < 0$. The first inequality implies $\hat{x}_{i,t} < \hat{x}_{i+1,t}$ if $i \in N_a \setminus \{n\}$ and $t \in \{1, \ldots, n-1\}$. The second inequality implies $\hat{x}_{i,t} \leq \hat{x}_{i,t+1}$ if $i \in N_a$ and $t \in \{1, \ldots, n-2\}$. Hence, if the algorithm drops player $i \in N_a \setminus \{n\}$ in step t and player $i+1$ in step t', then $t < t'$, which allows us to write $\hat{x}_i = \hat{x}_{i,t} < \hat{x}_{i+1,t} \leq \hat{x}_{i+1,t'} = \hat{x}_{i+1}$.

To prove $d(\hat{x}_i) \neq d(\hat{x}_j)$ for any pair of players $\{i, j\}$ with $i \neq j$, for $\forall t \in \{1, \ldots, n-1\}$,

$$\begin{aligned} d(\hat{x}_{i,t}) &= (x_i - x_m)(1 - 2\delta r_{t,b}) \quad \text{if} \quad i \in N_a \\ d(\hat{x}_{i,t}) &= (x_m - x_i)(1 - 2\delta r_{t,a}) \quad \text{if} \quad i \in N_b \end{aligned} \tag{A7}$$

and hence $d(\hat{x}_{i,t}) < d(\hat{x}_{i+1,t})$ if $i \in N_a \setminus \{n\}$ and $d(\hat{x}_{i,t}) < d(\hat{x}_{i-1,t})$ if $i \in N_b \setminus \{1\}$. In step $t \in \{1, \ldots, n-1\}$ of the algorithm, $\arg\min_{i \in \mathbb{P}_t} d(\hat{x}_{i,t})$ thus either includes unique player i' or pair of players $\{i', j'\}$ such that $i' \in N_a$ and $j' \in N_b$. In the former case, $\hat{x}_{i'} = \hat{x}_{i',t}$ and $d(\hat{x}_{i'}) < d(\hat{x}_{i,t}) \leq d(\hat{x}_{i,t+1})$ for $\forall i \in \mathbb{P}_t \setminus \{i'\}$, where the weak inequality follow from the fact that $r_{t,a}$ and $r_{t,b}$ are non-increasing in t and thus $d(\hat{x}_{i,t}) \leq d(\hat{x}_{i,t+1})$ for $\forall t \in \{1, \ldots, n-2\}$ for any $i \in N$. When the algorithm drops $i'' \in \mathbb{P}_{t+1} = \mathbb{P}_t \setminus \{i'\}$ in step $t+1$, $\hat{x}_{i''} = \hat{x}_{i'',t+1}$ and hence $d(\hat{x}_{i'}) < d(\hat{x}_{i''})$. In the latter case, suppose, without loss of generality, that i' is dropped. Then $\hat{x}_{i'} = \hat{x}_{i',t}$ and $d(\hat{x}_{i'}) < d(\hat{x}_{i,t}) \leq d(\hat{x}_{i,t+1})$ for $\forall i \in \mathbb{P}_t \setminus \{i', j'\}$. It thus suffices to show that $d(\hat{x}_{i',t}) < d(\hat{x}_{j',t+1})$, which follows from $d(\hat{x}_{i',t}) = d(\hat{x}_{j',t})$ and the fact that when $i' \in N_a$ is dropped, $r_{t,a} > r_{t+1,a}$ implies $d(\hat{x}_{i,t}) < d(\hat{x}_{i,t+1})$ for any $i \in N_b$, including $j' \in N_b$. □

A1.5 Proof of Proposition 2

We know from Lemma 1 that if $\hat{\mathbf{x}}$ induces SMPE σ, then $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$. From Lemma 3, the same is true for any $\hat{\mathbf{x}}$ produced by Algorithm 1. Lemma 2 thus applies when we refer to $\hat{\mathbf{x}}$ that constitutes an SMPE or is produced by Algorithm 1.

Case 1: When $\delta = 0$, clearly there exists unique $\hat{\mathbf{x}}$ that induces SMPE σ, $\hat{\mathbf{x}} = \mathbf{x}$, and we know from Lemma 3 part 1 that Algorithm 1 produces $\hat{\mathbf{x}} = \mathbf{x}$.

Case 2: When $\delta \in (0, 1)$ and $1 \leq 2\delta r_a$, by Lemma 3 part 2, we need to show that if $\hat{\mathbf{x}}$ induces SMPE σ, then it satisfies $\hat{x}_i = x_m$ for $\forall i \in N \setminus N_a$ and $\hat{x}_i = x_i$ for $\forall i \in N_a$. Note that $1 \leq 2\delta r_a$ implies $1 > 2\delta r_b$ as shown in the proof of Lemma 3. Fix $\hat{\mathbf{x}}$ and suppose it induces SMPE σ. We proceed with a series of claims.

First, we claim $\hat{x}_i > x_m$ for $\forall i \in N_a$. Suppose, towards a contradiction, that $\hat{x}_i = x_m$ for some $i \in N_a$. Using (A6) and $r_{nc,b}(x_m^+|\sigma) \leq r_b$, we have $U_i'(x_m^+|\sigma) = \frac{-2(x_m - x_i)(1-2\delta r_{nc,b}(x_m^+|\sigma))}{1-\delta r_{nc}(x_m^+|\sigma)} > 0$. Hence, there exists $\epsilon' > 0$ such that $U_i(x_m|\sigma) < U_i(x_m + \epsilon|\sigma)$ and, from $\hat{x}_i = x_m$, $p_i(x_m + \epsilon|\hat{x}_i) = x_m$ for $\forall \epsilon \in (0, \epsilon')$, which, because $x_m + \epsilon \in \mathcal{A}(x_m + \epsilon|\sigma)$ for $\forall \epsilon \in (0, \epsilon')$, contradicts $\hat{x}_i = x_m$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ.

Second, we claim $\hat{x}_i = x_m$ for $\forall i \in N_b$. Suppose, towards a contradiction, that $\hat{x}_i < x_m$ for some $i \in N_b$. Using (A6) and $r_{nc,a}(x_m^-|\sigma) = r_a \geq \frac{1}{2\delta}$, where the equality follows from $\hat{x}_j > x_m$ for $\forall j \in N_a$ proven in the previous claim, $U_i'(x_m^-|\sigma) = \frac{-2(x_m - x_i)(1-2\delta r_a)}{1-\delta r_{nc}(x_m^-|\sigma)} \geq 0$. Because $U_i''(x|\sigma) < 0$ for $\forall x \in \mathcal{D}(\sigma)$ by Lemma 2 part 4, there exists $\epsilon' > 0$ such that $U_i(x_m|\sigma) > U_i(x_m - \epsilon|\sigma)$ and, from $\hat{x}_i < x_m$, $p_i(x_m - \epsilon|\hat{x}_i) = x_m - \epsilon$ for $\forall \epsilon \in (0, \epsilon')$, which, because $x_m \in \mathcal{A}(x_m - \epsilon|\sigma)$ for $\forall \epsilon \in (0, \epsilon')$, contradicts $\hat{x}_i < x_m$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ.

Third, we claim $\hat{x}_i = x_i$ for $\forall i \in N_a$. Suppose, towards a contradiction, that $\hat{x}_i \neq x_i$ for some $i \in N_a$. By the first claim, this implies $\hat{x}_i \in (x_m, x_i) \cup (x_i, \infty)$. Using (A6) and $r_{nc,b}(x|\sigma) = 0$ for $\forall x \in \mathcal{D}(\sigma)$, where the equality follows from $\hat{x}_j = x_m$ for $\forall j \in N_b$ proven in the previous claim, $\operatorname{sgn}[U_i'(\hat{x}_i^-|\sigma)] = \operatorname{sgn}[U_i'(\hat{x}_i^+|\sigma)] = \operatorname{sgn}[x_i - \hat{x}_i]$. If $\hat{x}_i \in (x_m, x_i)$, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_i(\hat{x}_i|\sigma) < U_i(\hat{x}_i + \epsilon|\sigma)$, $p_i(\hat{x}_i + \epsilon|\hat{x}_i) = \hat{x}_i$ and $\hat{x}_i + \epsilon \in \mathcal{A}(\hat{x}_i + \epsilon|\sigma)$. If $\hat{x}_i \in (x_i, \infty)$, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_i(x_i|\sigma) > U_i(x_i + \epsilon|\sigma)$, $p_i(x_i + \epsilon|\hat{x}_i) = x_i + \epsilon$ and $x_i \in \mathcal{A}(x_i + \epsilon|\sigma)$. Each case contradicts $\hat{x}_i$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ.

Case 3: When $\delta \in (0,1)$ and $1 \leq 2\delta r_b$, by Lemma 3 part 2, we need to show that if $\hat{\mathbf{x}}$ induces SMPE σ, then it satisfies $\hat{x}_i = x_m$ for $\forall i \in N \setminus N_b$ and $\hat{x}_i = x_i$ for $\forall i \in N_b$. The proof is analogous to the proof of case 2 and is omitted.

Case 4: When $\delta \in (0,1)$, $1 > 2\delta r_a$ and $1 > 2\delta r_b$, we need to show that if $\hat{\mathbf{x}}$ induces SMPE σ, then $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$, where $\hat{\mathbf{X}}$ is the set of profiles of strategic bliss points produced by Algorithm 1. We start by proving several properties of $\hat{\mathbf{x}}$ that induces SMPE σ.

Lemma A1. *Assume $\delta \in (0,1)$, $1 > 2\delta r_a$ and $1 > 2\delta r_b$. If $\hat{\mathbf{x}}$ induces SMPE σ, then*

1. *$\hat{x}_i > x_m$ for $\forall i \in N_a$ and $\hat{x}_i < x_m$ for $\forall i \in N_b$;*
2. *$U_i'(\hat{x}_i^-|\sigma) = 0$ for $\forall i \in N_a$ and $U_i'(\hat{x}_i^+|\sigma) = 0$ for $\forall i \in N_b$;*
3. *$U_i'(x^-|\sigma) < U_{i+1}'(x^-|\sigma)$ and $U_i'(x^+|\sigma) < U_{i+1}'(x^+|\sigma)$ for $\forall x \in X$ and $\forall i \in N \setminus \{n\}$;*
4. *$\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N \setminus \{n\}$ and $d(\hat{x}_i) \neq d(\hat{x}_j)$ for $\forall i \in N$, $\forall j \in N$, $i \neq j$.*

Proof. To show part 1 of the lemma, note that $\hat{x}_i > x_m$ for $\forall i \in N_a$ follows from the first claim in case 2. The argument there relied only on $1 > 2\delta r_b$. An analogous argument can be used to prove $\hat{x}_i < x_m$ for $\forall i \in N_b$ if $1 > 2\delta r_a$.

To show part 2, we show $U_i'(\hat{x}_i^-|\sigma) = 0$ for $\forall i \in N_a$. The argument proving $U_i'(\hat{x}_i^+|\sigma) = 0$ for $\forall i \in N_b$ is analogous and omitted. Suppose, towards a first contradiction, that $U_i'(\hat{x}_i^-|\sigma) < 0$ for some $i \in N_a$. By part 1, $\hat{x}_i > x_m$. Hence, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_i(\hat{x}_i|\sigma) < U_i(\hat{x}_i - \epsilon|\sigma)$, $p_i(\hat{x}_i|\hat{x}_i) = \hat{x}_i$ and $\hat{x}_i - \epsilon \in \mathcal{A}(\hat{x}_i|\sigma)$, which contradicts $\hat{x}_i$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ. Suppose now, towards a second contradiction, that $U_i'(\hat{x}_i^-|\sigma) > 0$ for some $i \in N_a$. Using (A6) and $\hat{x}_i > x_m$,

$$\begin{aligned} U_i'(\hat{x}_i^-|\sigma) &= \tfrac{-2}{1-\delta r_{nc}(\hat{x}_i^-|\sigma)} \left[\hat{x}_i - x_i - 2\delta r_{nc,b}(\hat{x}_i^-|\sigma)(x_m - x_i)\right] \\ U_i'(\hat{x}_i^+|\sigma) &= \tfrac{-2}{1-\delta r_{nc}(\hat{x}_i^+|\sigma)} \left[\hat{x}_i - x_i - 2\delta r_{nc,b}(\hat{x}_i^+|\sigma)(x_m - x_i)\right]. \end{aligned} \tag{A8}$$

Because $r_{nc,b}(x^-|\sigma) \geq r_{nc,b}(x^+|\sigma)$ for any $x > x_m$, $U_i'(\hat{x}_i^-|\sigma) > 0$ implies $U_i'(\hat{x}_i^+|\sigma) > 0$. Hence, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_i(\hat{x}_i|\sigma) < U_i(\hat{x}_i+\epsilon|\sigma)$, $p_i(\hat{x}_i+\epsilon|\hat{x}_i) = \hat{x}_i$ and $\hat{x}_i + \epsilon \in \mathcal{A}(\hat{x}_i + \epsilon|\sigma)$, which contradicts $\hat{x}_i$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ.

For part 3, taking limits from below and from above in (A6) and differentiating with respect to x_i gives, for $\forall x \in X$,

$$\begin{aligned} \tfrac{\partial}{\partial x_i} U_i'(x^-|\sigma) &= \begin{cases} \tfrac{2}{1-\delta r_{nc}(x^-|\sigma)} \left[1 - 2\delta r_{nc,a}(x^-|\sigma)\right] & \text{if } x \leq x_m \\ \tfrac{2}{1-\delta r_{nc}(x^-|\sigma)} \left[1 - 2\delta r_{nc,b}(x^-|\sigma)\right] & \text{if } x > x_m \end{cases} \\ \tfrac{\partial}{\partial x_i} U_i'(x^+|\sigma) &= \begin{cases} \tfrac{2}{1-\delta r_{nc}(x^+|\sigma)} \left[1 - 2\delta r_{nc,a}(x^+|\sigma)\right] & \text{if } x < x_m \\ \tfrac{2}{1-\delta r_{nc}(x^+|\sigma)} \left[1 - 2\delta r_{nc,b}(x^+|\sigma)\right] & \text{if } x \geq x_m \end{cases} \end{aligned} \tag{A9}$$

which, by $r_{nc,a}(x|\sigma) \leq r_a < \frac{1}{2\delta}$ and $r_{nc,b}(x|\sigma) \leq r_b < \frac{1}{2\delta}$ for $\forall x \in \mathcal{D}(\sigma)$ and hence $r_{nc,g}(x^-|\sigma) \leq r_g$ and $r_{nc,g}(x^+|\sigma) \leq r_g$ for $\forall x \in X$ and $g \in \{a, b\}$, implies $\frac{\partial}{\partial x_i} U_i'(x^-|\sigma) > 0$ and $\frac{\partial}{\partial x_i} U_i'(x^+|\sigma) > 0$.

To show part 4, we first prove $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N \setminus \{n\}$. By part 1, $\hat{x}_i < x_m$ for $\forall i \in N_b$ and $\hat{x}_i > x_m$ for $\forall i \in N_a$. It thus suffices to prove $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N_a \setminus \{n\}$ and $\forall i \in N_b \setminus \{m-1\}$. We do so for $i \in N_a \setminus \{n\}$. For $i \in N_b \setminus \{m-1\}$ the argument is similar and omitted. Suppose, towards a first contradiction, that $\hat{x}_i = \hat{x}_{i+1}$ for some $i \in N_a \setminus \{n\}$. By part 1, $\hat{x}_i > x_m$, which by part 2 implies $U_i'(\hat{x}_i^-|\sigma) = 0$ and hence, by part 3, $U_{i+1}'(\hat{x}_i^-|\sigma) > 0$. The last inequality contradicts $U_{i+1}'(\hat{x}_i^-|\sigma) = 0$, which follows by part 2 and $\hat{x}_i = \hat{x}_{i+1}$. Suppose, towards a second contradiction, that $\hat{x}_{i+1} < \hat{x}_i$. By part 2, $U_{i+1}'(\hat{x}_{i+1}^-|\sigma) = 0$, which by part 3 implies $U_i'(\hat{x}_{i+1}^-|\sigma) < 0$. Because $\hat{x}_{i+1} > x_m$, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_i(\hat{x}_{i+1}|\sigma) < U_i(\hat{x}_{i+1} - \epsilon|\sigma)$, $p_i(\hat{x}_{i+1}|\hat{x}_i) = \hat{x}_{i+1}$ and $\hat{x}_{i+1} - \epsilon \in \mathcal{A}(\hat{x}_{i+1}|\sigma)$, which contradicts $\hat{x}_i$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ.

To prove $d(\hat{x}_i) \neq d(\hat{x}_j)$ for any pair of players $\{i, j\}$ such that $i \neq j$, because $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N \setminus \{n\}$, it suffices to rule out $d(\hat{x}_i) = d(\hat{x}_j)$ for $\forall i \in N_b$ and $\forall j \in N_a$. Suppose, towards contradiction, that there exists $i \in N_b$ and $j \in N_a$ such that $d(\hat{x}_i) = d(\hat{x}_j)$. By part 2, $U_j'(\hat{x}_j^-|\sigma) = 0$. Because $d(\hat{x}_i) = d(\hat{x}_j)$ and $i \in N_b$, $r_{nc,b}(\hat{x}_j^-|\sigma) > r_{nc,b}(\hat{x}_j^+|\sigma)$, which from (A8) implies $U_j'(\hat{x}_j^+|\sigma) > 0$. Hence, there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_j(\hat{x}_j|\sigma) < U_j(\hat{x}_j + \epsilon|\sigma)$, $p_j(\hat{x}_j + \epsilon|\hat{x}_j) = \hat{x}_j$ and $\hat{x}_j + \epsilon \in \mathcal{A}(\hat{x}_j + \epsilon|\sigma)$, which contradicts $\hat{x}_j$ being part of $\hat{\mathbf{x}}$ that induces SMPE σ. □

Returning to case 4, for any $\hat{\mathbf{x}}$ that constitutes an SMPE or is produced by Algorithm 1, define iteratively, for $t \in \{0, \ldots, n-1\}$ starting with $i_{\hat{\mathbf{x}}}(0) = m$,

$$i_{\hat{\mathbf{x}}}(t) = \operatorname*{arg\,min}_{i \in N \setminus \{i_{\hat{\mathbf{x}}}(0), \ldots, i_{\hat{\mathbf{x}}}(t-1)\}} d(\hat{x}_i) \tag{A10}$$

with the equal sign justified by $d(\hat{x}_i) \neq d(\hat{x}_j)$ for any pair of players $\{i, j\}$ in $\hat{\mathbf{x}}$ for which we define $i_{\hat{\mathbf{x}}}$. $i_{\hat{\mathbf{x}}}(t)$ is index of player with $(t+1)^{\text{th}}$ smallest $d(\hat{x}_i)$ in $\hat{\mathbf{x}}$, starting from $t = 0$. Using $i_{\hat{\mathbf{x}}}$ define for $t \in \{0, \ldots, n-1\}$

$$o(\hat{\mathbf{x}}, t) = (i_{\hat{\mathbf{x}}}(0), i_{\hat{\mathbf{x}}}(1), \ldots, i_{\hat{\mathbf{x}}}(t)) \tag{A11}$$

and write $o(\hat{\mathbf{x}}, t) = o(\hat{\mathbf{x}}', t)$ if and only if $i_{\hat{\mathbf{x}}}(k) = i_{\hat{\mathbf{x}}'}(k)$ for $\forall k \in \{0, \ldots, t\}$. $o(\hat{\mathbf{x}}, n-1)$ is the set of players in N ordered by $d(\hat{x}_i)$ in $\hat{\mathbf{x}}$, so that $d(\hat{x}_{i_{\hat{\mathbf{x}}}(k)}) < d(\hat{x}_{i_{\hat{\mathbf{x}}}(k+1)})$ for $\forall k \in \{0, \ldots, n-2\}$.

Lemma A2. *Assume $\delta \in (0,1)$, $1 > 2\delta r_a$ and $1 > 2\delta r_b$. If $\hat{\mathbf{x}}^o$ induces SMPE σ and $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$, then $o(\hat{\mathbf{x}}, t') = o(\hat{\mathbf{x}}^o, t')$ for some $t' \in \{0, \ldots, n-1\}$ implies $\hat{x}_{i_{\hat{\mathbf{x}}}(t)} = \hat{x}^o_{i_{\hat{\mathbf{x}}^o}(t)}$ for $\forall t \in \{0, \ldots, t'\}$.*

Proof. Fix $\hat{\mathbf{x}}^o$ that induces SMPE σ^o and $\hat{\mathbf{x}} \in \hat{\mathbf{X}}$ produced by Algorithm 1. Suppose $o(\hat{\mathbf{x}}^o, t') = o(\hat{\mathbf{x}}, t')$ for some $t' \in \{0, \ldots, n-1\}$. The proof proceeds by induction on t. For $t = 0$, we have $i_{\hat{\mathbf{x}}}(0) = i_{\hat{\mathbf{x}}^o}(0) = m$ and we know $\hat{x}^o_m = \hat{x}_m = x_m$. Suppose that $\hat{x}_{i_{\hat{\mathbf{x}}}(t'')} = \hat{x}^o_{i_{\hat{\mathbf{x}}^o}(t'')}$ for $\forall t'' \in \{0, \ldots, t\}$ for some $t < t'$. We need to show $\hat{x}_{i_{\hat{\mathbf{x}}}(t+1)} = \hat{x}^o_{i_{\hat{\mathbf{x}}^o}(t+1)}$.

Because $o(\hat{\mathbf{x}}^o, t') = o(\hat{\mathbf{x}}, t')$ and $t+1 \leq t'$, let us use only the $i_{\hat{\mathbf{x}}}$ indexing. Denote $j' = i_{\hat{\mathbf{x}}}(t)$ and $j'' = i_{\hat{\mathbf{x}}}(t+1)$. We need to show $\hat{x}_{j''} = \hat{x}^o_{j''}$. Assume that $j'' \in N_a$. When $j'' \in N_b$, the proof is similar and omitted. Denote $N_{j'} = \cup_{i=0}^{t} i_{\hat{\mathbf{x}}}(i)$ and $N_{j''} = N \setminus N_{j'}$.

By definition of j' and j'', $d(\hat{x}_{j'}) < d(\hat{x}_{j''})$ and $d(\hat{x}^o_{j'}) < d(\hat{x}^o_{j''})$. Because $\hat{x}_{i_{\hat{\mathbf{x}}}(t'')} = \hat{x}^o_{i_{\hat{\mathbf{x}}}(t'')}$ for $\forall t'' \in \{0, \ldots, t\}$, we know $\hat{x}_i = \hat{x}^o_i$ for $\forall i \in N_{j'}$, so that $d(\hat{x}_i) < d(\hat{x}_{j'})$ and $d(\hat{x}^o_i) < d(\hat{x}^o_{j'})$

for $\forall i \in N_{j'} \setminus \{j'\}$. From $o(\hat{\mathbf{x}}^o, t+1) = o(\hat{\mathbf{x}}, t+1)$, we know $j'' = i_{\hat{\mathbf{x}}}(t+1) = i_{\hat{\mathbf{x}}^o}(t+1)$, so that $d(\hat{x}_{j''}) < d(\hat{x}_i)$ and $d(\hat{x}^o_{j''}) < d(\hat{x}^o_i)$ for $\forall i \in N_{j''} \setminus \{j''\}$.

From these $r_{nc,a}(x|\sigma^o) = \sum_{i \in N_{j''} \cap N_a} r_i$ and $r_{nc,b}(x|\sigma^o) = \sum_{i \in N_{j''} \cap N_b} r_i$ for $\forall x \in (d_a(\hat{x}^o_{j'}), \hat{x}^o_{j''}) \subset \mathcal{D}(\sigma^o)$. Using (A6) and $U'_{j''}(\hat{x}^{o-}_{j''}|\sigma^o) = 0$ from Lemma A1 part 2, $\hat{x}^o_{j''} = x_{j''} + 2\delta \sum_{i \in N_{j''} \cap N_b} r_i \,(x_m - x_{j''})$.

To calculate $\hat{x}_{j''}$, Algorithm 1 drops player j' in step t, which means the algorithm uses, in step $t+1$ when j'' is dropped and $\hat{x}_{j''}$ set, $\mathbb{P}_{t+1} = N_{j''}$. This gives $r_{t+1,b} = \sum_{i \in N_{j''} \cap N_b} r_i$ and $\hat{x}_{j''} = x_{j''} + 2\delta \sum_{i \in N_{j''} \cap N_b} r_i \,(x_m - x_{j''})$. Clearly, $\hat{x}_{j''} = \hat{x}^o_{j''}$. □

Returning to case 4, fix $\hat{\mathbf{x}}^o$ that induces SMPE σ^o. We need to show $\hat{\mathbf{x}}^o \in \hat{\mathbf{X}}$. Suppose $\hat{\mathbf{x}}^o \notin \hat{\mathbf{X}}$. For $t \in \{0, \ldots, n-1\}$ define

$$\hat{\mathbf{X}}^t = \{\hat{\mathbf{x}} \in \hat{\mathbf{X}} | o(\hat{\mathbf{x}}, t) = o(\hat{\mathbf{x}}^o, t)\}. \tag{A12}$$

$\hat{\mathbf{X}}^t$ is the set of profiles of strategic bliss points from Algorithm 1 that satisfy $i_{\hat{\mathbf{x}}}(k) = i_{\hat{\mathbf{x}}^o}(k)$ for all $k \in \{0, \ldots, t\}$. By Lemma A2, if $\hat{\mathbf{x}} \in \hat{\mathbf{X}}^{t'}$, then $\hat{x}_{i_{\hat{\mathbf{x}}}(t)} = \hat{x}^o_{i_{\hat{\mathbf{x}}^o}(t)}$ for $\forall t \in \{0, \ldots, t'\}$. Clearly, $\hat{\mathbf{X}}^{t+1} \subseteq \hat{\mathbf{X}}^t$ for $\forall t \in \{0, \ldots, n-2\}$. Because $\hat{x}^o_m = x_m$ and $\hat{x}_m = x_m$ for $\forall \hat{\mathbf{x}} \in \hat{\mathbf{X}}$, $\hat{\mathbf{X}}^0 = \hat{\mathbf{X}}$. From $\hat{\mathbf{x}}^o \notin \hat{\mathbf{X}}$, we also have $\hat{\mathbf{X}}^{n-1} = \emptyset$; if $\hat{\mathbf{X}}^{n-1} \neq \emptyset$ we would have $o(\hat{\mathbf{x}}, n-1) = o(\hat{\mathbf{x}}^o, n-1)$ for $\hat{\mathbf{x}} \in \hat{\mathbf{X}}^{n-1}$ and hence, by Lemma A2, $\hat{\mathbf{x}} = \hat{\mathbf{x}}^o$.

Now pick t such that $\hat{\mathbf{X}}^t \neq \emptyset$ and $\hat{\mathbf{X}}^{t+1} = \emptyset$ and fix $\hat{\mathbf{x}} \in \hat{\mathbf{X}}^t$. Clearly, $t \in \{0, \ldots, n-2\}$ and $o(\hat{\mathbf{x}}, t) = o(\hat{\mathbf{x}}^o, t)$ follows from definition of $\hat{\mathbf{X}}^t$. Denote $j' = i_{\hat{\mathbf{x}}}(t) = i_{\hat{\mathbf{x}}^o}(t)$, $j''_a = i_{\hat{\mathbf{x}}}(t+1)$, $j''_o = i_{\hat{\mathbf{x}}^o}(t+1)$, $N_{j'} = \cup_{i=0}^{t} i_{\hat{\mathbf{x}}}(i)$ and $N_{j''} = N \setminus N_{j'}$.

By definition of j' and j''_a, $d(\hat{x}_i) < d(\hat{x}_{j'}) < d(\hat{x}_{j''_a}) < d(\hat{x}_j)$ for $\forall i \in N_{j'} \setminus \{j'\}$ and $\forall j \in N_{j''} \setminus \{j''_a\}$. Similarly, $d(\hat{x}^o_i) < d(\hat{x}^o_{j'}) < d(\hat{x}^o_{j''_o}) < d(\hat{x}^o_j)$ for $\forall i \in N_{j'} \setminus \{j'\}$ and $\forall j \in N_{j''} \setminus \{j''_o\}$. From Lemma A2 and $o(\hat{\mathbf{x}}, t) = o(\hat{\mathbf{x}}^o, t)$, we also know $\hat{x}_i = \hat{x}^o_i$ for $\forall i \in N_{j'}$.

From these $r_{nc,a}(x|\sigma^o) = \sum_{i \in N_{j''} \cap N_a} r_i$ and $r_{nc,b}(x|\sigma^o) = \sum_{i \in N_{j''} \cap N_b} r_i$ for $\forall x \in (d_b(\hat{x}^o_{j''_o}), d_b(\hat{x}^o_{j'})) \cup (d_a(\hat{x}^o_{j'}), d_a(\hat{x}^o_{j''_o})) \subset \mathcal{D}(\sigma^o)$. Also, Algorithm 1 drops player j' in step t, which means it uses, in step $t+1$ when j''_a is dropped and $\hat{x}_{j''_a}$ set, $\mathbb{P}_{t+1} = N_{j''}$. This gives $r_{t+1,a} = \sum_{i \in N_{j''} \cap N_a} r_i$ and $r_{t+1,b} = \sum_{i \in N_{j''} \cap N_b} r_i$.

We now show $d(\hat{x}_{j''_a}) = d(\hat{x}^o_{j''_o})$. Suppose, towards a first contradiction, that $d(\hat{x}^o_{j''_o}) < d(\hat{x}_{j''_a})$. From Lemma A1 part 2, $U'_{j''_o}(\hat{x}^{o-}_{j''_o}|\sigma^o) = 0$ if $j''_o \in N_a$ and $U'_{j''_o}(\hat{x}^{o+}_{j''_o}|\sigma^o) = 0$ if $j''_o \in N_b$. Using (A6), we get

$$\hat{x}^o_{j''_o} = \begin{cases} x_{j''_o} + 2\delta \sum_{i \in N_{j''} \cap N_b} r_i \,(x_m - x_{j''_o}) \text{ if } \ j''_o \in N_a \\ x_{j''_o} + 2\delta \sum_{i \in N_{j''} \cap N_a} r_i \,(x_m - x_{j''_o}) \text{ if } \ j''_o \in N_b \end{cases} \tag{A13}$$

Algorithm 1 in step $t+1$ calculates $\hat{x}_{j''_o,t+1}$ and $\hat{x}_{j''_a,t+1}$ and, since j''_a is dropped and $\hat{x}_{j''_a}$ set, we know $d(\hat{x}_{j''_a}) \leq d(\hat{x}_{j''_o,t+1})$. Because the algorithm in step $t+1$ uses $\mathbb{P}_{t+1} = N_{j''}$, clearly $\hat{x}_{j''_o,t+1} = \hat{x}^o_{j''_o}$ and hence $d(\hat{x}_{j''_a}) \leq d(\hat{x}^o_{j''_o})$, which yields the desired contradiction. Suppose now, towards a second contradiction, that $d(\hat{x}_{j''_a}) < d(\hat{x}^o_{j''_o})$. From Algorithm 1,

$$\hat{x}_{j''_a} = \begin{cases} x_{j''_a} + 2\delta \sum_{i \in N_{j''} \cap N_b} r_i \left(x_m - x_{j''_a}\right) \text{ if } \; j''_a \in N_a \\ x_{j''_a} + 2\delta \sum_{i \in N_{j''} \cap N_a} r_i \left(x_m - x_{j''_a}\right) \text{ if } \; j''_a \in N_b \end{cases} \tag{A14}$$

Because $d(\hat{x}^o_{j'}) = d(\hat{x}_{j'}) < d(\hat{x}_{j''_a})$, we can use $\hat{x}_{j''_a}$ in (A6) to show that $U'_{j''_a}(\hat{x}_{j''_a}|\sigma^o) = 0$. Assume $j''_a \in N_a$. When $j''_a \in N_b$ the argument is similar and omitted. From $j''_a \in N_{j''}$, we have $d(\hat{x}^o_{j''_o}) < d(\hat{x}^o_{j''_a})$ and hence $\hat{x}_{j''_a} < \hat{x}^o_{j''_a}$. $U'_{j''_a}(\hat{x}_{j''_a}|\sigma^o) = 0$ and $U''_{j''_a}(x|\sigma^o) < 0$ for $\forall x \in \mathcal{D}(\sigma^o)$ from Lemma 2 part 4 then imply that there exists $\epsilon' > 0$ such that, for $\forall \epsilon \in (0, \epsilon')$, $U_{j''_a}(\hat{x}_{j''_a}|\sigma^o) > U_{j''_a}(\hat{x}_{j''_a}+\epsilon|\sigma^o)$, $p_{j''_a}(\hat{x}_{j''_a}+\epsilon|\hat{x}^o_{j''_a}) = \hat{x}_{j''_a}+\epsilon$ and $\hat{x}_{j''_a} \in \mathcal{A}(\hat{x}_{j''_a}+\epsilon|\sigma^o)$, which contradicts $\hat{x}^o_{j''_a}$ being part of $\hat{\mathbf{x}}^o$ that induces SMPE σ^o.

Having shown $d(\hat{x}_{j''_a}) = d(\hat{x}^o_{j''_o})$, Algorithm 1 in step $t+1$ calculates $\hat{x}_{j''_a,t+1}$ and $\hat{x}_{j''_o,t+1}$ and, since j''_a is dropped and $\hat{x}_{j''_a}$ set, $d(\hat{x}_{j''_a}) = d(\hat{x}_{j''_a,t+1})$. Because the algorithm in step $t+1$ uses $\mathbb{P}_{t+1} = N_{j''}$, $\hat{x}_{j''_o,t+1} = \hat{x}^o_{j''_o}$ so that $d(\hat{x}_{j''_a,t+1}) = d(\hat{x}_{j''_o,t+1})$. Thus there exists $\hat{\mathbf{x}}' \in \hat{\mathbf{X}}$, such that $i_{\hat{\mathbf{x}}}(k) = i_{\hat{\mathbf{x}}'}(k)$ for $\forall k \in \{0, \dots, t\}$ and $i_{\hat{\mathbf{x}}'}(t+1) = j''_o$, created by dropping j''_o instead of j''_a in step $t+1$. Because $o(\hat{\mathbf{x}}, t) = o(\hat{\mathbf{x}}^o, t) = o(\hat{\mathbf{x}}', t)$ and $i_{\hat{\mathbf{x}}'}(t+1) = j''_o$, $o(\hat{\mathbf{x}}', t+1) = o(\hat{\mathbf{x}}^o, t+1)$, which implies $\hat{\mathbf{x}}' \in \hat{\mathbf{X}}^{t+1}$, a contradiction to $\hat{\mathbf{X}}^{t+1} = \emptyset$. □

A1.6 Proof of Lemma 4

By Lemma 3, it suffices to show the lemma only for $\delta \in (0,1)$, $1 > 2\delta r_a$ and $1 > 2\delta r_b$; if $\delta = 0$ or $\delta \in (0,1)$ and $1 \leq 2\delta r_g$ for some $g \in \{a, b\}$, then Algorithm 1 produces unique $\hat{\mathbf{x}}$. Fix $\hat{\mathbf{x}}$ from Algorithm 1 applied to $\mathcal{G}$ with $\mathbf{x}$ and assume another $\hat{\mathbf{x}}'$ produced by the algorithm exists.

We follow the steps of Algorithm 1 when producing $\hat{\mathbf{x}}$. In step 0, the algorithm sets $\hat{x}_m = x_m$. From $1 > 2\delta r_a$ and $1 > 2\delta r_b$, $\mathbb{R}_t = \emptyset$ for any remaining step $t \in \{1, \dots, n-1\}$. Because $\hat{\mathbf{x}}'$ exists, there must be step t' at which the algorithm calculates $\hat{x}_{i',t'}$ and $\hat{x}_{j',t'}$ with $d(\hat{x}_{i',t'}) = d(\hat{x}_{j',t'})$, drops i' and retains j'. Suppose t' is the first such step, that is in all steps $t \in \{0, \dots, t'-1\}$ the algorithm uniquely selects a player to drop. Assume $i' \in N_a$. When $i' \in N_b$ the argument is similar and omitted.

We start construction of the claimed perturbation $\mathbf{x}(\epsilon)$ by setting $x_i(\epsilon) = x_i$ for $\forall i \in N \setminus \{i'\}$ and $x_{i'}(\epsilon) = x_{i'} - \epsilon$.[39] Because $x_{i'-1} < x_{i'}$, there exists $\bar{\epsilon} > 0$ such that $x_{i'-1}(\epsilon) < x_{i'}(\epsilon)$

[39] If $i' \in N_b$ the perturbation required is $x_{i'}(\epsilon) = x_{i'} + \epsilon$.

for $\forall\epsilon \in (0,\bar{\epsilon})$. Clearly, $\lim_{\epsilon\to 0}\mathbf{x}(\epsilon) = \mathbf{x}$. We claim that there exists $\bar{\epsilon} > 0$ such that for $\forall\epsilon \in (0,\bar{\epsilon})$, Algorithm 1 applied to $\mathbf{x}(\epsilon)$ drops players in the same order as Algorithm 1 applied to $\mathbf{x}$, uniquely selects player i' to drop in step t', and produces $\hat{\mathbf{x}}(\epsilon)$ such that $\lim_{\epsilon\to 0}\hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.

To see that players are dropped in the same order for $\mathbf{x}$ and $\mathbf{x}(\epsilon)$, we know that in any step $t \in \{0,\dots,t'-1\}$ Algorithm 1 applied to $\mathbf{x}$ uniquely selects a player to drop and does not drop player i'. This implies that, for $\forall t \in \{0,\dots,t'-1\}$, there exists $i \in \mathbb{P}_t$ such that $d(\hat{x}_{i,t}) < d(\hat{x}_{i',t}) = d(x_{i'})(1-2\delta r_{t,b})$. Because the perturbation affects only the bliss point of player i', we have, for $\forall t \in \{0,\dots,t'-1\}$, $\hat{x}_{i,t}(\epsilon) = \hat{x}_{i,t}$ for $\forall i \in \mathbb{P}_t \setminus \{i'\}$ and $d(\hat{x}_{i',t}(\epsilon)) = (d(x_{i'}) - \epsilon)(1-2\delta r_{t,b})$. Clearly, there exists $\bar{\epsilon} > 0$ such that $\forall\epsilon \in (0,\bar{\epsilon})$ and $\forall t \in \{0,\dots,t'-1\}$, there exists $i \in \mathbb{P}_t$ such that $d(\hat{x}_{i,t}(\epsilon)) < d(\hat{x}_{i',t}(\epsilon))$. That is, players are dropped in the same order for $\mathbf{x}$ and $\mathbf{x}(\epsilon)$ in steps $t \in \{0,\dots,t'-1\}$. The same holds for steps $t \in \{t'+1,\dots,n-1\}$, because the perturbation does not affect the bliss points of any of the players still in the algorithm in these steps. What remains is to show that Algorithm 1 applied to $\mathbf{x}(\epsilon)$ drops player i' in step t'. To see this, we know that $d(\hat{x}_{i',t'}(\epsilon)) < d(\hat{x}_{i',t'})$, $d(\hat{x}_{i',t'}) = d(\hat{x}_{j',t'})$ and $d(\hat{x}_{j',t'}) = d(\hat{x}_{j',t'}(\epsilon))$. This implies $d(\hat{x}_{i',t'}(\epsilon)) < d(\hat{x}_{j',t'}(\epsilon))$ so that i' is dropped in step t'. Because $d(\hat{x}_{i',t'}(\epsilon)) < d(\hat{x}_{j',t'}(\epsilon))$, the algorithm uniquely selects player i' to drop in step t' and since the perturbation affects only the bliss point of player i', clearly $\lim_{\epsilon\to 0}\hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.

We followed Algorithm 1 when producing $\hat{\mathbf{x}}$ until step t', the first step at which the algorithm offers an option regarding the player to drop. At that point we constructed $\mathbf{x}(\epsilon)$ such that the algorithm applied to $\mathbf{x}(\epsilon)$ drops a unique player in step t' and the order of players dropped is the same for $\mathbf{x}$ and $\mathbf{x}(\epsilon)$. We can now proceed iteratively, find step $t'' > t'$, the second step of the algorithm applied to $\mathbf{x}$ at which it gives an option regarding the player to drop, and set $x_{i''}(\epsilon) = x_{i''} - \epsilon$ in $\mathbf{x}(\epsilon)$ for player i'' dropped in step t''. The order of players dropped again remains the same and the algorithm drops a unique player i'' in step t'' when constructing $\hat{\mathbf{x}}(\epsilon)$. □

A1.7 Proof of Proposition 3

From Definition 3 of SMPE, the profile of strategies $\hat{\sigma}$ constitutes an SMPE, by the one-stage-deviation principle, if $\hat{\sigma}$ induces $U_i(\hat{\sigma})$ for $\forall i \in N$ and $\mathcal{A}(\hat{\sigma})$ such that the set of optimal proposal strategies, arising from maximization of $U_i(\hat{\sigma})$ on $\mathcal{A}(\hat{\sigma})$ for any given status-quo, includes $\hat{\sigma}$.

Fix the profile of strategic bliss points $\hat{\mathbf{x}}$ from Algorithm 1 and the induced profile of strategies σ. Clearly, the voting strategies subsumed in σ are optimal for every player.

Because $\hat{\mathbf{x}}$ satisfies $\hat{x}_i \geq x_m$ for $\forall i \in N_a$, $\hat{x}_i \leq x_m$ for $\forall i \in N_b$ and $\hat{x}_m = x_m$, by Lemma 2, $p_i(x|\hat{x}_i) \in \mathcal{A}(x|\sigma)$ for $\forall x \in X$ and $\forall i \in N$. That is, proposals with zero probability of acceptance are never made. Also, for m we have $\hat{x}_m = x_m$, hence the proposal strategy of the median player is optimal by Lemma 2 part 5.

Now let us focus on player $i \in N_a$. The argument for $i \in N_b$ is symmetric and omitted. By Lemma 2 part 2, player i will never propose any policy $p < x_m$. Using the shape of $\mathcal{A}$ from Lemma 2 part 6, we need to make sure that proposing $d_a(x)$ for any $x \in [d_b(\hat{x}_i), d_a(\hat{x}_i)]$ and $\hat{x}_i$ otherwise is optimal for i. U_i making this proposal strategy optimal has to satisfy $U_i(x|\sigma) \leq U_i(y|\sigma)$ for any $x \in [x_m, \hat{x}_i]$ and $y \in [x_m, \hat{x}_i]$ such that $x < y$ and $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y > \hat{x}_i$. The first inequality follows from the way Algorithm 1 constructs the strategic bliss points; it generates $\hat{\mathbf{x}}$ such that $U_i'(\hat{x}_i^-|\sigma) = 0$ and $U_i'(x^-|\sigma) \geq 0$ for any $x \in \mathcal{ND}(\sigma) \cap (x_m, \hat{x}_i)$ which, combined with the piecewise strict concavity of U_i, shows the claim. To ensure the second inequality, notice that from (A6) we have $U_i'(x|\sigma) \leq 0$ for $x \in \mathcal{D}(\sigma)$ and $x \geq x_i$ so that $U_i(x_i|\sigma) \geq U_i(y|\sigma)$ for any $y > x_i$. Hence we need to make sure that $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in [\hat{x}_i, x_i]$ in order for σ to constitute an SMPE.

To prove that condition $\mathbb{S}$ is sufficient, part 1, first we note $U_i'(\hat{x}_i^+|\sigma) \leq 0$. When $\hat{x}_i = x_m$ Algorithm 1 drops i because $U_i'(\hat{x}_i^+|\sigma) \leq 0$. When $\hat{x}_i > x_m$ Algorithm 1 drops i because $U_i'(\hat{x}_i^-|\sigma) = 0$ and we have $U_i'(\hat{x}_i^-|\sigma) = U_i'(\hat{x}_i^+|\sigma)$ from (A6), the fact that exactly one player is dropped in any step of the algorithm, $d(\hat{x}_j) > d(\hat{x}_i)$ for any player j dropped subsequently, and from $r_{nc,b}(\hat{x}_i^-|\sigma) = r_{nc,b}(\hat{x}_i^+|\sigma)$ when $i \in N_a$ is dropped. Hence, by the piecewise strict concavity of U_i, we need to ensure that $U_i'(x^+|\sigma) \leq 0$ for $\forall x \in \mathcal{ND}(\sigma) \cap (\hat{x}_i, x_i) = \mathcal{S}_i(\sigma)$. Using (A6) this condition becomes $x - x_i - 2\delta r_{nc,b}(x^+|\sigma)(x_m - x_i) \geq 0$, which is what the condition $\mathbb{S}$ requires. Hence if $\mathbb{S}$ holds, we have $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in [\hat{x}_i, x_i]$ and σ constitutes an SMPE.

To prove that condition $\mathbb{N}$ is necessary and sufficient, part 2, we note that $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in [\hat{x}_i, x_i]$ is equivalent to $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in ((\mathcal{ND}(\sigma) \cup \mathcal{L}_i(\sigma)) \cap (\hat{x}_i, x_i)) \cup \{x_i, \hat{x}_i\} = \mathcal{N}_i(\sigma)$. To see this, take two adjacent elements of $\mathcal{ND}(\sigma)$ from $[\hat{x}_i, x_i]$, x' and x'', with $x' < x''$. If U_i has no local maximum on $[x', x'']$, that is when $[x', x''] \cap \mathcal{L}_i(\sigma) = \emptyset$, then $U_i(x'|\sigma) > U_i(x''|\sigma) \Leftrightarrow U_i(x'|\sigma) > U_i(y|\sigma)$ and $U_i(x'|\sigma) < U_i(x''|\sigma) \Leftrightarrow U_i(x'|\sigma) < U_i(y|\sigma)$ for any $y \in [x', x'']$ (equality cannot occur by the strict concavity of U_i). If U_i has local maximum on $[x', x'']$ then exactly one and we can set $x''' = [x', x''] \cap \mathcal{L}_i(\sigma)$ and proceed with a similar argument using x''' instead of x''.

To show that $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in \mathcal{N}_i(\sigma)$ is equivalent to $\mathbb{N}$, for any differentiable continuous function f, $f(x) - f(z) = [\int f'(a)da]_z^x$. When f is not differentiable at x, y, z with $x < y < z$ but possesses one-sided derivatives at x, y, z, we have $f(x) - f(z) = [\int f'(a)da]_{y^-}^{x^+} + [\int f'(a)da]_{z^-}^{y^+}$. Now, (A6) for $x > x_m$ can be rewritten as $U_i'(x|\sigma) = \frac{-2}{1-\delta r_{nc}(x|\sigma)}[x - c_i(x|\sigma)]$ where $c_i(x|\sigma) = x_i + 2\delta r_{nc,b}(x|\sigma)(x_m - x_i)$. Hence $\int U_i'(x|\sigma)dx = T_i(x|\sigma) = \frac{-2}{1-\delta r_{nc}(x|\sigma)}\left[\frac{x^2}{2} - c_i(x|\sigma)x\right]$ as $r_{nc,b}(x|\sigma)$ and $r_{nc}(x|\sigma)$ are both constant on any interval induced by $\mathcal{ND}(\sigma)$. Condition $\mathbb{N}$ then takes into account that $\mathcal{N}_i(\sigma)$ can have an arbitrary number of elements. When $\mathbb{N}$ holds, we have $U_i(\hat{x}_i|\sigma) \geq U_i(y|\sigma)$ for any $y \in [\hat{x}_i, x_i]$ and σ constitutes an SMPE. When $\mathbb{N}$ fails, we have $U_i(\hat{x}_i|\sigma) < U_i(y|\sigma)$ for some $y \in [\hat{x}_i, x_i]$ and σ cannot constitute an SMPE, as i would prefer to deviate to proposing y when the status-quo is y, as opposed to proposing $\hat{x}_i$ that σ requires. $\square$

A1.8 Proof of Proposition 4

Algorithm 1 in step t calculates

$$\begin{aligned} \hat{x}_{i,t} &= x_i + 2\delta r_{t,a}(x_m - x_i) \quad \text{if} \quad i \in N_b \\ \hat{x}_{i,t} &= x_i + 2\delta r_{t,b}(x_m - x_i) \quad \text{if} \quad i \in N_a \end{aligned} \tag{A15}$$

and drops $i \in \arg\min_{j \in \mathbb{P}_t} d(\hat{x}_{j,t})$ if $\mathbb{R}_t = \emptyset$. Throughout the proof let us assume $\delta \leq \frac{1}{2}$, so that $1 > 2\delta r_a$ and $1 > 2\delta r_b$, which implies $\mathbb{R}_t = \emptyset$.

Suppose first that $d(x_i) \neq d(x_j)$ for $\forall i \in N$ and $\forall j \in N$. Writing $d(\hat{x}_{i,t}) = d(x_i)(1 - 2\delta r_{t,a})$ for $i \in N_b$ and $d(\hat{x}_{i,t}) = d(x_i)(1 - 2\delta r_{t,b})$ for $i \in N_a$ shows that $d(\hat{x}_{i,t}) \in (d(x_i)(1 - 2\delta), d(x_i)]$ for $\forall i \in N \setminus \{m\}$ and $\forall t \in \{1, \ldots, n-1\}$. Hence there exists $\bar{\delta} \in (0, 1)$ such that for $\forall \delta \leq \bar{\delta}$, $d(x_j) < d(x_i)$ implies $d(x_j) < d(x_i)(1 - 2\delta)$ and hence $d(\hat{x}_{j,t}) < d(\hat{x}_{i,t})$ for $\forall t \in \{1, \ldots, n-1\}$. Since $d(x_i) \neq d(x_j)$ for any pair of players, Algorithm 1 for $\forall \delta \leq \bar{\delta}$ drops the player with the smallest $d(x_i)$ in step 0 and the player with the second smallest $d(x_i)$ in step 1. The algorithm continues in a similar manner, dropping the player with the t^{th} smallest $d(x_i)$ in step $t-1$, until step $n-1$ when it drops the player with the largest $d(x_i)$. Denote the profile of strategic bliss points produced for $\mathcal{G}$ with δ by $\hat{\mathbf{x}}(\delta)$ and the profile of strategies induced by $\sigma(\delta)$. Note that for $\forall \delta \leq \bar{\delta}$, $\hat{\mathbf{x}}(\delta)$ produced by Algorithm 1 is unique.

We now argue that, for $\forall \delta \leq \bar{\delta}$, $\hat{\mathbf{x}}(\delta)$ Algorithm 1 produces satisfies condition $\mathbb{S}$. Let i_t denote the player dropped in step $t \in \{0, \ldots, n-1\}$. For $i_0 = m$ we do not need to verify $\mathbb{S}$ since it does not apply to the median player. For i_{n-1}, $\hat{x}_{i_{n-1}} = x_{i_{n-1}}$ is easy to see from Algorithm 1 so that $\mathcal{S}_{i_{n-1}}(\sigma(\delta)) = \emptyset$ and condition $\mathbb{S}$ holds for i_{n-1}. For i_t with $t \in \{1, \ldots, n-2\}$, we know that $d(\hat{x}_{i_{t-1}}) \leq d(x_{i_{t-1}}) < d(\hat{x}_{i_t}) \leq d(x_{i_t}) < d(\hat{x}_{i_{t+1}}) \leq d(x_{i_{t+1}})$ for $\forall \delta \leq \bar{\delta}$ so that $\mathcal{S}_{i_t}(\sigma(\delta)) = \emptyset$ and condition $\mathbb{S}$ holds for i_t for any $t \in \{1, \ldots, n-2\}$.

Suppose now that a pair of players $\{i', j'\}$ with $d(x_{i'}) = d(x_{j'})$ exists. Without loss of generality let $i' \in N_b$ and $j' \in N_a$. If there are multiple such pairs, let $\{i', j'\}$ be the one with the largest i' and hence the smallest j'. By the preceding argument, there exists $\bar{\delta} \in (0,1)$, such that for $\forall \delta \leq \bar{\delta}$ Algorithm 1 drops players $\{i'+1, \ldots, j'-1\}$ in steps $t \in \{0, \ldots, j'-i'-2\}$, drops players i' and j' in steps $t' = j'-i'-1$ and $t'+1$, and drops players $\{1, \ldots, i'-1\} \cup \{j'+1, \ldots, n\}$ in steps $t \in \{t'+2, \ldots, n-1\}$. Moreover, for $\forall \delta \leq \bar{\delta}$, $d(x_i) < d(\hat{x}_{i'})$ and $d(x_i) < d(\hat{x}_{j'})$ for $\forall i \in \{i'+1, \ldots, j'-1\}$ and $d(x_{i'}) = d(x_{j'}) < d(\hat{x}_i)$ for $\forall i \in \{1, \ldots, i'-1\} \cup \{j'+1, \ldots, n\}$. This implies that condition $\mathbb{S}$ holds for $\forall i \in \{i'+1, \ldots, j'-1\}$ and that $\mathcal{S}_{i'}(\sigma(\delta))$ and $\mathcal{S}_{j'}(\sigma(\delta))$ include at most unique element $d_b(\hat{x}_{j'})$ and $d_a(\hat{x}_{i'})$ respectively.

We now need to verify condition $\mathbb{N}$ for i' and j'. Suppose i' has been dropped in step t' and j' in step $t'+1$. In step t' of the algorithm, $\mathbb{P}_{t'} = \{1, \ldots, i'\} \cup \{j', \ldots, n\}$, $r_{t',b} = \sum_{k=1}^{i'} r_k$ and $r_{t',a} = \sum_{k=j'}^{n} r_k$ and i' can be dropped only if $r_{t',b} \leq r_{t',a}$. This implies

$$\begin{aligned} \hat{x}_{i'} &= x_{i'} + 2\delta r_{t',a}(x_m - x_{i'}) \\ \hat{x}_{j'} &= x_{j'} + 2\delta(r_{t',b} - r_{i'})(x_m - x_{j'}) \end{aligned} \tag{A16}$$

which gives $d_b(\hat{x}_{j'}) = x_{i'} + 2\delta(r_{t',b} - r_{i'})(x_m - x_{i'})$ from $d(x_{i'}) = d(x_{j'}) \Leftrightarrow (x_m - x_{i'}) = -(x_m - x_{j'})$. Because $x_{i'} \leq d_b(\hat{x}_{j'}) < \hat{x}_{i'}$, it is easy to see that $d_a(\hat{x}_{i'}) < \hat{x}_{j'} \leq x_{j'}$. If $\hat{x}_{j'} = x_{j'}$, $\mathcal{S}_{i'}(\sigma(\delta)) = \mathcal{S}_{j'}(\sigma(\delta)) = \emptyset$ so that condition $\mathbb{S}$ and hence $\mathbb{N}$ holds for i' and j'. Suppose $\hat{x}_{j'} < x_{j'}$. Then $\mathcal{S}_{i'}(\sigma(\delta)) = \{d_b(\hat{x}_{j'})\}$ and $\mathcal{S}_{j'}(\sigma(\delta)) = \emptyset$ and we need to verify condition $\mathbb{N}$ for i'. Denote

$$\begin{aligned} z_0 &= x_{i'} + 2\delta r_{t',a}(x_m - x_{i'}) & z_2 &= x_{i'} + 2\delta(r_{t',a} - r_{j'})(x_m - x_{i'}) \\ z_1 &= x_{i'} + 2\delta(r_{t',b} - r_{i'})(x_m - x_{i'}) & z_3 &= x_{i'} \end{aligned}$$

and note that $z_0 = \hat{x}_{i'}$ and $z_1 = d_b(\hat{x}_{j'})$. From definitions of $r_{nc,a}$ and $r_{nc,b}$, $r_{nc,a}(x|\sigma(\delta)) = r_{t',a}$ for $\forall x \in (z_1, z_0)$, $r_{nc,a}(x|\sigma(\delta)) = r_{t',a} - r_{j'}$ for $\forall x \in (z_3, z_1)$ and $r_{nc,b}(x|\sigma(\delta)) = r_{t',b} - r_{i'}$ for $\forall x \in (z_3, z_1) \cup (z_1, z_0)$.

To verify condition $\mathbb{N}$ for i', we first verify condition $\mathbb{S}$, which suffices for $\mathbb{N}$, and only when it fails we directly verify $\mathbb{N}$. From $\mathcal{S}_{i'}(\sigma(\delta)) = \{d_b(\hat{x}_{j'})\}$, condition $\mathbb{S}$ for i' writes

$$d_b(\hat{x}_{j'}) - x_{i'} - 2\delta(r_{t',a} - r_{j'})(x_m - x_{i'}) \leq 0 \tag{A17}$$

which is equivalent to $2\delta(x_m - x_{i'})(r_{t',b} - r_{t',a} + r_{j'} - r_{i'}) \leq 0$. The condition holds if $r_{j'} \leq r_{i'}$ because $r_{t',b} \leq r_{t',a}$ and $x_m - x_{i'} > 0$. Assume $r_{j'} > r_{i'}$ and that condition $\mathbb{S}$ fails for i', that is $r_{t',b} - r_{t',a} + r_{j'} - r_{i'} > 0$. Because $r_{t',a} > r_{t',b} - r_{i'}$, we have $r_{t',a} > r_{t',b} - r_{i'} > r_{t',a} - r_{j'} \geq 0$ so that $z_0 > z_1 > z_2 \geq z_3$. To verify condition $\mathbb{N}$, $\mathcal{N}_{i'}(\sigma(\delta)) = \{z_0, z_1, z_2, z_3\}$ when $z_2 > z_3$

and $\mathcal{N}_{i'}(\sigma(\delta)) = \{z_0, z_1, z_2\}$ when $z_2 = z_3$ is easy to see from the definition of $\mathcal{N}_i$. Direct substitution of expressions for $r_{nc,a}$ and $r_{nc,b}$ into $T_{i'}(x|\sigma(\delta))$ gives

$$\begin{aligned} T_{i'}(x|\sigma(\delta)) &= -\tfrac{2}{1-\delta(r_{t',a'}+r_{t',b}-r_{i'})}\left[\frac{x^2}{2} - x\cdot z_0\right] \qquad \text{if } x \in (z_1, z_0) \\ T_{i'}(x|\sigma(\delta)) &= -\tfrac{2}{1-\delta(r_{t',a'}+r_{t',b}-r_{i'}-r_{j'})}\left[\frac{x^2}{2} - x\cdot z_2\right] \qquad \text{if } x \in (z_3, z_1). \end{aligned} \tag{A18}$$

Condition $\mathbb{N}$ writes $\sum_{j=1}^{J} T_{i'}(z_{j-1}^-|\sigma(\delta)) - T_{i'}(z_j^+|\sigma(\delta)) \geq 0$ for $J \in \{1,2,3\}$ when $z_2 > z_3$ and $J \in \{1,2\}$ when $z_2 = z_3$. Each of the (at most) three terms in the condition rewrites

$$\begin{aligned} T_{i'}(z_0^-|\sigma(\delta)) - T_{i'}(z_1^+|\sigma(\delta)) &= \frac{(z_0 - z_1)^2}{1-\delta(r_{t',a}+r_{t',b}-r_{i'})} \\ T_{i'}(z_1^-|\sigma(\delta)) - T_{i'}(z_2^+|\sigma(\delta)) &= \frac{-(z_1 - z_2)^2}{1-\delta(r_{t',a}+r_{t',b}-r_{i'}-r_{j'})} \\ T_{i'}(z_2^-|\sigma(\delta)) - T_{i'}(z_3^+|\sigma(\delta)) &= \frac{(z_2 - z_3)^2}{1-\delta(r_{t',a}+r_{t',b}-r_{i'}-r_{j'})} \end{aligned} \tag{A19}$$

When $z_2 > z_3$, the first and the third term are clearly positive. Condition $\mathbb{N}$ thus holds if $T_{i'}(z_0^-|\sigma(\delta)) - T_{i'}(z_1^+|\sigma(\delta)) + T_{i'}(z_1^-|\sigma(\delta)) - T_{i'}(z_2^+|\sigma(\delta)) \geq 0$. The same condition applies when $z_2 = z_3$ as the first term is positive and the third term is not part of condition $\mathbb{N}$. Dropping positive constants, the condition writes

$$\frac{(r_{t',a} - r_{t',b} + r_{i'})^2}{1-\delta(r_{t',a}+r_{t',b}-r_{i'})} - \frac{(r_{t',b} - r_{t',a} - r_{i'} + r_{j'})^2}{1-\delta(r_{t',a}+r_{t',b}-r_{i'}-r_{j'})} \geq 0. \tag{A20}$$

The denominator of the first term is smaller than the denominator of the second one, so the condition holds if

$$(r_{t',a} - r_{t',b} + r_{i'})^2 - (r_{t',b} - r_{t',a} - r_{i'} + r_{j'})^2 \geq 0 \tag{A21}$$

or $r_{i'} + r_{t',a} - r_{t',b} \geq \frac{r_{j'}}{2}$. Because $r_{t',a} \geq r_{t',b}$, $r_{i'} \geq \frac{r_{j'}}{2}$ suffices for $\mathbb{N}$ to hold for player i'.

To finish the proof, we know that if $r_{i'} \geq \frac{r_{j'}}{2}$, then condition $\mathbb{N}$ holds for i' and j' if $r_{t',a} \geq r_{t',b}$. For $r_{t',a} \leq r_{t',b}$, symmetric argument would lead to $r_{j'} \geq \frac{r_{i'}}{2}$, or $r_{i'} \leq 2r_{j'}$. These two conditions jointly require $r_{i'} \in [\frac{r_{j'}}{2}, 2r_{j'}]$. Finally, we assumed that $\{i', j'\}$ is pair of players with the largest i' among the pairs of player with $d(x_i) = d(x_j)$. The proof can now proceed to a pair of players $\{i'', j''\}$ such that $d(x_{i''}) = d(x_{j''})$ and $i'' < i'$. Identical argument gives $r_{i''} \in [\frac{r_{j''}}{2}, 2r_{j''}]$ and considering any further pair of players with $d(x_i) = d(x_j)$ leads to the very same condition. □

A1.9 Proof of Lemma 6

To prove part 1, that $\mathbb{G}_1$ implies $\mathbb{G}_2$ when $r_i \le r_{i+1}$ for $\forall i \in \{1, \dots, \frac{n-3}{2}\}$, we have for $\forall i \in \{1, \dots, \frac{n-3}{2}\}$ and $\forall j \in \{1, \dots, i\}$

$$\frac{1-2\delta r^e_{j-1}}{1-2\delta r^e_j} \overset{1}{\le} \frac{1-2\delta r^e_i}{1-2\delta r^e_{i+1}} \overset{2}{\le} \frac{x_m - x_i}{x_m - x_{i+1}} \overset{3}{\le} \frac{x_m - x_j}{x_m - x_{i+1}}. \tag{A22}$$

$\overset{2}{\le}$ is condition $\mathbb{G}_1$. $\overset{3}{\le}$ follows from $\frac{x_m - x_j}{x_m - x_{i+1}}$ decreasing in j. To see $\overset{1}{\le}$, note that $\frac{1-2\delta r^e_{i-1}}{1-2\delta r^e_i} \le \frac{1-2\delta r^e_i}{1-2\delta r^e_{i+1}}$ holds for $\forall i \in \{1, \dots, \frac{n-3}{2}\}$. It rewrites as $(r_{i+1} - r_i)(1 - 2\delta r^e_i) + 2\delta r_i r_{i+1} \ge 0$ for $i \in \{1, \dots, \frac{n-3}{2}\}$ and clearly holds when $r_i \le r_{i+1}$ for $\forall i \in \{1, \dots, \frac{n-3}{2}\}$. Subsequently $\overset{1}{\le}$ must hold for any $j \in \{1, \dots, i\}$. The outer inequality in (A22) is condition $\mathbb{G}_2$.

To prove part 2, that $\mathbb{G}_1$ implies $\mathbb{G}_2$ when $x_i - x_{i-1} \le x_{i+1} - x_i$ for $\forall i \in \{2, \dots, \frac{n-3}{2}\}$ and $\frac{1}{1-2\delta r_1} \le \frac{x_m - x_1}{x_m - x_2}$, we have for $\forall j \in \{2, \dots, \frac{n-3}{2}\}$ and $\forall i \in \{j, \dots, \frac{n-3}{2}\}$

$$\frac{1-2\delta r^e_{j-1}}{1-2\delta r^e_j} \overset{1}{\le} \frac{x_m - x_{j-1}}{x_m - x_j} \overset{2}{\le} \frac{x_m - x_j}{x_m - x_{j+1}} \overset{3}{\le} \frac{x_m - x_j}{x_m - x_{i+1}}. \tag{A23}$$

$\overset{1}{\le}$ is condition $\mathbb{G}_1$. $\overset{3}{\le}$ follows from $\frac{x_m - x_j}{x_m - x_{i+1}}$ increasing in i. To see $\overset{2}{\le}$, note that $\frac{x_m - x_{i-1}}{x_m - x_i} \le \frac{x_m - x_i}{x_m - x_{i+1}}$ holds for $\forall i \in \{2, \dots, \frac{n-3}{2}\}$. It rewrites as $(x_m - x_i)(d_{i+1} - d_i) + d_{i+1} d_i \ge 0$ for $i \in \{2, \dots, \frac{n-3}{2}\}$ where $d_i = x_i - x_{i-1}$ and clearly holds when $x_{i+1} - x_i = d_{i+1} \ge d_i = x_i - x_{i-1}$. The outer equality in (A23) is condition $\mathbb{G}_2$ except when $j = 1$ and $i \in \{1, \dots, \frac{n-3}{2}\}$. For these values of j and i, $\mathbb{G}_2$ reads $\frac{1}{1-2\delta r_1} \le \frac{x_m - x_1}{x_m - x_{i+1}}$ and holds by the virtue of $\frac{1}{1-2\delta r_1} \le \frac{x_m - x_1}{x_m - x_2}$ and the fact that the right hand side of the inequality is increasing in i. □

A1.10 Proof of Proposition 5

When $\delta = 0$ in part 1 clearly $\hat{\mathbf{x}} = \mathbf{x}$ so assume $\delta \in (0, 1)$. To show that there exist $2^{(n-1)/2}$ distinct sets of $\hat{\mathbf{x}}$ Algorithm 1 produces in a pairwise path and that any of these constitutes an SMPE, we first show that any $\hat{\mathbf{x}}$ produced has a special structure. Recall that the algorithm starts with step 0 in which it drops player m and that it finishes in $n-1$ steps. We want to show that, for any pairwise moderation inducing $\mathcal{G}$, the algorithm in every odd step $t \in \{1, 3, \dots, n-2\}$ gives an option to drop players $\{m - t', m + t'\}$ where $t' = \frac{t+1}{2}$. Dropping one of the players we want the other player to be dropped in the subsequent step $t+1$. This implies that in any odd step t, the number of players still in the algorithm is even and half come from N_a while the other half come from N_b.

Suppose the algorithm exhibited such behaviour in all steps until step $t \in \{1, 3, \dots, n-$

4} and hence already dropped players $\{m-t'+1,\dots,m+t'-1\}$. In t, the algorithm computes $\hat{x}_{i,t} = x_i + 2\delta r^e_{m-t'}(x_m - x_i)$ for all players still in the algorithm and gives an option to drop players $\{m-t', m+t'\}$. Assume, without loss of generality, that $m+t' \in N_a$ is dropped. Then in $t+1$, the algorithm computes, for the retained players,

$$\begin{aligned} \hat{x}_{i,t+1} &= x_i + 2\delta r^e_{m-t'}(x_m - x_i) && \text{if } \ i \in N_a \\ \hat{x}_{i,t+1} &= x_i + 2\delta r^e_{m-t'-1}(x_m - x_i) && \text{if } \ i \in N_b. \end{aligned} \tag{A24}$$

The algorithm at this point drops the player with $\hat{x}_{i,t+1}$ closest to x_m. There are two possible candidates, $m-t' \in N_b$ not dropped in t and $m+t'+1 \in N_a$. We want the algorithm to drop $m-t'$.[40] This will be the case whenever $x_m - \hat{x}_{m-t',t+1} \leq \hat{x}_{m+t'+1,t+1} - x_m$. This inequality rewrites as

$$\frac{1-2\delta r^e_{m-t'-1}}{1-2\delta r^e_{m-t'}} \leq \frac{x_m - x_{m-t'-1}}{x_m - x_{m-t'}} \tag{A25}$$

where we have already used $x_{m+t'+1} - x_m = x_m - x_{m-t'-1}$, which follows from the symmetry of $\mathcal{G}$. Setting $i = m-t'-1$ and using $t \in \{1,3,\dots,n-4\}$, we have $i \in \{1,\dots,\frac{n-3}{2}\}$. (A25) is thus equivalent to condition $\mathbb{G}_1$. The pairwise path through the algorithm from Definition 8 then ensures that the desired structure of $\hat{\mathbf{x}}$ arises even when (A25) holds with equality. As there are $\frac{n-1}{2}$ odd steps in the algorithm, each giving an option to drop one of two players, the multiplicity of $\hat{\mathbf{x}}$ evaluates at $2^{(n-1)/2}$.

To see that any $\hat{\mathbf{x}}$ produced constitutes an SMPE, we will show that it satisfies condition $\mathbb{S}$ when $\mathcal{G}$ induces pairwise moderation. Fix $\hat{\mathbf{x}}$ from Algorithm 1 produced for pairwise moderation inducing $\mathcal{G}$ and induced σ. Take player $i \in \{1,\dots,\frac{n-1}{2}\} = N_b$. For players in N_a the argument is symmetric and omitted. Suppose the algorithm dropped player i producing $\hat{x}_i$. The set of players dropped subsequently is $\{1,\dots,i-1\} \cup \{d^I_a(i),\dots,n\}$. Only these players can produce points in $\mathcal{ND}(\sigma)$ in the interval $[x_i, \hat{x}_i]$, that is points defining $\mathcal{S}_i(\sigma) = \mathcal{ND}(\sigma) \cap (x_i, \hat{x}_i)$ used in condition $\mathbb{S}$. Furthermore, from (A6) we know that for any $j' \in N_b$ and $i \in N_b$, $\operatorname{sgn}[U'_i(\hat{x}^-_j|\sigma)] = \operatorname{sgn}[U'_i(\hat{x}^+_j|\sigma)]$, so we will concern ourselves only with checking condition $\mathbb{S}$ for those points in $\mathcal{S}_i(\sigma)$ induced by players $j \in \{d^I_a(i),\dots,n\}$ being dropped by Algorithm 1. If condition $\mathbb{S}$ holds for these points, it must hold for all points in $\mathcal{S}_i(\sigma)$.

For any $j \in \{d^I_a(i),\dots,n\}$ Algorithm 1, by pairwise moderation, produces either $\hat{x}_j = x_j + 2\delta r^e_{d^I_b(j)}(x_m - x_j)$ or $\hat{x}_j = x_j + 2\delta r^e_{d^I_b(j)-1}(x_m - x_j)$. By the symmetry of $\mathcal{G}$ we can

[40] No condition is necessary for the final odd step, $n-2$, which is followed by the final step of the algorithm with only one player remaining.

map $\hat{x}_j$ below x_m into $d_b(\hat{x}_j) = x_{j'} + 2\delta r^e_{j'}(x_m - x_{j'})$ or $d_b(\hat{x}_j) = x_{j'} + 2\delta r^e_{j'-1}(x_m - x_{j'})$ for $j' = d^I_b(j) \in \{1, \ldots, i\}$. Condition $\mathbb{S}$ evaluated for $i \in N_b$ and $d_b(\hat{x}_j)$ becomes

$$\begin{aligned} x_{j'} + 2\delta r^e_{j'}(x_m - x_{j'}) - x_i - 2\delta r^e_{j'-1}(x_m - x_i) &\leq 0 \\ x_{j'} + 2\delta r^e_{j'-1}(x_m - x_{j'}) - x_i - 2\delta r^e_{j'-1}(x_m - x_i) &\leq 0 \end{aligned} \tag{A26}$$

where we used $r_{nc,a}(d_b(\hat{x}_j)^-|\sigma) = r^e_{j'-1}$; when j is dropped by the algorithm, $j' - 1$ players in N_a remain on the non-constant part of their strategy as we approach $d_b(\hat{x}_j)$ from below.

When $j' = i$, i must have been dropped by Algorithm 1 first out of pair $\{i, d^I_a(i)\}$ of players. This implies $d_b(\hat{x}_j) = x_{j'} + 2\delta r^e_{j'-1}(x_m - x_{j'})$ so that only the second line of (A26) applies and the left hand side equals 0. When $j' < i$ both lines of (A26) apply but from $r^e_{j'}(x_m - x_{j'}) > r^e_{j'-1}(x_m - x_{j'})$, if the first line holds the second one must hold as well. The first line rewrites as

$$\frac{1 - 2\delta r^e_{j'-1}}{1 - 2\delta r^e_{j'}} \leq \frac{x_m - x_{j'}}{x_m - x_i} \tag{A27}$$

and needs to hold for $i \in \{2, \ldots, \frac{n-1}{2}\}$ and $j' \in \{1, \ldots, i-1\}$, where we have already adjusted for the fact that we only need to take care of cases when $i > j'$. Rewriting the condition as

$$\frac{1 - 2\delta r^e_{j-1}}{1 - 2\delta r^e_{j}} \leq \frac{x_m - x_j}{x_m - x_{i+1}} \tag{A28}$$

for $\forall i \in \{1, \ldots, \frac{n-3}{2}\}$ and $\forall j \in \{1, \ldots, i\}$, we get condition $\mathbb{G}_2$.

To summarize, when $\mathcal{G}$ induces pairwise moderation, conditions $\mathbb{G}_1$ and $\mathbb{G}_2$ hold by Definition 7. Condition $\mathbb{G}_1$ implies that any $\hat{\mathbf{x}}$ produced by a pairwise path through Algorithm 1 has a special structure that allowed us to use condition $\mathbb{G}_2$ to show that condition $\mathbb{S}$ holds, which by Proposition 3 implies that σ induced by $\hat{\mathbf{x}}$ constitutes an SMPE.

What remains is to show that U_i is single peaked on X for $\forall i \in N$. For m we already know the claim is true by Lemma 2 part 5. Consider $i \in N_a$ omitting again the symmetric argument for players in N_b. By condition $\mathbb{S}$, U_i is single peaked for $x \geq x_m$. For $x \leq x_m$ and any $x \in \mathcal{D}(\sigma)$, from (A6) we need $x - x_i - 2\delta r_{nc,a}(x|\sigma)(x_m - x_i) \leq 0$. This follows from $x \leq x_m$ and $1 - 2\delta r_{nc,a}(x|\sigma) > 0$ as $r_{nc,a}(x|\sigma) \leq \frac{1}{2}$ for any symmetric $\mathcal{G}$. □

A1.11 Proof of Proposition 6

Fix $\hat{\mathbf{x}}$ produced by a pairwise path through Algorithm 1. Denote by t_i for $\forall i \in N$ step of the algorithm at which i has been dropped. Note that t_i is decreasing in i for $i \in N_b \cup \{m\}$ and increasing in i for $i \in N_a \cup \{m\}$. We construct the perturbation of $\mathbf{x}$ by $\epsilon > 0$, $\mathbf{x}(\epsilon)$, the proposition postulates as

$$\mathbf{x}(\epsilon) = \{x_1 + \tfrac{\epsilon}{t_1}, \ldots, x_{m-1} + \tfrac{\epsilon}{t_{m-1}}, x_m, x_{m+1} - \tfrac{\epsilon}{t_{m+1}}, \ldots, x_n - \tfrac{\epsilon}{t_n}\} \tag{A29}$$

where $\lim_{\epsilon \to 0} \mathbf{x}(\epsilon) = \mathbf{x}$ is immediate. Note also that there exists $\bar{\epsilon}$ such that, for $\forall \epsilon \leq \bar{\epsilon}$, $x_{m-1}(\epsilon) < x_m < x_{m+1}(\epsilon)$ and hence $x_i(\epsilon) < x_{i+1}(\epsilon)$ for $\forall i \in N \setminus \{n\}$.

We now show that Algorithm 1 for $\mathcal{G}(\epsilon) = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X \rangle$ produces unique $\hat{\mathbf{x}}(\epsilon)$ and that the order in which players are dropped during construction of $\hat{\mathbf{x}}(\epsilon)$ and $\hat{\mathbf{x}}$ is the same. Recall that, when producing $\hat{\mathbf{x}}$, Algorithm 1 in step $t \in \{1, 3, \ldots, n-2\}$ dropped one of players from $\{m - t', m + t'\}$ where $t' = \frac{t+1}{2}$ and the other player in step $t+1$. We need to show the algorithm (uniquely) mimics this behaviour when constructing $\hat{\mathbf{x}}(\epsilon)$.

Assume the algorithm has done so until step $t \in \{1, 3, \ldots, n-2\}$ and hence has already dropped players $\{m - t' + 1, \ldots, m + t' - 1\}$. In t, the algorithm computes $\hat{x}_{i,t}(\epsilon) = x_i + 2\delta r^e_{m-t'}(x_m - x_i) + \frac{\epsilon}{t_i}(1 - 2\delta r^e_{m-t'})$ for $\forall i \in N_b$ and $\hat{x}_{i,t}(\epsilon) = x_i + 2\delta r^e_{m-t'}(x_m - x_i) - \frac{\epsilon}{t_i}(1 - 2\delta r^e_{m-t'})$ for $\forall i \in N_a$. Only players $m - t' \in N_b$ or $m + t' \in N_a$ can be dropped in t and we need to show the former is dropped if $t_{m-t'} < t_{m+t'}$ and the latter is dropped if $t_{m-t'} > t_{m+t'}$. Calculating $d(\hat{x}_{m-t',t}(\epsilon))$ and $d(\hat{x}_{m+t',t}(\epsilon))$,

$$\begin{aligned} d(\hat{x}_{m-t',t}(\epsilon)) &= d(x_{m-t'})(1 - 2\delta r^e_{m-t'}) - \tfrac{\epsilon}{t_{m-t'}}(1 - 2\delta r^e_{m-t'}) \\ d(\hat{x}_{m+t',t}(\epsilon)) &= d(x_{m+t'})(1 - 2\delta r^e_{m-t'}) - \tfrac{\epsilon}{t_{m+t'}}(1 - 2\delta r^e_{m-t'}). \end{aligned} \tag{A30}$$

Because $d(x_{m-t'}) = d(x_{m+t'})$ and $1 - 2\delta r^e_{m-t'} > 0$, $t_{m+t'} < t_{m-t'}$ implies required $d(\hat{x}_{m+t',t}(\epsilon)) < d(\hat{x}_{m-t',t}(\epsilon))$ and $t_{m+t'} > t_{m-t'}$ implies required $d(\hat{x}_{m+t',t}(\epsilon)) > d(\hat{x}_{m-t',t}(\epsilon))$.

We now show that from the pair of players $\{m - t', m + t'\}$, the one not dropped in step t is uniquely dropped in step $t+1$. Assume, without loss of generality, that $m + t' \in N_a$ is dropped in step t. In step $t+1$ the algorithm computes, for the retained players,

$$\begin{aligned} \hat{x}_{i,t+1}(\epsilon) &= x_i + 2\delta r^e_{m-t'-1}(x_m - x_i) + \tfrac{\epsilon}{t_i}(1 - 2\delta r^e_{m-t'-1}) \quad \text{if } \ i \in N_b \\ \hat{x}_{i,t+1}(\epsilon) &= x_i + 2\delta r^e_{m-t'}(x_m - x_i) - \tfrac{\epsilon}{t_i}(1 - 2\delta r^e_{m-t'}) \qquad\quad \text{if } \ i \in N_a. \end{aligned}$$

which, for the pair of players $\{m - t', m + t' + 1\}$ that can be dropped, gives

$$\begin{aligned} d(\hat{x}_{m-t',t+1}(\epsilon)) &= d(x_{m-t'})(1 - 2\delta r^e_{m-t'-1}) - \tfrac{\epsilon}{t_{m-t'}}(1 - 2\delta r^e_{m-t'-1}) \\ d(\hat{x}_{m+t'+1,t+1}(\epsilon)) &= d(x_{m+t'+1})(1 - 2\delta r^e_{m-t'}) - \tfrac{\epsilon}{t_{m+t'+1}}(1 - 2\delta r^e_{m-t'}). \end{aligned}$$

We know $d(x_{m-t'})(1 - 2\delta r^e_{m-t'-1}) \leq d(x_{m+t'+1})(1 - 2\delta r^e_{m-t'})$ because $\mathcal{G}$ induces pairwise moderation. To show $d(\hat{x}_{m-t',t+1}(\epsilon)) < d(\hat{x}_{m+t'+1,t+1}(\epsilon))$, it thus suffices to show that

$\frac{1-2\delta r^e_{m-t'-1}}{t_{m-t'}} > \frac{1-2\delta r^e_{m-t'}}{t_{m+t'+1}}$, which follows from $t_{m-t'} < t_{m+t'+1}$ and $r^e_{m-t'-1} < r^e_{m-t'}$.

Because Algorithm 1, when constructing $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}(\epsilon)$, dropped players in the identical order, we have, for any $i \in N_a$, $\hat{x}_i = x_i + 2\delta r'(x_m - x_i)$ and $\hat{x}_i(\epsilon) = x_i + 2\delta r'(x_m - x_i) - \frac{\epsilon}{t_i}(1 - 2\delta r')$, where r' is the probability the algorithm used in step t_i. Clearly $\lim_{\epsilon\to 0} \hat{x}_i(\epsilon) = \hat{x}_i$ for $\forall i \in N_a$. Using a similar argument for $i \in N_b$ and noting $\hat{x}_m = \hat{x}_m(\epsilon)$ shows $\lim_{\epsilon\to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.

To show that $\hat{\mathbf{x}}(\epsilon)$ satisfies condition $\mathbb{S}$, take player $i \in \{1, \ldots, \frac{n-1}{2}\} = N_b$. For players in N_a the argument is symmetric and omitted. The set of players subsequently dropped is $\{1, \ldots, i-1\} \cup \{d^I_a(i), \ldots, n\}$. Using a similar argument as in the proof of Proposition 5, when $\hat{\mathbf{x}}(\epsilon)$ induces $\sigma(\epsilon)$, we only need to check condition $\mathbb{S}$ for those points in $\mathcal{S}_i(\sigma(\epsilon))$ induced by players $j \in \{d^I_a(i), \ldots, n\}$ being dropped by Algorithm 1. For any $j \in \{d^I_a(i), \ldots, n\}$ Algorithm 1 produces either $\hat{x}_j(\epsilon) = x_j + 2\delta r^e_{d^I_b(j)}(x_m - x_j) - \frac{\epsilon}{t_j}(1 - 2\delta r^e_{d^I_b(j)})$ or $\hat{x}_j(\epsilon) = x_j + 2\delta r^e_{d^I_b(j)-1}(x_m - x_j) - \frac{\epsilon}{t_j}(1 - 2\delta r^e_{d^I_b(j)-1})$. Mapping $\hat{x}_j(\epsilon)$ below x_m and using $j' = d^I_b(j)$ gives $d_b(\hat{x}_j(\epsilon)) = x_{j'} + 2\delta r^e_{j'}(x_m - x_{j'}) + \frac{\epsilon}{t_j}(1 - 2\delta r^e_{j'})$ or $d_b(\hat{x}_j(\epsilon)) = x_{j'} + 2\delta r^e_{j'-1}(x_m - x_{j'}) - \frac{\epsilon}{t_j}(1 - 2\delta r^e_{j'-1})$. Condition $\mathbb{S}$ evaluated for $i \in N_b$ and $d_b(\hat{x}_j(\epsilon))$ becomes

$$\begin{aligned} x_{j'} + 2\delta r^e_{j'}(x_m - x_{j'}) - x_i - 2\delta r^e_{j'-1}(x_m - x_i) +& \\ \epsilon\left[\tfrac{1-2\delta r^e_{j'}}{t_j} - \tfrac{1-2\delta r^e_{j'-1}}{t_i}\right] &\leq 0 \\ x_{j'} + 2\delta r^e_{j'-1}(x_m - x_{j'}) - x_i - 2\delta r^e_{j'-1}(x_m - x_i) +& \\ \epsilon\left[\tfrac{1-2\delta r^e_{j'-1}}{t_j} - \tfrac{1-2\delta r^e_{j'-1}}{t_i}\right] &\leq 0 \end{aligned} \tag{A31}$$

and we know, since $\mathcal{G}$ induces pairwise moderation, that it holds for $\forall i \in \{1, \ldots, \frac{n-1}{2}\}$ and $\forall j' \in \{1, \ldots, i\}$ when $\epsilon = 0$. Noting that $t_i < t_j$ and $r^e_{j'-1} < r^e_{j'}$, each of the terms in the square brackets in the condition is non-positive, showing that condition $\mathbb{S}$ holds for $\hat{\mathbf{x}}(\epsilon)$ as well. □

A1.12 Proof of Proposition 7

Throughout the proof assume $\mathcal{G}$ is strongly symmetric with $n \geq 5$ and $\delta \geq \frac{n}{n+1}$. Algorithm 1 in step 0 sets $\hat{x}_m = x_m$ and in step 1 gives an option to drop one of the players in $\{m-1, m+1\}$. For these two players

$$\begin{aligned} \hat{x}_{m-1,1} &= x_{m-1} + 2\delta \tfrac{n-1}{2}\tfrac{1}{n}(x_m - x_{m-1}) \\ \hat{x}_{m+1,1} &= x_{m+1} + 2\delta \tfrac{n-1}{2}\tfrac{1}{n}(x_m - x_{m+1}) \end{aligned} \tag{A32}$$

and $d(\hat{x}_{m-1,1}) = d(\hat{x}_{m+1,1})$ follows from the strong symmetry of $\mathcal{G}$. Assume the algorithm drops $m-1$. The argument for $m+1$ is symmetric and omitted.

We claim the algorithm in steps $t \in \{2, \dots, \frac{n-1}{2}\}$ drops all the remaining players from N_b. Suppose the algorithm has been dropping players from N_b until step $t-1 \in \{1, \dots, \frac{n-3}{2}\}$. We need to show that it drops the player from N_b in step $t \in \{2, \dots, \frac{n-1}{2}\}$. From $\mathbb{P}_t = N_a \cup \{1, \dots, i'\}$ where $i' = \frac{n+1}{2} - t$, $r_{t,a} = \frac{n-1}{2}\frac{1}{n}$ and $r_{t,b} = i'\frac{1}{n}$. Since only players in $\{i', m+1\}$ can be dropped, we need to show $d(\hat{x}_{m+1,t}) \geq d(\hat{x}_{i',t})$ for

$$\begin{aligned} \hat{x}_{m+1,t} &= x_{m+1} + 2\delta\tfrac{i'}{n}(x_m - x_{m+1}) \\ \hat{x}_{i',t} &= x_{i'} + 2\delta\tfrac{n-1}{2n}(x_m - x_{i'}) \end{aligned} \tag{A33}$$

for $\forall i' \in \{1, \dots, \frac{n-3}{2}\}$. Denoting $x_i - x_{i+1} = l > 0$ for $\forall i \in N \setminus \{n\}$, $d(\hat{x}_{m+1,t}) = l(1 - 2\delta\frac{i'}{n})$ and $d(\hat{x}_{i',t}) = (\frac{n+1}{2} - i')l(1 - \delta\frac{n-1}{n})$ so that $d(\hat{x}_{m+1,t}) \geq d(\hat{x}_{i',t})$ is equivalent to $\delta \geq \frac{n}{n+1}$. When $\delta = \frac{n}{n+1}$, the algorithm gives an option to drop i' or $m+1$ and we assume the former player is dropped. When $\delta > \frac{n}{n+1}$ the algorithm uniquely selects player i' to drop.

Because the algorithm drops all players from N_b in steps $t \in \{1, \dots, \frac{n-1}{2}\}$, in steps $t \in \{\frac{n-1}{2} + 1, \dots, n-1\}$ it drops all players from N_a. The resulting $\hat{\mathbf{x}}$ thus satisfies $\hat{x}_m = x_m$, $\hat{x}_i = x_i$ for $\forall i \in N_a$ and $\hat{x}_i = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for $\forall i \in N_b$. To finish the proof of part 1, what remains is to show that $d(\hat{x}_i) \in (0, d(x_{m-1}))$ for $\forall i \in N_b$. Because $d(\hat{x}_i) = d(x_i)(1 - \delta\frac{n-1}{n})$, $d(\hat{x}_i) > 0$ for $\forall i \in N_b$ is immediate. To show $d(\hat{x}_i) < d(x_{m-1}) = d(x_{m+1})$, it suffices to show $d(\hat{x}_1) < d(x_{m+1})$ since $d(\hat{x}_i) \leq d(\hat{x}_1)$ for $\forall i \in N_b$. $d(\hat{x}_1) < d(x_{m+1})$ then follows from the fact that in step $\frac{n-1}{2}$ the algorithm dropped player 1 due to $d(\hat{x}_{1,\frac{n-1}{2}}) \leq d(\hat{x}_{m+1,\frac{n-1}{2}}) < d(x_{m+1})$.

Fix $\hat{\mathbf{x}}$ with $\hat{x}_m = x_m$, $\hat{x}_i = x_i$ for $\forall i \in N_a$ and $\hat{x}_i = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for $\forall i \in N_b$ and σ it induces. By the strong symmetry of $\mathcal{G}$ we have $\mathcal{ND}(\sigma) = \{x_1, \dots, x_{m-1}, \hat{x}_1, \dots, \hat{x}_{m-1}, x_m, \dots\}$ and from definitions $r_{nc,a}(x|\sigma) = \frac{j}{n}$ and $r_{nc,b}(x|\sigma) = 0$ for $\forall x \in (x_j, x_{j+1})$ where $j \in \{1, \dots, \frac{n-3}{2}\}$. Furthermore, $r_{nc,a}(x|\sigma) = \frac{n-1}{2n}$ for $\forall x \in (x_{m-1}, x_m) \setminus \mathcal{ND}(\sigma)$. To prove part 2, we need to show σ constitutes an SMPE.

For players in $N_a \cup \{m\}$, the optimality of their strategies is easy to see; x_m is clearly optimal for m and for any $i \in N_a$, $\mathcal{S}_i(\sigma) = \emptyset$ follows from $x_i = \hat{x}_i$ and hence condition $\mathbb{S}$ holds for any $i \in N_a$. Because condition $\mathbb{S}$ in general fails for players in N_b, we need to check condition $\mathbb{N}$ for these players.

We now argue that it suffices to check condition $\mathbb{N}$ for player 1. To see this we first claim that $U_i(\hat{x}_1|\sigma) \leq U_i(\hat{x}_i|\sigma)$ for $\forall i \in N_b$. The claim follows from the piecewise strict concavity of U_i proven in Lemma 2 part 4, $\hat{x}_i < \hat{x}_{i+1}$ for $\forall i \in N_b$, $U_i'(\hat{x}_i^+|\sigma) = U_i'(\hat{x}_i^-|\sigma) = 0$ for $\forall i \in N_b$, which follows from the way Algorithm 1 constructs $\hat{x}_i$, and $\operatorname{sgn}[U_i'(\hat{x}_j^+|\sigma)] =$

$\text{sgn}\,[U_i'(\hat{x}_j^-|\sigma)]$ for $\forall i \in N_b$ and $\forall j \in N_b$, which follows from $r_{nc,a}(x|\sigma) = \frac{n-1}{2n}$ for $\forall x \in (x_{m-1}, x_m) \setminus \mathcal{ND}(\sigma)$ and inspection of (A6). Suppose now that condition $\mathbb{N}$ holds for player 1. This means $U_1(x|\sigma) \leq U_1(\hat{x}_1|\sigma)$ for $\forall x \leq \hat{x}_1$. From Lemma 2 part 5 and $\hat{x}_1 < x_m$ we know $U_m(x|\sigma) \leq U_m(\hat{x}_1|\sigma)$ for $\forall x \leq \hat{x}_1$. Using an argument similar to the one used to prove Proposition 1, we thus have $U_i(x|\sigma) \leq U_i(\hat{x}_1|\sigma)$ for $\forall x \leq \hat{x}_1$ and $\forall i \in N_b$, or, using the claim above, $U_i(x|\sigma) \leq U_i(\hat{x}_1|\sigma) \leq U_i(\hat{x}_i|\sigma)$. Thus, if condition $\mathbb{N}$ holds for player 1 it must hold for all players in N_b.

What remains is to show that condition $\mathbb{N}$ holds for player 1. The set of points in $[x_1, \hat{x}_1]$ at which U_1 is not differentiable is $\{x_1, x_2, \ldots, x_{m-1}, \hat{x}_1\}$. We first show that, for $j \in \{1, \ldots, \frac{n-3}{2}\}$, U_1 has a unique local maximizer on (x_j, x_{j+1}), which we denote by x_j'. Using $r_{nc,a}(x|\sigma) = \frac{j}{n}$ and $r_{nc,b}(x|\sigma) = 0$ for $\forall x \in (x_j, x_{j+1})$ and $\forall j \in \{1, \ldots, \frac{n-3}{2}\}$ in (A6) gives

$$U_1'(x|\sigma) = -\frac{2}{1-\delta\frac{j}{n}}\left[x - x_1 - 2\delta\tfrac{j}{n}(x_m - x_1)\right]. \tag{A34}$$

$x_j' = x_1 + 2\delta\frac{j}{n}(x_m - x_1)$ is the local maximizer of U_1 if $x_j' \in (x_j, x_{j+1})$. We will show that this is the case for $\forall j \in \{1, \ldots, \frac{n-3}{2}\}$. Noting $x_1 = x_m - \frac{n-1}{2}l$ and $x_j = x_m - (\frac{n+1}{2} - j)l$, straightforward algebra shows $x_j' < x_{j+1} \Leftrightarrow \delta < \frac{n}{n-1}$ and $x_j < x_j' \Leftrightarrow \delta > \frac{n}{n-1}\frac{j-1}{j}$. The first inequality clearly holds. The right hand side of the second inequality is increasing in j so it must hold for any j if it holds for $j = \frac{n-3}{2}$. Evaluation gives $\delta > \frac{n}{n-1}\frac{n-5}{n-3}$ and because, as is easily checked, $\frac{n}{n+1} > \frac{n}{n-1}\frac{n-5}{n-3}$, shows that the inequality holds. We thus have $\mathcal{N}_1(\sigma) = \{x_1, x_1', x_2, x_2', \ldots, x_{m-2}, x_{m-2}', x_{m-1}, \hat{x}_1\}$. We now make two claims that jointly imply that in order to check condition $\mathbb{N}$ for player 1, it suffices to ensure $U_1(x_{m-2}'|\sigma) \leq U_1(\hat{x}_1|\sigma)$, that is, to check condition $\mathbb{N}$ only for $J = 2$.

First, we claim that $\frac{x_j + x_{j+1}}{2} \leq x_j'$ for $\forall j \in \{1, \ldots, \frac{n-3}{2}\}$. The condition rewrites as $\delta \geq \frac{n}{n-1}\frac{j-\frac{1}{2}}{j}$, its right hand side is increasing in j and evaluated at $j = \frac{n-3}{2}$ reads $\delta \geq \frac{n}{n-1}\frac{n-4}{n-3}$. Below we will show that $\delta \geq \frac{n}{n-1}\frac{n-4}{n-3}$ indeed holds when $\delta \geq \bar{\delta}(n)$.

Second, we claim that $U_1'(x|\sigma) > U_1'(x + l|\sigma)$ for $\forall x \in (x_j, x_{j+1})$ and $\forall j \in \{1, \ldots, \frac{n-5}{2}\}$. Using (A34) the condition is

$$-\frac{2\left[x - x_1 - 2\delta\frac{j}{n}(x_m - x_1)\right]}{1-\delta\frac{j}{n}} > -\frac{2\left[x + l - x_1 - 2\delta\frac{j+1}{n}(x_m - x_1)\right]}{1-\delta\frac{j+1}{n}} \tag{A35}$$

and rewrites as $d(x) + d(x_1) < l(\frac{n}{\delta} - j)$. Because $d(x) \leq (\frac{n+1}{2} - j)l$ and $d(x_1) = \frac{n-1}{2}l$, $d(x) + d(x_1) \leq (n - j)l$, so the inequality holds.

From the second claim, for $\forall j \in \{1, \ldots, \frac{n-5}{2}\}$ and any $y \in [x_j, x_{j+1}]$,

$$\begin{aligned} U_1(x_{j+1}|\sigma) - U_1(y|\sigma) &= \int_y^{x_{j+1}} U_1'(z|\sigma)dz \\ &\geq \int_y^{x_{j+1}} U_1'(z+l|\sigma)dz \\ &= \int_{y+l}^{x_{j+2}} U_1'(w|\sigma)dw \\ &= U_1(x_{j+2}|\sigma) - U_1(y+l|\sigma). \end{aligned} \tag{A36}$$

From the first claim, for $\forall j \in \{1, \ldots, \frac{n-3}{2}\}$, because U_1 is quadratic on (x_j, x_{j+1}) and hence symmetric about $x_j' \geq \frac{x_j + x_{j+1}}{2}$, $U_1(x_j|\sigma) \leq U_1(x_{j+1}|\sigma)$. Combining the inequalities we get

$$0 \geq U_1(x_{j+1}|\sigma) - U_1(x_{j+2}|\sigma) \geq U_1(y|\sigma) - U_1(y+l|\sigma) \tag{A37}$$

so that $U_1(y+l|\sigma) \geq U_1(y|\sigma)$ for $\forall y \in [x_j, x_{j+1}]$ and $\forall j \in \{1, \ldots, \frac{n-5}{2}\}$. Since $U_1(x_{m-2}'|\sigma) \geq U_1(y|\sigma)$ for $\forall y \in [x_{m-2}, x_{m-1}]$ and $y + l \in [x_{m-2}, x_{m-1}]$ when $y \in [x_{m-3}, x_{m-2}]$, it must be the case that $U_1(x_{m-2}'|\sigma) \geq U_1(y|\sigma)$ for $\forall y \in [x_1, x_{m-1}]$. Hence, if we prove $U_1(x_{m-2}'|\sigma) \leq U_1(\hat{x}_1|\sigma)$, we can conclude that $U_1(x|\sigma) \leq U_1(\hat{x}_1|\sigma)$ for $\forall x \in \mathcal{N}_1(\sigma)$, that is, that condition $\mathbb{N}$ holds for player 1.

To prove $U_1(x_{m-2}'|\sigma) \leq U_1(\hat{x}_1|\sigma)$, we evaluate condition $\mathbb{N}$ for $J = 2$. For $x \in (x_{m-1}, \hat{x}_1)$, $c_1(x|\sigma) = \hat{x}_1$ and $T_1(x|\sigma) = -\frac{2}{1-\delta\frac{n-1}{2n}}\left[\frac{x^2}{2} - c_1(x|\sigma)x\right]$. For $x \in (x_{m-2}, x_{m-1})$, $c_1(x|\sigma) = x_{m-2}'$ and $T_1(x|\sigma) = -\frac{2}{1-\delta\frac{n-3}{2n}}\left[\frac{x^2}{2} - c_1(x|\sigma)x\right]$. Substitution into the condition gives

$$\begin{aligned} \Big[T_1(x|\sigma)\Big]_{x_{m-1}^+}^{\hat{x}_1^-} &= \frac{1}{1-\delta\frac{n-1}{2n}}(\hat{x}_1 - x_{m-1})^2 \\ \Big[T_1(x|\sigma)\Big]_{x_{m-2}'^+}^{x_{m-1}^-} &= -\frac{1}{1-\delta\frac{n-3}{2n}}(x_{m-1} - x_{m-2}')^2 \end{aligned} \tag{A38}$$

and $\hat{x}_1 - x_{m-1} = -l(\frac{n-3}{2} - 2\delta\frac{n-1}{2n}\frac{n-1}{2})$, $x_{m-2}' - x_{m-1} = -l(\frac{n-3}{2} - 2\delta\frac{n-3}{2n}\frac{n-1}{2})$. The condition reads $\Big[T_1(x|\sigma)\Big]_{x_{m-1}^+}^{\hat{x}_1^-} + \Big[T_1(x|\sigma)\Big]_{x_{m-2}'^+}^{x_{m-1}^-} \geq 0$, which after some algebra rewrites as $\delta \geq \delta'(n) = \frac{n}{n-3}\left[2\frac{n-2}{n-1} - \sqrt{\frac{n^3-n^2-n-7}{(n-1)^3}}\right]$. Checking that $1 > \delta'(n) > \frac{n}{n-1}\frac{n-4}{n-3}$ for $\forall n \geq 5$ is routine algebra. The proposition claims the first inequality and we required $\delta \geq \frac{n}{n-1}\frac{n-4}{n-3}$ above. Because $\delta'(n) < \frac{n}{n+1}$ holds if and only if $n = 5$, $\delta \geq \bar{\delta}(n) = \max\{\frac{n}{n+1}, \delta'(n)\}$ guarantees that condition $\mathbb{N}$ holds for all players in N_b and hence σ constitutes an SMPE. □

A1.13 Proof of Proposition 8

Throughout the proof assume $\mathcal{G}$ is strongly symmetric with $n = 5$ and $\delta > \frac{n}{n+1}$ or with $n \geq 7$ and $\delta \geq \bar{\delta}(n)$.[41] From Proposition 7, there exist two $\hat{\mathbf{x}}$ from Algorithm 1 with $\hat{x}_i = x_i$ for $\forall i \in N_g \cup \{m\}$ and $\hat{x}_i = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for $\forall i \in N \setminus (N_g \cup \{m\})$, where $g \in \{a, b\}$. From the proof of that proposition, there exists no other $\hat{\mathbf{x}}$ produced by Algorithm 1.

Fix $\hat{\mathbf{x}}$ with $g = a$, that is $\hat{\mathbf{x}}$ with $\hat{x}_i = x_i$ for $\forall i \in N_a \cup \{m\}$ and $\hat{x}_i = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for $\forall i \in N_b$. For the other profile the argument is similar and omitted. We construct the perturbation of $\mathbf{x}$ by $\epsilon > 0$, $\mathbf{x}(\epsilon)$, the proposition postulates as

$$\mathbf{x}(\epsilon) = \{x_1, \ldots, x_{m-2}, x_{m-1} + \epsilon, x_m, x_{m+1}, x_{m+2}, \ldots, x_n\} \tag{A39}$$

where $\lim_{\epsilon \to 0} \mathbf{x}(\epsilon) = \mathbf{x}$ is immediate.[42] Clearly, there exists $\bar{\epsilon}$ such that, for $\forall \epsilon \leq \bar{\epsilon}$, $x_{m-1}(\epsilon) < x_m$.

We now show that Algorithm 1 for $\mathcal{G} = \langle n, \mathbf{x}(\epsilon), \mathbf{r}, \delta, X \rangle$ produces a unique $\hat{\mathbf{x}}(\epsilon)$ with $\hat{x}_{m-1}(\epsilon) = x_{m-1} + \delta\frac{n-1}{n}(x_m - x_{m-1}) + \epsilon(1 - \delta\frac{n-1}{n})$ and $\hat{x}_i(\epsilon) = \hat{x}_i$ for $\forall i \in N \setminus \{m-1\}$. In step 1, the algorithm calculates, for the players in $\{m-1, m+1\}$ that can in principle be dropped,

$$\begin{aligned} \hat{x}_{m-1,1}(\epsilon) &= x_{m-1} + \delta\tfrac{n-1}{n}(x_m - x_{m-1}) + \epsilon(1 - \delta\tfrac{n-1}{n}) \\ \hat{x}_{m+1,1}(\epsilon) &= x_{m+1} + \delta\tfrac{n-1}{n}(x_m - x_{m+1}). \end{aligned} \tag{A40}$$

Because $d(x_{m-1}) = d(x_{m+1})$, $m - 1$ is dropped with $\hat{x}_{m-1}(\epsilon) = \hat{x}_{m-1,1}(\epsilon)$. From the arguments presented in the proof of Proposition 7, it follows that the algorithm uniquely drops all the remaining players from N_b in steps $t \in \{2, \ldots, \frac{n-1}{2}\}$ and all the players from N_a in steps $t \in \{\frac{n-1}{2} + 1, \ldots, n-1\}$. This implies $\hat{x}_i(\epsilon) = x_i + \delta\frac{n-1}{n}(x_m - x_i)$ for $\forall i \in N_b \setminus \{m-1\}$ and $\hat{x}_i(\epsilon) = x_i$ for $\forall i \in N_a$. Clearly, $\hat{x}_i(\epsilon) = \hat{x}_i$ for $\forall i \in N \setminus \{m-1\}$ and since $\lim_{\epsilon \to 0} \hat{x}_{m-1}(\epsilon) = \hat{x}_{m-1}$, $\lim_{\epsilon \to 0} \hat{\mathbf{x}}(\epsilon) = \hat{\mathbf{x}}$.

What remains is to show that $\hat{\mathbf{x}}(\epsilon)$ induces $\sigma(\epsilon)$ that supports SMPE. The argument is essentially identical to the one used to prove Proposition 7 and for space consideration we include here only the key steps. For any $i \in N_a$ condition $\mathbb{S}$ holds as $\mathcal{S}_i(\sigma(\epsilon)) = \emptyset$ and x_m is optimal for m. That $U_i(\hat{x}_1(\epsilon)|\sigma(\epsilon)) \leq U_i(\hat{x}_i(\epsilon)|\sigma(\epsilon))$ for $\forall i \in N_b$ follows by the same argument as in the proof of proposition 7 and hence condition $\mathbb{N}$ only needs to hold

[41] The perturbation of $\mathbf{x}$ we are about to construct is extremely simple when $\delta > \frac{n}{n+1}$ as Algorithm 1 produces exactly two $\hat{\mathbf{x}}$ for the unperturbed $\mathbf{x}$, giving an option regarding which player to drop only in the first step. When $\delta = \frac{n}{n+1}$, we would have to construct more complex perturbation of $\mathbf{x}$. Since $\delta \geq \bar{\delta}(n)$, which we need to show that condition $\mathbb{N}$ holds, implies $\delta > \frac{n}{n+1}$ for any $n \geq 7$, for $n = 5$ we assume $\delta > \frac{n}{n+1}$.

[42] The perturbation required for $\hat{\mathbf{x}}$ from Proposition 7 with $g = b$ would be identical except for $x_{m-1}(\epsilon) = x_{m-1}$ and $x_{m+1}(\epsilon) = x_m - \epsilon$.

for player 1 for $\sigma(\epsilon)$ to support SMPE. That the condition indeed holds when $\delta \geq \bar{\delta}(n)$ follows again by the argument used in the proof of Proposition 7. The entire argument there relied only on the derivative of U_1 and it is easy to see that $U_1'(x|\sigma(\epsilon)) = U_1'(x|\sigma)$, where σ is induced by $\hat{\mathbf{x}}$, for $\forall x \in [x_1, \hat{x}_1] \setminus \mathcal{ND}(\sigma) = [x_1(\epsilon), \hat{x}_1(\epsilon)] \setminus \mathcal{ND}(\sigma(\epsilon))$. □

A1.14 Proof of Proposition 9

Take $\mathcal{G}$ that induces pairwise moderation. From the proof of Proposition 5, we know that for any pair of players $\{i, i'\}$ with $i \in \{1, \ldots, \frac{n-1}{2}\}$ and $i' = d_a^I(i)$, a pairwise path through Algorithm 1 produces one of the following pairs of SMPE strategic bliss points

$$\begin{aligned}(\hat{x}_i, \hat{x}_{i'}) &= (x_i + 2\delta r_{i-1}^e(x_m - x_i), x_{i'} + 2\delta r_i^e(x_m - x_{i'})) \\ (\hat{x}_i', \hat{x}_{i'}') &= (x_i + 2\delta r_i^e(x_m - x_i), x_{i'} + 2\delta r_{i-1}^e(x_m - x_{i'}))\end{aligned} \tag{A41}$$

and we have, by symmetry of $\mathcal{G}$,

$$\begin{aligned}d(\hat{x}_i) = d(\hat{x}_{i'}') &= (x_m - x_i)(1 - 2\delta r_{i-1}^e) \\ d(\hat{x}_{i'}) = d(\hat{x}_i') &= (x_m - x_i)(1 - 2\delta r_i^e).\end{aligned} \tag{A42}$$

Denote the profile of strategic bliss points related to the first pair in (A41) by $\hat{\mathbf{x}}$ with associated σ, and associate $\hat{\mathbf{x}}'$ and σ' with the second pair. Assume $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ differ only in terms of the strategic bliss points of $\{i, i'\}$ and note that because $\mathcal{G}$ is symmetric, $r_i = r_{i'}$. If, for some status-quo x, both i and i' propose their strategic bliss points, we have $r_i d(\hat{x}_i) + r_{i'} d(\hat{x}_{i'}) = r_i d(\hat{x}_i') + r_{i'} d(\hat{x}_{i'}')$ and hence $\mathbb{E}[d(p(x|\sigma))] = \mathbb{E}[d(p(x|\sigma'))]$. The same equality holds if x is such that i and i' propose $d_b(x)$ and $d_a(x)$ respectively under σ, they propose the same policies under σ'. If x is such that i and i' propose $d_b(x)$ and $\hat{x}_{i'}$ respectively under σ, the only remaining case possible as $d(\hat{x}_i) > d(\hat{x}_{i'})$, they propose $\hat{x}_i'$ and $d_a(x)$ under σ' and we have $r_i d(d_b(x)) + r_{i'} d(\hat{x}_{i'}) = r_i d(\hat{x}_i') + r_{i'} d(d_a(x))$. If $\hat{\mathbf{x}}$ and $\hat{\mathbf{x}}'$ differ in terms of other pairs of players, we repeat the same argument. Hence $\mathbb{E}[d(p(x|\sigma))] = \mathbb{E}[d(p(x|\sigma'))]$.

Finally, from $d(\hat{x}_i) = d(x_i)(1-2\delta r_{i-1}^e)$ and $d(\hat{x}_{i'}) = d(x_i)(1-2\delta r_i^e)$, it is straightforward that $\mathbb{E}[d(p(x|\sigma))]$ is non-increasing in δ and r_i for $i \neq m$ and non-decreasing in $d(x_i)$. □

A1.15 Proof of Proposition 10

Part 1 follows from the shape of the acceptance set $\mathcal{A}(x|\sigma) = [d_b(x), d_a(x)]$ for any status-quo $x \in X$ and any SMPE σ from Proposition 5. To see part 2, note that under the simple proposal strategies from Definition 4, every player $i \in N$ for any status-quo $x \in X$ proposes either her strategic bliss point $\hat{x}_i$ or policy in $\{d_b(x), d_a(x)\}$. The claim then follows from $d(x) = d(d_b(x)) = d(d_a(x))$. For part 3, we have $p_i(x|\sigma) \neq x_m$ for $\forall i \in N \setminus \{m\}$ and $\forall x \in X \setminus \{x_m\}$. Hence $\mathbb{P}[d(p_t) > 0]$ is equal to the probability that, starting with status-quo $x \neq x_m$, m has not been recognized to propose in periods $\{0, 1, \ldots, t\}$, which is $(1 - r_m)^{t+1}$. For part 4, as $d(p_t)$ is non-increasing in t for any path of proposer identities (part 1), the number of players proposing, for status-quo p_{t-1}, p_t with $d(p_t) = d(p_{t-1})$ is non-decreasing and so is the sum of their recognition probabilities. Finally, part 5 follows from the fact that for any status-quo $x \neq x_m$, all the players in N_a propose policy strictly above x_m and all the players in N_b propose policy strictly below x_m. □

A1.16 Proof of Proposition 11

To prove part 1, we need to consider several cases.

Case 1: When $d(x_1) = d(x_3)$, Algorithm 1 produces a profile of strategic bliss points $\hat{\mathbf{x}}$ either with $\hat{x}_1 = x_1$ and $\hat{x}_3 = \max\{x_2, x_3 + 2\delta r_1(x_2 - x_3)\}$ or with $\hat{x}_1 = \min\{x_2, x_1 + 2\delta r_3(x_2 - x_1)\}$ and $\hat{x}_3 = x_3$ (when $r_1 = r_3$ both are possible, when $r_1 \neq r_3$ only one is). In either case $\mathcal{S}_i(\sigma) = \emptyset$ for $i \in \{1, 3\}$, condition $\mathbb{S}$ holds and σ induced by $\hat{\mathbf{x}}$ constitutes an SMPE.

Case 2: When $d(x_1) \neq d(x_3)$ and $d(x_e)(1 - 2\delta r_{-e}) > d(x_{-e})(1 - 2\delta r_e)$, Algorithm 1 produces $\hat{\mathbf{x}}$ either with $\hat{x}_e = x_e$ and $\hat{x}_{-e} = x_{-e} + 2\delta r_e(x_m - x_{-e})$ or with $\hat{x}_e = x_e$ and $\hat{x}_{-e} = x_m$ (when $\delta r_e < \frac{1}{2}$ the former applies and when $\delta r_e \geq \frac{1}{2}$ the latter applies). In either case $\mathcal{S}_i(\sigma) = \emptyset$ for $i \in \{1, 3\}$, condition $\mathbb{S}$ holds and σ induced by $\hat{\mathbf{x}}$ constitutes an SMPE.

Case 3: When $d(x_1) \neq d(x_3)$ and $d(x_e)(1 - 2\delta r_{-e}) = d(x_{-e})(1 - 2\delta r_e)$, Algorithm 1 produces $\hat{\mathbf{x}}$, due to $1 - 2\delta r_i > 0$ for $i \in \{1, 3\}$ and implied $\delta r_{-e} < \frac{1}{2}$, either with $\hat{x}_e = x_e$ and $\hat{x}_{-e} = x_{-e} + 2\delta r_e(x_m - x_{-e})$ or with $\hat{x}_e = x_e + 2\delta r_{-e}(x_m - x_e)$ and $\hat{x}_{-e} = x_{-e}$. In the former case $\mathcal{S}_i(\sigma) = \emptyset$ for $i \in \{1, 3\}$, condition $\mathbb{S}$ holds and σ induced by $\hat{\mathbf{x}}$ constitutes an SMPE.[43] In the latter case, easy argument shows that condition $\mathbb{S}$ fails and we need to check condition $\mathbb{N}$ for σ induced by $\hat{x}_e = x_e + 2\delta r_{-e}(x_m - x_e)$ and $\hat{x}_{-e} = x_{-e}$. Assume that $e = 3$. The argument when $e = 1$ is symmetric and omitted. Because $\hat{x}_1 = x_1$ and $\hat{x}_3 = x_3 + 2\delta r_1(x_2 - x_3)$, we have $\mathcal{S}_1(\sigma) = \emptyset$ and $\mathcal{N}_3(\sigma) = \{\hat{x}_3, d_a(x_1), x_3\}$. That $d_a(x_1) \in (\hat{x}_3, x_3)$ follows from the conditions defining this case $d(x_3)(1 - 2\delta r_1) = d(x_1)(1 - 2\delta r_3) < d(x_1)$ and $d(x_3) > d(x_1)$ and $\mathcal{L}_3(\sigma) \cap (\hat{x}_3, x_3) = \emptyset$ follows from $U_3'(x|\sigma) < 0$ for $\forall x \in (\hat{x}_3, d_a(x_1))$ and $U_3'(x|\sigma) > 0$ for $\forall x \in (d_a(x_1), x_3)$. To evaluate condition $\mathbb{N}$ for player 3, we have $T_3(x|\sigma) = \frac{-2}{1-\delta r_1}\left[\frac{x^2}{2} - \hat{x}_3 x\right]$ for $x \in (\hat{x}_3, d_a(x_1))$ and $T_3(x|\sigma) = -2\left[\frac{x^2}{2} - x_3 x\right]$

[43] If case 3 applies and condition $\mathbb{E}$ fails, Proposition 12 part 2 obtains. That $\hat{x}_e = x_e$ with $\hat{x}_{-e} = x_{-e} + 2\delta r_e(x_m - x_{-e})$ constitutes an SMPE follows by $\mathcal{S}_i(\sigma) = \emptyset$ for $i \in \{1, 3\}$.

for $x \in (d_a(x_1), x_3)$. Condition $\mathbb{N}$ then rewrites as

$$\begin{aligned} &\frac{-2}{1-\delta r_1}\left[\frac{x^2}{2}-\hat{x}_3 x\right]_{d_a(x_1)^-}^{\hat{x}_3^+} \geq 0 \\ &\frac{-2}{1-\delta r_1}\left[\frac{x^2}{2}-\hat{x}_3 x\right]_{d_a(x_1)^-}^{\hat{x}_3^+} - 2\left[\frac{x^2}{2}-x_3 x\right]_{x_3^-}^{d_a(x_1)^+} \geq 0. \end{aligned} \tag{A43}$$

The first inequality rewrites as $\frac{1}{1-\delta r_1}[d(\hat{x}_3)-d(x_1)]^2 \geq 0$ and clearly holds. The second inequality rewrites as $\frac{1}{1-\delta r_1}[d(\hat{x}_3)-d(x_1)]^2 - [d(x_1)-d(x_3)]^2 \geq 0$, can be expressed as condition $\mathcal{B}_e$ for $\delta r_1 < \frac{1}{2}$ and hence holds.

Case 4: When $d(x_1) \neq d(x_3)$ and $d(x_e)(1-2\delta r_{-e}) < d(x_{-e})(1-2\delta r_e)$, Algorithm 1 produces $\hat{\mathbf{x}}$ either with $\hat{x}_e = x_e + 2\delta r_{-e}(x_m - x_e)$ and $\hat{x}_{-e} = x_{-e}$ or with $\hat{x}_e = x_m$ and $\hat{x}_{-e} = x_{-e}$ (when $\delta r_{-e} < \frac{1}{2}$ the former applies and when $\delta r_{-e} \geq \frac{1}{2}$ the latter applies). Condition $\mathbb{S}$ fails in both cases and we need to check condition $\mathbb{N}$ for σ induced by $\hat{x}_e$ and $\hat{x}_{-e}$. Assume that $e = 3$. The argument when $e = 1$ is symmetric and omitted. Because $\hat{x}_1 = x_1$ and $\hat{x}_3 = \max\{x_2, x_3 + 2\delta r_1(x_2 - x_3)\}$, we have $\mathcal{S}_1(\sigma) = \emptyset$ and $\mathcal{N}_3(\sigma) = \{\hat{x}_3, d_a(x_1), x_3\}$. That $d_a(x_1) \in (\hat{x}_3, x_3)$ follows from similar argument as in the previous case. To evaluate condition $\mathbb{N}$ for player 3, when $\delta r_1 < \frac{1}{2}$, we have the same expressions for $T_3(x|\sigma)$ as in the previous case and condition $\mathbb{N}$ thus holds by similar argument. When $\delta r_1 \geq \frac{1}{2}$, $T_3(x|\sigma) = \frac{-2}{1-\delta r_1}\left[\frac{x^2}{2} - (x_3 + 2\delta r_1(x_2 - x_3))x\right]$ for $x \in (\hat{x}_3, d_a(x_1))$ and $T_3(x|\sigma) = -2\left[\frac{x^2}{2} - x_3 x\right]$ for $x \in (d_a(x_1), x_3)$. Condition $\mathbb{N}$ rewrites as

$$\begin{aligned} &[T_3(x|\sigma)]_{d_a(x_1)^-}^{x_2^+} \geq 0 \\ &[T_3(x|\sigma)]_{d_a(x_1)^-}^{x_2^+} + [T_3(x|\sigma)]_{x_3^-}^{d_a(x_1)^+} \geq 0. \end{aligned} \tag{A44}$$

The first inequality rewrites as $\frac{d(x_1)}{1-\delta r_1}[d(x_1) - 2d(x_3)(1-2\delta r_1)] \geq 0$ and clearly holds as $1 - 2\delta r_1 \leq 0$. The second inequality rewrites as

$$\frac{d(x_1)}{1-\delta r_1}[d(x_1) - 2d(x_3)(1-2\delta r_1)] - [d(x_1) - d(x_3)]^2 \geq 0, \tag{A45}$$

can be expressed as condition $\mathcal{B}_e$ for $\delta r_1 \geq \frac{1}{2}$ and hence holds.

We leave proof of part 2, the existence of SMPE in adjusted simple proposal strategies, for the proof of Proposition 12. There we deal with the adjusted simple strategies in full detail (see footnote 45).

To prove part 3, we note that single-peakedness of U_1 on $\{x \in X | x \leq x_m\}$ and of U_3 on $\{x \in X | x \geq x_m\}$ obtains when condition $\mathbb{S}$ holds for both players for $\hat{\mathbf{x}}$ that induces

SMPE σ. Reviewing the cases above, condition $\mathbb{S}$ holds in case 1 ($d(x_1) = d(x_3)$), case 2 ($d(x_e)(1 - 2\delta r_{-e}) > d(x_{-e})(1 - 2\delta r_e)$) and in case 3 ($d(x_e)(1 - 2\delta r_{-e}) = d(x_{-e})(1 - 2\delta r_e)$) when $\hat{x}_e = x_e$, that is when the Algorithm 1 drops player $-e$ in step 1. From (A6) we then have single-peakedness of U_1 and U_3 on X when $\delta r_1 \leq \frac{1}{2}$ and $\delta r_3 \leq \frac{1}{2}$ respectively. $\square$

A1.17 Proof of Proposition 12

We start by observing that we have already proved part 2 of the proposition as part of the process of proving Proposition 11 (see footnote 43). What remains is part 1. Since condition $\mathbb{E}$ fails, $\mathcal{A}_e$ holds and $\mathcal{B}_e$ fails. Because $\mathcal{A}_e$ holds, $d(x_1) \neq d(x_3)$ and $d(x_e)(1 - 2\delta r_{-e}) \leq d(x_{-e})(1 - 2\delta r_e)$. Thus Algorithm 1 produces (dropping e in step 1, if given an option) $\hat{\mathbf{x}}$ either with $\hat{x}_e = x_e + 2\delta r_{-e}(x_m - x_e)$ and $\hat{x}_{-e} = x_{-e}$ or with $\hat{x}_e = x_m$ and $\hat{x}_{-e} = x_{-e}$.

Assume $e = 3$. When $e = 1$ the argument is symmetric and omitted. Then we have $\hat{x}_1 = x_1$ and $\hat{x}_3 = \max\{x_2, x_3 + 2\delta r_1(x_2 - x_3)\}$. Denote by σ' profile of strategies induced by $\hat{\mathbf{x}} = \{x_1, x_2, \max\{x_2, x_3 + 2\delta r_1(x_2 - x_3)\}\}$. Trivially, condition $\mathbb{S}$ holds for player 1 and it is easy to see that it fails for player 3. Using similar arguments as in the proof of case 3 of Proposition 11, we have $\mathcal{N}_3(\sigma') = \{\hat{x}_3, d_a(x_1), x_3\}$ with $d_a(x_1) \in (\hat{x}_3, x_3)$. Denote by $\sigma'' = (x_1, x_2, (\hat{x}_3, x_a))$ profile of simple adjusted proposal strategies, with x_a from Definition 10. Note that $d_a(x_1) < x_a$ and $x_a < x_3$. The former because the inequality is equivalent to $d(x_1) - 2d(x_3)(1 - 2\delta r_1) > 0$ and $[d(\hat{x}_3) - d(x_1)]^2 > 0$ when $\delta r_1 \geq \frac{1}{2}$ and $\delta r_1 < \frac{1}{2}$ respectively. The latter follows from failure of $\mathcal{B}_e$. We need to show that σ'' constitutes an SMPE.

Lemma A3. *Suppose $\sigma'' = (x_1, x_2, (\hat{x}_3, x_a))$ where x_a is as in Definition 10 and $\hat{x}_3 = \max\{x_2, x_3 + \delta r_1(x_m - x_3)\}$. Then*

1. *$U_3(x|\sigma'')$ is continuous, $U_3'(x|\sigma'') > 0$ for $\forall x \in (x_2, \hat{x}_3) \cup (d_a(x_1), x_3)$ and $U_3'(x|\sigma'') < 0$ for $\forall x \in (\hat{x}_3, d_a(x_1)) \cup (x_3, \sup\{X\})$;*

2. *$U_2(x|\sigma'')$ is continuous on $X \setminus \{d_b(x_a), d_a(x_a)\}$, $U_2(d_b(x_a)^-|\sigma'') < U_2(d_b(x_a)|\sigma'')$ and for $\forall x \in (\inf\{X\}, x_2) \setminus \{d_b(x_3), d_b(x_a), x_1, d_b(\hat{x}_3)\}$, $U_2'(x|\sigma'') > 0$;*

3. *$U_1(x|\sigma'')$ is continuous on $X \setminus \{d_b(x_a), d_a(x_a)\}$, $U_1(d_b(x_a)^-|\sigma'') < U_1(d_b(x_a)|\sigma'')$, $U_1'(x|\sigma'') > 0$ for $\forall x \in (\inf\{X\}, x_1) \setminus \{d_b(x_3), d_b(x_a)\}$ and $U_1'(x|\sigma'') < 0$ for $\forall x \in (x_1, d_b(\hat{x}_3)) \cup (d_b(\hat{x}_3), x_2)$.*

Proof. We start by deriving x_a given in Definition 10. x_a is implicitly defined by $U_3(\hat{x}_3|\sigma') = U_3(x_a|\sigma')$. It can be found by solving

$$[T_3(x|\sigma')]_{d_a(x_1)^-}^{\hat{x}_3^+} + [T_3(x|\sigma')]_{x_a^-}^{d_a(x_1)^+} = 0 \tag{A46}$$

where $T_3(x|\sigma') = \frac{-2}{1-\delta r_1}\left[\frac{x^2}{2} - (x_3 + 2\delta r_1(x_2 - x_3))x\right]$ for $x \in (\hat{x}_3, d_a(x_1))$ and $T_3(x|\sigma') = -2\left[\frac{x^2}{2} - x_3 x\right]$ for $x \in (d_a(x_1), x_3)$.[44] Carrying out the straightforward algebra gives x_a from Definition 10. By Lemma 2 part 5 we also have $U_2(\hat{x}_3|\sigma') > U_2(x_a|\sigma')$ and by implication $U_1(\hat{x}_3|\sigma') > U_1(x_a|\sigma')$, using a similar argument to the one used to prove Proposition 1.

Next we note $V_i(x|\sigma') = V_i(x|\sigma'')$ and thus $U_i(x|\sigma') = U_i(x|\sigma'')$ for $\forall x \in [d_b(x_a), d_a(x_a)]$ and $\forall i \in \{1, 2, 3\}$. This follows from the fact that σ' and σ'' induce identical proposed policies for any status-quo $x \in [d_b(x_a), d_a(x_a)]$ and that any proposed policy for status-quo $x \in [d_b(x_a), d_a(x_a)]$ falls within the $[d_b(x_a), d_a(x_a)]$ interval.

To establish the claimed continuity properties, that $U_i(x|\sigma'')$ is continuous for $\forall i \in \{1, 2, 3\}$ and $\forall x \in X \setminus \{d_b(x_a), d_a(x_a)\}$ can be shown using similar arguments as in proof of Lemma 2 part 3. For x_a the previous paragraph implies $V_i(d_b(x_a)|\sigma'') = V_i(d_b(x_a)^+|\sigma'')$. What remains is then $V_i(d_b(x_a)^-|\sigma'') < V_i(d_b(x_a)|\sigma'')$ for $i \in \{1, 2\}$ and $V_3(d_b(x_a)^-|\sigma'') = V_3(d_b(x_a)|\sigma'')$. By the symmetry of V_i for $\forall i \in \{1, 2, 3\}$ about x_2, this will imply $V_i(d_a(x_a)|\sigma'') > V_i(d_a(x_a)^+|\sigma'')$ for $i \in \{1, 2\}$ and $V_3(d_a(x_a)|\sigma'') = V_3(d_a(x_a)^+|\sigma'')$. Denote $\mathcal{T}_i(\sigma'') = \sum_{j\in\{1,2\}} r_j[u_i(x_j) + \delta V_i(x_j|\sigma'')]$. Then

$$\begin{aligned} V_3(d_b(x_a)|\sigma'') &= r_3[u_3(\hat{x}_3) + \delta V_3(\hat{x}_3|\sigma'')] + \mathcal{T}_3(\sigma'') \\ &\overset{1}{=} r_3[u_3(\hat{x}_3) + \delta V_3(\hat{x}_3|\sigma')] + \mathcal{T}_3(\sigma'') \\ &\overset{2}{=} r_3[u_3(x_a) + \delta V_3(x_a|\sigma')] + \mathcal{T}_3(\sigma'') \\ &\overset{3}{=} r_3[u_3(x_a) + \delta V_3(x_a|\sigma'')] + \mathcal{T}_3(\sigma'') \\ &\overset{4}{=} \frac{r_3 u_3(x_a) + \mathcal{T}_3(\sigma'')}{1 - \delta r_3} \end{aligned} \tag{A47}$$

where $\overset{1}{=}$ follows from $\hat{x}_3 \in [d_b(x_a), d_a(x_a)]$, $\overset{2}{=}$ follows from definition of x_a, $\overset{3}{=}$ follows from $x_a \in [d_b(x_a), d_a(x_a)]$ and $\overset{4}{=}$ follows from $x_a = d_a(x_a)$ and $V_3(d_a(x_a)|\sigma'') = V_3(d_b(x_a)|\sigma'')$.

[44] The equation is condition $\mathbb{N}$ with the last evaluation point being x_a instead of x_3. x_a can be thought of as being the largest point in $(d_a(x_1), x_3)$ such that $U_3(x|\sigma') \leq U_3(\hat{x}_3|\sigma')$ holds. That x_a is unique follows from $U_3'(x|\sigma') > 0$ on $(d_a(x_1), x_3)$.

Now for any $x \in [d_b(x_3), d_b(x_a))$ we have

$$\begin{aligned} V_3(x|\sigma'') &= r_3[u_3(d_a(x)) + \delta V_3(d_a(x)|\sigma'')] + \mathcal{T}_3(\sigma'') \\ &= \frac{r_3 u_3(d_a(x)) + \mathcal{T}_3(\sigma'')}{1-\delta r_3} \\ V_3(d_b(x_a)^-|\sigma'') &= \frac{r_3 u_3(d_a(d_b(x_a)^-)) + \mathcal{T}_3(\sigma'')}{1-\delta r_3} = V_3(d_b(x_a)|\sigma'') \end{aligned} \tag{A48}$$

by continuity of u_3 and d_a. Similarly for $i \in \{1,2\}$

$$\begin{aligned} V_i(d_b(x_a)|\sigma'') &= r_3[u_i(\hat{x}_3) + \delta V_i(\hat{x}_3|\sigma'')] + \mathcal{T}_i(\sigma'') \\ &\overset{1}{=} r_3[u_i(\hat{x}_3) + \delta V_i(\hat{x}_3|\sigma')] + \mathcal{T}_i(\sigma'') \\ &\overset{2}{>} r_3[u_i(x_a) + \delta V_i(x_a|\sigma')] + \mathcal{T}_i(\sigma'') \\ &\overset{3}{=} r_3[u_i(x_a) + \delta V_i(x_a|\sigma'')] + \mathcal{T}_i(\sigma'') \\ &\overset{4}{=} \frac{r_3 u_i(x_a) + \mathcal{T}_i(\sigma'')}{1-\delta r_3} \end{aligned} \tag{A49}$$

where $\overset{1}{=}$, $\overset{3}{=}$ and $\overset{4}{=}$ follow from similar arguments as above and $\overset{2}{>}$ follows from $U_i(\hat{x}_3|\sigma') > U_i(x_a|\sigma')$ for $i \in \{1,2\}$. Again for any $x \in [d_b(x_3), d_b(x_a))$ and $i \in \{1,2\}$ we have

$$\begin{aligned} V_i(x|\sigma'') &= r_3[u_i(d_a(x)) + \delta V_i(d_a(x)|\sigma'')] + \mathcal{T}_i(\sigma'') \\ &= \frac{r_3 u_i(d_a(x)) + \mathcal{T}_i(\sigma'')}{1-\delta r_3} \\ V_i(d_b(x_a)^-|\sigma'') &= \frac{r_3 u_i(d_a(d_b(x_a)^-)) + \mathcal{T}_i(\sigma'')}{1-\delta r_3} < V_i(d_b(x_a)|\sigma'') \end{aligned} \tag{A50}$$

by continuity of u_i and d_a.

To establish the sign inequalities on $U_i'(x|\sigma'')$, for $x \in [d_b(x_a), d_a(x_a)]$ and when the derivative exists, we can use (A6). The claim is then immediate from

$$\begin{aligned} r_{nc,a}(x|\sigma'') &= \begin{cases} r_3 & \text{for } \forall x \in (x_2, \hat{x}_3) \\ 0 & \text{for } \forall x \in (\hat{x}_3, d_a(x_1)) \cup (d_a(x_1), x_a) \end{cases} \\ r_{nc,b}(x|\sigma'') &= \begin{cases} r_1 & \text{for } \forall x \in (x_2, \hat{x}_3) \cup (\hat{x}_3, d_a(x_1)) \\ 0 & \text{for } \forall x \in (d_a(x_1), x_a) \end{cases} \end{aligned} \tag{A51}$$

using the symmetry of $r_{nc,a}$ and $r_{nc,b}$ about x_2. For $x \notin [d_b(x_a), d_a(x_a)]$, $U_i'(x|\sigma'')$ can still be computed as in (A6) except when the derivative does not exist, that is except at

$\{d_b(x_3), d_a(x_3)\}$. The claim is again immediate using $r_{nc,a}(x|\sigma'') = r_3$ for $\forall x \in (x_a, x_3)$, $r_{nc,a}(x|\sigma'') = 0$ for $x > x_3$ and $r_{nc,b}(x|\sigma'') = 0$ for $\forall x \in (x_a, x_3) \cup (x_3, \sup\{X\})$. $\square$

From Lemma A3 we know that $\mathcal{A}(x|\sigma'') = [d_b(x), d_a(x)]$. The same lemma implies that for any $x \in X$, the solution to $\max_{z \in \mathcal{A}(x|\sigma'')} U_2(z|\sigma'')$ is x_2. The solution to $\max_{z \in \mathcal{A}(x|\sigma'')} U_1(z|\sigma'')$ is easily seen to be $d_b(x)$ for $\forall x \in [d_b(x_1), d_a(x_1)]$ and x_1 for $\forall x \notin [d_b(x_1), d_a(x_1)]$. The best response of players 1 and 2 to $\sigma'' = (x_1, x_2, (\hat{x}_3, x_a))$ are thus $\hat{x}_1 = x_1$ and $\hat{x}_2 = x_2$ respectively. Again from Lemma A3, the solution to $\max_{z \in \mathcal{A}(x|\sigma'')} U_3(z|\sigma'')$ is $d_a(x)$ for $\forall x \in [x_2, \hat{x}_3] \cup (x_a, x_3)$, $\hat{x}_3$ for $\forall x \in (\hat{x}_3, x_a)$ and x_3 for $x \geq x_3$. At x_a, player 3 is indifferent between proposing $\hat{x}_3$ and x_a as $U_3(\hat{x}_3|\sigma'') = U_3(x_a|\sigma'')$, both of which solve her optimization problem. Her best response to σ'' can thus be described by $\vec{\sigma}_3 = (\hat{x}_3, x_a)$. As a result σ'' constitutes an SMPE.[45] $\square$

A1.18 Proof of Proposition 13

The proposition is an implication of Banks and Duggan (2006b). We present full proof in order to demonstrate dependence of the result on the Euclidean utilities used. The key to the argument is that for any vector of random variables $\vec{z}$ with vector of means $\vec{\mu}_z$ and variances $\vec{\sigma}_z^2$ and for Euclidean utility with bliss point $\vec{x}_i$, $\mathbb{E}[-(\vec{z}-\vec{x}_i)'(\vec{z}-\vec{x}_i)] = -[\iota'\vec{\sigma}_z^2 + (\vec{\mu}_z - \vec{x}_i)'(\vec{\mu}_z - \vec{x}_i)]$, where ι is n' vector of ones. Note also $\frac{\partial}{\partial \vec{x}_i}[-[\iota'\vec{\sigma}_z^2 + (\vec{\mu}_z - \vec{x}_i)'(\vec{\mu}_z - \vec{x}_i)]] = 2(\vec{\mu}_z - \vec{x}_i)$, which is linear in $\vec{x}_i$.

Now fix any profile of pure stationary Markov strategies $\hat{\sigma}$. Consider two policies $\vec{p}_0$ and $\vec{q}_0$ generating stochastic sequence, via $\hat{\sigma}$, of policies $\vec{\mathbf{p}} = \{\vec{p}_0, \vec{p}_1, \ldots\}$ and $\vec{\mathbf{q}} = \{\vec{q}_0, \vec{q}_1, \ldots\}$ respectively. The utility of player i from voting either for $\vec{p}_0$ or $\vec{q}_0$ is

$$\begin{aligned} U_i(\vec{p}_0|\hat{\sigma}) &= \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (\vec{p}_t - \vec{x}_i)'(\vec{p}_t - \vec{x}_i)\right] \\ U_i(\vec{q}_0|\hat{\sigma}) &= \mathbb{E}\left[\sum_{t=0}^{\infty} -\delta^t (\vec{q}_t - \vec{x}_i)'(\vec{q}_t - \vec{x}_i)\right]. \end{aligned} \tag{A52}$$

[45] When $\mathcal{A}_e$ holds and $\mathcal{B}_e$ holds with equality, we are in Proposition 11 part 2. $\mathcal{B}_e$ satisfied with equality means $x_a = x_e$. Algorithm 1 produces $\hat{\mathbf{x}}$ either with $\hat{x}_e = x_m$ and $\hat{x}_{-e} = x_{-e}$ or with $\hat{x}_e = x_e + 2\delta r_{-e}(x_m - x_e)$ and $\hat{x}_{-e} = x_{-e}$. That $\sigma'' = (x_{-e}, x_m, (\hat{x}_e, x_e))$ constitutes an SMPE then follows from a similar argument to the one just presented. The only difference is that, using $e = 3$, (x_a, x_3) interval does not exist and $p_3(x|\hat{x}_3, x_a) = \hat{x}_3$ for $\forall x \in [\hat{x}_3, x_3]$ and $p_3(x|\hat{x}_3, x_a) = x_3$ for $x > x_3$, that is, player 3 switches from proposing $\hat{x}_3$ directly to proposing x_3 at $x_a = x_3$.

Differentiating the difference in utility from the two policies with respect to $\vec{x}_i$ gives

$$\frac{\partial [U_i(\vec{p}_0|\hat{\sigma}) - U_i(\vec{q}_0|\hat{\sigma})]}{\partial \vec{x}_i} = \mathbb{E}\left[2\sum_{t=0}^{\infty} -\delta^t(\vec{q}_t - \vec{p}_t)\right] \tag{A53}$$

which is independent of $\vec{x}_i$ and hence $U_i(\vec{p}_0|\hat{\sigma}) - U_i(\vec{q}_0|\hat{\sigma})$ is linear in $\vec{x}_i$. As a consequence, for any pair of players $\{i, i^r\}$ for $\forall i \in N \backslash \{m\}$, which exists by radial symmetry, there exists at least one player $i' \in \{i, i^r\}$, such that $U_m(\vec{p}_0|\hat{\sigma}) \geq U_m(\vec{q}_0|\hat{\sigma})$ implies $U_{i'}(\vec{p}_0|\hat{\sigma}) \geq U_{i'}(\vec{q}_0|\hat{\sigma})$ and $U_m(\vec{p}_0|\hat{\sigma}) < U_m(\vec{q}_0|\hat{\sigma})$ implies $U_{i'}(\vec{p}_0|\hat{\sigma}) < U_{i'}(\vec{q}_0|\hat{\sigma})$.

Now assume $U_m(\vec{p}_0|\hat{\sigma}) \geq U_m(\vec{q}_0|\hat{\sigma})$. Then by the argument just made, there are at least $\frac{n+1}{2}$ players with $U_i(\vec{p}_0|\hat{\sigma}) \geq U_i(\vec{q}_0|\hat{\sigma})$ and $\vec{p}_0$ is accepted. Conversely, if $U_m(\vec{p}_0|\hat{\sigma}) < U_m(\vec{q}_0|\hat{\sigma})$, then there are at least $\frac{n+1}{2}$ players with $U_i(\vec{p}_0|\hat{\sigma}) < U_i(\vec{q}_0|\hat{\sigma})$ and $\vec{q}_0$ is rejected. This implies that $\vec{p}_0$ is accepted if and only if $U_m(\vec{p}_0|\hat{\sigma}) \geq U_m(\vec{q}_0|\hat{\sigma})$, that is, when the median player (weakly) prefers $\vec{p}_0$ to $\vec{q}_0$. □

A1.19 Proof of Lemma 8

To see part 1, for $\forall \vec{x} \in X$ and $\forall \vec{y} \in X$ with $||\vec{x}|| = ||\vec{y}||$, we have $\vec{p}_i(\vec{x}|\hat{k}_i) = \vec{p}_i(\vec{y}|\hat{k}_i)$ for $\forall i \in N$ and any $\hat{k}_i \geq 0$. Because

$$V_i(\vec{x}|\sigma) = \sum_{j\in N} r_j \left[u_i(\vec{p}_j(\vec{x}|\hat{k}_j)) + \delta V_i(\vec{p}_j(\vec{x}|\hat{k}_j)|\sigma)\right] \tag{A54}$$

where σ is induced by $\hat{\mathbf{k}}$, $V_i(\vec{x}|\sigma) = V_i(\vec{y}|\sigma)$ for $\forall i \in N$ follows.[46]

For part 2, $U_i(\vec{x}|\sigma) = u_i(\vec{x}) + \delta V_i(\vec{x}|\sigma)$ for any $\vec{x} \in X$. Because $V_i(\vec{x}|\sigma)$ is constant on any hypersphere in X by part 1 and since (strict) maximizer of $u_i(\vec{x})$ on any hypersphere in X lies on i-ray when $i \in N \setminus \{m\}$, we have $U_i(k\vec{x}_i|\sigma) > U_i(\vec{y}|\sigma)$ for any $\vec{y} \in X$ such that $k||\vec{x}_i|| = ||\vec{y}||$ but $k\vec{x}_i \neq \vec{y}$.

For part 3, fix $\hat{\mathbf{k}}$ with $\hat{k}_i \geq 0$ for $\forall i \in N \setminus \{m\}$ and $\hat{k}_m = 0$ and the induced profile of strategies σ. Proving that $U_i(\vec{x}|\sigma) = u_i(\vec{x}) + \delta V_i(\vec{x}|\sigma)$ is continuous on X is equivalent to proving that $V_i(k\vec{x}_i|\sigma)$ is continuous in k on $[0, \infty)$. From $||\vec{p}_i(\vec{x}|\hat{k}_i)|| = ||\vec{x}||$ for $\forall \vec{x} \in \{\vec{x} \in X | \, ||\vec{x}|| \in \mathcal{D}(\sigma)\}$ and $\forall i \in \mathcal{NC}(||\vec{x}|| \, |\sigma)$, combined with part 1, we can rewrite (A54) for

[46] We can use (A54) since, when $\hat{k}_i \geq 0$ for $\forall i \in N$, any proposal generated by the simple proposal strategy $\vec{p}_i$ of any $i \in N$ is always accepted, which in turn follows from the properties of the social acceptance correspondence $\mathcal{A}$ proved in part 6. For now, we conjecture that part 6 holds and then confirm it is the case.

$\forall k||\vec{x}_i|| \in \mathcal{D}(\sigma)$

$$V_i(k\vec{x}_i|\sigma) = \frac{\begin{array}{c}\sum_{j\in N} r_j u_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)) \\ +\delta \sum_{j\in\mathcal{C}(k||\vec{x}_i||\,|\sigma)} r_j V_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)|\sigma)\end{array}}{1-\delta r_{nc}(k||\vec{x}_i||\,|\sigma)} \tag{A55}$$

which is continuous in k, for $\forall i \in N$, by continuity of $\vec{p}_j(k\vec{x}_i|\hat{k}_j)$ for $\forall j \in N$, constancy of $\vec{p}_j(k\vec{x}_i|\hat{k}_j)$ for $\forall j \in \mathcal{C}(k||\vec{x}_i||\,|\sigma)$ and by local, that is on any interval induced by $\mathcal{ND}(\sigma)$, constancy of $\mathcal{C}(k||\vec{x}_i||\,|\sigma)$ and $r_{nc}(k||\vec{x}_i||\,|\sigma)$.

What remains is, for $\forall i \in N$, $V_i(\vec{x}_i k^-|\sigma) = V_i(k\vec{x}_i|\sigma) = V_i(\vec{x}_i k^+|\sigma)$ for any $k \geq 0$ such that $k||\vec{x}_i|| \in \mathcal{ND}(\sigma)$ (the first equality not at $k=0$).[47] For $k=0$ we have $\vec{p}_j(\vec{x}_i 0^+|\hat{k}_j) = \vec{x}_m$ for $\forall j \in N$ so that $V_i(\vec{x}_i 0^+|\sigma) = \frac{u_i(\vec{x}_m)}{1-\delta} = V_i(0\vec{x}_i|\sigma)$.

For k such that $k||\vec{x}_i|| \in \mathcal{ND}(\sigma)\setminus\{0\}$, we first notice that $\vec{p}_j(\vec{x}_i k^-|\hat{k}_j) = \vec{p}_j(k\vec{x}_i|\hat{k}_j) = \vec{p}_j(\vec{x}_i k^+|\hat{k}_j)$ for $\forall j \in N$ and any $k>0$ so that the first sum in the numerator of (A55) is continuous in k. Now use, for any $k>0$ such that $k||\vec{x}_i|| \in \mathcal{ND}(\sigma)$, i) $V_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)|\sigma) = V_i(\vec{p}_j(\vec{x}_i k^+|\hat{k}_j)|\sigma) = V_i(\vec{x}_i k^-|\sigma)$ for $\forall j \in \mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)$ (players who switch from non-constant to constant part of their strategy at $k||\vec{x}_i||$ distance), ii) $\mathcal{C}(||\vec{x}_i||k^-|\sigma)\cap\mathcal{C}(||\vec{x}_i||k^+|\sigma) = \mathcal{C}(||\vec{x}_i||k^-|\sigma)$ (players switch to proposing constant policy at $k||\vec{x}_i||$), iii) $r_{nc}(||\vec{x}_i||k^-|\sigma) = r_{nc}(||\vec{x}_i||k^+|\sigma) + \sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j$ and iv) $V_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)|\sigma) = V_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)|\sigma) = V_i(\vec{p}_j(\vec{x}_i k^+|\hat{k}_j)|\sigma)$ for $\forall j \in \mathcal{C}(||\vec{x}_i||k^-|\sigma)\cap\mathcal{C}(||\vec{x}_i||k^+|\sigma)$ (players that propose constant policy in the neighbourhood, below and above, of $k||\vec{x}_i||$) to rewrite (A55), for any $i \in N$,

$$\begin{aligned} V_i(\vec{x}_i k^+|\sigma) &= \frac{\begin{array}{c}\sum_{j\in N} r_j u_i(\vec{p}_j(\vec{x}_i k^+|\hat{k}_j)) \\ +\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)} r_j V_i(\vec{p}_j(\vec{x}_i k^+|\hat{k}_j)|\sigma)\end{array}}{1-\delta r_{nc}(||\vec{x}_i||k^+|\sigma)} \\ &= \frac{\begin{array}{c}\sum_{j\in N} r_j u_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)) \\ +\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j V_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)|\sigma) \\ +\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j V_i(\vec{x}_i k^-|\sigma)\end{array}}{1-\delta r_{nc}(||\vec{x}_i||k^-|\sigma)+\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j} \\ &= \frac{\begin{array}{c}V_i(\vec{x}_i k^-|\sigma)(1-\delta r_{nc}(||\vec{x}_i||k^-|\sigma) \\ +V_i(\vec{x}_i k^-|\sigma)\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j\end{array}}{1-\delta r_{nc}(||\vec{x}_i||k^-|\sigma)+\delta\sum_{j\in\mathcal{C}(||\vec{x}_i||k^+|\sigma)\setminus\mathcal{C}(||\vec{x}_i||k^-|\sigma)} r_j} \\ &= V_i(\vec{x}_i k^-|\sigma). \end{aligned} \tag{A56}$$

[47] $V_i(\vec{x}_i k^-|\sigma)$ and $V_i(\vec{x}_i k^+|\sigma)$ denote one-sided limits along the i-ray approaching $||\vec{x}_i||k$ distance from origin from below and above respectively.

To prove $V_i(k\vec{x}_i|\sigma) = V_i(\vec{x}_i k^-|\sigma)$, we have, from $V_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)|\sigma) = V_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)|\sigma)$ for $\forall j \in \mathcal{C}(||\vec{x}_i||k^-|\sigma)$ and $V_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)|\sigma) = V_i(k\vec{x}_i|\sigma)$ for $\forall j \in \mathcal{NC}(||\vec{x}_i||k^-|\sigma)$,

$$
\begin{aligned}
V_i(k\vec{x}_i|\sigma) &= \sum_{j\in N} r_j \left[u_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)) + \delta V_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)|\sigma)\right] \\
&= \sum_{j\in N} r_j u_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)) + \delta \sum_{j\in\mathcal{C}(||\vec{x}_i||k^-|\sigma)} V_i(\vec{p}_j(\vec{x}_i k^-|\hat{k}_j)|\sigma) \\
&+ \delta r_{nc}(||\vec{x}_i||k^-|\sigma) V_i(k\vec{x}_i|\sigma) \\
&= V_i(\vec{x}_i k^-|\sigma)(1 - \delta r_{nc}(||\vec{x}_i||k^-|\sigma)) \\
&+ \delta r_{nc}(||\vec{x}_i||k^-|\sigma) V_i(k\vec{x}_i|\sigma)
\end{aligned}
\tag{A57}
$$

and the claim, for any $i \in N$, follows.

To prove part 4, $\frac{\partial^2}{\partial^2 k}[U_i(k\vec{x}_i|\sigma)] < 0$ for $k \geq 0$ such that $k||\vec{x}_i|| \in \mathcal{D}(\sigma)$ for $\forall i \in N$, we first show the result for $\forall i \in N \setminus \{m\}$. Note that, for any $j \in \mathcal{NC}(k||\vec{x}_i||\,|\sigma)$,

$$
u_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)) = -k^2||\vec{x}_i||^2 + 2k||\vec{x}_i||^2 \frac{\vec{x}_j'\vec{x}_i}{||\vec{x}_j|| \cdot ||\vec{x}_i||} - \vec{x}_i'\vec{x}_i \tag{A58}
$$

and hence $\frac{\partial}{\partial k} u_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)) = -2||\vec{x}_i||^2(k - \cos(i,j))$ and $\frac{\partial^2}{\partial^2 k} u_i(\vec{p}_j(k\vec{x}_i|\hat{k}_j)) = -2||\vec{x}_i||^2$. Using (A55), along with the fact that $\vec{p}_j(k\vec{x}_i|\hat{k}_j)$ is constant in k for $\forall j \in \mathcal{C}(k||\vec{x}_i||\,|\sigma)$ and that both $\mathcal{C}(k||\vec{x}_i||\,|\sigma)$ and $r_{nc}(k||\vec{x}_i||\,|\sigma)$ are both locally, on any interval induced by $\mathcal{ND}(\sigma)$, constant, we have

$$
\frac{\partial U_i(k\vec{x}_i|\sigma)}{\partial k} = \frac{2||\vec{x}_i||^2}{1 - \delta r_{nc}(k||\vec{x}_i||\,|\sigma)} \left[1 - k - \delta \sum_{j\in\mathcal{NC}(k||\vec{x}_i||\,|\sigma)} r_j[1 - \cos(i,j)]\right] \tag{A59}
$$

for $\forall i \in N \setminus \{m\}$. The desired result now follows easily. For m it follows from proof of part 5.

For part 5, we need to show that, along arbitrary z-ray, $\frac{\partial}{\partial k} U_m(k\vec{x}_z|\sigma) < 0$ for $k \geq 0$ such that $k||\vec{x}_z|| \in \mathcal{D}(\sigma)$. From $\frac{\partial}{\partial k} u_m(\vec{p}_j(k\vec{x}_z|\hat{k}_j)) = -2k||\vec{x}_z||^2$ for any $j \in \mathcal{NC}(k||\vec{x}_z||\,|\sigma)$, we have

$$
\frac{\partial U_m(k\vec{x}_z|\sigma)}{\partial k} = -\frac{2k||\vec{x}_z||^2}{1 - \delta r_{nc}(k||\vec{x}_z||\,|\sigma)} \tag{A60}
$$

and the claim, using continuity of U_m from part 3, follows. Part 6 is then direct consequence of part 5 and of Proposition 13. □

A1.20 Proof of Proposition 14

From Definition 3 of SMPE, the profile of strategies $\hat{\sigma}$ constitutes an SMPE, by the one-stage-deviation principle, if $\hat{\sigma}$ induces $U_i(\hat{\sigma})$ for $\forall i \in N$ and $\mathcal{A}(\hat{\sigma})$ such that the set of optimal proposal strategies, arising from maximization of $U_i(\hat{\sigma})$ on $\mathcal{A}(\hat{\sigma})$ for any given status-quo, includes $\hat{\sigma}$.

Fix the profile of strategic bliss points $\hat{\mathbf{k}}$ from Algorithm 2 and the induced profile of strategies σ. Clearly, the voting strategies subsumed in σ are optimal for every player. Because $\hat{\mathbf{k}}$ satisfies $\hat{k}_i \geq 0$ for $\forall i \in N \setminus \{m\}$ and $\hat{k}_m = 0$, by Lemma 8, $\vec{p}_i(\vec{x}|\hat{k}_i) \in \mathcal{A}(\vec{x}|\sigma)$ for $\forall \vec{x} \in X$ and $\forall i \in N$. That is, proposals with zero probability of acceptance are never made. Also, for m we have $\hat{k}_m = 0$, hence the proposal strategy of the median player is optimal by Lemma 8 part 5.

Now let us focus on $i \in N \setminus \{m\}$. By Lemma 8 part 2, policy maximizing dynamic utility U_i of player i, for any status-quo $\vec{x} \in X$, lies on the i-ray. Using the shape of $\mathcal{A}$ from Lemma 8 part 6, we need to make sure that proposing $\frac{||\vec{x}||}{||\vec{x}_i||}\vec{x}_i$ for any $\vec{x} \in X$ with $\frac{||\vec{x}||}{||\vec{x}_i||} \in [0, \hat{k}_i]$ and $\hat{k}_i\vec{x}_i$ otherwise is optimal for i. U_i making this proposal strategy optimal has to satisfy $U_i(k\vec{x}_i|\sigma) \leq U_i(l\vec{x}_i|\sigma)$ for any $k \in [0, \hat{k}_i]$ and $l \in [0, \hat{k}_i]$ such that $k < l$ and $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k > \hat{k}_i$. The first inequality follows from the way Algorithm 2 constructs the strategic bliss points; it generates $\hat{\mathbf{k}}$ such that, denoting the derivative of $U_i(k\vec{x}_i|\sigma)$ with respect to k by $U'_i(k\vec{x}_i|\sigma)$, $U'_i(\vec{x}_i\hat{k}_i^-|\sigma) = 0$ and $U'_i(\vec{x}_i\hat{k}_j^-|\sigma) \geq 0$ for any j such that $\hat{k}_j||\vec{x}_j|| \in [0, \hat{k}_i||\vec{x}_i||)$, which, combined with the piecewise strict concavity of U_i, shows the claim. To ensure the second inequality, notice that from (A59) we have $U'_i(k\vec{x}_i|\sigma) \leq 0$ for any $k \geq 1$ such that $k||\vec{x}_i|| \in \mathcal{D}(\sigma)$, so that $U_i(\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k > 1$. Hence we need to make sure that $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k \in [\hat{k}_i, 1]$ in order for σ to constitute an SMPE.

To prove that condition $\mathbb{S}'$ is sufficient, part 1, we have $U'_i(\vec{x}_i\hat{k}_i^+|\sigma) \leq 0$. When $\hat{k}_i = 0$ Algorithm 2 drops i because $U'_i(\vec{x}_i\hat{k}_i^+|\sigma) \leq 0$. When $\hat{k}_i > 0$ Algorithm 2 drops i because $U_i(\vec{x}_i\hat{k}_i^-|\sigma) = 0$ and we have $U_i(\vec{x}_i\hat{k}_i^-|\sigma) = U_i(\vec{x}_i\hat{k}_i^+|\sigma)$ from (A59), the fact that exactly one player is dropped in any step of the algorithm and from $1 - \cos(i, i) = 0$. Hence, by the piecewise strict concavity of U_i, we need to ensure that $U'_i(\vec{x}_ik^+|\sigma) \leq 0$ for $\forall k||\vec{x}_i|| \in \mathcal{ND}(\sigma) \cap (\hat{k}_i||\vec{x}_i||, ||\vec{x}_i||)$ or, equivalently, $\forall k \in \mathcal{ND}_i(\sigma) \cap (\hat{k}_i, 1) = \mathcal{S}_i(\sigma)$. Using (A59) this condition becomes $1 - k - \delta\sum_{j\in\mathcal{NC}_i(k^+|\sigma)} r_j[1 - \cos(i, j)] \leq 0$ (we have used $\mathcal{NC}_i(x) = \mathcal{NC}(x||\vec{x}_i||)$ in the expression), which is what the condition $\mathbb{S}'$ requires. Hence if $\mathbb{S}'$ holds, we have $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k \in [\hat{k}_i, 1]$ and σ constitutes an SMPE.

To prove that condition $\mathbb{N}'$ is necessary and sufficient, part 2, we note that $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k \in [\hat{k}_i, 1]$ is equivalent to $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(k\vec{x}_i|\sigma)$ for any $k \in ((\mathcal{ND}_i(\sigma)\cup$

$\mathcal{L}_i(\sigma)) \cap (\hat{k}_i, 1)) \cup \{\hat{k}_i, 1\} = \mathcal{N}_i(\sigma)$. To see this, take two adjacent elements of $\mathcal{ND}_i(\sigma)$ from $[\hat{k}_i, 1]$, k' and k'', with $k' < k''$. If U_i has no local maximum on $[k', k'']$, that is when $[k', k''] \cap \mathcal{L}_i(\sigma) = \emptyset$, then $U_i(k'\vec{x}_i|\sigma) > U_i(k''\vec{x}_i|\sigma) \Leftrightarrow U_i(k'\vec{x}_i|\sigma) > U_i(y\vec{x}_i|\sigma)$ and $U_i(k'\vec{x}_i|\sigma) < U_i(k''\vec{x}_i|\sigma) \Leftrightarrow U_i(k'\vec{x}_i|\sigma) < U_i(y\vec{x}_i|\sigma)$ for any $y \in [k', k'']$ (equality cannot occur by the strict concavity of U_i). If U_i has local maximum on $[k', k'']$ then exactly one and we can set $k''' = [k', k''] \cap \mathcal{L}_i(\sigma)$ and proceed with similar argument using k''' instead of k''.

To show that $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(y\vec{x}_i|\sigma)$ for any $y \in \mathcal{N}_i(\sigma)$ is equivalent to $\mathbb{N}'$, for any differentiable continuous function f, $f(x) - f(z) = [\int f'(a)da]_z^x$. When f is not differentiable at x, y, z with $x < y < z$ but possesses one-sided derivatives at x, y, z, we have $f(x) - f(z) = [\int f'(a)da]_{y^-}^{x^+} + [\int f'(a)da]_{z^-}^{y^+}$. Now, (A59) can be rewritten as $U_i'(k\vec{x}_i|\sigma) = \frac{-2||\vec{x}_i||^2}{1-\delta \sum_{j \in \mathcal{NC}_i(k|\sigma)} r_j}[k - c_i(k|\sigma)]$ where $c_i(k|\sigma) = 1 - \delta \sum_{j \in \mathcal{NC}_i(k|\sigma)} r_j[1 - \cos(i, j)]$. Hence $\int U_i'(k\vec{x}_i|\sigma)dk = T_i(k|\sigma) = \frac{-2||\vec{x}_i||^2}{1-\delta \sum_{j \in \mathcal{NC}_i(k|\sigma)} r_j}\left[\frac{k^2}{2} - c_i(k|\sigma)k\right]$ as $\mathcal{NC}_i(k|\sigma)$ is constant on any interval induced by $\mathcal{ND}_i(\sigma)$. Condition $\mathbb{N}'$ then takes into account that $\mathcal{N}_i(\sigma)$ can have an arbitrary number of elements. When $\mathbb{N}'$ holds, we have $U_i(\hat{k}_i\vec{x}_i|\sigma) \geq U_i(y\vec{x}_i|\sigma)$ for any $y \in [\hat{k}_i, 1]$ and σ constitutes an SMPE. When $\mathbb{N}'$ fails, we have $U_i(\hat{k}_i\vec{x}_i|\sigma) < U_i(y\vec{x}_i|\sigma)$ for some $y \in [\hat{k}_i, 1]$ and σ cannot constitute an SMPE, as i would prefer to deviate to proposing $y\vec{x}_i$ when the status-quo is $y\vec{x}_i$, as opposed to proposing $\hat{k}_i\vec{x}_i$ that σ requires. □

A1.21 Proof of Proposition 15

First index players such that $m = n$ and $i^r = i + \frac{n-1}{2}$ modulo $n - 1$ for $\forall i \in N \setminus \{m\}$ so that $N = \{1, 2, \ldots, \frac{n-1}{2}, 1^r, 2^r, \ldots, \frac{n-1}{2}^r, m\}$. We denote the first and the second half of the non-median players by $H_1 = \{1, 2, \ldots, \frac{n-1}{2}\}$ and $H_2 = \{1^r, 2^r, \ldots, \frac{n-1}{2}^r\}$ respectively. We claim that Algorithm 2 selects players to drop such that it drops all the players from H_1 in steps $\{1, \ldots, \frac{n-1}{2}\}$ and then all the players from H_2 in steps $\{\frac{n-1}{2} + 1, \ldots, n - 1\}$. To show that the claim is true, we will show that when the algorithm, in a generic step, still includes i, i^r and j but not j^r, then j cannot be dropped. Assume $i \in \mathbb{P}_t$, $j \in \mathbb{P}_t$, $i^r \in \mathbb{P}_t$ and $j^r \notin \mathbb{P}_t$ in step t of Algorithm 2. We need to compare $1 - \delta \sum_{s \in \mathbb{P}_t} r_s[1 - \cos(j, s)]$ with $1 - \delta \sum_{s \in \mathbb{P}_t} r_s[1 - \cos(i, s)] = 1 - \delta \sum_{s \in \mathbb{P}_t} r_s[1 - \cos(i^r, s)]$. For j to be dropped it must the case that $\sum_{s \in \mathbb{P}_t} \cos(j, s) - \cos(i, s) \leq 0$. Now $\cos(j, j) = 1$, $\cos(j, s) = 0$ for $\forall s \in \mathbb{P}_t \setminus \{j\}$, $\cos(i, i) = 1 = -\cos(i, i^r)$ and $\cos(i, s) = 0$ for $\forall s \in \mathbb{P}_t \setminus \{i, i^r\}$ so that the left hand side of the inequality is equal to unity and j cannot be dropped.

To see part 1, when $\delta = 0$ it is obvious. When $\delta \in (0, 1)$, Algorithm 2 gives an option to drop one of $n - 1$ players in step $t = 1$. In step $t = 2$, the option is among $n - 3$ players. The two players not considered are the one dropped in the previous step, i, and

i^r, who can be dropped only at a later step. The algorithm proceeds in this manner until it includes pairs of players $\{j, j^r\}$, until step $t = \frac{n-1}{2}$. In step $t = \frac{n-1}{2} + 1$ the algorithm gives an option to drop one of $\frac{n-1}{2}$ players, in step $t = \frac{n-1}{2} + 2$ one of $\frac{n-3}{2}$ and so on, until the final step. The number of different profiles of strategic bliss points is then $\prod_{i=1}^{\frac{n-1}{2}} (n+1-2i) \prod_{i=1}^{\frac{n-1}{2}} \left(\frac{n+1-2i}{2}\right) = 2^{(n-1)/2} \left(\frac{n-1}{2}!\right)^2$.

For part 2 we need to show that any profile of strategic bliss points from Algorithm 2 satisfies condition $\mathbb{S}'$. Suppose the algorithm, in step t with players in $\mathbb{P}_t$ still in the algorithm, has dropped player i'. Then the strategic bliss point of player i' is $\hat{k}_{i'} = 1 - \delta \sum_{j \in \mathbb{P}_t} r_j [1 - \cos(i', j)]$ and condition $\mathbb{S}'$ reads $1 - \hat{k}_{i'} - \delta \sum_{s \in \mathcal{NC}_i(\hat{k}^+_{i'}|\sigma)} r_s [1 - \cos(s, i)] \leq 0$ for $\forall i \in N \setminus \{\mathbb{P}_t \cup m\}$. Using $\mathcal{NC}_i(\hat{k}^+_{i'}|\sigma) = \mathbb{P}_t \setminus \{i'\}$ and $1 - \cos(i', i') = 0$ the condition rewrites as $\sum_{s \in \mathbb{P}_t \setminus \{i'\}} \cos(s, i) - \cos(s, i') \leq 0$ for $\forall i \in N \setminus \{\mathbb{P}_t \cup m\}$.[48]

To see that the condition holds, $i' \notin \mathbb{P}_t \setminus \{i'\}$ and, since $i \in N \setminus \{\mathbb{P}_t \cup m\}$, $i \notin \mathbb{P}_t \setminus \{i'\}$. Thus $\cos(s, i) \in \{0, -1\}$ and $\cos(s, i') \in \{0, -1\}$ for $\forall s \in \mathbb{P}_t \setminus \{i'\}$. Now suppose $i' \in H_2$. Then $\cos(s, i') = 0$ for $\forall s \in \mathbb{P}_t \setminus \{i'\}$ and condition $\mathbb{S}'$ holds. Now suppose $i \in H_1$ and $i' \in H_1$. Then $\cos(s, i) = -1$ for exactly one $s \in H_2 \subseteq \mathbb{P}_t \setminus \{i'\}$ and $\cos(s, i') = -1$ for exactly one $s \in H_2 \subseteq \mathbb{P}_t \setminus \{i'\}$ and condition $\mathbb{S}'$ holds. Since we do not need to consider the remaining case, $i \in H_2$ and $i' \in H_1$, due to i having been dropped earlier than i', we have just shown that condition $\mathbb{S}'$ holds, for all the previously dropped players, when Algorithm 2 drops player i'. Repeating the argument for any step of the algorithm proves that the profile of strategic bliss points it produces induces σ that constitutes an SMPE.

Part 3, single-peakedness of $U_i(k\vec{x}_i|\sigma)$ in k on $\mathbb{R}_{\geq 0}$, is direct consequence of condition $\mathbb{S}'$ being satisfied for $i \in N \setminus \{m\}$ and of Lemma 8 part 5. □

A1.22 Proof of Proposition 16

Recall that for equiangular $\mathcal{G}$ on a circle we index players such that $\vec{x}_1 = (b, 0)$, $\cos(i, 1) = \cos((i-1)\alpha)$ for $i \in \{1, \ldots, n-1\}$ where $\alpha = \frac{2\pi}{n-1}$, $\vec{x}_i$ are arranged, with increasing i, counter-clockwise on a circle of radius b and $m = n$. With this notation we have $\cos(i, j) = \cos{(i-j)\alpha}$. Without loss of generality we set $b = 1$ as $\mathcal{G}$ is scale invariant. Throughout the proof, we use, without further notice, the well known trigonometric identities $\sin -\theta = -\sin\theta$, $\cos -\theta = \cos\theta$, $\sin 2\theta = 2\sin\theta\cos\theta$, $\sin\theta + \sin\varphi = 2\sin\frac{\theta+\varphi}{2}\cos\frac{\theta-\varphi}{2}$, $\cos\theta - \cos\varphi = -2\sin\frac{\theta+\varphi}{2}\sin\frac{\theta-\varphi}{2}$ and Lagrange's trigonometric identity $\sum_{i=1}^{n} \cos n\theta = -\frac{1}{2} + \frac{1}{2}\csc\frac{\theta}{2}\sin{(n+\frac{1}{2})\theta}$.

[48] This is not fully precise as we are still in step t of the algorithm so σ is not yet fully specified. It is obvious this is purely a matter of exposition; we can finish specification of σ and then look at i' dropped in step t and players dropped before i', $N \setminus \{\mathbb{P}_t \cup m\}$.

To see part 1, when $\delta = 0$ it is obvious. When $\delta \in (0, 1)$, we claim Algorithm 2 gives an option to drop one of $n-1$ players in step 1 and gives an option to drop one of two players in any of the remaining steps $t \in \{2, \ldots, n-2\}$, except for the last one. This produces $2^{(n-3)}(n-1)$ profiles of strategic bliss points. The key to our claim is that, with $\mathbb{P}_t$ players still in the algorithm for $t \in \{2, \ldots, n-2\}$, the option regarding which player to drop is between the pair of players $\{\min \mathbb{P}_t, \max \mathbb{P}_t\}$. This, in any step $t \in \{1, \ldots, n-1\}$ of the algorithm, creates $\mathbb{P}_t$ that is 'convex'; if it includes players i and j with $i \leq j \leq n-1$, then it also includes all the players $\{i, \ldots, j\}$.

Consider general step t of the algorithm with the set of players still considered $\mathbb{P}_t$ and denote $j' = \min \mathbb{P}_t$ and $j'' = \max \mathbb{P}_t$. Suppose $\mathbb{P}_t = \{j', \ldots, j''\}$ where $1 \leq j' \leq j'' \leq n-1$. We need to show the algorithm drops player j' or j''. The player to drop in step t will be the player with the smallest $\hat{k}_{i,t}$ where

$$\begin{aligned} \hat{k}_{i,t} &= 1 - \tfrac{\delta}{n} \sum_{j \in \{j', \ldots, j''\}} 1 - \cos{(i-j)\alpha} \\ &= 1 - \tfrac{\delta}{n}(j'' + 1 - j') \\ &+ \tfrac{\delta}{n} \csc \tfrac{\alpha}{2} \left[\sin\left((j'' + 1 - j')\tfrac{\alpha}{2}\right) \cos\left((j'' + j' - 2i)\tfrac{\alpha}{2}\right)\right] \end{aligned} \tag{A61}$$

which is minimized for $i = j'$ or $i = j''$, due to $\csc \frac{\alpha}{2} > 0$, $\frac{\alpha}{2}(j'' + 1 - j') \in [\frac{\pi}{n-1}, \pi]$ and $\frac{\alpha}{2}(j'' + j' - 2i) \in [-\pi\frac{n-2}{n-1}, \pi\frac{n-2}{n-1}]$.

For part 2, we need to show that any profile of strategic bliss points from Algorithm 2 satisfies condition $\mathbb{S}'$. Suppose that in step t with $\mathbb{P}_t$ still in the algorithm, $j' = \min \mathbb{P}_t$ is dropped. When $j'' = \max \mathbb{P}_t$ is dropped the argument is symmetric and omitted. The strategic bliss point of j' is

$$\hat{k}_{j'} = 1 - \tfrac{\delta}{n}\left(j'' - j' + \tfrac{1}{2}\right) + \tfrac{\delta}{2n} \csc \tfrac{\alpha}{2} \sin\left(\alpha\left(j'' - j' + \tfrac{1}{2}\right)\right) \tag{A62}$$

and we need to check condition $\mathbb{S}'$ for the players dropped previously, that is for $i \in \{1, \ldots, j'-1\} \cup \{j''+1, \ldots, n-1\}$. Condition $\mathbb{S}'$ reads

$$1 - \hat{k}_{j'} - \tfrac{\delta}{n} \sum_{j \in \mathcal{NC}_i(\hat{k}^+_{j'}|\sigma)} 1 - \cos(i, j) \leq 0 \tag{A63}$$

which, using $\mathcal{NC}_i(\hat{k}^+_{j'}|\sigma) = \{j'+1, \ldots, j''\}$, rewrites as

$$-4sin\left(\tfrac{\alpha}{2}\left(i - j'' - 1\right)\right) \sin\left(\tfrac{\alpha}{2}\left(i - j'\right)\right) \sin\left(\tfrac{\alpha}{2}\left(j'' - j'\right)\right) \leq 0. \tag{A64}$$

To see that the inequality holds, we note $\frac{\alpha}{2}(j''-j') \in [0, \pi\frac{n-2}{n-1}]$, if $i \in \{1, \ldots, j'-1\}$ then $\frac{\alpha}{2}(i-j') \in [-\pi\frac{n-2}{n-1}, -\frac{\pi}{n-1}]$ and $\frac{\alpha}{2}(i-j''-1) \in [-\pi, -\frac{2\pi}{n-1}]$ and if $i \in \{j''+1, \ldots, n-1\}$ then $\frac{\alpha}{2}(i-j') \in [\frac{\pi}{n-1}, \pi\frac{n-2}{n-1}]$ and $\frac{\alpha}{2}(i-j''-1) \in [0, \pi\frac{n-3}{n-1}]$. We have just shown that condition $\mathbb{S}'$ holds, for all the previously dropped players, when Algorithm 2 drops player j'. Repeating the argument for any step of the algorithm proves that the profile of strategic bliss points it produces induces σ that constitutes an SMPE.

Part 3, single-peakedness of $U_i(k\vec{x}_i|\sigma)$ in k on $\mathbb{R}_{\geq 0}$, is direct consequence of condition $\mathbb{S}'$ being satisfied for $i \in N \setminus \{m\}$ and of Lemma 8 part 5.

For part 4, we use expression for $\hat{k}_{j'}$ from (A62). When $\frac{\gamma}{2\pi}$ fraction of players has already been dropped, we have $j'' = n-1$ and $j' = \frac{\gamma}{2\pi}(n-1)$, so that $j''-j' = (n-1)(1-\frac{\gamma}{2\pi})$. Then $\lim_{n\to\infty} \frac{\delta}{n}(j''-j'+\frac{1}{2}) = \delta(1-\frac{\gamma}{2\pi})$, $\lim_{n\to\infty} \frac{\delta}{2n}\csc\frac{\alpha}{2} = \frac{\delta}{2\pi}$ and $\lim_{n\to\infty} \sin\alpha(j''-j'+\frac{1}{2}) = -\sin\gamma$. Combining these expressions we get $\lim_{n\to\infty} \hat{k}_{j'} = 1-\delta+\delta\left[\frac{\gamma-\sin\gamma}{2\pi}\right]$. The expression used to generate Figure 3a is then $\lim_{\delta\to 1}\lim_{n\to\infty} \hat{k}_{j'} = \frac{\gamma-\sin\gamma}{2\pi}$. For Figure 3b, for angle, with horizontal axis, γ fraction 2γ of players has already been dropped and $\frac{2\gamma-\sin 2\gamma}{2\pi} = \frac{\gamma-\sin\gamma\cos\gamma}{\pi}$. □

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